

PATTERNS OF PROPAGATING PULSES*

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This paper is dedicated to Edward L. Reiss on the occasion of his 60th birthday.

Abstract. The complex dynamics that arise in certain nonlinear partial differential equations in time and in one space dimension are studied. In the general case considered, the equation admits a solitary wave in the form of a pulse tailing off exponentially, fore and aft, with possibly oscillatory character. Complicated solutions are described by a superposition of many such solitary structures in interaction. The description is asymptotic in terms of a parameter that becomes exponentially small as the ratio of typical pulse separation to pulse width becomes large. The outcome is a set of dynamical equations for the motion of the individual pulses with nearest neighbor interactions. This system of ordinary differential equations (ODEs) admits a wide range of patterns, both regular and chaotic. The stability theory of such patterns is sketched and the continuum limit of the lattice-dynamical equations of the pulses is given.

Key words. partial differential equations, solitary waves, homoclinic orbits, chaos, pattern theory, defects, waves, nonlinear dynamics

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1. Introduction. Pulse-like solutions are found in many different nonlinear partial differential equations (PDEs) of macroscopic physics, chemical reaction theory, and neurophysics. They frequently arise as exact traveling wave solutions. Along with shocks (or fronts), these localized structures, as they are sometimes called, are among the simplest solutions found in nonlinear PDEs, and it is worth exploring their usefulness as building blocks of more complex solutions. In this paper we study trains of such pulses.

The kinds of equations for which our approach is most useful are either model equations, like the FitzHugh–Nagumo system of neurophysics [28], or equations that come from coarse-graining more difficult problems. An instance of the latter case is convection. Some physicists call the Navier–Stokes equations “microscopic equations” because they think of the Bénard cells as being like the cells or atoms of literally microscopic problems. In the face of such detail, it may not be fruitful to look directly for localized structures, but to work on the system only after it has been coarsened by two-timing or multiscaling. This typically leads to reduced systems such as the various forms of the Ginzburg–Landau equation [17].

In some problems, we do not yet know how to do a good job on coarse graining. For example, in fluid turbulence, the coherent structures appear to be patches of small-scale excitation. We can try to model turbulence with equations of motion for these coherent objects themselves. But how do we derive such equations explicitly? That question [30] is one of the motivations for the present work, although we realize that we are as yet far from realizing this aim. Even a greatly reduced example like the one-dimensional, complex Ginzburg–Landau equation, a standard model for the dynamics of packets of excitation, produces complications that are not easily understood by analytic means. Numerical simulations on that equation reveal pulses that

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tend to split in two and to develop important amplitude and velocity variations during strong interactions [3]. We generally do not know how to deal with such creation and destruction processes of solitary structures.

On the other hand, there are even tamer examples with which we can do something. In simulations on the FitzHugh–Nagumo equations [28], generic situations arise where the pulses do not come very close to each other. There we can use the so-called dilute gas approximation, where the separations of the structures greatly exceed their widths. This regime has been developed in the study of defect dynamics, particularly for interacting kinks and antikinks [4], [5]. It is basically the same approach as has been used in a variety of nonlinear field theories [27] where the particles are the pulses that we are talking about. This is the point of view that we want to develop here as an asymptotic procedure.

A solution in the form of an individual pulse may be relatively easy to find numerically, but accurate solutions representing trains of many pulses require much more effort. However, multipulse solutions do appear in numerical simulations and we could hope to treat them approximately. Of course, for many integrable systems, pulse trains can be obtained as solutions by the nonlinear superposition achieved in the inverse scattering transform [1]. But here we are interested in dissipative systems, such as reaction-diffusion equations [15], which may also be subject to instabilities. The methods that we use for the study of trains of interacting solitary waves are therefore approximate, but, as we shall explain, they have an asymptotic character based on the dilute approximation. Hence they are amenable to systematic study as we shall describe in what follows. They also tie in nicely with the methods of dynamical systems theory.

Although we have reported elsewhere some first results from this kind of analysis [9], we have not yet properly described the method for getting them. To do this here, we shall speak generally about a system of partial differential equations for a state vector, $\mathbf{U}(x, t)$, depending on only one space coordinate x , all of whose components are scalars or pseudoscalars. We adopt the general form of PDE,

$$(1.1) \quad \partial_t \mathbf{U} = \mathcal{L}(\partial_x, \Gamma) \mathbf{U} + \mathcal{N}(\mathbf{U}).$$

Here \mathcal{L} is a partial differential operator in x , depending on the parameters of the system, say $\Gamma = (\gamma_1, \gamma_2, \dots)$, and \mathcal{N} is a nonlinear operator, which may also depend on ∂_x . Neither operator depends on t or ∂_t . We assume that (1.1) is invariant under translation. Equation (1.1) clearly contains a number of well-known special cases, of which we have already mentioned some familiar examples. For simplicity, we further assume that the translation group is the only continuous group of (1.1); in particular, we rule out scale invariance and Galilean invariance.

The form (1.1) represents systems whose state vectors evolve under the joint influences of nonlinearity and spatial coupling. We expect (1.1) to display various kinds of complicated solutions in different parameter ranges. The particular behavior that we are going to study here comes from the interactions of many coherent structures, or solitary waves. The asymptotics of this problem relies on the interactions among pulses becoming quite weak when the interpulse spacing is large. When we take advantage of this weak coupling, we are able to replace (1.1) by a solution in the form of a pulse train together with a discrete set of ordinary differential equations for the motions of the individual pulses. Thus, we reduce the PDE (1.1) to a simple N -body problem, whose particles are the solitary structures we describe in § 2. Then, in § 3, we outline the reductions that lead to the equations of motion of these structures when

we allow for nearest neighbor interactions and neglect tidal distortions of pulses. Finally, we say something about the content of these equations in § 4.

