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A state dependent reinsurance model

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ABSTRACT

We consider the surplus of an insurance company that employs reinsurance. The reinsurer covers part of the claims, but in return it receives a certain part of the income from premiums of the insurance company. In addition, the reinsurer receives some of the dividends that are withdrawn when a certain surplus level b is reached.

A special feature of our model is that both the fraction of the premium that goes to the reinsurer and the fraction of the claims covered by the reinsurer are state-dependent. We focus on five performance measures, viz., time to ruin, deficit at ruin, the dividend withdrawn until ruin, and the amount of money transferred to the reinsurer, respectively covered by the reinsurer.

1. Introduction

An effective way for an insurance company to reduce risk is to buy a reinsurance. According to the reinsurance contract, part of the expenditure burden caused by claims is covered by the reinsurer, and in return the insurance firm transfers part of its income premium to the reinsurer. In addition part of the dividends, that are withdrawn when a certain surplus level b is reached is also transferred to the reinsurer.

The reinsurance may be assumed to be provided instantaneously. In practice, big institutions such as corporations of several big insurance companies, governments or national banks may cover the losses of the insurance firm.

In our model the input is a fluid stream of premiums with general *state-dependent* input rate, and the output is generated by negative *state-dependent* jumps corresponding to the claims that are partially covered by the reinsurer (in a state-dependent way). When the surplus reaches a certain level b (which could be a decision variable) the extra input from premiums is taken as a dividend, so that the surplus is bounded by b. Then the withdrawal of dividend is stopped once the surplus drops below b (at the time of a claim) and so forth.

Let $\tilde{\mathbf{R}} = \{\tilde{R}(t) : t \geq 0\}$ be the *risk-type* process, whose content level is the surplus cash where both the input and the output are state-dependent.

Input: We assume without loss of generality and without any impact on the analysis, that the gross input rate is the constant c, but the net input rate (the dominant factor in the analysis) is a general deterministic function, say $0 < \alpha_R(x) < c$. We modify the process $\tilde{\mathbf{R}}$ as follows: when level b is reached all the extra input from premiums are taken as dividends. Let $\mathbf{R}$ be the modified process. Clearly, $\mathbf{R} \leq b$ and during a dividend period, say I, $\alpha_R(b-)I$ represents the net income from dividend that is taken by the insurance firm, while the part $[c - \alpha_R(b-)]I$ of the dividend is transferred to the reinsurer. Overall, $\alpha_R(x)dx$ for $0 < x \leq b$ is the net amount of infinitesimal input added to the cash of the insurance firm, whenever the state is x.

Output: The net infinitesimal output rate $\beta_R(x)dx$ is a general deterministic function where $0 < \beta_R(x) < 1$; it means that $\beta_R(x)dx$ is the net infinitesimal loss that is subtracted from the content level of the cash, whenever x is downcrossed at moments of claims (negative jumps); the infinitesimal amount $[1-\beta_R(x)]dx$ is covered by the reinsurer.

The policy described above provides a general framework for state dependent claim payments. For a better understanding of how this policy can be implemented consider the following special

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case: suppose that an arriving claim finds the surplus below a certain threshold level γ, or alternatively, it brings the surplus below that level. Then, the reinsurer covers a certain part of the claim, i.e. $\beta(x) = \beta_0$ for $x < \gamma$. But, whenever the arriving claim does not find or bring the surplus below level γ, the reinsurer pays nothing. In this case $\beta(x) = \beta_0$ for $x < \gamma$ and $\beta(x) = 1$ for $x > \gamma$. In fact, this policy has been introduced in Boxma et al. (2017), but for the discounted model. A natural extension of the latter dichotomous case is to take $\beta(x)$ as a *step function*. That is, let $0 = \gamma_0 < \gamma_1 < \gamma_2 < \cdots < \gamma_n$ and $\beta(x) = \beta_i$ for $\gamma_{i-1} < x < \gamma_i$. In words, whenever an arriving claim finds the surplus between γ_{i-1} and γ_i or for part of the claim that is between γ_{i-1} and γ_i the reinsurer covers $1 - \beta_i$ of the claim. Then $\beta(x) = \beta_i$ in this strip. In the present study we introduce the general case of arbitrary function β. This arbitrary function includes the special cases of the models mentioned above.

The dynamics described above is a natural procedure of a risk sharing model. However, in order to ease explanations we explain it as a type of reinsurance.

