Commitments in Extensive Form Games

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Abstract

We consider an implementation problem in perfect information extensive form games faced by a mediator who is ignorant about to the payoff structure of the game. We study a class of *commitment protocols* in which players are given the opportunity to commit to an action in every given decision node before the game is played, in accordance with a predetermine order. Commitments are public, optional, and irreversible. The order by which players announce their commitments defines a *commitment game*. We ask under which conditions pareto efficiency is implementable, i.e., can be guaranteed as the unique *sub game perfect equilibrium outcome* of the commitment game, independently of the actual payoffs. Our first main result demonstrates a family of *depth first search* commitment protocols for which pareto efficiency is implementable in *every* two-player extensive form game. We further show that in general efficiency is not implementable if the number of players in the game is at least four. Lastly, we consider the class of *quitting games* and show that within this class pareto efficiency is implementable for every number of players.

1 Introduction

A main social puzzle is how a set of participants finds its way to a Pareto-efficient outcome while each participant maximizes his own utility. In his seminal book Schelling [9] introduced several concepts that may serve this purpose. While Shelling emphasizes the existence of multiple equilibria and the concept of focal point, the main ideas and concepts carry out to other contexts

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as well. As noted in Myerson [7] two of the main issues discussed in Schelling's book that led to central developments in game theory one may further exploit, are *credible commitments* and *legitimate authority*. Commitments can be used in order to promise cooperation or threatening against deviation from cooperation by other participants, but must subscribe to sequential rationality. A legitimate authority can be used in order to facilitate moving the society from one equilibrium to another.

Considering a setting of a game, and the potential use of credible commitments obeying sequential rationality, and of legitimate authority in imposing such commitments, one may ask what are the capabilities of the legitimate authority: can it impose behavior? does it know the players' utilities? can it enrich the set of actions, by e.g. communicating with the participants or make monetary transfers? Of central interest is the question of whether by only offering services of imposing voluntary commitments, the legitimate authority can lead to Pareto-efficient outcome even if it does not know the participants' utilities.

Consider an interaction that is modeled as an extensive form game with complete information. Assume that the game tree structure is known also to a mediator (i.e. legitimate authority) but the utility functions (modeled as payoffs at the tree leaves) are unknown to him. The mediator can however provide a voluntary commitments protocol before the actual play starts. In such protocol the participants are approached with asks for voluntary commitments in different nodes of the tree, and in a well-defined order. Namely, the nodes of the game tree are ordered, and at each point after hearing the previous commitments, the corresponding player can choose to commit to one of the actions in that node, or make no commitment. The role of the mediator is to enforce the voluntary commitments made when the actual game is played. Notice that the voluntary commitments stage, followed by the play of the original game subject to the commitments made, defines a new extensive form game[1] where we can require sequential rationality: as from the players' perspectives this is yet again an extensive form game with complete information we will be interested in sub-game perfect equilibrium. A major question expanding on Schelling concepts, manifested in extensive form games, can be now fully formalized: can the mediator lead that way to Pareto-optimal outcome?

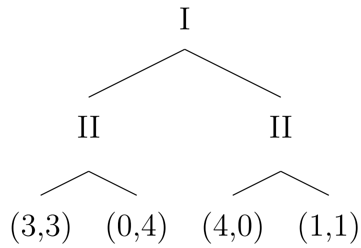
Notice that the above setup is vastly different from approaches such as the ones discussed in

[1]Using Hurwicz [3] terminology the second stage "true game" includes legal and non-legal moves according to the commitments enforced by the "guardian".

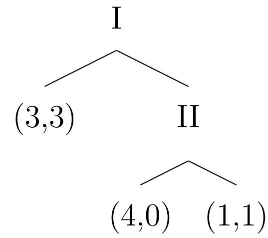
the theory of mechanism design and implementation theory, where in order to implement desired outcomes new games where new actions (such as communication of messages of various forms, monetary transfers, etc.) are added with the aim to overcome the information asymmetry while yielding economic efficiency (as in auction theory [6]) or to overcome the existence of multiple equilibria (as in classical implementation theory for games with complete information [4]). It is also vastly different from the approaches discussed in the literature on conditional commitments implementing threats and promises by mediators in complete information games [5, 10]. The setting discussed above is perhaps the most minimal bridge one may consider in connecting the concepts originated by Schelling of credible commitments and legitimate authority who wishes to lead to desired sequentially rational behavior. Commitments are voluntary on one hand, and are not conditioned on any event; in the same time, the mediator is trustful but does not have the private information held by the players about their utilities.

In the same spirit as this work, Hamilton and Slutsky [1, 2], van Damme and Hurkens [11], and Renou [8] consider unconditional commitment mechanism. In these works players are engaged in a preplay stage that enable them to commit simultaneously to a strategy (or in the case of Renou, a subset of strategies). Similarly to the conditional commitment case, this commitment device may enrich the set of equilibria of the underline game or in some cases generates an effective tool to select among multiple equilibria. In any of these works, however, Pareto efficiency is not guaranteed as a unique outcome of a sub-game perfect equilibrium.

To gain more intuition on our settings of commitments consider the following variant of the prisoner's dilemma.



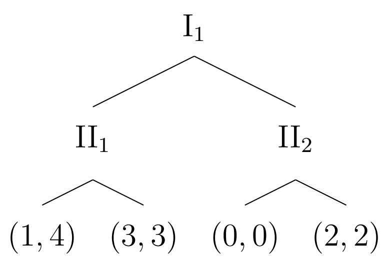
The game is played as follows. Player 1 decides whether to cooperate (play $left$) or to defect (play $right$). Player 2 observes the decision of player 1 and as a function of 1's choice can also cooperate (play $left$) or defect (play $right$). As is well known, this game comprises a unique equilibrium, which is also *subgame perfect equilibrium* where every player defect in each of his decision nodes and the resulting outcome is the Pareto dominated outcome $(1, 1)$. Consider a preplay stage that allows players to commit to an action in some decision nodes prior to the game being played. If player II would commit to play $left$ in his left hand decision node, given that player I did not commit to an action the resulting game tree would be



with $(3,3)$ as a unique equilibrium. Hence the opportunity to commitment may lead to efficiency.

As mentioned above, we consider a general class of commitment protocols that are induced by orders over the decision nodes implemented by a mediator who is ignorant about the payoffs of the game. Every order induces a new game that we shall call *the commitment game.* The commitment game comprises with two stages: *the commitment stage* where players decide sequentially whether to commit to an action, and *the play stage* where players are playing the game subject to the commitments that have been made in the commitment stage. Such family of protocols is natural in the context of extensive form games since it is loyal to the sequential nature of the game. Instead of playing the game in the usual way, top to bottom, we allow player to commit in a predetermined order.

In the above prisoner's dilemma game, in fact, every order over the decision nodes induces a commitment game for which $(3,3)$ is the unique subgame perfect equilibrium outcome.[2] This highlights that the opportunity to commit may induce an efficient outcome. But the above reasoning strongly depends on the payoff structure. Consider for example the following game.



[2]The outcome $(3,3)$ is implementable also using simultaneous commitment, but not as the unique subgame perfect equilibrium. The inefficient outcome $(1,1)$ is also a subgame perfect equilibrium in this case. For further discussion on simultaneous commitments, and an example where *all* equilibria are inefficient see Section 5 Comment 2.

Consider a commitment protocol under which player 1 has the opportunity to commit first to an action and then player 2 may commit to an action first in his right hand side decision node and thereafter in his left hand side decision node. If player 1 chooses not to commit to an action in the first step of the commitment stage then player 2 may force the outcome $(1, 4)$ as a subgame equilibrium outcome in the play stage of the commitment game. To see this note that player 2 can commit to play $left$ in his two decision nodes. Doing this he allows player 1 to choose between two actions in the play stage: $left$ that yields $(1, 4)$ or *right* that yields $(0, 0)$. Since player 1 prefers the *right* option he must choose *right* in the play stage, which yields $(1, 4)$. Knowing the above, player 1 may alternatively commit to *right* in his first decision node and as a result guarantee 2. Hence, the subgame perfect equilibrium outcome of this commitment game is inefficient. This highlights that achieving efficiency as an outcome of the commitment game without knowing the payoff structure is not obvious.

Say that Pareto efficiency is *implementable* for a particular game structure if there exists an order over the decision nodes for which the subgame perfect equilibrium outcome of the commitment game is Pareto efficient *no matter what the payoffs are.* The question that naturally arises is whether for any game structure Pareto efficiency is implementable.

We provide inconclusive answer to this question. For two players games we provide a positive answer. Particularly, we show that the *depth first search order* (henceforth DFS) post-order (a well known order in computer science and algorithms, see Section 2.3) implements Pareto efficiency for every game structure. For the case of four players or more we provide a negative answer. We show an extensive form game structure of four players for which Pareto efficiency is not implementable.[3]

Following our sharp negative result one may ask whether Pareto efficiency is implementable in some classes of (beyond 2-person) extensive form games. We consider the class of *quitting games* where the decision nodes can be ordered in such a way that at every node but the last the corresponding player has two alternatives: either stop the game and force a particular outcome, or to stay and pass the decision to the next node. The two alternative that are available to the last node stop the game and each alternative yields a certain outcome. As an example the well known *centipede game* by Rosenthal is a quitting game. Lastly we show that Pareto efficiency is implementable for the class of quitting games.

[3]The case of three-player games remains an open question.