2. Basic structures. Equation (1.1) is assumed invariant under the transformation $x \rightarrow \chi = x + a$, where a is a constant. If now $a = ct$, where c is another constant, the transformation still leaves the right-hand side of (1.1) unchanged. However, the left-hand side is altered in this case, for $\partial_t \rightarrow \partial_t + c\partial_\chi$, where

$$(2.1) \quad \chi = x + ct.$$

The building blocks of our description are constructed by using this transformation to look for traveling waves. We seek solutions of (1.1) of the form

$$(2.2) \quad \mathbf{U}(x, t) = \mathbf{H}(\chi).$$

By direct substitution into (1.1), we obtain the ODE

$$(2.3) \quad [\mathcal{L}(D) - c\mathcal{E}D]\mathbf{H} + \mathcal{N}(\mathbf{H}) = 0,$$

where D is the ordinary derivative operator with respect to χ and \mathcal{E} is the unit matrix.

Solution of (2.3) immediately leads to traveling wave solutions of (1.1). We will not pursue these in any detail here since, even if we could find them easily, it is usually difficult to decide which are stable solutions of (1.1). This question has been discussed, along the lines of slowly varying wave theory, for reaction diffusion waves [14]. We will see below how the techniques described here lead to another version of this stability problem. However, our immediate interest in the isolated structures arising in (2.3) is that they provide us with elementary objects from which to construct approximate solutions of (1.1).

There are so many problems where solitary waves do exist that we do not worry about their existence issue in general, although this may come up in particular applications. Generally, to find the solitary wave of the PDE, we look for a solution of infinite period in the phase space associated with (2.3). For our purposes, we do not need to know details about the internal structures of these solitary solutions, except for some measure of their widths. However, we do need to know their behavior at large distances from their main bodies.

Infinite period (in χ) results when the solution tends to a fixed point as $|\chi| \rightarrow \pm\infty$. As the motion follows a trajectory in the phase space of (2.3) toward or away from a fixed point, it goes so slowly near this point that an arbitrarily long time is required, where χ is the “time” of this system. The solitary pulse that is central to this work corresponds to a trajectory that connects the fixed point to itself. We adapt a word from Poincaré and call this a *homoclinic* orbit; similarly, we call a trajectory that joins two different fixed points *heteroclinic*. The latter are the shock-like structures such as we find in the Burgers equation. For brevity, we leave these out here and concentrate on the homoclinic orbits, the pulses.

It is convenient to locate the origin of phase space at the fixed point that is joined to itself by the homoclinic orbit. As $\chi \rightarrow \infty$, the behavior is controlled by the linearized version of (2.1), which admits solutions proportional to $\exp(s\chi)$. We generally have no difficulty in finding a characteristic value equation for s . The values obtained will depend on the parameters of the problem Γ , normally specified by physical conditions, and on c .

A most important property of the characteristic value s is the sign of its real part. Various cases may be distinguished according to the spectrum of these values and, to shorten this discussion, we exclude the following two atypical (though potentially

interesting) situations: (a) purely imaginary roots for s and (b) multiple roots for s or the associated possibility of algebraic behavior.

For $\chi \rightarrow \infty$, the solution will asymptotically approach $\mathbf{0}$ on a path lying in the subspace of the phase space that is spanned by those characteristic vectors of the linear problem that are associated with the eigenvalues with $\Re s < 0$. Were we to mix in a characteristic vector with positive $\Re s$, that would drive the trajectory away from $\mathbf{0}$. In the final approach to $\mathbf{0}$, the trajectory in phase space is further squeezed into the subspace spanned by only the characteristic vector (or complex conjugate pair) with the *smallest* value of $-\Re s$. We call these the dominant eigenvalue and the dominant eigenvector(s) and we shall use the notation $\sigma = -\Re s$. In this last phase of the approach to the origin, the behavior is either pure exponential decay or exponential decay with a superposed oscillation.

When the representative point in the phase space is very near to the origin, that corresponds to being far from the structure itself in x in the original space. This translates into the simple nature of the forward edge of the solitary wave in original space: far ahead of the main body of the pulse the structure is like $\exp(-\sigma\chi)$, with or without superposed oscillations.

A similar discussion applies as we take the solution backward into negative χ . Here, we are concerned only with the case when the trajectory goes back to the origin $\mathbf{0}$. But this time, we are interested in the characteristic value for the linear problem about $\mathbf{0}$ that has smallest *positive* real part, ρ , say. When the appropriate characteristic value is real, the behavior near to the fixed point is proportional to $\exp(\rho\chi)$ times the eigenvector that goes with this eigenvalue. When the dominant eigenvalues (in the sense of being closest to the imaginary axis in root space) are a complex pair, the approach to $\mathbf{0}$ is oscillatory and occurs in the subspace spanned by the corresponding two complex eigenvectors. In short, as we move away from the pulse in either direction (in χ) the decay to the fixed value of \mathbf{H} at $\pm|\chi|$ is exponential, with or without superposed oscillations.

We know then how the trajectory leaves the fixed point and how it returns. What we need to know in addition is the condition for these two asymptotic motions (in χ) to meet sufficiently smoothly at some finite χ to form a single orbit. If the values of the system parameters Γ are fixed, we need to choose a value $c = c_0(\Gamma)$ such that we do have a homoclinic trajectory for (2.3). For the FitzHugh–Nagumo equation, for example, it has been shown that such values of c_0 exist [13]. Similar results exist for many well-studied problems and homoclinic orbits have played a fundamental role in dynamical systems. They have been used especially in the theory of chaos [31], largely because they do arise often for reasonable parameter values. What makes them usable is that they can be found numerically, to good accuracy, with quite modest means.