In this study we are interested in analyzing the problem from the point of view of the insurance firm and the reinsurer.

The most interesting five performance measures of this model are (i) the time to ruin, (ii) the deficit at ruin, (iii) the dividend reinsurer withdrawn until ruin, (iv) the amount of money transferred to the until ruin and (v) the total insurance coverage until ruin whose source is the reinsurer. In this paper we shall study the functionals and measures associated with all these five performance measures. An important feature of the paper is the fact that the net premium rate and the net claim sizes are *state-dependent* in a quite general way, giving us considerable modeling flexibility. However, this comes at a price; for example, we only determine the *mean* value of the time to ruin. When more explicit assumptions are being made about the rate functions $\alpha_R(\cdot)$ and $\beta_R(\cdot)$, one might also be able to determine the Laplace transform of the time to ruin (see Boxma et al., 2017).

Related literature

Reinsurance in principle gives rise to multidimensional risk processes. However, despite their obvious relevance, exact analytic studies of multidimensional risk processes are scarce in the insurance literature. An early attempt to assess multivariate risk measures can be found in Sundt (1999), where multivariate Panjer recursions are developed which are then used to compute the distribution of the aggregate claim process, assuming simultaneous claim events and discrete claim sizes. Other approaches are deriving integro-differential equations for the various measures of risk and then iterating these equations to find numerical approximations (Chan et al., 2003; Gong et al., 2012), or computing bounds for the different types of ruin probabilities that can occur in a setting where more than one insurance line is considered (Cai and Li, 2005, 2007). In Badila et al. (2014) a two-dimensional functional equation is taken as a departure point. The authors show how one can find transforms of ruin related performance measures by solving a Riemann–Hilbert type boundary value problem. It is also shown that the boundary value problem has an explicit solution in terms of transforms, if the claim sizes are ordered. In Badila et al. (2015) this is generalized to the case in which the claim amounts are also correlated with the time elapsed since the previous claim arrival.

A special, important case is the setting of proportional reinsurance, which was studied in Avram et al. (2008). There it is assumed that there is a single arrival process, and the claims are proportionally split among two reserves. The two-dimensional exit (ruin) problem then becomes a *one*-dimensional first-passage problem above a piece-wise linear barrier. Badescu et al. (2011) have extended this model by allowing a dedicated arrival stream of claims into only one of the insurance lines. They show that the transform of the time to ruin of at least one of the reserve processes can be derived by applying similar ideas as in Avram et al. (2008).

Bivariate models where one company can transfer its capital to the other have also been considered in the literature. Recently, Avram et al. (2015) proposed a model of an insurance company which splits its premiums between a reinsurance/investment fund and a reserves fund necessary for paying claims. In their setting only the second fund receives claims, and hence all capital transfers are one way: from the first fund to the second. Another example is the capital-exchange agreement in Chapter 4 of Lautscham (2013), or Albrecher and Lautscham (2015) where two insurers pay dividends according to a barrier strategy and the dividends of one insurer are transferred to the other unless the other is also fully capitalized. This work led to systems of integro-differential equations for the expected time of ruin and expected discounted dividends, which are hard to solve even in the case of exponential claims.

In Ivanovs and Boxma (2015) a bivariate risk process is considered with the feature that each insurance company agrees to cover the deficit of the other. Under the assumptions that capital transfers between companies incur a certain proportional cost, and that ruin occurs when neither company can cover the deficit of the other, the survival probability is studied as a function of initial capitals. The bivariate transform of the survival probability is determined, in terms of Wiener–Hopf factors associated with two auxiliary compound Poisson processes. The case of a non-mutual agreement, i.e., reinsurance, is also discussed in Ivanovs and Boxma (2015).

Like the present paper, Boxma et al. (2017) is also devoted to a reinsurance model with an infinitely rich reinsurer, who pays part of the claim when it would bring the surplus below a certain threshold. The focus in that paper is on the discounted case, and on the Gerber–Shiu penalty function.

The features of having a dividend barrier, and of having state-dependent premium rates, appear in quite a few papers in the insurance literature. The following is a far from exhaustive list: Boxma et al. (2010a,b, 2011b), Kyprianou and Loeffen (2010), Lin and Pavlova (2006), Wan (2007) and Zhang et al. (2006).

Finally, we would like to point out that, methodologically, when it comes to studying the density of the surplus capital, this paper bears some relationship to Boxma et al. (2005). The latter paper is concerned with a dam process, and does not consider insurance risk performance measures.