2 Settings

2.1 Extensive form games

An extensive-form perfect information game $G = (N, T, d, u)$ is a tuple comprises:

- A set of players N.
- A rooted directed tree $T = (V, E)$. For every node v let $A_v = \{w \in V : (v, w) \in E\}$ denotes the *sons* of v. The *terminal nodes* are denoted by $C = \{v \in V : A_v = \emptyset\}$. The non-terminal nodes $D := V \setminus C$ are called *decision nodes.* For every node v we denote by T_v the rooted directed sub-tree (of T) with the root v. The nodes in T_v excluding v itself are called *descendants* of v.
- A labelling of the decision nodes $d : D \to N$. We say that $d(v)$ is *the acting player at node* v. For every player $i \in N$ let $D_i := \{v \in V : d(v) = i\}$ be the decision nodes of i.
- Payoff function $u : C \to \mathbb{R}^{|N|}$.

The first three components (without the payoffs) are called the *game structure.*

A strategy of player i in the game is a choice of a son at each one of his decision nodes (i.e., $s_i = (a_v)_{v \in D_i}$ where $a_v \in A_v$). Every profile of strategies $s = (s_i)_{i \in N}$ induces a unique terminal node $c(s)$. The payoffs of the players for the profile s are given by $u(c(s))$.

A game is called *generic* if for every two terminal nodes $x, y \in C$ if $u_i(x) = u_i(y)$ for some player i then $u(x) = u(y)$. I.e., the set of outcomes is generic, but we allow the same outcome to appear several times. For simplicity, during the paper we focus on *generic* games only. Genericity guarantees uniqueness of subgame perfect equilibrium outcome[4] which will be denoted by $\mathsf{Val}(G)$, and will be called *the value of the game* G.

An outcome $u(x)$ *Pareto dominates* the outcome $u(y)$ if $u_i(x) > u_i(x)$ for every $i \in N$. An outcome $u(y)$ is *Pareto efficient* if there is no outcome $u(x)$ that Pareto dominates $u(y)$.

2.2 Commitment Protocols

In this section we formalise the idea of commitment protocols.

[4]Note that the subgame perfect equilibrium path may not be unique.

Given an extensive form game G and an order over all the decision nodes $\overline{v} = (v_1, v_2, ..., v_n)$ such that $\{v_1, v_2, ..., v_n\} = D$, the *commitment game* $Com(G, \overline{v})$ is the perfect information game where first players have the option to commit to an action at every decision node v_i, according to the order $\overline{v}$, and thereafter the game G is played subject to the commitments that have been made at the first stage. Formally, the game $Com(G, \overline{v})$ has two stages which are described below.

The Commitment Stage

In this stage, each player is asked to commit to an action at a node according to the order $\overline{v}$. Commitment is optional but irreversible. That is, first player $d(v_1)$ may commit to an action at v_1. Then player $d(v_2)$ observes the commitment of player $d(v_1)$ at v_1 and decides whether to commit to an action at v_2, and so forth. Formally, at every time $1 \leq i$ the player $d(v_i)$ may commit to an action $a_{v_i} \in A_{v_i}$ at node v_i, as a function of the commitment history up to time i. Thus the decision available to $d(v_i)$ at time i may be seen as a mapping $\mathbf{a}_i : \prod_{k<i}(A_{v_k} \cup \{\phi_k\}) \rightarrow A_{v_i} \cup \{\phi_i\}$ that assigns an action $a_{v_i} \in A_{v_i}$ or the null action ϕ_i, as a function of the commitment history up to time i.

The Play Stage

Let $l = (l_{v_1}, \ldots, l_{v_n})$ be the vector of commitments from the commitment stage, where for every $1 \leq i \leq n$, $l_{v_i} \in A_{v_i} \cup \{\phi_i\}$. In the play stage the original game tree T is being played with the distinction that if a node $v \in D$ is reached such that $l_v \in A_v$ then l_v is played.

Note that any vector of commitments h defines a new game tree T' which is a subtree of the original game tree T. Essentially in the first stage players, using commitments, determine the game that they are going to play in the play stage.

For a generic game G, the commitment game $Com(G, \overline{v})$ is also a generic extensive form game, and therefore has a value $\mathsf{Val}(Com(G, \overline{v}))$. In cases where it is obvious what is the order $\overline{v}$, we will simply write $\mathsf{Val}(Com(G))$. We now turn to our notion of implementation.

Definition 1. We say that pareto efficiency is *implementable* for the game structure (N, T, d) if there exists an order $\overline{v}$ such that *for every* generic game $G = (N, T, d, u)$ the subgame game perfect equilibrium outcome $\mathsf{Val}(Com(G, \overline{v}))$ is pareto efficient.

Our notion of implementation is compatible with the fact that the mediator is ignorant of the actual payoff u. We therefore require that pareto efficiency should be resulted regardless of the actual payoff function.

We shall next turn to some central notions that will be used in the sequel.

Definition 2. Given an order $\overline{v} = (v_1, ..., v_n)$, a node v_i is called *pre-terminal with respect to* $\overline{v}$ if $i \geq j$ for every $v_j \in T_{v_i}$. Namely, v_i appears in the order after all its descendants. An order $\overline{v}$ is *pre-terminal* if all nodes are pre-terminal with respect to $\overline{v}$.

The following observation will be useful to exclude the option of not committing in some cases:

Lemma 1. *Let G be a game, and let v_i be a pre-terminal node with respect to $\overline{v} = (v_1, \ldots, v_n)$. There exists a subgame prefect equilibrium of the game $Com(G, \overline{v})$ where at node v_i player $d(v_i)$ commits to some action $a_{v_i} \in A_v$ (i.e., does not choose ϕ_i).*

Proof. The idea is that choosing the action ϕ_i at the commitment stage is equivalent to a commitment to the subgame perfect equilibrium action of the subtree with the root v_i. More formally, let T'_{v_i} be the game subtree with the root v_i at the moment player $d(v_i)$ is asked to commit at the node v_i (after taking into account all commitments made up to this point). Since v_i is pre-terminal no additional commitments will be made at T'_{v_i}. Let a_{v_i} be a subgame perfect equilibrium action of the game T'_{v_i} (without commitments). We argue that the action ϕ_i (i.e., not committing to any action) leads to the same subgame perfect equilibrium outcome as the one that is obtained by committing to a_{v_i}. This follows since all nodes in T_{v_i} have already decided if and to what action they are committed to. Therefore, if in the play stage the node v_i will be reached the action a_{v_i} is the action that will be eventually played by $d(v_i)$ at v_i under the subgame perfection assumption. Hence, if in a subgame perfect equilibrium ϕ_i is chosen, then there exists another subgame perfect equilibrium in which a_{v_i} is chosen instead of ϕ_i. $\square$

From Lemma 1 we deduce the following useful remark.

Remark 2.1. In what follow, we will study $\mathsf{Val}(Com(G, \overline{v}))$ as a function of the order orders $\overline{v}$. Since $\mathsf{Val}(Com(G, \overline{v}))$ independent of the sub game perfect equilibrium strategy, we can apply Lemma 1 and assume that players at pre-terminal nodes always commit to an action.

2.3 DFS orders

Definition 3. An order $\overline{v} = (v_1, \ldots, v_n)$ is called a *depth first search* (DFS) order if every node v_i appears in $\overline{v}$ precisely after all of its descendants. Namely, if $w_1, ..., w_m$ are the descendants of v_i, then $\{v_{i-1}, v_{i-2}, ..., v_{i-m}\} = \{w_1, ..., w_m\}$.

DFS order is a fundamental notion in computer science and algorithms (see [Classic book ref for DFS]). There are three types of DFS orders: pre-order, in-order, and post-order. Our notion of DFS is identical to the post-order.

The following properties of a DFS orders are easily verified:

- Every DFS order is a pre-terminal order.
- Every DFS order induced a DFS order over any sub tree T_v of T for every node $v \in D$.
- Once we set an order of the sons $(w^i_1, w^i_2, ..., w^i_{m(i)})$ of every vertex v_i, there exists a unique DFS order that satisfies the property: w^i_j is visited before w^i_l for every i and every $j < l$.

3 Two player games

Our main positive result shows that pareto efficiency is implementable for every game structure of two players. More particularly we will show that in every two player extensive form game every DFS order induces a pareto efficient subgame perfect equilibrium outcome.

Theorem 1. *For every generic two player extensive form game G and every DFS order $\overline{v}$, the unique subgame perfect equilibrium outcome of the commitment game $Com(G, \overline{v})$ is Pareto efficient.*

In particular, Theorem 1 shows that every two player game structure is implementable.

Note that, since *any* DFS order implements pareto efficiency the designer of the protocol may be ignorant, in addition to the payoffs, also with respect to the identity of the players that act at any given node.

Outline of the proof of Theorem 1: Since DFS order is pre-terminal Remark 2.1 allows us to consider an equivalent simpler case where players always commit to some action at every decision node. In such a case the play-stage of the commitment game is redundant, since players already committed to an action at all nodes. This in particular implies that for every node $v \in D$, when v's turn to commit has reached every action of v, along with the previous commitments, defines a unique outcome. Hence at every time of the commitment stage we can identify any action of every node v with a unique outcomes. Therefore, we can view the commitment game as follows: At every node v, player $d(v)$ decides which among the available outcomes would replace the subtree T_v in the remaining game tree.

The proof is by induction on the number of decision nodes. Let v be the root node. Let $w_1, w_2, ..., w_m$ be the non-terminal sons of v. The DFS order goes over all nodes in T_{w_1}, then in T_{w_2},..., then in T_{w_m}, and finally at v. The idea is to identify the commitment game with the following two-steps procedure.

Step 1: Players in T_{w_1} commit to an action. By the above considerations the commitment of all nodes in T_{w_1} defines a unique outcome. This outcome may be viewed as the outcome that "replace" the sub-tree T_{w_1} in the remaining subgame.

Step 2: Players commit on the remaining parts of the tree (i.e., $T_{w_2} \cup ... \cup T_{w_m} \cup \{v\}$).