When the homoclinic orbit is stable, there is no problem in finding it numerically. To find an unstable homoclinic orbit, a simple approach is to choose parameter values for which the problem is greatly reduced and homoclinic orbits are easily found. Then the parameters are incremented in small steps back to the desired values, and a new homoclinic orbit is calculated at each step, using the orbit from the previous step as a first guess [2]. Such homotopic methods have been used in dynamical problems with success, so we may consider that this technical part of the problem has been resolved for the rest of this paper. We can then go on to see how to use these pulses.

Another issue that ought to be mentioned is the prospect of finding many homoclinic orbits issuing from the same fixed point. This can arise for us because c is not given a priori but is sort of a nonlinear eigenvalue that goes with the homoclinic orbit. A typical nonuniqueness arises with the occurrence of many discrete eigenvalues of c

corresponding to homoclinic solutions in the form of pulses, double pulses, triple pulses, and so on [11], [12]. As we are able to approximate the multipulse solutions as several nonlinearly superposed single pulses, we shall simplify at the outset by considering only the single-pulse coherent structure in our scheme.

3. Dynamics of pulses. We will approach the problem of interacting coherent structures from the point of view of singular perturbation theory. So we begin by defining a small parameter. Let L be the typical separation of the pulses of the system. Pulses tail off on the scale σ^{-1} in the positive χ direction, while in the negative χ direction their decay scale is ρ^{-1} , as discussed in § 2. In this work we shall confine our attention to cases where the parameter, $\delta = \sigma/\rho$, is of order unity. So, we can think of σ as the effective range of the pulses in their interaction. Our small parameter for this work is $\varepsilon = \exp(-\sigma L)$.

A single pulse is given by the homoclinic solution, $\mathbf{H}(\chi)$ discussed in § 2, where $\chi = x + ct$ where c is a constant. By translational invariance, $\mathbf{H}(\chi - a)$ is also a solution, where a is another constant. We choose the origin of χ so that the maximum of the excursion of \mathbf{H} from the fixed point—the peak of the pulse—occurs near $\chi = 0$. Let us look for an N -pulse solution with individual pulses at $\chi = \chi_n$, where $\chi_n(t) = nL + \phi_n(\tau)$ and where $\tau = \varepsilon t$. We seek solutions in the form of a train of pulses:

$$(3.1) \quad \mathbf{U} = \sum_{n=1}^N \mathbf{H}(\chi - nL - \phi_n(\tau)) + \varepsilon \mathbf{R}(\chi, \tau),$$

where N may be any *fixed* integer. Since a superposition even of widely spaced pulses cannot be an exact solution, we have made the accommodation of letting the pulses move slowly; the ϕ_i depend on εt . Even then, the solution cannot be exact, so we allow for that by including a remainder term, $\varepsilon \mathbf{R}$.

We substitute (3.1) into (1.1), and get, on using the definition of \mathbf{H} ,

$$(3.2) \quad c\varepsilon \partial_\chi \mathbf{R} + \varepsilon^2 \partial_\tau \mathbf{R} - \varepsilon \mathcal{L} \mathbf{R} = \varepsilon \sum_{n=1}^N \dot{\phi}_n \mathbf{H}'_n + \mathcal{N} \left(\sum_{i=1}^N \mathbf{H}_i + \varepsilon \mathbf{R} \right) - \sum_{i=1}^N \mathcal{N}(\mathbf{H}_i),$$

where $\mathbf{H}_n = \mathbf{H}(\chi - nL - \phi_n)$. Next, we begin the development of the perturbation theory with the expansion to $O(\varepsilon)$ of the nonlinear term:

$$(3.3) \quad \mathcal{N}(\mathbf{P} + \varepsilon \mathbf{Q}) = \mathcal{N}(\mathbf{P}) + \varepsilon \mathbf{G} \mathcal{N}'(\mathbf{P}) \cdot \mathbf{Q} + \dots,$$

where \mathbf{G} denotes *functional* gradient with respect to the argument, taken at $\varepsilon = 0$. Thus, we obtain the perturbation equation for \mathbf{R} :

$$(3.4) \quad (c\partial_\chi - \mathcal{L}) \mathbf{R} - \mathbf{G} \mathcal{N}' \left(\sum_{n=1}^N \mathbf{H}_n \right) \cdot \mathbf{R} = \varepsilon^{-1} \left[\mathcal{N} \left(\sum_{n=1}^N \mathbf{H}_n \right) - \sum_{n=1}^N \mathcal{N}(\mathbf{H}_n) \right] + \sum_{n=1}^N \dot{\phi}_n \mathbf{H}'_n + O(\varepsilon).$$

Now we introduce a simple approximation on nonlinear functions of sums of widely spaced pulses. Let $\mathcal{N}(X) = X^n$. Suppose that $X = \sum_{m=1}^N H_m$, where the H_m peak at widely spaced points, $\chi = \chi_m := mL + \phi_m$. The value of H_{m+1} is $O(\varepsilon)$ at χ_m and is $O(\varepsilon^2)$ at χ_{m-1} , and so on. We see that $\mathcal{N}(\sum_{m=1}^N H_m) = \sum_{m=1}^N \mathcal{N}(H_m) + O(\varepsilon)$. When we keep only the leading terms in such a development, we shall call that the superposed pulse approximation, or Spa. In the examples, for which we use the Spa here, we deal with vector valued functions like \mathcal{N} , and these are functions of the components of \mathbf{U} and possibly its χ derivatives. We shall assume that these functions are (possibly

infinite) sums of monomials in those components and their derivatives. Then, in (3.4) in particular, we have

$$(3.5) \quad \mathbf{GN}\left(\sum_{m=1}^N \mathbf{H}_m\right) = \sum_{m=1}^N \mathbf{GN}(\mathbf{H}_m) + O(\varepsilon).$$

