Spatiotemporal Complexity in Traveling Patterns

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This is a study of the nonlinear interactions of solitary waves or impulses in homogeneous, extended media. We obtain the set of ordinary differential equations for the positions of the impulses in the nearest-neighbor approximation. Solutions with constant velocity lead to pattern maps that give the successive spacings of the impulses. From among the infinitely many metastable patterns given by the maps, the system chooses one asymptotically through a hierarchical evolution.

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In this Letter we study spatiotemporal complexity in traveling-wave patterns. We consider extended systems with translational invariance that admit the propagation of solitary waves at constant velocity. Behind our work lie theories of defects in systems with broken, discrete symmetries,[1,2] of wave patterns in excitable media,[3] and of dynamical systems.[4] This work is especially applicable to excitable systems such as the Belousov-Zhabotinskii medium,[5] and cardiac fibers and axons that propagate action potentials.[6]

We illustrate the questions of interest with a simple model of an excitable medium[7-9]

$$\epsilon v_t = -w + 3v - v^3 + v_{xx}, \quad w_t = v - \mu, \tag{1}$$

where the subscripts denote partial derivatives and ϵ is a small parameter. The steady, uniform solution of (1) is unstable for $|\mu| < 1$, and stable for $|\mu| > 1$. For $|\mu|$ slightly greater than unity, the system has threshold behavior: Large enough perturbations lead to the propagation of solitary waves through the medium.

Equations (1) are reduced to an ordinary differential equation upon transformation to a moving coordinate system and on the assumption that the solutions depend on $\chi = x + ct$ alone. Then

$$v''' - \epsilon c v'' + 3(1 - v^2)v' - c^{-1}(v - \mu) = 0, \tag{2}$$

where the prime denotes differential with respect to χ. The stable, steady solution of Eqs. (1) becomes a saddle point or a saddle focus in the flow (2).[10] We concentrate on the latter case, which is far richer. For $c = c_0$, a homoclinic solution, $H(\chi)$, occurs,[11] corresponding to a solitary-wave solution of Eq. (1).

Three-dimensional flows, such as (2), passing near a saddle focus have been extensively studied.[4,10,12] A notable theorem of Shil'nikov[13,14] states that under the homoclinic condition, $c = c_0$, and with suitable eigenvalues, Eq. (2) admits an infinite number of unstable periodic orbits. The wealth of behavior implied by this result carries over to the constant-speed, traveling-wave solutions of Eqs. (1). Yet, solutions found in this way are special in having a constant velocity, and their stability in (1) is not apparent from (2) alone. In addressing these issues, we are led into the study of spatiotemporal complexity in traveling patterns.

Consider a general system described by the translationally invariant set of partial differential equations

$$\partial_t \mathbf{U} = \mathbf{L}(\partial_x)\mathbf{U} + \mathbf{N}(\mathbf{U}), \tag{3}$$

where $\mathbf{U}$ is a vector of state variables, $\mathbf{L}$ is a linear differential operator, and $\mathbf{N}$ stands for nonlinear terms. We assume that Eq. (3), of which (1) is a special case, admits a solitary-wave solution, $H(\chi)$, that corresponds to a homoclinic orbit biasymptotic to a saddle focus for the ordinary differential equation in the moving frame.[14] A complex wave pattern is conveniently represented by interacting, coherent features in nonlinear wave theory. This technique has been used in particle physics[15] and in condensed-matter theory,[16] and has also proved valuable for dissipative systems.[1] Here we use it to approximate a general traveling-wave-train solution of Eq. (3) as a superposition of solitary waves, or *impulses:*

$$\mathbf{U} = \textstyle\sum_j \mathbf{H}(\chi - \chi_j(t)) + \mathbf{R}, \tag{4}$$

where χ_j is the position of the jth impulse and $\mathbf{R}$ is a correction term. Figure 1 illustrates the notation.

We assume that the waves are widely spaced with little overlap. This approximation is realistic for excitable media, where the refractory tails[6,8,9] of the impulses keep them apart. If this is to be true, $\mathbf{R}$ should be small. We can then use perturbation theory to calculate it.

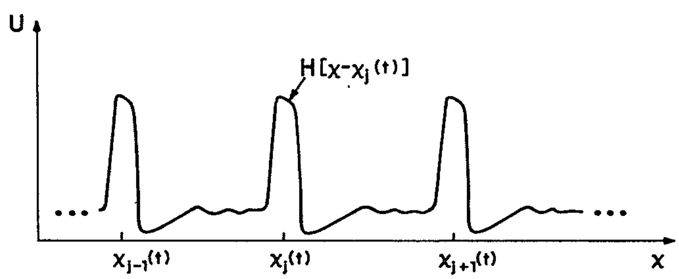

FIG. 1. Schematic illustration of a train of solitary waves with oscillatory tails in the moving frame, $\chi = x + ct$.