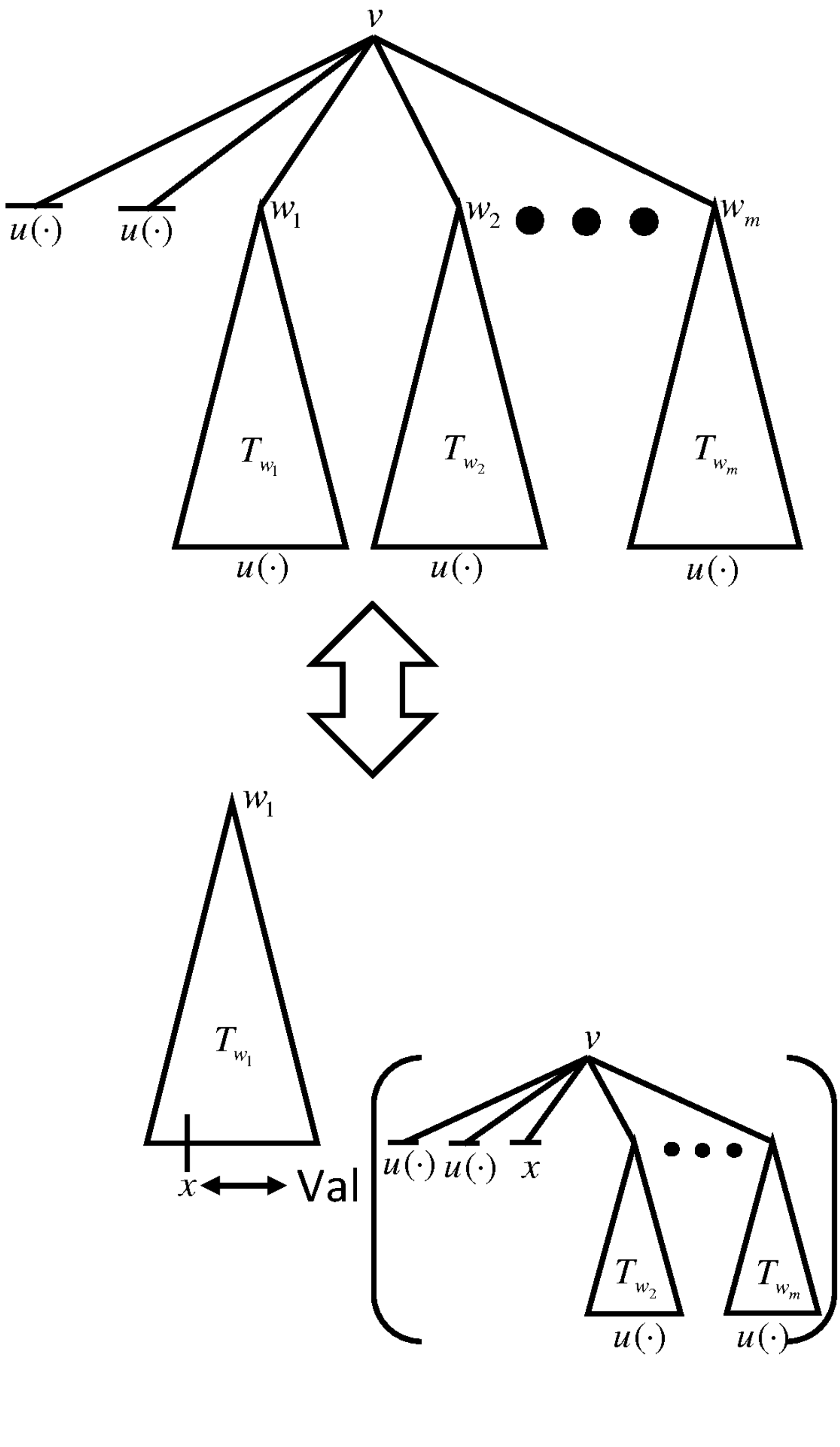
It is obvious that step 2 is a commitment game of smaller size. An interesting observation is that Step 1 can also be viewed as a commitment game over the subtree T_{w_1}, when we replace each outcome x by the resulting SPE outcome of Step 2 where x replace T_{w_1}, see Figure 3. Moreover, the value of G is equal to the value of the new commitment game defined on T_{w_1}.

Assume by way of contradiction that the the value of the game is Pareto dominated by some outcome c. The outcome c cannot appear in $T_{w_2} \cup ... \cup T_{w_m}$ because, as mentioned above, we may view the subgame at step 2 as a smaller size commitment game, and by the induction hypothesis the outcome of this subgame is Pareto efficient. We argue that c cannot appear in the sub-tree T_{w_1} either. In order to prove it, we compare the actual outcome of the game with the outcome that would have occurred in SPE of step 2 if at the end of step 1 the commitment of the players in T_{w_1} led to the outcome c. The following lemma states that in this case the outcome of the second step is better for both players.

Lemma 2. *Let G be a game such that the root v has at least one terminal son $t \in C$. Let $\tilde{G}(c)$ be the game G where we replace the outcome $u(t)$ by the outcome c. Let $\overline{v} = (v_1, ..., v_n)$ be a pre-terminal order. If $c_i > \mathsf{Val}_i(Com(G, \overline{v}))$ for both players $i =$ I,II, then $\mathsf{Val}_i(Com(\tilde{G}(c), \overline{v})) \geq c_i$ for both players $i =$ I,II.*

In simple words, the Lemma states that replacing the outcome at a terminal sun of the root, with an outcome that is better for both players, improves the final outcome of the commitment game for both of them. Counter-intuitively, this lemma is *incorrect* for games with four players.

Figure 1: Division of the commitment game into two steps.



Example 1 demonstrates it, where the original game G has a terminal son with the outcome $(1,1,1,1)$, and we replace it by $(3,3,3,3)$.

To complete the proof, we consider the commitment game that occur at Step 1, and use the induction hypothesis for the sub-tree T_{w_1} with the replaced outcomes.

3.1 Proof of Theorem 1

We start with the proof of Lemma 2. The proof is based on a general lemma regarding general two player extensive form games.

Lemma 3. *Let G be a generic game where at every pre-terminal node the acting player is I. Let $c = (c_I, c_{II})$ be an outcome such that $c_I > \mathsf{Val}_I(G)$. Let G' be any game that is obtained from G when we replace by c exactly one of the terminal outcomes of every pre-terminal node. Then $\mathsf{Val}_i(G') \geq c_i$ for $i = I, II$.*

Proof. We prove the result using induction on the depth of the tree. For a game G of depth one, by assumption, only player I gets to choose and it clearly holds that $\mathsf{Val}(G') = c$. Assume the result holds for all games with a depth smaller than k. Let G be of depth k. We shall consider two cases.

Case 1: Player I is the acting player at the root. Let $w_1, \ldots, w_m$ be the sons of the root, and denote by $G_{w_1}, ..., G_{w_m}$ the corresponding sub-games. Since $\mathsf{Val}_I(G) < c_I$, and since $\mathsf{Val}_I(G) = \max_{1 \leq j \leq m} \mathsf{Val}_I(G_{w_j})$ we must have that $\mathsf{Val}_I(G_{w_j}) < c_1$, for every $1 \leq j \leq m$. Hence we can apply the induction hypothesis to deduce that for every $1 \leq j \leq m$ it holds that $\mathsf{Val}_i(G'_{w_j}) \geq c_i$ for $i = I, II$. And therefore $\mathsf{Val}_i(G') \geq c_i$ for $i = I, II$.

Case 2: Player II is the acting player at the root. As before let $w_1, \ldots, w_m$ be the sons of the root, and $G_{w_1}, ..., G_{w_m}$ the corresponding sub-games. Since $\mathsf{Val}_I(G) < c_I$ we must have l for which $\mathsf{Val}_I(G_{w_l}) < c_I$. Hence by the induction hypothesis $\mathsf{Val}_i(G'_{w_l}) \geq c_i$ for $i = I, II$. Since player II controls the root, we must have that $\mathsf{Val}_{II}(G') \geq c_{II}$. Furthermore, since player I controls all pre-terminal nodes we must have that $\mathsf{Val}_I(G'_{w_j}) \geq c_I$ for every $1 \leq j \leq m$. Hence we must also have that $\mathsf{Val}_I(G') \geq c_I$. This concludes the proof of Lemma 3. $\square$

Remark 2.1 allows us to focus on a simpler equivalent procedure, instead of focusing on the original commitment procedure. We recall that $Com(G)$ denote the original commitment game. We denote by $Act(G)$ the equivalent game (in terms of values) where at each step of the commitment procedure players must commit to an action. After all commitments has been made, the game terminates and the outcome is the outcome determined by the actions that players have committed to.

Remark 3.1. Remark 2.1 states that for every pre-terminal order $\overline{v}$ it holds that $\mathsf{Val}(Com(G,\overline{v})) = \mathsf{Val}(Act(G,\overline{v}))$.

Now, Lemma 2 follows immediately from Lemma 3 and Remark 3.1.

Proof of Lemma 2. By Remark 3.1 the lemma is equivalent to the following statement: If $c_i > \mathsf{Val}_i(Act(G,\overline{v}))$ for both players i =I,II, then $\mathsf{Val}_i(Act(\tilde{G}(c),\overline{v})) \geq c_i$ for both players i =I,II.