The Spa also tells us that the right-hand side of (3.4) is really $O(1)$, despite appearances. Now we can rewrite (3.4) as

$$(3.6) \quad \mathcal{H}\mathbf{R} = \varepsilon^{-1} \left[\mathcal{N}\left(\sum_{n=1}^N \mathbf{H}_n\right) - \sum_{n=1}^N \mathcal{N}(\mathbf{H}_n) \right] + \sum_{n=1}^N \dot{\phi}_n \mathbf{H}'_n + O(\varepsilon),$$

where

$$(3.7) \quad \mathcal{H} = c\partial_\chi - \mathcal{L} - \sum_{m=1}^N \mathbf{GN}(\mathbf{H}_m) \cdot \cdot$$

The linear operator \mathcal{H} has a form that is reminiscent of the Hamiltonian operator of the time-independent Schrödinger equation for an electron moving in a lattice. That is, it is the sum of the linear differential operator, $c\partial_\chi - \mathcal{L}$, and the “lattice potential” $\mathcal{V} = \sum_{m=1}^N \mathbf{GN}(\mathbf{H}(\chi - mL - \phi_m))$, which need not be periodic. The approximations that we have used here correspond to what is called *tight binding* in solid state physics, in which the wave function has greatest intensity near the atoms of the lattice. Here, \mathbf{R} is the analogue of the wave function, and we may similarly expect it to be concentrated near the pulses.

There are also differences from the solid state situation: our differential operator need be neither second-order nor self-adjoint. Nevertheless, some of the approaches used in the solid state analogue are helpful to us. Thus, we observe that for widely separated pulses the eigenfunctions of \mathcal{H} are well approximated by eigenfunctions of the operator

$$(3.8) \quad \mathcal{H}_j = c\partial_\chi - \mathcal{L} - \mathbf{GN}(\mathbf{H}_j) \cdot \cdot$$

As we see on differentiating (2.3) with respect to χ , this operator has *zero* as an eigenvalue for the eigenfunction $\mathbf{R} = \mathbf{H}'_j$, where the prime means differentiation with respect to the argument. But the significant reason for the existence of this neutral mode is the translational invariance of the original problem. Indifference to transformation with respect to the parameter of any group (here translation) makes the first variation of the homoclinic solution a mode of zero eigenvalue.

All this leads us to expect \mathcal{H} to have an eigenvalue near to zero. If we seek a solution to the equation

$$(3.9) \quad \mathcal{H}\mathbf{Q} = E\mathbf{Q},$$

where E is an eigenvalue, we can approximate the solution by

$$(3.10) \quad \mathbf{Q} = \sum_{i=1}^N A_i \mathbf{H}'_i$$

where the A_i are constants. We can write a stationary for E in the usual way. When \mathcal{H} is self-adjoint, the stationary expression for E actually bounds it from above, and we have a systematic procedure for improving our estimate of E . But it is sufficient for our purposes that we can use the stationary expression to estimate that $E = O(\varepsilon)$.

Now, if it is to be true that a solution to the problem in the form of widely separated, interacting pulses exists, then the correction term, $\varepsilon\mathbf{R}$ in (3.1), must be

small. To ensure this, we require \mathbf{R} to be finite everywhere. To close this problem, we impose this condition of finiteness on \mathbf{R} as a solvability condition on (3.6). But here, we take this only to leading order in ε .

We assume the usual inner product of vector analysis and use integration over the infinite range of χ for functional products. Then we can define an adjoint linear operator \mathcal{H}^\dagger through the relation

$$(3.11) \quad \int_{-\infty}^{\infty} \mathbf{P} \cdot \mathcal{H}\mathbf{R} \, d\chi = \int_{-\infty}^{\infty} (\mathcal{H}^\dagger\mathbf{P}) \cdot \mathbf{R} \, d\chi,$$

with a similar definition for \mathcal{H}_j^\dagger . Thus,

$$(3.12) \quad \mathcal{H}_j^\dagger = -c\partial_\chi - \mathcal{L}^\dagger - [\mathbf{G}\mathcal{N}]^\dagger(\mathbf{H}_j) \cdot$$

Very often, the existence of a null vector of \mathcal{H} means that we can expect a solution of

$$(3.13) \quad \mathcal{H}^\dagger\mathbf{P} = \mathbf{0}$$

to exist. But when \mathcal{H}_j is not self-adjoint that is not a foregone conclusion, although it is usually a safe one. For example, the ground state of the lowering operator for the harmonic oscillator is an eigenstate of eigenvalue zero. Yet its adjoint, the raising operator, does not have zero as eigenvalue. However, this arises because the harmonic oscillator is in an infinitely deep well, which we do not have here. We will proceed here on the assumption that a solution to (3.13) exists.

For the same reasons as for \mathbf{H} , \mathbf{P} must decay exponentially when we go far from $\chi = \chi_j$. We therefore can conveniently denote the adjoint homogeneous solution of (3.13) as $\mathbf{P}_j = \mathbf{P}(\chi - jL - \phi_j)$. Further, by the same arguments as we used for \mathbf{H}_n , we see that

$$(3.14) \quad \mathcal{H}^\dagger\mathbf{P}_j = O(\varepsilon).$$

If we take the inner product of (3.6) with \mathbf{P}_j for any j , we get on the left-hand side a factor that is $O(\varepsilon)$ times $\int d\chi \mathbf{P}_j \cdot \mathbf{R}$. If \mathbf{R} is finite everywhere, this integral is finite, since \mathbf{P}_j decays exponentially. Hence the inner product of any of the \mathbf{P}_j with the right-hand side of (3.6) must be $O(\varepsilon)$. Since the right side of (3.6) is $O(1)$, by the Spa, we are driven to a solvability condition in leading order straightaway.