Organization of the paper

In Section 2 we provide some background on the level crossing technique, which is heavily used in the rest of the paper.

The model under consideration is described in Section 3. We there introduce not only the surplus cash model, but also a strongly related dam process (taking $D(t) = b - R(t)$), as well as an other, regenerative, dam process. The five key performance measures mentioned above are studied in Section 5, by relating the surplus cash process to those dam processes. Our results are mostly expressed in the steady-state density of the amount of cash, or of the dam content. That density is determined in Section 4. For the model in full generality, that density is expressed in the form of a Neumann series which is the solution of a Volterra integral equation of the second kind. Under specific assumptions on the claim size distribution and the functions $\alpha_R(\cdot)$ and $\beta_R(\cdot)$, more explicit formulas for the density of the surplus and the five key performance measures can be obtained. In Section 6 we consider the case that the claim arrivals do not follow a Poisson process, but in which the gross negative jump sizes are exponentially distributed. We subsequently consider not only the dam model with $D(t) = b - R(t)$, but we also construct a model that is in a sense dual to that dam model, applying a similar duality as exists between the $M/G/1$ queue and the $G/M/1$ queue (where interarrival and service times are swapped).

2. The level crossing technique

In our analysis we heavily make use of the *level crossing technique* (LCT). In the late seventies, Brill and Posner (1977) and Cohen (1977) independently developed the LCT, which is based on a balancing argument, equating the numbers of up and downcrossings of a particular queue length or workload level. This technique is quite often applied in queueing and storage theory, but is not commonly used in insurance risk. For that reason, and also because we introduce some extensions to the LCT, we devote a separate introductory section to this topic. The basic LCT theory can be found in the comprehensive book by Brill (2008).

Basic facts about LCT (Brill, 2008)

Consider the following integral equation:

$$f(x) = \lambda \int_{0-}^{x} [1 - G(x-w)]dF(w), \quad x > 0, \tag{1}$$

which is sometimes called the *Pollaczek–Khinchine* integral equation associated with the *work process* of the $M/G/1$ queue (see Neuts, 1986); there λ is the arrival rate of customers and $G(\cdot)$ is their service time distribution. The argument to derive (1) for the $M/G/1$ queue is as follows. Let $F(\cdot)$ denote the distribution of the steady-state workload (amount of work) V in the $M/G/1$ queue, which by PASTA (*Poisson Arrivals See Time Averages*, see Wolff, 1989, Sections 5–16) is in distribution equal to that of the waiting time, and let $f(x)$ denote its density for all $x > 0$. The level crossing approach says that the long run average number of downcrossings of level $x > 0$ is the steady-state density $f(x)$, and that it should equal the long run average number of upcrossings of that level. Thus, the balance equation (1) is obtained by equating the rates to cross level $x > 0$ from above and from below. It is well known that the steady-state distribution $F(\cdot)$ has an atom at level 0 ($\pi = \Pr(V = 0)$), while $F(\cdot)$ has a density for $x > 0$. Hence (1) can be written as

$$f(x) = \lambda \int_{0-}^{x} [1 - G(x-w)]f(w)dw + \lambda\pi[1 - G(x)],$$
$$x > 0. \tag{2}$$

The balance equation (1) can be extended in several directions. One direction is the case in which the workload of the $M/G/1$ queue is interpreted as the content level of a dam with general release rate function $r(\cdot)$. LCT applied to dam models can be found in Kaspi et al. (1997), Boxma et al. (2005) and Perry and Posner (2002). Another direction is the case where the constant arrival rate λ is replaced by a state-dependent arrival rate $\lambda(\cdot)$. When both extensions occur, the balance equation becomes (cf. Theorem 2.1 of Bekker et al., 2004):

$$r(x)f(x) = \int_{0-}^{x} \lambda(w)[1 - G(x-w)]dF(w), \quad x > 0. \tag{3}$$

For example, consider the $M/G/1+G$ queue, the last G denoting that customers have some general impatience distribution; more specifically, an arriving customer who meets an amount of work w enters the system with probability $1 - H(w)$, where $H(\cdot)$ is the distribution of his patience time. Then $\lambda(w) = \lambda(1 - H(w))$ (Boxma et al., 2011c,a).

