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Parametric excitation of multimode dissipative systems

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A general approach to the study of parametric excitation of multimode dissipative systems is proposed. It is based on the derivation of normal form equations and on the reduction of these equations to a flow of lower dimensionality. General results for single- and double-mode systems are presented. The implications for the system of parametrically excited surface waves are discussed; it is predicted that the onset of waves can be either smooth or hysteretic depending on the excitation frequency, and an explanation is given for the experimentally observed mode-suppression effect, occurring in double-mode systems.

Parametrically driven systems are very common in physics; they arise when a parameter in the equations of motion is allowed to vary periodically in time and thereby to act as an energy feed source.[1,2] Examples are available from mechanics,[1,2] hydrodynamics,[1,3–5] and electronics,[6] as well as from many other fields. Considerable amount of work has been devoted to the study of nonlinear phenomena in parametrically excited *single*-mode systems.[2,7] Much less is known, however, on the dynamics of *several* interacting modes even in weakly forced systems.[8] The motivation to study multimode systems derives from recent experimental studies which reveal rich nonlinear behavior in such systems.[9,10] Since the complexity of the system increases rapidly with the number of modes, it becomes desirable to have a simple and general approach for the analysis of multimode systems. The aim of this Rapid Communication is to propose such an approach for the case of weakly forced systems.

We shall consider the set of equations

$$\ddot{\zeta}_i+2\lambda_i\dot{\zeta}_i+[\Omega_{0i}^2+\varepsilon f_i(t)]\zeta+g_i(\zeta_1,\zeta_2,\ldots,\zeta_N)=0\ , \quad i=1,\ldots,N\ , \tag{1}$$

where N is the number of excited modes, $f_i(t)$ is a 2π-periodic function, and g_i is strictly nonlinear. Our analysis will be general in the sense that we shall not be concerned with the specific forms of f_i and g_i. This generality is achieved by considering the normal form of Eqs. (1).[11] It then appears that one can factor out the periodic time-dependent terms *without* resorting to any method of approximation (such as multiple time scales, averaging, etc.[2]). The resulting autonomous equations can be easily analyzed to obtain "static" information about the system (steady-state solutions). It will be shown, however, that in the regime where all modes are excited one can reduce the number of active degrees of freedom by a half, and thereby obtain useful "dynamic" information. I will demonstrate this approach for single- and double-mode systems and discuss the implications for the system of parametrically excited surface waves. In particular, I will explain the experimentally observed mode-suppression effect between a pair of interacting modes, and predict that the onset of surface waves can be either supercritical (smooth) or subcritical (hysteretic) depending on the excitation frequency.

It is instructive to first consider the damped Mathieu equation[2]

$$\ddot{\zeta}+2\lambda\dot{\zeta}+(\Omega_0^2+\varepsilon\cos t)\zeta=0\ . \tag{2}$$

The ε-Ω plane, where $\Omega\equiv\sqrt{\Omega_0^2-\lambda^2}$, contains "tongue"-like domains as shown in Fig. 1. Outside the tongues the equilibrium solution, $\zeta=0$, prevails. On the boundaries of the tongues periodic motion sets in, while inside them the solutions are unstable. The tongues lie above (and approach, as $\lambda\rightarrow 0$) an infinite sequence of points, $\{\Omega_k=k/2;\ k=1,2,\ldots\}$ on the Ω axis. The period of the motion on the boundaries is 2π (4π) if the tongue lies above an integer (half an integer) number. We note that Ω can be interpreted as the ratio between the natural frequency, ω_i, of the mode and the forcing frequency ω $(\Omega=\omega_i/\omega)$. In the case of multimode systems it is useful to consider the ε-ω plane, since it contains information about all modes; the infinite sequence of tongues in the ϵ-Ω plane pertaining to each mode falls now in a finite interval, $[0,2\omega_i]$, on the ω axis. It is well known from the theory of the Mathieu equation[2,11] that higher tongues (higher k) are narrower and lie higher above the ω (or Ω) axis (see also Fig. 1). Therefore for small ε values (weakly forced systems) one should not worry about the accumulation of tongues at small ω values. Moreover, only the first few ones are easily observable and therefore of practical interest.

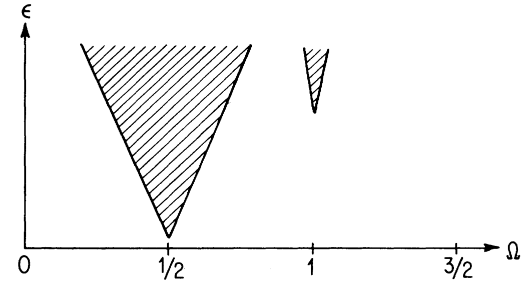

FIG. 1. The ε-Ω plane for the damped Mathieu equation. Shaded regions (tongues) correspond to unstable solutions. Higher tongues are narrower and lie higher above the Ω axis (the third tongue above $\frac{3}{2}$ is already not seen).