Now, we need to calculate the $O(\varepsilon)$ term in (3.5). To get it, we again consider $\mathcal{N}(\{H_j\}) = (\sum_{m=1}^N H_m)^n$. We obtain $\mathcal{N} = \sum_{m=1}^N (H_m)^n + \sum_{m=1}^N \sum_{j \neq m} n(H_m)^{n-1} H_j$, plus terms $O(\varepsilon^2)$. The first term is the Spa and, in the next term, we get as single factors pulses whose peaks are far from χ_m , for each m . The factors in the second term generated by the distant peaks are produced by their exponentially small tails and have local amplitude $O(\varepsilon)$. In the case where \mathcal{N} does not contain derivatives, we have

$$(3.15) \quad \mathcal{N}\left(\sum_{i=m}^N \mathbf{H}_m\right) = \sum_{m=1}^N \mathcal{N}(\mathbf{H}_m) + \sum_{m=1}^N \sum_{j \neq m} \mathbf{G}\mathcal{N}(\mathbf{H}_m) \cdot \mathbf{H}_j + O(\varepsilon^2).$$

We can do a similar thing when \mathcal{N} contains derivatives, but it is not useful to write that out here.

If we now project the right-hand side of (3.6) onto \mathbf{P}_j , we get, in leading order,

$$(3.16) \quad (\mathbf{P}_k | \mathbf{H}'_k) \dot{\phi}_k + \varepsilon^{-1} \left(\mathbf{P}_k \left| \sum_{m=1}^N \sum_{j \neq m} \mathbf{G}\mathcal{N}(\mathbf{H}_m) \cdot \mathbf{H}_j \right. \right) = O(\varepsilon).$$

Since the inner product involves integration over χ , the second term in (3.16), when written out, is

$$(3.17) \quad \varepsilon^{-1} \int d\chi \sum_{m=1}^N \sum_{j \neq m} \mathbf{P}_k \cdot \mathbf{G}\mathcal{N}(\mathbf{H}_m) \cdot \mathbf{H}_j.$$

Because \mathbf{P}_k is sharply peaked, only terms linear in $H_{k\pm 1}$ contribute to this expression, since more distant peaks contribute only terms of order ε^2 . So, in leading order, (3.17) becomes

$$(3.18) \quad \varepsilon^{-1} \int d\chi \mathbf{P}_k \cdot \mathcal{F}(\mathbf{H}_k) \cdot (\mathbf{H}_{k+1} + \mathbf{H}_{k-1}).$$

where \mathcal{F} is a suitable nonlinear function.

Now, $\mathbf{H}_{k\pm 1} \propto \Re \exp \{s_{\pm}[\chi - (k \pm 1)L - \phi_{k\pm 1}]\}$ in the neighborhood of χ_k , where we recall that $\Re s_+ = \rho$ and $\Re s_- = -\delta\rho$. If we set $y = \chi - kL - \phi_k$, we find that, for instance, $\mathbf{H}_{k+1} \propto \Re \exp [s_+ y - s_+ L - s_+(\phi_{k+1} - \phi_k)]$. Since $\varepsilon = \exp(-\rho L)$, (3.18) reduces to two terms, one of which is proportional to

$$(3.19) \quad \exp[-\rho(\phi_{k+1} - \phi_k)] \cos[\omega_+(\phi_{k+1} - \phi_k - \Phi_+)] \Re \int \mathbf{P}_k(y) \cdot \mathbf{G}\mathcal{N}(\mathbf{H}_k(y)) e^{\rho y} dy.$$

We can see that the integral converges and, like ω and Φ , is a parameter of the system.

The other term from (3.18) has a similar integral as a factor, which does not matter here; the interesting factor is

$$(3.20) \quad \varepsilon^{\delta-1} \exp[-\rho\delta(\phi_k - \phi_{k-1})] \cos[\omega_-(\phi_k - \phi_{k-1} - \Phi_-)].$$

For δ close to unity, we have $\varepsilon^{\delta-1} = 1 + (\delta - 1) \ln \varepsilon + \dots$. Hence, for $\delta \rightarrow 1$, if we neglect $O(\varepsilon)$ and $O(\delta - 1)$, we have for the solvability condition,

$$(3.21) \quad \begin{aligned} \dot{\phi}_k &= a_0 e^{-\rho_0(\phi_{k+1} - \phi_k)} \cos[\omega_0(\phi_{k+1} - \phi_k) - \Phi_0] \\ &+ a_1 e^{-\rho_1(\phi_k - \phi_{k-1})} \cos[\omega_1(\phi_k - \phi_{k-1}) - \Phi_1], \end{aligned}$$

where the overdot denotes differentiation with respect to τ and a_I, ω_I , and Φ_I are constants with $I = 0, 1$ and $\rho_0 = \rho$. Some constants have been renamed in an evident way for cosmetic reasons, as in $\rho_1 = \sigma = \delta\rho$.