Extensions of the LCT

The following variant of the LCT occurs a few times in the present paper, and does not occur in Brill (2008). Consider a queue or storage process that evolves as an ordinary $M/G/1$ queue, except for the following feature: Each time that the system becomes empty, immediately a new cycle begins at a certain level, say a. In every cycle of this model the number of upcrossings is equal to the number of downcrossings for $x > a$, and hence the long run rate

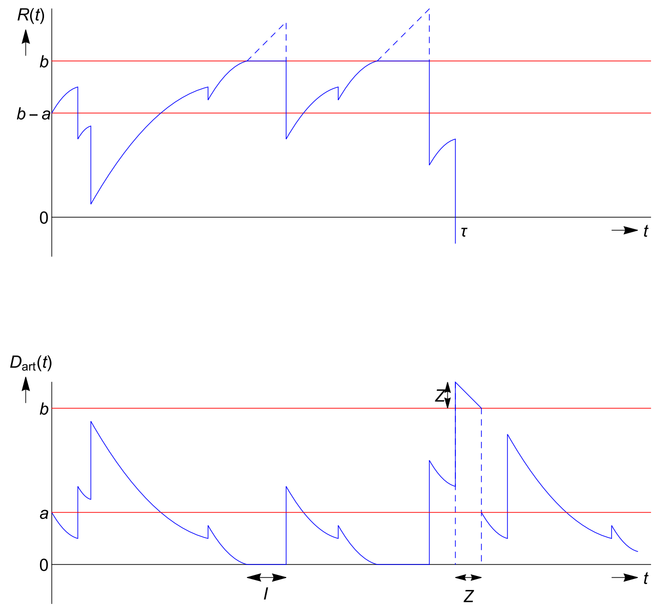

Fig. 1. The surplus cash process **R** and the artificial dam process $\mathbf{D_{art}}$.

of upcrossings equals the long run rate of downcrossings for every $x > a$. However, when $x \le a$, the number of downcrossings minus the number of upcrossings *is equal to 1 in each cycle*. Since level 0 is reached only once during a cycle (at the end of the cycle), the latter fact means that the long run rate of downcrossings minus the long run rate of upcrossings is equal to the long run rate of downcrossings of level 0.

Two other classes of models which are not discussed in Brill (2008) are the class of so-called mountain models (Boxma et al., 1999, 2005; Perry and Stadje, 2003; Boxma et al., 2015) and the class of models with state-dependent jumps. We shall make use of LCT for mountain models in Section 6, and of LCT for models with state-dependent jumps in Section 4; there we shall explain the precise working of the LCT for such models.

3. The model

The surplus cash process $\tilde{\mathbf{R}}$ is a *risk-type* process with general fluid state-dependent input of rate $\alpha_R(x) \in (0, c)$, with $c - \alpha_R(x)$ the rate of transferring funds to the reinsurer; see Fig. 1. We assume that $R(0) = b - a$, where $b - a$ is the initial investment of the insurance firm. The claims arrive according to a Poisson process with rate λ (can be extended as in Boxma et al., 2005 to $\lambda(x)$). The successive gross claims form a sequence of i.i.d. random variables whose generic size is $\tilde{Z}$ and whose generic distribution is $G(\cdot)$. However, the *net* jump size which is the amount subtracted from the cash is less than $\tilde{Z}$, since a certain part of $\tilde{Z}$ is covered by the reinsurer. More specifically: when the cash drops below level $x > 0$ due to a claim, the infinitesimal gross payment due to that claim equals dx but the net payment is $\beta_R(x)dx$ ($0 < \beta_R(x) < 1$) and $(1-\beta_R(x))dx$ is covered by the reinsurer. Hence the negative jumps of process $\tilde{\mathbf{R}}$ are also state-dependent.

Formally, the dynamics of the process $\tilde{\mathbf{R}}$ is

$$\tilde{R}(t) = b - a + \int_0^t \alpha_R(s)ds - \sum_{j=1}^{N(t)} Z_j \tag{4}$$

where $N(t)$ is the Poisson arrival process of the negative jumps and Z_j is the size of the jth negative jump. That is, $N(t) = \sup\{j : T_j \le t\}$ where T_j is the arrival time of the jth jump. Note that *effective* jumps $Z_1, Z_2, \ldots$ are not i.i.d. random variables because they are state dependent. The claim sizes $\tilde{Z}_1, \tilde{Z}_2, \ldots$ are i.i.d. random variables and

$$Z_j = \int_0^t 1_{\{T_j \in ds\}} ds \int_0^{\tilde{Z}_j} \beta(R(s) - u) du. \tag{5}$$