We note that $Act(G,\overline{v})$ is also an extensive form game. Consider its game tree. Since player $d(v)$ that controls the root v is the last to make a commitment in $Act(G,\overline{v})$ ($v = v_n$) it follows that in the game tree of $Act(G,\overline{v})$, player $d(v)$ is acting at all pre-terminal nodes. Replacing the outcome $u(t)$ by the outcome c is translated to replacing by c exactly one of the terminal outcomes of every pre-terminal node. Lemma 3 states that indeed the value of the new game (after the replacement) is weakly above c_i for both players i =I,II. □

We recall the notations from the outline of the proof, and present several others to formalize the ideas presented there. The node v is the root node, $\overline{v}$ denotes the DFS order. The non-terminal sons of v are $w_1, w_2, ..., w_m$, where w_1 (and the sub-tree T_{w_1}) is visited before all other sons by the DFS order $\overline{v}$. Recall that the order $\overline{v}$ induces a DFS orders on the nodes of T_{w_1}. In addition, if we let T^{w_1} be the tree that is obtained when we replace T_{w_1} with a terminal node, then $\overline{v}$ also induces a DFS order on T^{w_1}. We denote these two induced orders on T_{w_1} and on T^{w_1} by $\overline{v}_*$ and $\overline{v}^*$ respectively.

We denote by $\tilde{G}(x)$ the game over the tree T^{w_1} where the outcome at the terminal node replacing T_{w_1} is x, and all other outcomes are identical to the outcomes in G. The value of the commitment game on $\tilde{G}(x)$ with the order $\overline{v}^*$ is denoted by $r(x) := \mathsf{Val}(Com(\tilde{G}(x),\overline{v}^*))$.

We denote by $\widehat{G}$ the game on the tree T_{w_1} where we replace every outcome x of the game G with $r(x)$ and denote by $\mathsf{Val}(Com(\tilde{G}(x),\overline{v}^*))$ the value of the commitment game on $\tilde{G}$ with the order $\overline{v}_*$. We claim that

Lemma 4. *$\mathsf{Val}(Com(G,\overline{v})) = \mathsf{Val}(Com(\widehat{G},\overline{v}_*))$*

Proof. By Remark 3.1 it is sufficient to prove $\mathsf{Val}(Act(G, \overline{v})) = \mathsf{Val}(Act(\widehat{G}, \overline{v}_*))$.

Fix a subgame perfect equilibrium of the game $Act(G, \overline{v})$. Consider the commitment game $Act(G, \overline{v})$ up to the point where player $d(w_1)$ has to commit to an action at the node w_1 (this is exactly the point when the order completes all nodes in the tree T_{w_1}). By assumption, any choice of $d(w_1)$ at w_1 determines a unique outcome x. The remaining subgame after the choice of $d(w_1)$ is equivalent to $Act(\tilde{G}(x), \overline{v}^*)$. Therefore, by definition, the resulting subgame perfect equilibrium outcome in $Act(G, \overline{v})$ after the choice of $d(w_1)$ would be $r(x) = \mathsf{Val}(Com(\tilde{G}(x), \overline{v}^*)) = \mathsf{Val}(Act(\tilde{G}(x), \overline{v}^*))$. Hence every outcome x that results from the commitments of the players in T_{w_1} in the game $Act(G, \overline{v})$ induces the outcome $r(x)$. Hence since in $Act(\widehat{G}, \overline{v}_*)$, we replace every outcome x by $r(x)$, players in T_{w_1} have exactly the same incentives as in $Act(G, \overline{v})$, and their choices in $Act(\widehat{G}, \overline{v}_*)$ induces the same subgame perfect equilibrium outcome as in $Act(G, \overline{v})$. □

Proof of Theorem 1. As before, by Remark 3.1 it is sufficient to prove that $\mathsf{Val}(Act(G, \overline{v})$ is Pareto efficient.

The proof is by induction on the number of decision nodes in T. Clearly, for $n = 1$ the result holds, because the acting player maximizes his own payoff. Assume that $\mathsf{Val}(Act(G, \overline{v})$ is Pareto efficient for every game with less than n nodes, we prove it for games with n nodes.

Consider a sub-game perfect equilibrium of the game $Act(G, \overline{v})$, and denote by b the outcome that is chosen in the sub-tree T_{w_1}. We have

$$\mathsf{Val}(Act(\widehat{G}, \overline{v}_*)) = \mathsf{Val}(Act(G, \overline{v})) = \mathsf{Val}(Act(\tilde{G}(b), \overline{v}^*)) = r(b).$$

The first equality follows from Lemma 4, the second is by considering the remaining game after the order visited all nodes in T_{w_1}, and the third is by the definition of $r(\cdot)$.

It is easy to see that both games $\widehat{G}$ and $\tilde{G}(b)$ are games of size strictly smaller than n.

Assume by way of contradiction that $\mathsf{Val}(Act(G, \overline{v})) = r(b)$ is Pareto dominated by an outcome c.

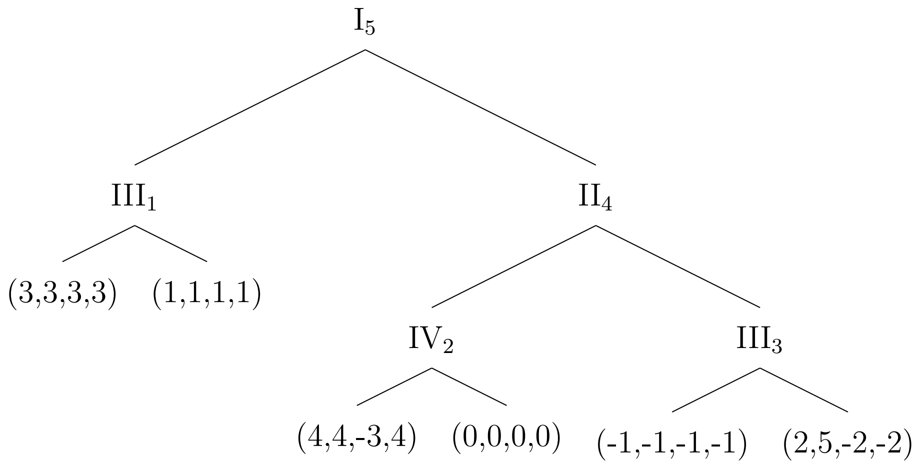
Case 1: The outcome c lies in the tree T_{w_1}. Consider the game $Act(\widehat{G}, \overline{v}_*)$. By Lemma 2 the outcome $r(c)$ is weakly better than c for both players. This is a contradiction to the induction hypothesis applied to the game $\widehat{G}$, because its value is $r(b)$ and there exists a better outcome for both players $r(c)$.

Case 2: The dominating action is not in T_{w_1}, which means that it lies in T^{w_1}. This stands in contradiction to the induction hypothesis applied to the game $\tilde{G}(b)$, because the value is $r(b)$ whereas c is a better outcome for both players. $\square$

4 Multi-Player Games

Theorem 1 shows that any DFS commitment protocol implements a Pareto efficient outcome in every two-player game. The following example demonstrates that the DFS protocol fails in four-player games.

Example 1. The DFS order over the decision vertices appears at the bottom-right side of the players.



(1)

Proposition 1. *The outcome of the commitment protocol in Example 1 is* $(1,1,1,1)$.

Before providing the formal proof we outline the intuition for it. The idea is that player III wants to avoid the outcome $(4,4,-3,4)$. His only way to avoid it is by committing to the outcome $(2,5,-2,-2)$ at the third decision node (because this way III would convince II not to choose $(4,4,-3,4)$).

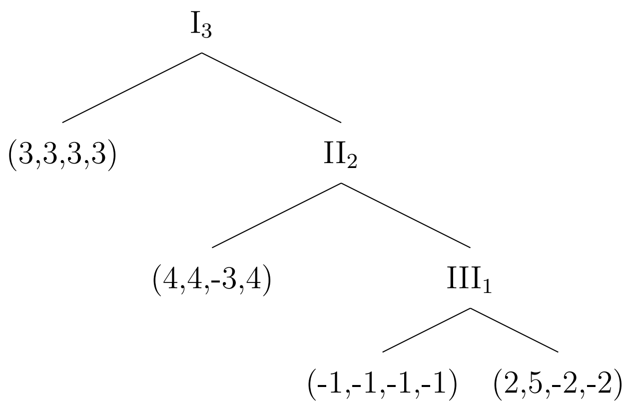
In case III commits to $(1,1,1,1)$ at the first decision node, III has threatening power on IV: "If you (IV) will commit to $(4,4,-3,4)$ then the best option for me (III) would be to commit to $(2,5,-2,-2)$, which also will be chosen as the final outcome, and this is bad for both of us". Therefore IV has to commit to (0,0,0,0), and this eventually leads to the outcome $(1,1,1,1)$.

In case III commits to $(3,3,3,3)$ at the first decision node, he looses his threatening power: IV no longer afraid of $(2,5,-2,-2)$ because it will not be chosen by I. Therefore, IV commits to $(4,4,-3,4)$ and it is chosen as a final outcome.

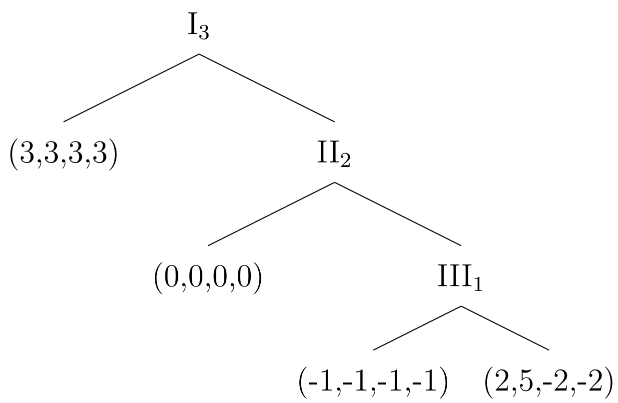
Hence, III prefers to commit to $(1,1,1,1)$ at the first decision node, moreover, eventually it is selected as the final outcome.