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When we introduce (4) into (3), we get

$$\mathcal{L}_j \mathbf{R} = \mathbf{H}_j' \dot{\chi}_j + \mathbf{N}\left[\sum_i \mathbf{H}_i\right] - \sum_i \mathbf{N}(\mathbf{H}_i) + \text{smaller terms}, \tag{5}$$

where $\mathcal{L}_j = c_0 \partial_\chi - \mathbf{L} - \nabla\mathbf{N}(\mathbf{H}_j)$, $\mathbf{H}_j := \mathbf{H}(\chi - \chi_j)$, and the prime and overdot denote differentiation.

A consequence of translational invariance is that $\mathcal{L}_j$ has a null vector (or zero mode): $\mathcal{L}_j \mathbf{H}_j' = 0$. We suppose that, correspondingly, there exists a solution of $\mathcal{L}_j^\dagger \mathbf{P}_j = 0$, where $\mathcal{L}_j^\dagger$ is the adjoint operator defined in the usual way. The right-hand side of (5) must be orthogonal to $\mathbf{P}_j$ and when we allow for the peakedness of the impulses, we obtain equations of motion for them. In the case where the homoclinic solution drops off at comparable rates for $\chi \to \pm\infty$, we get

$$\dot{\chi}_j = b\exp[-\xi(\chi_{j+1} - \chi_j)] + a\exp[-\eta(\chi_j - \chi_{j-1})]\cos[\nu(\chi_j - \chi_{j-1}) + \phi]. \tag{6}$$

Here, ξ and $-\eta \pm i\nu$ are the eigenvalues associated with the saddle focus, a and ϕ are constants which depend on the specific form of Eq. (3), and the overdot stands for differentiation with respect to an appropriately scaled time variable. This scaling may be adjusted to make $b = \pm 1$; for definiteness, we work with $b = 1$.

In the derivation of Eq. (6), only nearest-neighbor interactions have been included, in keeping with the assumption of small overlap between successive impulses. The structure of the right-hand side of (6) depends on the assumed form of $\mathbf{H}$ with a decaying tail on one side and a decaying oscillation on the other.[17] More general evolution equations are obtained with less restrictive conditions on the homoclinic structure, as can be seen by a detailed asymptotic analysis of the solvability conditions.

Equation (6) has separable solutions in the form of a function of t plus a function of j. These special solutions have $\dot{\chi}_i = \Delta c$, and correspond to patterns traveling with constant velocity $c = c_0 - \Delta c$. If we now let

$$Z_i = \exp[-\xi(\chi_i - \chi_{i-1})], \tag{7}$$

(6) becomes a *pattern map*:

$$Z_{i+1} = f(Z_i) := \Delta c - aZ_i^\delta \cos(\omega \ln Z_i - \phi), \tag{8}$$

where $\delta = \eta/\xi$ and $\omega = \nu/\xi$. This is just the first return map for the flow (2) in a small neighborhood of the saddle focus.[4,18] We recover the solutions of (2), but now we can both study their stability as solutions of (1) and consider more general patterns.

The simplest solutions of (8) are the fixed points, $Z_i = Z^*$, corresponding to a train of impulses with constant spacing λ, where $Z^* = \exp(-\xi\lambda)$. For any spacing, we can use (8) to compute Δc. We then have a velocity-spacing relation $c(\lambda)$ that agrees with results from numerical experiments[19] on (1) and, for $\delta < 1$, with previous analytic work.[20] To study the stability of these simplest patterns, we write $\chi_j = j\lambda + \Delta c\, t + \theta_j(t)$. Then, on keeping only terms linear in θ_i, we have

$$d\theta_j/dt = \xi Z^*[-\alpha\theta_{j-1} + (1+\alpha)\theta_j - \theta_{j+1}], \tag{9}$$

where $\alpha = f'(Z^*)$ is the slope of the map at Z^*. For $|\alpha| < 1$, fixed points of the map are stable.

Equation (9) admits solutions of the form $\theta_j = A_k \times \exp(\sigma_k t + ikj) + \text{c.c.}$, where c.c. means complex conjugate, k ranges over integral multiples of $2\pi/N$, and N is the (large) number of impulses in the pattern. This leads to the dispersion relation

$$\sigma_k = \xi Z^*[(1+\alpha)(1-\cos k) \pm i(1-\alpha)\sin k]. \tag{10}$$

For $k = 0$ or $Z^* \to 0$ the regular lattice of impulses is neutrally stable, as expected from translational invariance.