For the sake of simplicity the approach presented here will be illustrated using the example of a single-mode system [$N=1$ in Eq. (1)]. We assume that both ε and λ are much smaller than Ω. For the purpose of finding the normal form it is convenient to consider the periodic force as an additional degree of freedom, $a_f \equiv \exp(it)$. We can now expand the forcing term, $f(t)$, in Fourier series and write it as

$$f(t)=a_f+\bar{a}_f+\sum_{n=2}^{\infty}(d_n a_f^n+\bar{d}_n \bar{a}_f^n) , \tag{3}$$

where the bars denote complex conjugates. It is also useful to introduce the dynamical complex variable $a=(2\Omega)^{-1}[(\Omega-i\lambda)\zeta-i\dot{\zeta}]$. In terms of a and a_f Eq. (1) (for $N=1$) reads

$$\dot{a}=(-\lambda+i\Omega)a+i\gamma(a_f+\bar{a}_f)(a+\bar{a})+h(\boldsymbol{a}) , \tag{4a}$$

$$\dot{a}_f=ia_f , \tag{4b}$$

where $\gamma=\varepsilon/(4\Omega)$ and h is a nonlinear function of the set of variables $\boldsymbol{a}=(a,\bar{a},a_f,\bar{a}_f)$. We now apply a nonlinear transformation to the variables $\boldsymbol{a}$, of the form $\boldsymbol{a}=\mathbf{A}+\boldsymbol{\psi}(\mathbf{A})$ [where $\boldsymbol{\psi}$ is strictly nonlinear and $\mathbf{A}=(A,\bar{A},a_f,\bar{a}_f)$], such that the equations for $\mathbf{A}$ are in normal form. For the case $\Omega \approx \frac{1}{2}$ (first tongue) we get[12]

$$\dot{A}=(-\lambda+i\Omega)A+i\gamma e^{it}\bar{A}+i\sigma|A|^2A+\text{h.o.t.} , \tag{5}$$

where the coefficient σ is a complex constant with small imaginary part (of order λ) whose specific form depends on that of h [we have neglected in Eq. (5) a small contribution of $O(\varepsilon^2)$ to λ] and h.o.t. stands for higher-order terms. Substituting now the transformation

$$A(t)=a(t)e^{it/2} , \tag{6}$$

in Eq. (5) we obtain the *autonomous* equation

$$\dot{a}=(-\lambda+i\phi)a+i\gamma\bar{a}+i\sigma|a|^2a+\text{h.o.t.} , \tag{7}$$

where $\phi \equiv \Omega-\frac{1}{2}$.

The stability of the equilibrium solution of Eq. (7) is determined by the eigenvalues

$$s^{\pm}=-\lambda \pm D , \tag{8}$$

where $D \equiv \sqrt{\gamma^2-\phi^2}$. Equation (7) has in addition the steady-state solutions

$$r_1=\pm\sqrt{(-\phi-\Delta)/\sigma},\ 2\theta_1=\arccos(\Delta/\gamma) ,$$

and

$$r_2=\pm\sqrt{(-\phi+\Delta)/\sigma},\ 2\theta_2=\arccos(-\Delta/\gamma) ,$$

where r and θ are the modulus and phase of a, $\Delta \equiv \sqrt{\gamma^2-\lambda^2}$ and we have assumed that σ is real. The stability of these solutions is determined by the eigenvalues $p_1^{\pm}=-\lambda\pm\sqrt{\lambda^2-4\Delta(\phi+\Delta)}$ and $p_2^{\pm}=-\lambda \pm\sqrt{\lambda^2+4\Delta(\phi-\Delta)}$, respectively. The phase diagram in the ε-Ω plane corresponding to the solutions for σ positive is shown in Fig. 2. For $\Omega>\frac{1}{2}$ the equilibrium–steady-state transition is continuous (second order) and described by a supercritical pitchfork bifurcation. For $\Omega<\frac{1}{2}$ the transition is discontinuous (first order) and occurs via a subcritical pitchfork bifurcation.[13] A similar phase diagram is obtained for σ negative except that it is now reflected about the $\Omega=\frac{1}{2}$ line. The inclusion of a small $[O(\lambda)]$ imaginary part in σ as well as higher order terms in Eq. (7) may slightly shift the middle point (at $\Omega=\frac{1}{2}$) which separates the line of smooth excitation from that of hysteretic excitation (see below). It may also deform the line $\varepsilon=4\lambda\Omega$. We note that the significance of Eq. (6), where a represents a steady state, is that everywhere inside the tongue the motion is locked to the external frequency.