We have not given any higher terms in (3.21), but we should say that there is only one term $O(\delta - 1)$ and a diversity of $O(\varepsilon)$ terms. Provided that ε^δ stays larger than ε , we are able simply to absorb the $\varepsilon^{\delta-1} - 1$ terms into the factor a_1 , with a gain in the range of usefulness of (3.21). Another change in (3.21) makes it easier to work with. We reintroduce the explicit positions of individual pulses,

$$(3.22) \quad \chi_k(t) = kL + \phi_k(\tau).$$

Then, with $\tilde{\Phi}_I = \Phi_I + \omega_I L$, (3.21) becomes

$$(3.23) \quad \begin{aligned} \dot{\chi}_k &= a_0 e^{-\rho_0(\chi_{k+1} - \chi_k)} \cos[\omega_0(\chi_{k+1} - \chi_k) - \tilde{\Phi}_0] \\ &+ a_1 e^{-\rho_1(\chi_k - \chi_{k-1})} \cos[\omega_1(\chi_k - \chi_{k-1}) - \tilde{\Phi}_1]. \end{aligned}$$

4. Pattern dynamics.

4.1. Lattice forces. We have taken a rather general PDE with translational invariance that admits pulse-like solutions and have looked for N -pulse solutions. Those can be found, provided that the pulses remain far enough apart and move in accordance with (3.23). For that kind of solution, we can reduce the PDE to this relatively simple system of ODEs. To conclude, we want to sketch some features of this dynamical system.

It is convenient to write (3.23) in abbreviated form, letting

$$(4.1) \quad \dot{\chi}_k = F_R(\chi_{k+1} - \chi_k) + F_L(\chi_k - \chi_{k-1}),$$

where F_R is the force exerted by the pulse to the right and F_L is exerted by the pulse to the left, respectively, and we make the convention that $\chi_{k+1} > \chi_k$. The subscripts R and L seem easier to keep in mind than 0 and 1 in this context. So we shall, in (3.23), let $I = 0 = R$ and $I = 1 = L$. Then, we can write the general form of the force as

$$(4.2) \quad F_I(\Delta_k) = a_I e^{-\rho_I \Delta_k} \cos(\omega_I \Delta_k \Omega_I)$$

where

$$(4.3) \quad \Delta_k = \chi_k - \chi_{k-1}.$$

The various constants, like a_I , are renamed from (3.23), whose form is, however, kept intact. As before, $\rho_L/\rho_R = \delta \approx 1$. We have previously reported on results obtained by simulations on a particular case of this system [9].

The dynamics implied by (4.1) are simple. In a space of dimension d , a Coulomb-type force between two particles a distance r apart goes like $r^{-(d-1)}$. For $d = 1$, the force is constant, as it is here, except for the exponentials. The monotonic parts of the exponentials describe the ranges of the forces and, because the pulses need not be symmetric fore and aft, the ranges may be different on the two sides. The oscillatory part of the range has no obvious analogue in traditional dynamics. The term on the left of (4.1) ought to be moved to the right and thought of as a frictional drag. And, or course, there is no inertial term because we disallowed Galilean invariance.

The exponential nature of the interpulse forces is a result of the great separation of pulses, which exists by assumption. To get an idea of the risk of violating this condition, let us study what happens when particles k and $k-1$ develop a smaller than average separation. Let $\Delta_{k+1} \gg \Delta_k$ and $\Delta_{k-1} \gg \Delta_k$. Then we easily derive the approximate equations

$$(4.4) \quad \dot{\Delta}_k = F_L(\Delta_k) - F_R(\Delta_k)$$

and

$$(4.5) \quad \dot{\Xi} = \frac{1}{2} F_L(\Delta_k) + \frac{1}{2} F_R(\Delta_k),$$

where $\Xi = \frac{1}{2}(\chi_k + \chi_{k-1})$.

Equation (4.4) says that the pulse separation will adjust until the left and right forces are equal and then the particles will lock in, with constant separation. In cases where an oscillatory tail separates the pulses, we can expect them to be kept apart by this locking effect, at least long enough for the results to be interesting, for at least some of the locked states will be stable. The approach of a third and a fourth particle to the pair could be disruptive and often is. But this need not cause close approaches. Though there are parameter choices in some problems where the world lines of the pulses do cross, as these remarks suggest, there are many cases where no problem arises [9]. Equation (4.5) tells us that the velocities in the various locked states are constant and different from that in the single particle state. (These locked states may in fact approximate double-humped homoclinic solutions of the associated PDE, as we know from computations on specific examples that will be elaborated elsewhere [7].)

4.2. Pattern maps. We see that (4.1) has a solution with $\dot{\chi}_k = V$ where V is a constant velocity in the frame of the single pulse, independent of k . In this case, pulse

spacings do not change in time, for the pulses are locked together as for the pair we just discussed. We obtain the *pattern map*,

$$(4.6) \quad V = F_R(\Delta_{k+1}) + F_L(\Delta_k).$$

Given a spacing between two neighboring pulses, we can deduce the spacing between the next pair over from (4.6) and so use (4.6) to map out the entire steady pattern. The simplest pattern has uniform spacing, $\Delta_k = \Delta$, and we get the dispersion relation $V(\Delta) = F_R(\Delta) + F_L(\Delta)$. We thus get qualitatively useful results from the general theory without extensive computation. The stability of the uniform pattern is straightforward to investigate as we shall see presently, but first, let us look at (4.6) a bit more.

Maps like (4.6) are dynamical systems. They may arise as first return maps for ODEs like (2.3). There are really three distinct cases of (4.6). We can have (1) purely exponential tails on both sides, (2) one oscillating and one monotonous, or (3) both oscillatory. Each of these cases gives rise to different pattern dynamics, and we shall illustrate the content of this work with a brief discussion of case (2). Case (1) is too simple to be intriguing and case (3) is normally an implicit map, too complicated for a brief discussion. It arises in certain fourth-order ODEs and has recently been investigated by Fowler and Sparrow (preprint).