Equation (10) implies that stable fixed points of the map (8) correspond to unstable traveling patterns ($\mathrm{Re}\sigma_k > 0$). However, the converse is not true. Stable, equally spaced traveling waves correspond to unstable fixed points of the map only for $\alpha < -1$. Thus $\alpha < -1$ is the necessary and sufficient conditon for stability of the traveling-wave solutions that arise as fixed points of the map. For general b, this condition implies $dc/d\lambda > 2b\xi Z^*$. In the limit of large λ this reduces to a known criterion for instability, $dc/d\lambda < 0$.[21] Another vision of such results is provided by the continuum limit of (6), which is a point of contact with work on phase instability.[22,23]

Equation (8) is a cornucopia of constant-velocity solutions at small Δc. The simplest are the fixed points shown in Fig. 2 for the homoclinic case $\Delta c = 0$ with

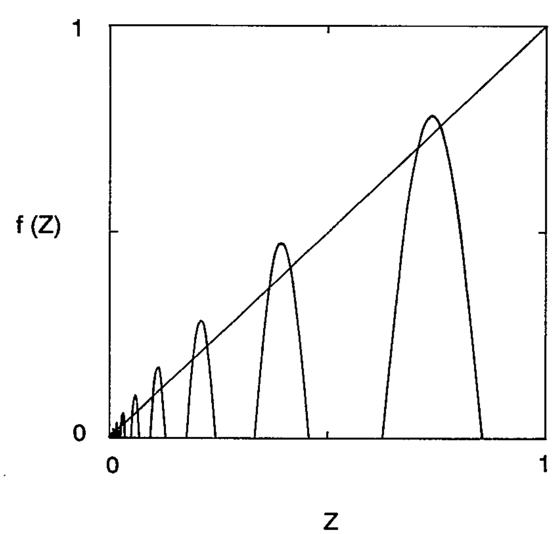

FIG. 2. The map (8) for the homoclinic case, $\Delta c = 0$, with $\delta < 1$. The intersections of the diagonal with the map correspond to equally spaced traveling-wave solutions of Eq. (1).

$\delta < 1$. There are infinitely many of them with $\alpha < -1$ accumulating at $Z = 0$. Hence there are infinitely many metastable, equally spaced patterns, traveling with velocity c_0, with spacings λ ranging up to infinity. In addition, we have the fixed points of the iterates of (8) of higher and higher orders and multipulsed homoclinic solutions.[3,20] The former correspond to nonuniformly spaced periodic trains, while the latter represent finite trains. The stability problem of a wave train corresponding to a fixed point of the iterated map in any order can be reduced to the analysis of a characteristic polynomial with the Routh-Hurwitz criterion. Sufficient conditions for instability can be written simply, but the necessary conditions are complicated. We leave the details aside for now since they do not influence the essential conclusion that for an infinite system, *there exist infinitely many metastable periodic traveling patterns with all length scales.*

Numerical simulations on (6) reveal that finite systems relax to a pattern with uniform velocity, the particular pattern selected being sensitive to initial conditions. We conclude that the relaxation is toward one of the many possible metastable states already mentioned. Starting from a rather uniform, unstable initial pattern, the relaxation proceeds with the formation of larger and larger patches of impulses moving at nearly the same velocity, and the evolution proceeds on longer and longer time scales. We have followed this evolution numerically, starting from such equally spaced unstable patterns, with $\chi_j(0) = j\lambda$, and $\alpha = f'[\exp(-\xi\lambda)] > -1$. In Fig. 3, we present some of the world lines, $\chi_j(t)$, obtained by numerical integration of (6).

We see that the uniform train breaks up into groups of impulses, each with a local value of α smaller than -1. This leads to the creation of larger spacings, or defects, seen in Fig. 3 as white stripes.[24] These larger spacings introduce exponentially longer relaxation times, as suggested by Eq. (10). Our interpretation is that, as the subsequent evolution of the pattern proceeds on a longer time scale, it is governed by the interaction between the groups of impulses. The whole process repeats itself hierarchically until the asymptotic state of a pattern with uniform velocity is reached. In this final state the position of every pulse is determined by the pattern map. Hence χ_k is a function of χ_1: The correlation length has increased to be the size of the system.