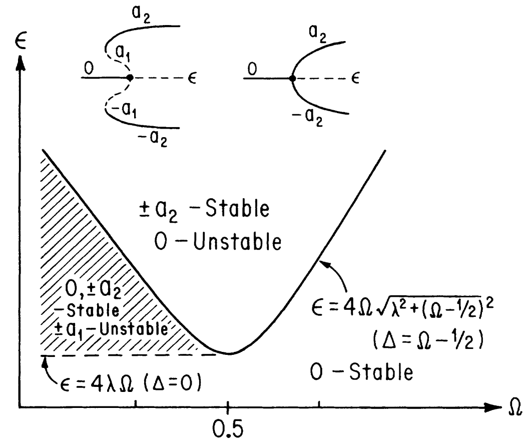

FIG. 2. Supercritical vs subcritical excitation. The symbols 0 and $\pm a_i$ $(i=1,2)$ refer to the equilibrium solution and the steady-state solutions $(\pm r_i,\theta_i)$, respectively. The shaded region corresponds to a case of tristability. The top figures are bifurcation diagrams for constant Ω: $\Omega>\frac{1}{2}$ (right) and $\Omega<\frac{1}{2}$ (left).

Equation (7) represents a two-dimensional flow. It is evident from Eq. (8) that on the boundary of the tongue, defined by $D=\lambda$, one degree of freedom is marginal while the other one is stable. We can therefore reduce the dynamics on the boundary (and inside the tongue as well) to a one-dimensional flow. The reduction is done here in the spirit of Refs. 4 and 14. The reduced equation reads[12]

$$\dot{x}=s^{+}x-\eta x^3+\text{h.o.t.},\quad \eta\equiv\frac{\gamma}{2D}\left[\frac{\phi}{D}\text{Re}\sigma+\text{Im}\sigma\right] . \tag{9}$$

For σ real (and say positive) we recover the result of the "static" analysis, namely, for $\Omega>\frac{1}{2}$ $(\phi>0)$ the bifurcation to a steady state is supercritical $(\eta>0)$ while for $\Omega<\frac{1}{2}$ it is subcritical $(\eta<0)$. The inclusion of a small imaginary part in σ shifts the border point from $\Omega=\frac{1}{2}$ to $\Omega=\frac{1}{2}+D\,\text{Im}\sigma/\text{Re}\sigma$.

Let us consider now a nondegenerate double-mode system, by which we mean a system which satisfies Eq. (1) with $N=2$ and $\Omega_1\neq\Omega_2$ (where $\Omega_i\equiv\sqrt{\Omega_{0i}^2-\lambda_i^2}$). We shall consider the case $\Omega_1<\frac{1}{2}<\Omega_2$ which corresponds to an excitation of two modes, having natural frequencies ω_1 and ω_2, in their first tongues. The corresponding phase diagram in the ε-ω plane is shown in Fig. 3. We consider first the case where the equations of motion are invariant under the transformations $(\zeta_1,\zeta_2)\rightarrow(-\zeta_1,\zeta_2)$, $(\zeta_1,-\zeta_2)$, $(-\zeta_1,-\zeta_2)$. The normal form [after making the equa-

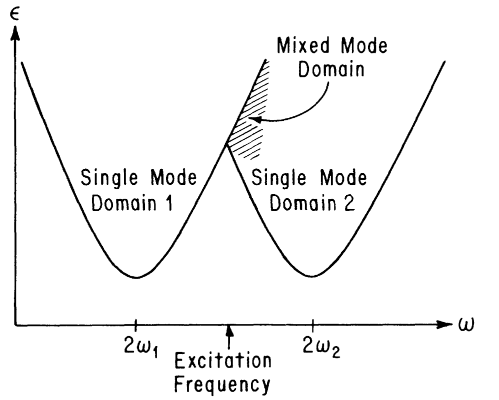

FIG. 3. Schematic phase diagram (in the ε-ω plane) for a double-mode system with $\Omega_1 < \frac{1}{2} < \Omega_2$. Only one mode prevails in the region where the two tongues overlap.