Formal proof of Proposition 1. Consider the following four games:

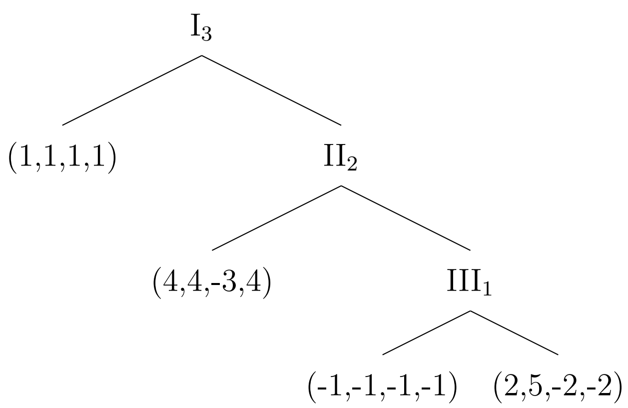
G_1 :



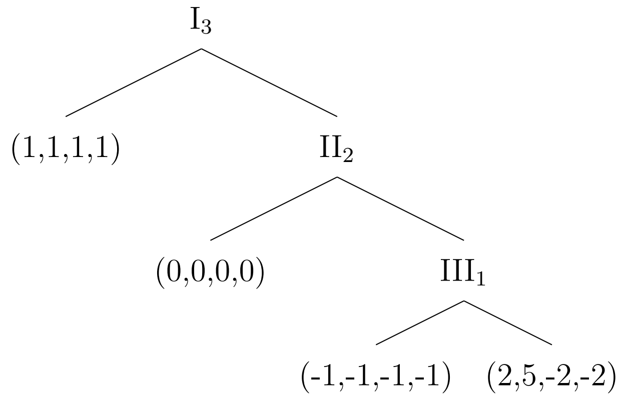
G_2 :



G_3 :



G_4 :



It is easy to see that $\mathsf{Val}(Com(G_1)) = (4, 4, -3, 4)$, $\mathsf{Val}(Com(G_2)) = (3, 3, 3, 3)$, $\mathsf{Val}(Com(G_3)) = (2, 5, -2, -2)$, and $\mathsf{Val}(Com(G_4)) = (1, 1, 1, 1)$. Therefore, a choice of $(3, 3, 3, 3)$ by III, will be proceeded by a choice of $(4, 4, -3, 4)$ by IV (because $\mathsf{Val}_{\mathrm{IV}}(Com(G_1)) > \mathsf{Val}_{\mathrm{IV}}(Com(G_2)))$. A choice of $(1, 1, 1, 1)$ by III, will be proceeded by a choice of $(0, 0, 0, 0)$ by IV (because $\mathsf{Val}_{\mathrm{IV}}(Com(G_4)) > \mathsf{Val}_{\mathrm{IV}}(Com(G_3)))$. When III takes it into account, he prefers to choose $(1, 1, 1, 1)$ and we have $\mathsf{Val}(Com(G)) = \mathsf{Val}(Com(G_4)) = (1, 1, 1, 1)$. $\square$

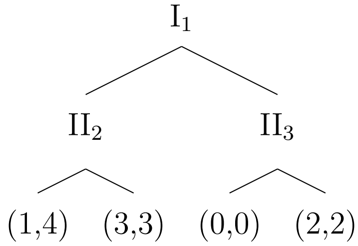
Proposition 1 demonstrates that pareto efficiency is not implementable by any DFS order, as in the two player case, where the number of players is greater than 3. However it doesn't rule out implementation, as there still may exists some order that implements pareto efficiency in these games. The following Theorem excludes such a possibility and shows that pareto efficiency cannot be implementable in every extensive form games with 4 players or more.

We recall that *extensive form game structure* (N, T, d) comprises with all components of a game but for the payoffs (i.e., the set of players N, a game tree T, and a labelling of the decision nodes d) and an order $\overline{v}$ implements pareto efficiency in a game structure (N, T, d) if for every payoff function u the resulting outcome of the commitment game is pareto efficient.

Theorem 2. *There exists a four-player extensive form game structure (N, T, d), such that for every order $\overline{v}$ there exist a game $G = (N, T, d, u)$ for which $\mathsf{Val}(Com(G, \overline{v}))$, the subgame perfect equilibrium outcome of the commitment game, is Pareto dominated.*

Idea of the proof of Theorem 2. The proof is based on two examples. The first is Example 1. For the proof of the theorem we have to extend the set of orders for which Example 1 leads to the inefficient outcome $(1, 1, 1, 1)$. It turns out (see Lemma 6) that $(1, 1, 1, 1)$ is the outcome of the commitment game not only for the DFS order presented in Example 1 but also for much richer set of orders. The second example we shall use is the following example, which was discussed also in the introduction.

Example 2.



Lemma 5. *The outcome of the game in Example 2 is* $(2,2)$.

Proof. If I does not commit at the first step, then II will commit to play $left$ in both nodes (i.e., $(1,4)$ and $(0,0)$), and the outcome will be $(1,4)$. Same outcome is the result of a commitment to $left$ of player I at the first step. On the other hand, if I commits to right at the first step the outcome will be $(2,2)$. Therefore $(2,2)$ is the pareto dominated outcome of this commitment game. $\square$

We say that an order over vertices *admits uncertainty in two sub-trees* if there exists a vertex that appears before two of his descendants in two different sub-trees, or formally, if there exists a triple $i < j < k$ such that $v_j \in T_{w_1}, v_k \in T_{w_2}, w_1 \neq w_2$ and w_1, w_2 are sons of v_i.

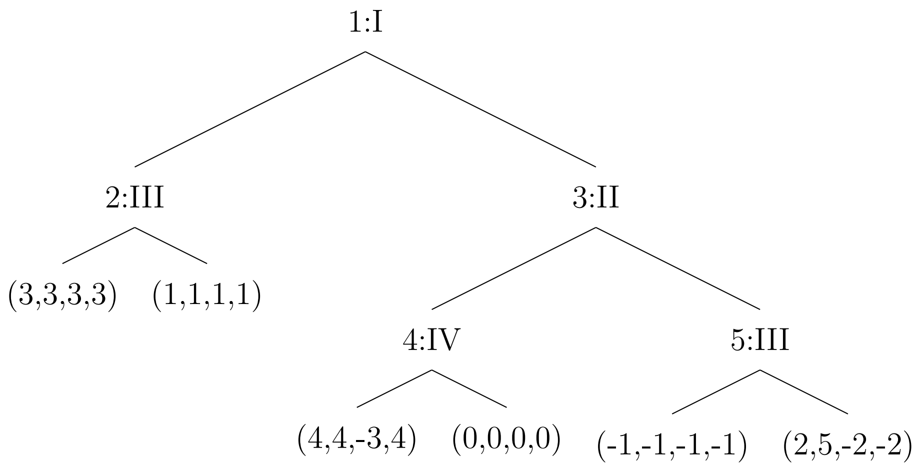
The idea is to use Example 2 to argue that every order that admits uncertainty in two sub-trees fails for some payoffs. This is because we can "plant" the payoffs of Example 2 in the relevant leafs of v_j and v_k, and make all other payoffs irrelevant. This allows us to focus only on orders that do not admit uncertainty in two sub-trees.

Recall that in Example 1 the order in which players commit is (III, IV, III, II, I). Define the game structure to be a complete binary tree of depth five where players I, II, III, IV, III are making the decision in *all* nodes of depth 1,2,3,4,5 respectively. This structure, together with the fact that the order does not admit uncertainty in two sub-trees allows us to focus only on orders that satisfy the following conditions:

- III is asked to commit first (at depth 5).

- In both sub-trees that starts at depth 2, IV is asked to commit at depth 4 before II, III are asked to commit at depths 2,3.

It turns out that these two properties are sufficient for "planting" the payoffs of Example 1 into the large tree (i.e., the depth five tree), such that the induced order on the relevant nodes of the large tree fits the set of orders in Lemma 6 for which Example 1 fails efficiency.



(2)

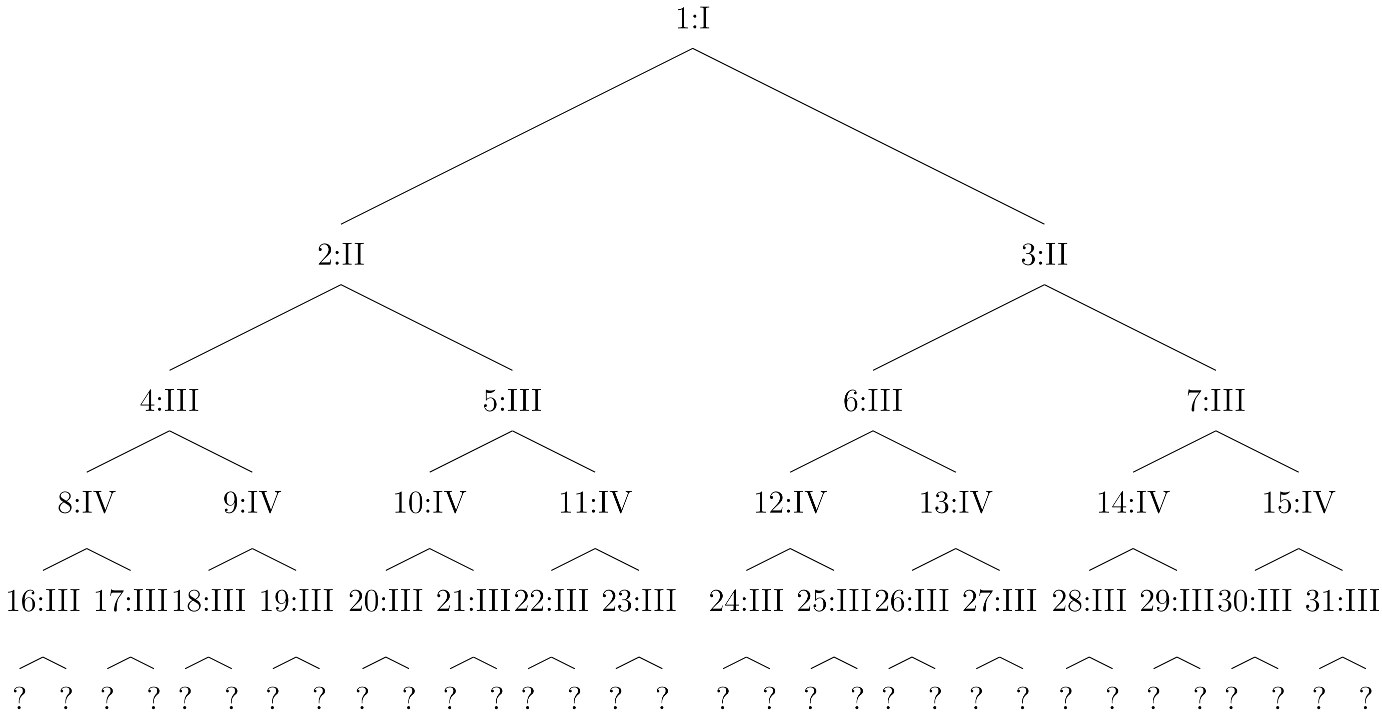
Lemma 6. *Let G be the extensive form game of Example 1 (see* (2)*). For every order that satisfies:*

- *Node 2 appears first.*
- *Node 4 appears before 3 and before 5.*

the outcome of the commitment game is $(1, 1, 1, 1)$.