We assume that the insurance firm applies a dividend policy such that whenever the cash reaches level b the whole excess input is taken as a dividend. We denote the modified process by **R**. Let τ be the time of ruin of the modified process **R**. That is, $\tau = \inf\{t : R(t) < 0\}$, so that $R(\tau) < 0$ and if we define $\beta_R(x) = 1$ for $x \le 0$, $|R(\tau)| > 0$ is the deficit at ruin. Every interval of time that **R** spends at level b is called a *dividend period*. Then, during an arbitrary dividend period of length I the dividends are divided between the insurance firm and the reinsurer according to the proportion of $\alpha_R(b-)/c$ and $1-\alpha_R(b-)/c$, respectively (we assume that $\alpha_R(b) = 0$ but $\alpha_R(b-) > 0$). That is, during I an income of $\alpha_R(b-)I$ goes to the insurance firm and $[c-\alpha_R(b-)]I$ is transferred to the reinsurer.

Clearly, by the above dividend policy **R** never upcrosses level b until τ, so that for all $t < \tau$, $0 \le R(t) \le b$.

It is natural to assume that

$$\int_x^y \frac{1}{\alpha_R(w)} dw < \infty \quad \text{and} \quad \int_x^y \frac{1}{\beta_R(w)} dw < \infty,$$

for every $0 < x < y \le b$. The former integral represents the time it takes to go from any level x up to level y, if no jumps occur in between. This means that the boundary b will be reached in a finite amount of time from any level $x < b$. The latter condition, combined with the finiteness of the upper boundary b, implies that ruin will occur within a finite time.

A related dam process

A dam process $\tilde{D}(t)$ describes the content of a dam at time t, fluid inflow arrives according to a Poisson process and the fluid release rate depends on the content of the dam. We consider a dam model with capacity b, where the oveflow of the fluid above b is lost. Let **D** be the modified process with upper barrier b. Clearly, $D(t) = b - R(t)$; see Fig. 1. **D** is called the *dam* version of the risk process **R** and by definition the analysis of **D** is equivalent to the analysis of **R**. It is clear that by definition $D(0) = a$ and $\tau = \inf\{t : D(t) > b\}$. In addition, if $\alpha_R(\cdot)$ is the increase rate of **R**, then $\alpha_D(\cdot)$ where $\alpha_D(x) = \alpha_R(b-x)$ can be interpreted as the release rate function with respect to the dam **D** and the integral $\int_x^y [1/\alpha_D(w)]dw$ represents the time it takes to go from state y down to state x (where $x < y$), if no jumps occur, in **D**. The so-called dry periods of the dam (where $D(t) = 0$) are the dividend periods of the risk process and by definition $D(\tau) - b$ is interpreted as the *deficit* at time of ruin of the surplus in **R**. The jumps in **D** are positive and state-dependent. In a similar way to $\alpha_R(\cdot)$ and $\alpha_D(\cdot)$ we define $\beta_D(x) = \beta_R(b-x)$. Then, every negative jump of size $\tilde{Z}$ in the risk process can be interpreted as a positive jump of the same size in the dam process. Notice that a net jump in **D** has distribution

$$P(\text{net jump size} < x - w | \text{jump from level } w)$$
$$= G\left(\int_w^x \frac{1}{\beta_D(y)} dy\right). \tag{6}$$

In a similar way to the description of $\tilde{\mathbf{R}}$ in (4) the dynamics of the dam process $\tilde{\mathbf{D}}$ is defined by:

$$\tilde{D}(t) = a - \int_0^t \alpha_{\mathbf{D}}(\tilde{D}(s))ds + \sum_{j=0}^{N(t)} Z_j(t), \tag{7}$$

where Z_j is as per (5).