In case (2), let $Z_k = \exp(-\rho_R \Delta_k)$. Then (4.6) takes the form

$$(4.7) \quad Z_{k+1} = C - AZ_k^\delta \cos[\omega \ln Z_k - \Omega],$$

where the constants have been renamed to streamline the formula. This map has both regular and chaotic solutions [2]. In parallel with the familiar period-doubling sequence of chaos theory [6], there are sequences of patterns with trains of paired pulses, paired pairs, and so on. All of the complications maps of the line are here. The possibilities seem endless and, to begin to sort them out, it will help to study their stability.

4.3. Stability theory. Now we let

$$(4.8) \quad \chi_k = k\Delta + Vt + \theta_k$$

and treat θ_k as a small perturbation to the pattern with constant spacing, Δ . The linear equation is

$$(4.9) \quad \frac{d\theta_k}{dt} = F'_R(\Delta)[\alpha\theta_{k-1} - (1 + \alpha)\theta_k + \theta_{k+1}],$$

where $\alpha = -F'_L(\Delta)/F'_R(\Delta)$.

Basically, α measures the slope of the pattern map, so it is clearly related to the stability of the map. But that is not the same thing as the stability of the pattern in the PDE; that is governed by (4.9). Another way to think about α is as a measure of dispersion. If we define an individual pulse velocity $V_k = \dot{\chi}_k$, we find from (4.1)

$$(4.10) \quad \frac{\partial V_k}{\partial \chi_k} = -F'_R(\Delta_{k+1}) + F'_L(\Delta_k).$$

For constant Δ_k , this quantity is $-F'_R(1 + \alpha)$, a sort of tidal force on pulses.

We look for solutions of (4.9) in the form $\theta_k = \Theta_l \exp[\sigma_l t + 2\pi ikl/N] + \text{c.c.}$ where N is the total number of pulses and we find

$$(4.11) \quad \sigma_l = -F'_R(\Delta) \left[(1 + \alpha) \left(1 - \cos \frac{2\pi l}{N} \right) \pm i(1 - \alpha) \sin \frac{2\pi l}{N} \right].$$

This says that the necessary and sufficient condition for stability of the pattern with constant spacing Δ is that

$$(4.12) \quad F'_R(\Delta) - F'_L(\Delta) > 0.$$

Comparison with formula (4.10) for dispersion suggests that, should the pulses be moving faster when they are farther downstream, instability develops. When $F'_R(\Delta) + F'_L(\Delta) \neq 0$, the instability is in the form of a growing oscillation.

If we have more complicated patterns, the stability theory proceeds in the same way, leading to more complicated instability criteria. For trains of pulse pairs, or pairs of pairs, the treatment is simple and comes down to application of the Routh-Hurwitz criterion. We have decided to omit these details, for the main outcome is that there are typically many linearly stable solutions. The message is that these are metastable [9] and that the final state depends on initial conditions and the level of noise in the background. Such a system resembles what physicists call a spin glass, but it is not a potential system. It is not clear what kind of statistical mechanics to invent for this problem as yet.

4.4. Large-scale dynamics. The pattern theory of this section is founded on the idea that the pulses are widely separated compared to their widths. Within this restriction, (3.23) or (4.1) gives a detailed description of the pattern. The last question we touch on is this: how should we formulate the question of organization on scales very large compared to the typical pulse separation, L ? In that limit, L seems small, and we may think of a continuous medium of pulses. From that vantage point, we no longer distinguish individual pulses but worry about some measure of their density. This limit has been extensively studied by Rosenau [29] for lattice dynamics like (4.1). We next apply his approach to this case, making the needed, but evident, generalizations for asymmetric pulses.

Let $u_k = \Delta_k/L$. This quantity is a measure of the local spacing of pulses, hence it is an inverse pulse density. From (4.1), we can derive an equation for the evolution of u_k :

$$(4.13) \quad L\dot{u}_k = F_R(Lu_{k+1}) - F_R(Lu_k) + F_L(Lu_k) - F_L(Lu_{k-1}).$$

In the limit $L \rightarrow 0$, we replace the discrete variable k by the continuous variable $x = kL$. Then $u_k(t)$ goes over into $u(x, t)$.

Now we introduce $\mathcal{F}_l(u) = F_l(Lu)$, and (4.13) becomes

$$(4.14) \quad L\partial_t u = (e^{L\partial_x} - 1)\mathcal{F}_R(u) - (e^{-L\partial_x} - 1)\mathcal{F}_L(u).$$

To study the stability of the uniformly spaced solution, we set $u = 1 + v$ where v is a small perturbation that we may take to vary like $\exp(\sigma_k t + ikj) + \text{c.c.}$. Then, the linear perturbation equation gives us back the dispersion relation (4.11).