The proof of the Lemma is relegated to Appendix A.

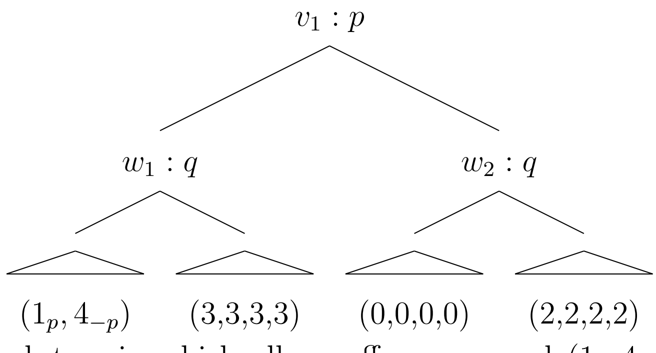
Proof of Theorem 2. Let the structure of the game be a complete binary tree of depth five where player I, II, III, IV, III act at all nodes of depth 1,2,3,4,5 respectively. We enumerate the decision nodes as follows:



Let $v_1, v_2, ..., v_{31}$ be an order of the decision nodes.

If $v_1 \in \{1, 2, ..., 15\}$, we denote $p = d(v_1)$ the acting player at v_1, and q the acting player at one depth below v_1. We plant the payoffs of Example 2 as follows:

On the sub-tree with the root v_1 we set the payoffs to be



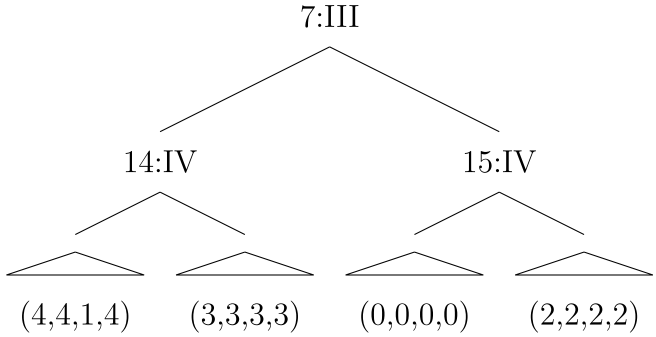
where $\underset{(u)}{\triangle}$ denotes a sub-tree in which all payoffs are u, and $(1_p, 4_{-p})$ denotes the payoff profile where player p gets 1 and all other players get 4.

Out of the sub-tree with the root v_1 we set the payoffs to be $(-1, -1, -1, -1)$.

The outcome of this game is $(2, 2, 2, 2)$, because in all decision nodes other than v_1, w_1, w_2 players will commit on (or equivalently will not commit but play this action at the final stage of playing the game) actions that lead to v_1 (this is their only option to avoid the payoff -1). By the arguments in the proof of Lemma 5 at node v_1 player p commits on w_2, which essentially leads to the outcome (2,2,2,2).

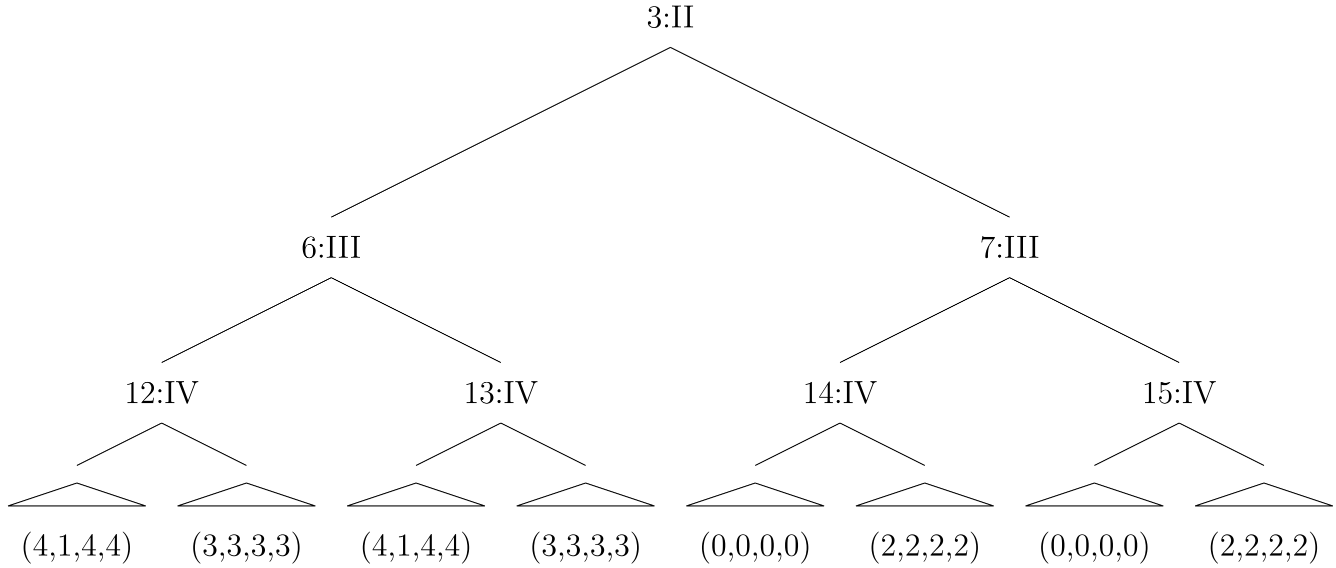
Therefore, we can focus only on the case where $v_1 \in \{16, 17, ..., 31\}$. With no lose of generality, assume that $v_1 = 16$ and that 12 appears in the order before 13,14, and 15.

If 7 appears in the order before 12, then we set the payoffs to be



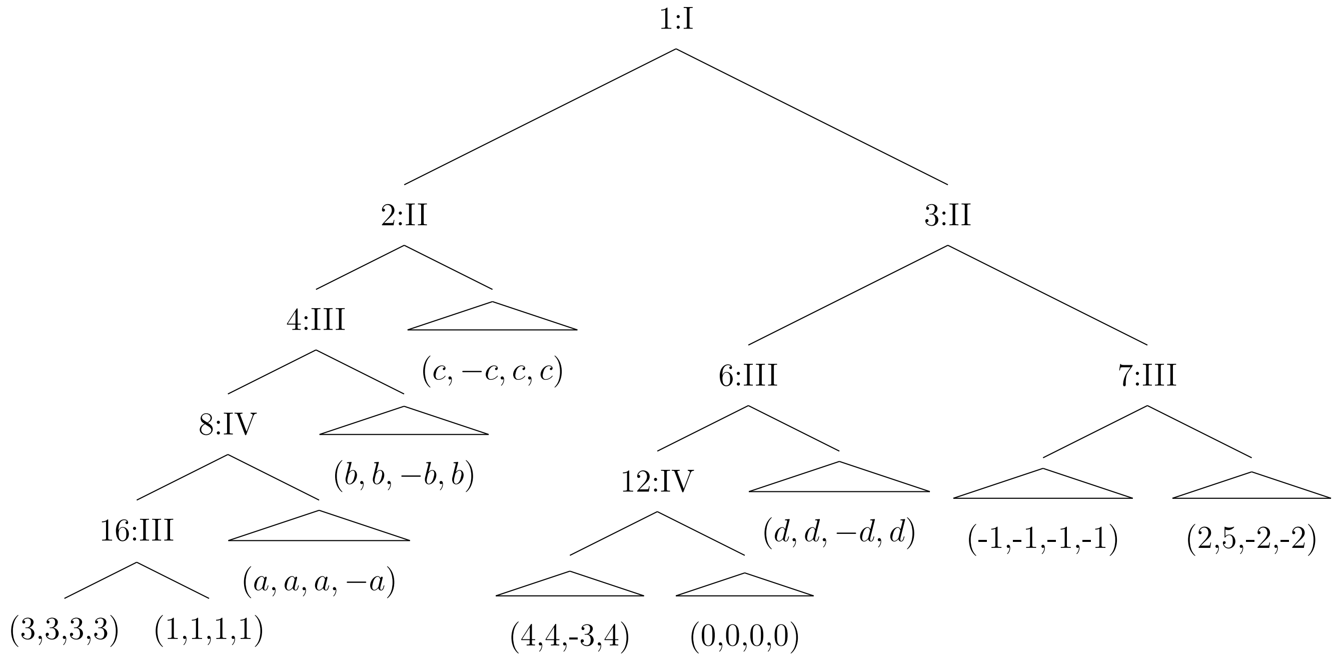
and $(-1, -1, -1, -1)$ everywhere else. By similar arguments, since 14 and 15 appears after 7 in the order, the outcome of the game is $(2, 2, 2, 2)$.