A regenerative dam process

To analyze the 5 performance measures mentioned in the introduction we construct an artificial dam process $\mathbf{D_{art}} = \{D_{art}(t) : t \ge 0\}$ from **D** as follows (see also Fig. 1). Recall that **D** is a stopped process that is terminated when it upcrosses b. We construct the process $\mathbf{D_{art}}$ such that it is a regenerative process. For the first cycle we replicate the process **D** until time τ—where an overshoot above b occurs. Then, from time τ the artificial process $\mathbf{D_{art}}$ decreases at rate 1 until level b is reached. At that moment, the cycle of $\mathbf{D_{art}}$ is terminated and a new cycle restarts from level a. In other words, $\{D(t) : 0 \le t \le \tau\}$ and $\{D_{art}(t) : 0 \le t \le \tau\}$ are equal up to time τ, but $D_{art}(t)$ continues after τ, going down at rate 1 until it reaches b. Then the cycle of $\mathbf{D_{art}}$ ends, so that the cycle length of $\mathbf{D_{art}}$ is $T = \tau + D_{art}(\tau) - b$. The dam $\mathbf{D_{art}}$ can be interpreted as a special version of a Markov regenerative dam with general release rate $\alpha_D(w)$, state dependent jumps rate $\beta_D(w)$ and *pseudo finite* capacity b. The *pseudo finiteness* of the dam is introduced with the convention that the content of the dam is not finite, but jumps that arrive and find the content level above b are not admitted to the buffer. As will be seen below, the performance measures (i)–(v) introduced in the introduction can be expressed in the steady state distribution of $\mathbf{D_{art}}$.

4. The density of the artificial dam process

In this section we shall analyze the steady state probability law of the regenerative process $\mathbf{D_{art}}$. That law will be used in the next section for determining the five key performance measures listed in the introduction.

Let π denote the probability mass of $\mathbf{D_{art}}$ at 0. Remember that $\alpha_D(x) = \alpha_R(b-x)$ for $0 \le x \le b$, and $\beta_D(x) = \beta_R(b-x)$. Furthermore, define $\alpha_D(x) := 1$ for $x \ge b$, because by construction $D_{art}(t)$ decreases at rate 1 above level b. Also, define $\beta_D(x) := 1$ for $x \ge b$, because the reduction of the gross claim sizes by $\beta_R(x) \in (0, 1)$ only applies as long as the surplus $x > 0$. For simplicity of representation we define

$$B(x) = \int_0^x \frac{1}{\beta_D(y)} dy, \quad x > 0. \tag{8}$$

The next theorem lays the groundwork for the analysis of the model.

Theorem 1. *The density $f(\cdot)$ of the stationary distribution of the Markov process $\mathbf{D_{art}}$ satisfies the integral equation*

$$\alpha_D(x)f(x) = \begin{cases} \lambda \displaystyle\int_0^x [1 - G(B(x) - B(w))]f(w)dw & 0 < x < a, \\ + \lambda\pi[1 - G(B(x))], & \\ \lambda \displaystyle\int_0^x [1 - G(B(x) - B(w))]f(w)dw & a \le x < b, \\ + \lambda\pi[1 - G(B(x))] - f(b), & \\ \lambda \displaystyle\int_0^b [1 - G(B(x) - B(w))]f(w)dw & x \ge b. \\ + \lambda\pi[1 - G(B(x))], & \end{cases} \tag{9}$$

Proof. Eq. (9) is derived by the level crossing theory (LCT) introduced in Section 2. $\alpha_D(x)f(x)$ is the rate of downcrossing of any level $x > 0$ (see Cohen, 1977; Brill, 2008; Brill and Posner, 1977; Perry and Asmussen, 1995). To obtain the rate of upcrossing note that the arrival rate of upward jumps is λ, and given that the jump starts at $w \in (0, x \wedge b)$ the probability to jump above x is $[1 - G(B(x) - B(w))]$. To explain this consider a certain effective jump size in $\mathbf{D_{art}}$ that starts at level w and ends at level x, and construct an artificial process $\mathbf{X} = \{X(t) : t \ge 0\}$ from $\mathbf{D_{art}}$ such that each positive jump of $\mathbf{D}_{\mathbf{art}}$ is replaced by a piece of trajectory that increases continuously at rate $\beta_{\mathbf{D}}(y)$ for all $w < y < x$ in $\mathbf{X}$. Specifically, if the jump size in $\mathbf{D}_{\mathbf{art}}$ starts at level w and ends at level x, the piece of trajectory in $\mathbf{X}$ starts at the same level w and ends at the same level x, but in between w and x the modified process $\mathbf{X}$ increases as a fluid with rate $\beta_{\mathbf{D}}(y)$ for all $w < y < x$. Clearly, $B(x) - B(w)$ (see Harrison and Resnick, 1976, 1978) is the time it takes for the fluid in $\mathbf{X}$ to go from level w up to level x. Thus, $[1 - G(B(x) - B(w))]$ is the probability of the event that the fluid time of the trajectory $\mathbf{X}$ is greater than $B(x) - B(w)$. The term $\lambda\pi[1 - G(B(x))]$ corresponds to a jump above x starting from 0. By construction of $\mathbf{X}$ from $\mathbf{D}_{\mathbf{art}}$ every upcrossing of level x in the cycle of $\mathbf{X}$ is accompanied by an upcrossing of level x in a cycle of $\mathbf{D}_{\mathbf{art}}$ and vice versa. This implies that the number of upcrossings during one cycle in $\mathbf{X}$ is equal to the number of upcrossings during one cycle of $\mathbf{D}_{\mathbf{art}}$.