In the nonlinear regime, we can replace the exponentiated operator by its first few terms. Once that is done, it is possible to regroup those few terms in the manner of Padé. Thus, we write the expansion

$$(4.15) \quad e^{\pm L\partial_x} - 1 = \frac{1}{2}L^2(1 + \frac{1}{12}L^2\partial_x^2 + \dots)\partial_x^2 \pm L(1 + \frac{1}{6}L^2\partial_x^2 + \dots)\partial_x,$$

and regroup these terms to give the approximation

$$(4.16) \quad e^{\pm L\partial_x} - 1 = \frac{1}{2} \frac{\partial_x^2}{1 - \frac{1}{12}\partial_x^2} \pm \frac{\partial_x}{(1 - \frac{1}{12}\partial_x^2)^2},$$

where $X = x/L$. Then (4.14) becomes, after some rearrangement,

$$(4.17) \quad L\partial_t u = \frac{1}{6}L(1 - \frac{1}{24}\partial_x^2)\partial_x^2\partial_t u + \frac{1}{2}(1 - \frac{1}{12}\partial_x^2)\partial_x^2(\mathcal{F}_R - \mathcal{F}_L) + \partial_x(\mathcal{F}_R + \mathcal{F}_L).$$

Suppose that we are close to an equally spaced solution with $L = \Delta$ and write $u = 1 + v/L$. Then, (4.17) becomes

$$(4.18) \quad \begin{aligned} \partial_t v = & \frac{1}{6} (1 - \frac{1}{24} \partial_x^2) \partial_x^2 \partial_t v + \frac{1}{2} (1 - \frac{1}{12} \partial_x^2) \partial_x^2 (\alpha_2 v + \frac{1}{2} \beta_2 v^2 + \dots) \\ & + \partial_x (\alpha_1 v + \beta_1 v^2 + \dots) \end{aligned}$$

where $\alpha_k = \mathcal{F}'_R(1) - (-1)^k \mathcal{F}'_L(1)$ and $\beta_k = -\mathcal{F}''_R(1) - (-1)^k \mathcal{F}''_L(1)$.

Next, we should look for a solitary wave solution to (4.18), but perhaps this is not the moment. What we do see from this reduction to another PDE is how the procedure we have outlined connects with what some people call phase dynamics [10], the theory behind various extensions of the Burgers equation [15]. Those theories consider large-scale modulations of simple waves. In looking at our train of pulses in a coarse-grained way, we get the sort of modulational equation associated with those developments.

5. Conclusion. In many fields, complicated solutions to nonlinear field equations can be reduced to superposed, interacting solitary structures. The motion of these objects is usually governed by dynamical equations. Examples include particle physics [27] and the theory of defects [5]. In this work, we have outlined a general approach to such questions in terms of singular perturbation theory. Other approaches have been used in this kind of problem, the most familiar being forms of gauge field theory such as phase dynamics [15]. A relation between the approach we have taken and phase dynamics was indicated at the end of the last section. We would like to conclude by trying to clarify this connection.

In perturbation theory, we develop an approximation around a special solution at a special value, Γ_0 , of the system parameter, Γ . The special solution is either exact or very accurate at Γ_0 . For small $\varepsilon^2 = \Gamma - \Gamma_0$, we look for approximate solutions that are the special solution plus small corrections. Here, smallness is measured in terms of ε . Sometimes this works straightforwardly in the sense of ordinary perturbation theory. At other times, the development fails and leads to singularity.

The latter case is interesting when the special solution contains one or more arbitrary constants. These constants normally enter because the system has invariances either for all Γ or at Γ_0 . The practice in physics is to identify the constants as parameters of continuous groups, so it is possible to see in advance whether these constants will enter. If the system generally, or for $\Gamma = \Gamma_0$, has invariances, arbitrary constants may be expected in the special solution.

When the ordinary perturbation theory blows up in your face, you can try to remove the singularity by choosing the constants in the special solution to remove the trouble. In deriving the amplitude equation known as Landau's equation, you can let the arbitrary amplitude depend slowly on time in such a way as to remove secular terms in the perturbation theory. The idea is that if the original try is not good enough, the easiest adjustment is through the variation of the parameters, to whose values the system is indifferent. In the case of the Landau equation, the approximate linear system is scale invariant, so the amplitude is allowed to vary.

Sometimes it is not enough to let the constants depend on time. So you can try letting them depend weakly on position as well. This is where our departure from conventional multiscaling occurs. Instead of letting the group parameter of our problem depend slowly on position, we let it depend on a discrete index. As we indicated briefly in concluding the last section, when the discrete index becomes continuous, the results look like the sort of equation that ordinary multiscaling produces. The choice of approach depends on the circumstances.

If you find a simple wave solution to your problem, the phase dynamical approach, or slowly varying wave theory, works well. Here the phases are the group parameters that will depend slowly on space and time. When the system supports localized structures, especially those tailing off exponentially, you can describe large-scale patterns by superposing such structures, as we have done here. But, as we saw, even in this discrete case, you can go over to a phase equation in the continuous limit.

Naturally, there are several extensions of such a theory that may be studied. Other invariances, the most common being Galilean and scale invariance, can be included [7], [8]. Higher dimensional problems are also interesting [16], such as vortex interaction in two dimensions. Those extensions, while not obvious, are feasible. But there is a nagging problem that needs to be confronted. It is the choice of N , the number of pulses in the solution. Here, we have specified N , but that choice should be made by the system. So far, we do not know how to allow that. It may be difficult to do this as long as the pulses are constrained to stay far apart, except for integrable systems. Perhaps the right approach may be found in the study of near-integrable systems.

In the meantime, we are interested in testing the theory. We have mentioned the FitzHugh–Nagumo equation [28] as a paradigm and have been studying it with Rinzel along the lines described here. As always, there are some technical details involved in the comparison with experiments, whether real or numerical. Experimental systems are finite, and that has to be allowed for. But the main confrontation with reality arises because we have been thinking of an initial value problem, in which the pulse positions are all given for $t = 0$. In biological experiments one frequently uses pacemakers, that is, one gives the times for creating pulses at $x = 0$. That can be handled, but we shall postpone the discussion to another occasion.

In this work, ε is our small parameter and we have tried to put it in all the right places, while δ is $O(1)$. Nevertheless, we attach great importance to their product, especially in the quantity $\varepsilon\delta\rho\varepsilon\sigma\sigma$. May he continue to operate the lights for another sixty years.

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