If 3 appears in the order before 12, then we set the payoffs to be



and $(-1, -1, -1, -1)$ everywhere else. Again by similar arguments, since 12,13,14, and 15 appears after 3, only commitment on right of player II leads to a payoff above 1 for him (because player IV can commit on left in all vertices 12,13,14,15), and the outcome is $(2, 2, 2, 2)$.

The remaining case is the one where the first node in the order is 16, and 12 appears before 3 and 7 in the order. In such a case we set the payoffs as follows:



where $a, b, c, d > 5$. We order the values of a, b, c, d to be in opposite order according to which the vertices $8, 4, 2, 6$ are visited: for example, if along the order we first visit 8, after that 6, after that 4, and after that 2, then we set the values to satisfy $a > d > b > c > 5$. We argue that in nodes $2, 4, 6, 8$ players II, III, III, IV players do not commit to the outcomes $(a, a, a, -a), ..., (d, d, -d, d)$ respectively and do not play it in the play stage of the commitment game. Consider the first vertex among $2, 4, 6, 8$ in the order, assume w.l.o.g that it is node 8. Note that $-a$ is the worst payoff player IV can get in the game, a is the best payoff players I, II, III can get in the game. If IV commits to $(a, a, a, -a)$ then it also will be chosen as a final outcome. Therefore IV would not commit to $(a, a, a, -a)$ and obviously won't play it at the final stage of playing the game. Consider the second node among $2, 4, 6, 8$ in the order, assume w.l.o.g that it is vertex 6. Note that $-d$ is the worst payoff for III, d is the best payoff for players I, II, IV because we already ruled out the possibility of IV receiving the payoff a. If III commits to $(d, d, -d, d)$ then it also will be chosen as a final outcome. Therefore III does not commit to $(a, a, a, -a)$ and obviously does not play it in the play stage of the commitment game. We proceed similarly for the third and the forth nodes in the order. Therefore we can ignore the subtrees where the outcomes are $(a, a, a, -a), ..., (d, d, -d, d)$ and we get a tree that has exactly the same outcomes as in Example 1. Finally, the order over nodes 1,3,7,12,16 satisfies:

1. 16 if the first.

2. 12 is before 3 and 7.

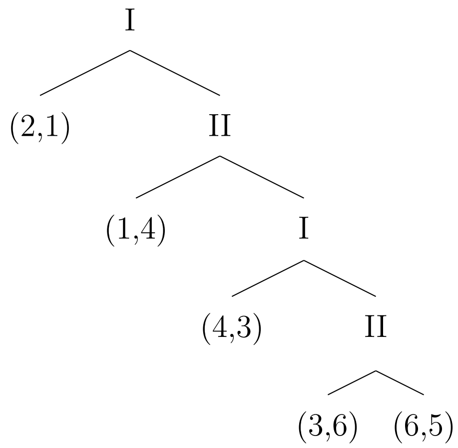
By Lemma 6 the outcome of the game for every such order is the pareto dominated outcome $(1, 1, 1, 1)$. $\square$

4.1 Quitting games

In this section we study a special class of extensive form games that is called *quitting games*. Our main result shows that pareto efficiency is implementable in this class of games.

Definition 4. An extensive-form perfect information game G is called a *quitting game* if $D = \{v_1, \ldots, v_n\}$ such that every node v_j has two actions $\{exit, stay\}$, a choice of *stay* by $d(v_j)$ for $j < n$ leads to v_{j+1} and a choice of *exit* leads to a terminal node. Both choices at v_n lead to a terminal node.

A well known example of a quitting game is the *centipede game* below.



A slightly more intricate example of a quitting game is the three player centipede game studied by Rapoport et al. For the above centipede game, we know from Theorem 1 that the unique pre-terminal order $(v_n, v_{n-1}, ..., v_1)$, which is also a DFS order, leads to Pareto efficiency. The following result shows that this is in fact true for every multi-player quitting games.

Theorem 3. *For every quitting game G the unique pre-terminal order $\overline{v} = (v_n, \ldots, v_1)$ leads to Pareto efficiency; I.e., $\mathsf{Val}(Com(G, \overline{v}))$ is Pareto efficient.*

Proof. The proof is obtained using simple induction on n. Clearly, for $n = 1$ the theorem holds. We assume the result for $n - 1$ and prove it for n.

We denote the acting player at v_1 by $d(v_1) = j$. We denote by y the outcome after *exit* at node v_1. We use the same notations as in Theorem 1. We recall that $\widehat{G}$ is the game defined on

T_{v_2} when we replace every outcome x with $r(x)$, where $r(x)$ is the value of the game $\tilde{G}(x)$ that is obtained when we replace the subtree T_{v_2} with the outcome x. In the quitting games case, $\tilde{G}(x)$ translates to a simple decision of player j between two options x or y. Therefore, $r(x)$ gets a particular simple form:

$$r(x) = \begin{cases} x & \text{if } x_j \geq y_j \\ y^1 & \text{if } x_j < y_j. \end{cases} \tag{3}$$

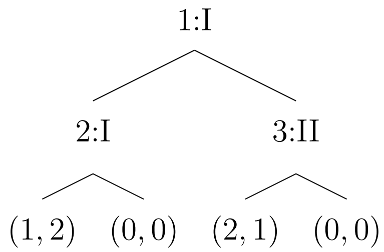
By Lemma 4, we know that $\mathsf{Val}(Com(G, \overline{v})) = \mathsf{Val}(Com(\widehat{G}, \overline{v}_*))$ where in this case $\overline{v}_* = (v_n, \ldots, v_2)$. So applying the induction hypothesis on $Com(\widehat{G}, \overline{v}_*)$ we can deduce that $\mathsf{Val}(Com(G, \overline{v}))$ is Pareto efficient with respect to the outcomes $\{r(x) : x \in T_{v_2}\}$. If, by contradiction, $\mathsf{Val}(Com(G, \overline{v}))$ is Pareto dominated by some outcome x' (in the original game), then

$$x'_j > \mathsf{Val}_j(Com(G, \overline{v})) \geq y_j$$

where the second inequality follows from the fact that player j can guarantee y_j. Therefore, by equation (3) we have $r(x') = x'$. This contradicts the fact that the value is Pareto efficient with respect to $\{r(x) : x \in T_{v_2}\}$. $\square$

5 Comments

1. The dependence of the outcome on the DFS order. Theorem 1 shows that every DFS order of commitments leads to a Pareto efficient outcome. It is natural to ask whether *every* DFS order leads to the same outcome in every two player extensive form game. The answer is negative, as demonstrated by the following example.



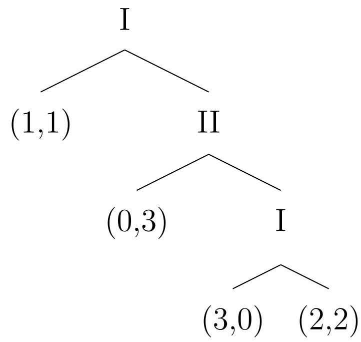
For the DFS order (3,2,1) player II commits to the right action (i.e., the outcome $(0, 0)$), which enforces player I to choose left in both of his nodes, and the outcome is $(1, 2)$.

For the DFS order (2,3,1) player I commits to $(0,0)$, which leads player II to choose $(2,1)$, and it is also the final outcome.

2. On the necessity of sequentiality The commitment protocols that we consider in this paper are sequential in two aspects. First, players commit sequentially (and not simultaneously). Second, each player is asked to commit on a single decision node at each step. We argue that both aspects of sequentiality are necessary for efficiency to be implemented, even in two-player games.

Regarding the first aspect, consider a commitment protocol where players simultaneously commit to actions, and then turn to the play stage. This is the model that is discussed in the previous works on commitments[1, 2, 11, 8]. The sequential prisoner's dilemma demonstrates that in such a case inefficient outcomes may be obtained as a subgame perfect equilibrium outcome. The following example demonstrates the existence of an extensive form games where *all* equilibria outcomes are inefficient.

Example 3.



In the commitment stage player II has three actions: Committing to left (L), committing to right (R), or not committing (ϕ). Similarly, player I has nine actions which represent a choice of an action (among the three) in each one of his decision nodes.

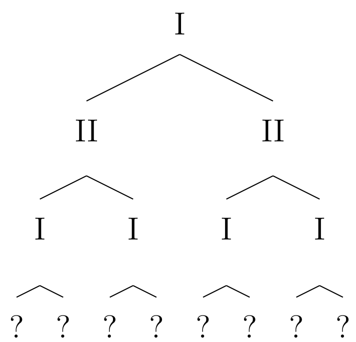
After the simultaneous commitment, both players will follow the sub-game perfect equilibrium of the remaining game. Therefore the simultaneous commitment game is equivalent to the following one-shot game:

It is easy to check, using weak dominance arguments that the unique Nash equilibrium outcome (pure or mixed) of this game is (1,1), which is inefficient.