When $x \in (0; a) \cup [b; \infty)$ the number of upcrossings is equal to the number of downcrossings in every cycle of $\mathbf{D}_{\mathbf{art}}$. Thus, by renewal theory, the rates of up and downcrossings are also equal. However, when $x \in [a; b)$ the number of downcrossings minus the number of upcrossings in each cycle is 1, since each cycle starts to decrease at level a. But level b is upcrossed only once during one cycle (in the last jump of the cycle) and after the last jump $\mathbf{D}_{\mathbf{art}}$ decreases continuously down to level b. This means that level $b+$ is also downcrossed only once during a cycle (at the end of the cycle). By renewal theory the rate of downcrossings of level $x \in [a; b)$ minus the rate of upcrossings is equal to the rate of downcrossings of level $b+$. But by LCT the latter rate is $\alpha(b+)f(b)$. In our model $\alpha(b+) = 1$ by definition. The proof is complete. □

To solve for $f(\cdot)$ we apply the technique introduced by Harrison and Resnick (1976, 1978). Define

$$K(x, w) = \frac{\lambda\left[1 - G\left(\int_w^x \frac{1}{\beta_D(y)}dy\right)\right]}{\alpha_D(x)}$$

to get the integral equation

$$f(x) = \begin{cases} \int_0^x K(x, w)f(w)dw + \pi K(x, 0), & 0 < x < a, \\ \int_0^x K(x, w)f(w)dw + \pi K(x, 0) \\ \quad - \frac{f(b)}{\alpha_D(x)}, & a \le x < b, \\ \int_0^b K(x, w)f(w)dw + \pi K(x, 0), & x \ge b. \end{cases} \tag{10}$$

Formula (10) is a Volterra integral equation of the second kind on $(0, a)$ and on $[a, b)$. This equation is known to be uniquely solvable by a Neumann series (in the space of continuous functions), see Harrison and Resnick (1976). We shall now provide that solution; See for example Ch.1 of Tricomi (1970). For other applications featuring Neumann series equations see Boxma et al. (2011a, 2015, 1999, 2011c) and Kaspi and Perry (1989).

Define

$$\begin{aligned} K_{n+1}(x, w) &= \int_w^x K_n(x, y)K_1(y, w)dy \\ &= \int_w^x K_1(x, y)K_n(y, w)dy \end{aligned}$$

where $K_1(x, w) := K(x, w)$ for $0 < x < a$,

$$f(x) = \pi \sum_{n=1}^{\infty} K_n(x, 0) =: \pi K^*(x, 0), \quad 0 < x < a, \tag{11}$$

where it is easy to show that the Neumann series $\sum_{n=1}^{\infty} K_n(x, 0) = K^*(x, 0)$ in (11) converges for every $x > 0$.

Next, for $a \le x < b$,

$$\begin{aligned} f(x) &= \int_0^a K(x, w)f(w)dw \\ &\quad + \int_a^x K(x, w)f(w)dw + \pi K(x, 0) - \frac{f(b)}{\alpha_D(x)}. \end{aligned} \tag{12}$$

Note that $f(x)$ is known on $(0, a)$ except for the factor π. Let

$$l(x) = \int_0^a K(x, w)f(w)dw + \pi K(x, 0) - \frac{f(b)}{\alpha_D(x)}, \quad x \in [a, b).$$

$l(\cdot)$ is a known function except for the constants π and $f(b)$. Also note that only given functions and parameters and $f(\cdot)$ restricted to $(0, a)$ appear in the definition of $l(\cdot)$. We have

$$f(x) = l(x) + \int_a^x K(x, w)f(w)dw, \quad a \le x < b. \tag{13}$$

Iterating, we get another Neumann series: $f(\cdot)$ can be written for $x \in [a, b)$ as

$$\begin{aligned} f(x) &= l(x) + \int_a^x K(x, w)f(w)dw \\ &= l(x) + \sum_{n=1}^{\infty} \int_a^x K_n(x, w)l(w)dw. \end{aligned}$$

To determine $f(\cdot)$ for $x \ge b$ one can simply substitute the solution of $f(\cdot)$ for $x < b$ into (10).