tions autonomous by applying the transformation $A_j(t) = a_j(t)\exp(it/2)$] is

$$\dot{a}_1 = (-\lambda_1 + i\phi_1)a_1 + i\gamma_1\bar{a}_1 + i\sigma_1|a_1|^2 a_1 + i\rho_1|a_2|^2 a_1 , \tag{10a}$$

$$\dot{a}_2 = (-\lambda_2 + i\phi_2)a_2 + i\gamma_2\bar{a}_2 + i\sigma_2|a_2|^2 a_2 + i\rho_2|a_1|^2 a_2 , \tag{10b}$$

where $\phi_i = \Omega_i - \frac{1}{2}$ and $\gamma_i = \varepsilon/(4\Omega_i)$. We note that in the degenerate case ($\Omega_1 = \Omega_2$) additional terms are resonant[11] and should be included in the normal form [i.e., $a_2^2\bar{a}_1$ and $a_1^2\bar{a}_2$ in Eqs. (10a) and (10b), respectively]. For the sake of simplicity we shall assume that the coefficients σ_i, ρ_i are all real. At the intersection point of the two tongues (see Fig. 3) the modes are excited simultaneously. The two eigenvalues, $s_i^+ = -\lambda_i + D_i$, $i = 1,2$, at this point are marginal while the remaining two, $s_i^- = -\lambda_i - D_i$, are negative. Equations (10) are therefore reducible to a two-dimensional flow which takes the form[12]

$$\dot{x}_1 = s_1^+ x_1 - \eta_1 x_1^3 - \mu_1 x_2^2 x_1 + \text{h.o.t.} , \tag{11a}$$

$$\dot{x}_2 = s_2^+ x_2 - \eta_2 x_2^3 - \mu_2 x_1^2 x_2 + \text{h.o.t.} , \tag{11b}$$

where

$$\eta_i = \frac{\gamma_i \sigma_i \phi_i}{2D_1 D_2}, \quad i = 1,2 , \tag{12a}$$

$$\mu_1 = \frac{\gamma_2 \rho_1 \phi_1}{2D_1 D_2}, \quad \mu_2 = \frac{\gamma_1 \rho_2 \phi_2}{2D_1 D_2} . \tag{12b}$$

We expect Eqs. (11) to provide a good approximation in the region where the two tongues overlap each other and in a small neighborhood around the region. We assume now that the coefficients ρ_1 and ρ_2 in Eqs. (10) have the same sign. This is a reasonable assumption since simultaneously excited modes have natural frequencies which do not differ much. In that case the main observation here is that the coefficients μ_1 and μ_2 in Eqs. (11) have opposite signs. This stems from the fact that, by the nature of the problem, ϕ_1 and ϕ_2 have opposite signs. The significance is that one mode acts to *suppress* the other, while the later acts to enhance the former. The obvious result is that in the region of two overlapping tongues only one mode prevails. Mixed-mode states exist only in a single-tongue domain (see Fig. 3).

A classical physical example of multimode parametric excitation is the system of surface waves;[3,4,9] basically, a cell containing a fluid layer is oscillated in the vertical direction and the wave modes which develop at the free surface of the fluid are detected. The spatial wave pattern of each mode is characterized by two "quantum" numbers which determine the number of nodes along two independent directions. The time dependence of each mode is described, at the linear level, by a damped Mathieu equation.[4] The symmetry requirements which led to Eqs. (10) are met by any pair of modes of the form $(2n, 2m+1)$ and $(2m+1, 2n)$ in rectangular cell, or of the form $(2n, i)$ and $(2m+1, j)$ in cylindrical geometry.[4] The peculiar nature of a single-mode excitation (i.e., the occurrence of different types of excitations on the left and right sides of the $\Omega = \frac{1}{2}$ line) has not been observed yet in surface-wave experiments. The mode-suppression effect in double-mode systems, however, has been observed by Ciliberto and Gollub in a cylindrical cell.[9] We note that there might be pairs of modes for which only the $(\zeta_1, \zeta_2) \rightarrow (-\zeta_1, -\zeta_2)$ symmetry requirement should be satisfied [for example, the pair $(2n, 2m)$ and $(2m, 2n)$ in rectangular geometry]. In this case new terms might appear in Eqs. (10), the significance of which will be discussed elsewhere.[12]

The extension of our analysis to higher mode systems, as well as to systems where the interaction involves different kinds of tongues, is straightforward. We stress that the normal form can be easily inferred from the *linear* problem by inspecting the spectrum of eigenvalues. In three and higher mode systems the reduced equations may prove to be particularly useful, since they are simpler in structure and contain apparent information (through the coefficients) which is not readily available from the original equations. Finally we note that in regimes other than that treated here (where all modes are excited) the equations are not always reducible to a flow of lower dimensionality. An example of such regime is analyzed in Ref. 15.

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