Regarding the second aspect of sequentiality, consider the class of commitment protocols where the order is not on decision nodes but on players. Player i, at his turn, is allowed to commit on actions in all his decision nodes. This is the model that is discussed in the existing

	ϕ	L	R
ϕ,ϕ	(1,1)	(1,1)	(3,0)
ϕ,L	(1,1)	(1,1)	(3,0)
ϕ,R	(1,1)	(1,1)	(2,2)
L,ϕ	(1,1)	(1,1)	(1,1)
L,L	(1,1)	(1,1)	(1,1)
L,R	(1,1)	(1,1)	(1,1)
R,ϕ	(0,3)	(0,3)	(3,0)
R,L	(0,3)	(0,3)	(3,0)
R,R	(0,3)	(0,3)	(2,2)

literature [1, 2, 11, 8]. We do not assume that player has an option to commit only once; i.e., the order may contain player i several times. A game structure that excludes the possibility of pareto efficient implementation is the following:



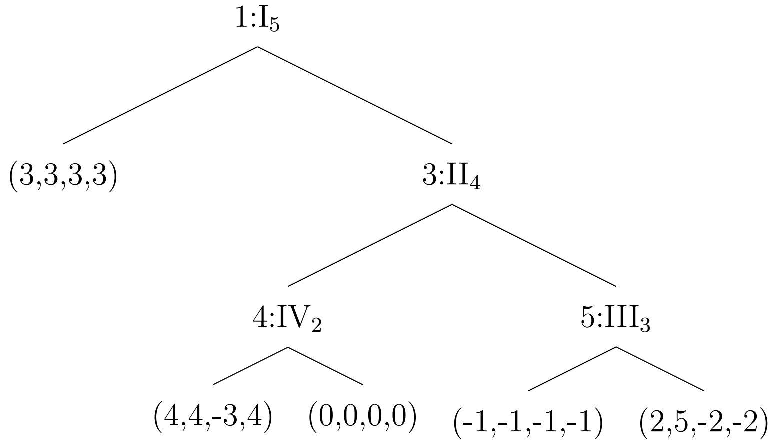
Depending on the player that commits first (player I or player II) we can "plant" into this structure the game of Example 2 to get inefficiency.

Here, we demonstrate inefficiency in the extreme case, where each player is allowed to commit in *all* of his nodes. Theorem 2 demonstrated inefficiency in the other extreme case, where players are restricted to commit to a single node at each step. We believe that the negative examples developed in this paper may also show inefficiency in four-player games for much richer class of commitment protocols. In particular, intermediate protocols between the two extremes, where players are allowed to commit on subsets of nodes, and the option to commit at a certain node may appear several times along the protocol.

A Appendix: Proof of Lemma 6

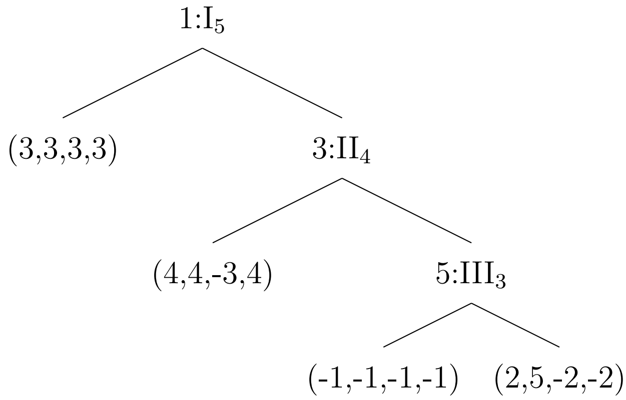
Player III commits first at node 2. By Remark 2.1 we can assume that he commits to an action. We consider first the case where he commits to (3,3,3,3). We argue that for every order over the vertices 1,3,4,5 the outcome of the commitment game:

G_1:



is (4,4,-3,4). The outcome cannot be below 3 for player I, because I can guarantee 3 by choosing left (irrespective of commitments). Therefore the outcome is either (3,3,3,3) or (4,4,-3,4). By Remark 2.1 we can assume that player IV commits to an action at node 4. A commitment on (0,0,0,0) by Player IV (at any stage of the commitment procedure) guarantees that the outcome will be (3,3,3,3) (because it excludes the possibility of (4,4,-3,4)), therefore it is weakly better for player IV to choose (4,4,-3,4). Therefore it is sufficient to focus on the game:

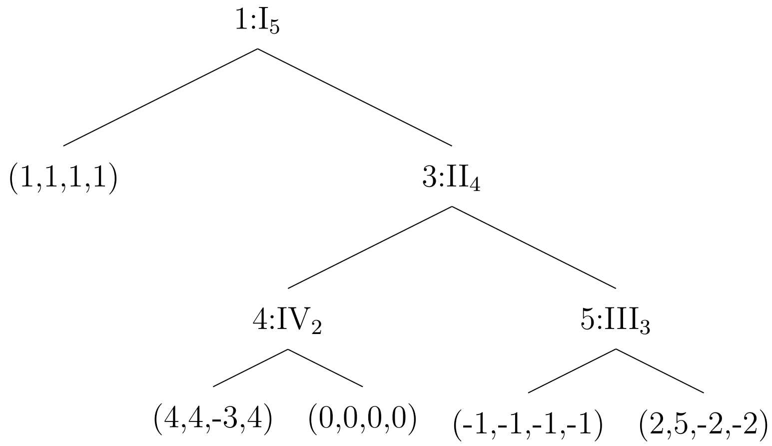
G_1':



We leave as an exercise to the reader to check that every order over the vertices 1,3,5 leads to the outcome (4,4,-3,4), where in equilibrium player I always chooses not to commit, and player 2 commits to the outcome (4,4,-3,4).

Since -3 is the worst outcome for player III, he never will choose to commit to (3,3,3,3) at node 2. It must be the case that at the first step player III commits to (1,1,1,1). So we remain with the game

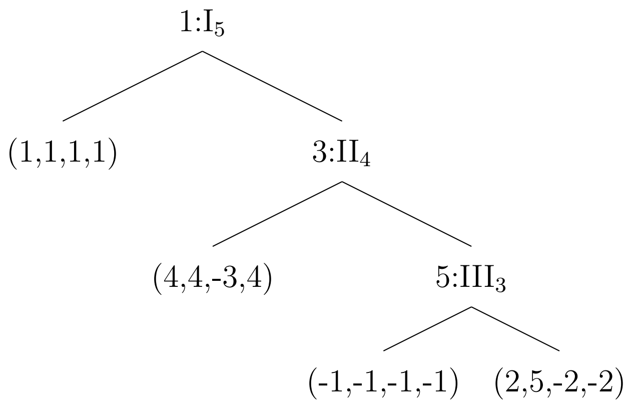
G_2:



We argue that for every order where IV commits before II and before III the outcome is (1,1,1,1).

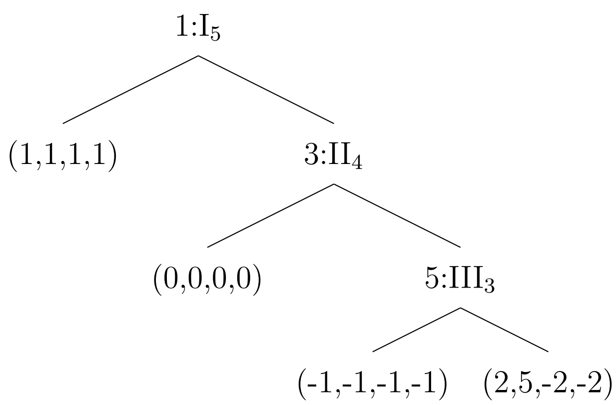
Case 1: Player IV commits first. Again, by Remark 2.1 we can assume that IV commits to an action. If IV commits to (4,4,-3,4), then we remain with the game:

G_2':



We leave as an exercise to the reader to check that every order over the vertices 1,3,5 leads to the outcome (2,5,-2,-2), where in equilibrium player III always commits on (2,5,-2,-2). Since -2 is the worst outcome for player IV, he does not commits on (4,4,-3,4) at node 4, and therefore commits on (0,0,0,0). So we remain with the game

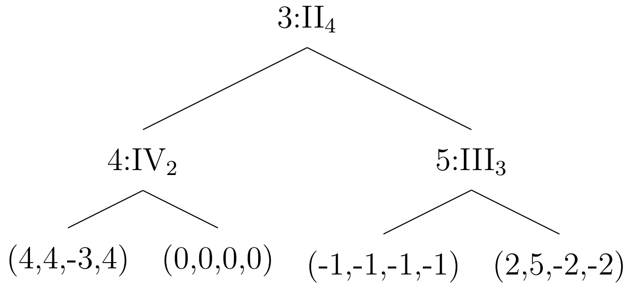
G_2':



where (2,5,-2,-2) is the worst outcome for player III. Moreover if he chooses it, then it will be selected (because it is the best outcome for both players I and II). Therefore, III always commits on (-1,-1,-1,-1) and then (1,1,1,1) is the best outcome for player I, and it is selected.

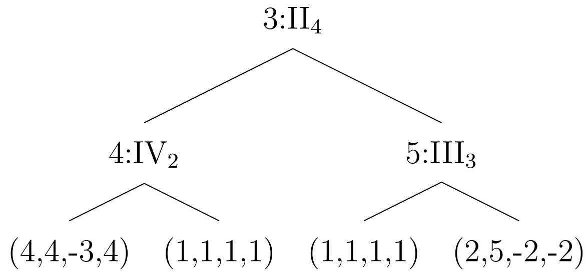
Case 2: Player I acts first. If I commits on (1,1,1,1) we are done. If I commits on right, then we remain with the game:

G_3:



Player IV is the first player to commit. If he commits on (4,4,-3,4), then the outcome will be (2,5,-2,-2), because in equilibrium II always can choose not to commit while III always commit on (2,5,-2,-2). Therefore IV commits on (0,0,0,0). In such a case, in equilibrium III always commits on (-1,-1,-1,-1), which leads to the outcome (0,0,0,0). Therefore I will never commits on right. The remaining option of I is to choose not to commit. In such a case, by the arguments in the proof of Theorem 3, we get that the renaming game is equivalent to the game:

G_4:



Here, similar arguments to those of the analysis of game G_3 proves that the outcome of the game is $(1, 1, 1, 1)$.

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