Another vision of this process is provided by Fig. 4(a) which shows the spacing between successive impulses as a function of the impulse number j for an asymptotic metastable pattern. By the time that the largest group of correlated impulses has attained the size of the system, and the asymptotic form has been reached, four generations of relaxation have occurred, represented by the four lengths in the figure. The stabilization process that the system performs is better understood in the light of Fig. 4(b) which shows $\beta(j) := Z_j[1+\alpha(Z_j)]$. This function is a measure of the local growth rate of perturbations for a given spacing [see Eq. (10)]. It arises in the stability criteria for nonuniform, periodic patterns. We suggest that β plays a role in the characterization of the approach to an asymptotic state of the system. From

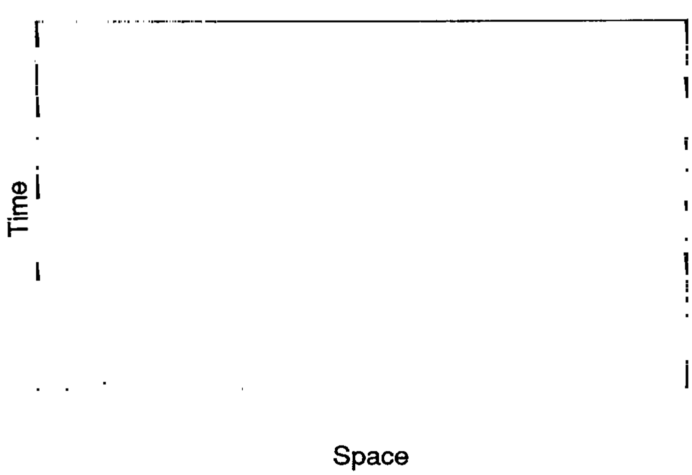

FIG. 3. World lines, $\chi_j(t)$, of an initially unstable, equally spaced pattern. The pattern, computed with (6), contains 200 impulses but only a portion of it is displayed. Parameter values used in this simulation are $\xi = 1.0$, $\eta = 0.8$, $\nu = 10.0$, $\alpha = 1.0$, and $\phi = 0.0$. Initial spacing is 2.204.

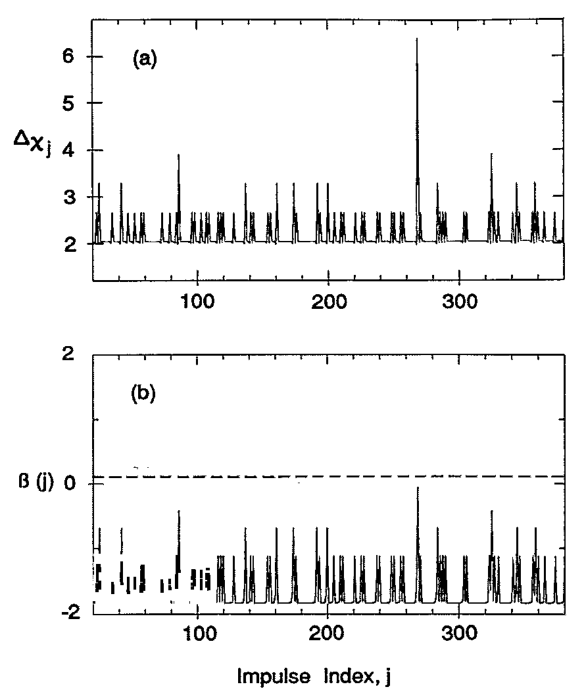

FIG. 4. (a) $\Delta\chi_j := \chi_j - \chi_{j-1}$ and (b) $\beta(j) := Z_j[1+\alpha(Z_j)]$ as functions of j for an asymptotic, metastable pattern of 1000 impulses of which only a 360 are shown. The dashed line in (b) represents the initial, equally spaced ($\lambda = 2.191$) unstable pattern. Parameter values are as in Fig. 3.

a constant *positive* value (dashed line in the figure) for the uniform unstable initial pattern, it tends to an *everywhere negative,* irregular asymptotic form in the final metastable traveling pattern.

These numerical results are perforce limited to rather small systems. Nevertheless, they qualitatively suggest the development of scaling structure that we propose to elaborate in future work. We also hope to return to the subject of experimental tests elsewhere. In particular, some of our results can be tested in experiments where the initial spacings are forced. For a system prepared with uniform spacing, we would expect to find windows (as a function of spacing) of regularly spaced patterns interspersed with complex patterns. Further, Eq. (6) can be modified to allow for the effect of a periodic stimulation of the medium by a pacemaker. This variant of the problem is related to a recent study of traveling patterns in stimulated His-Perkinje fibers (of the heart system) where the kind of complexity considered here has been observed.[25]

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