We have determined $f(\cdot)$ except for the constants π and $f(b)$. $f(b)$ is obtained as follows: By substituting $x = b-$ and $x = b$ into (10) (or in (9)) we get that

$$f(b-) = 0, \tag{14}$$

Thus by substituting $x = b-$ into (12), using $\alpha_D(b) = 1$, is tantamount to

$$f(b) = \pi K(b, 0) + \int_0^b K(b, w)f(w)dw.$$

Finally use the normalizing condition

$$\int_0^{\infty} f(x)dx = 1 - \pi, \tag{15}$$

and $f(\cdot)$ is found for all $x > 0$.

Example 1. Special Case—Exponential Jumps

In this subsection we consider the special case of exponentially distributed example gross jumps: $G(x) = 1 - e^{-\mu x}$, $x \ge 0$. Now a direct solution for $f(\cdot)$ is possible. Next to $B(\cdot)$ introduced in (8) we also define:

$$A(x) = \int_0^x \frac{1}{\alpha_D(y)}dy. \tag{16}$$

We get in (9):

$$\alpha_D(x)f(x) = \begin{cases} \lambda \int_0^x e^{-\mu[B(x)-B(w)]}f(w)dw \\ \quad +\lambda\pi e^{-\mu B(x)}, & 0 < x < a, \\ \lambda \int_0^x e^{-\mu[B(x)-B(w)]}f(w)dw \\ \quad +\lambda\pi e^{-\mu B(x)} - f(b), & a \le x < b, \\ \lambda \int_0^b e^{-\mu[B(x)-B(w)]}f(w)dw \\ \quad +\lambda\pi e^{-\mu B(x)}, & x \ge b. \end{cases} \tag{17}$$

Now multiply both sides of (17) by $e^{\mu B(x)}$. We get

$$\alpha_D(x)e^{\mu B(x)}f(x) = \begin{cases} \lambda \int_0^x e^{\mu B(w)} f(w)dw \\ \quad +\lambda\pi, & 0 < x < a, \\ \lambda \int_0^x e^{\mu B(w)} f(w)dw \\ \quad +\lambda\pi - e^{\mu B(x)} f(b), & a \le x < b, \\ \lambda \int_0^b e^{\mu B(w)} f(w)dw \\ \quad +\lambda\pi, & x \ge b. \end{cases} \tag{18}$$

Solving for $f(\cdot)$ in (18) we get

$$f(x) = \begin{cases} \frac{k_0}{\alpha_D(x)} e^{-\mu B(x)+\lambda A(x)}, & 0 < x < a, \\ \frac{k_1}{\alpha_D(x)} e^{-\mu B(x)+\lambda A(x)} \\ \quad -\frac{f(b)}{\alpha_D(x)} e^{-\mu B(x)+\lambda A(x)} \\ \quad \times \int_0^x \frac{e^{\mu B(y)-\lambda A(y)}}{\beta_D(y)} dy, & a \le x < b, \\ k_2 e^{-\mu x}, & x \ge b, \end{cases} \tag{19}$$

where k_0, k_1, and k_2 are constants. To find the constants we have the following:

(1) We have $k_0 = \lambda\pi$. To see this, substitute $x = 0$ into both (17) and (19).

(2) Clearly $f(b-) = 0$ (see also (14)); this may be seen by comparing the second and third equations of (17). It implies that $k_1 = f(b) \int_0^b \frac{e^{\mu B(y)-\lambda A(y)}}{\beta_D(y)} dy$.

(3) Level a is point of discontinuity for $f(\cdot)$. We have (cf. (17)):

$\alpha_D(a)f(a-) = \alpha_D(a)f(a) + f(b)$.

By substituting $x = a$ into (19) we get

$$k_0 = k_1 - f(b) \int_0^a \frac{e^{\mu B(y)-\lambda A(y)}}{\beta_D(y)} dy.$$

(4) To compute k_2 recall that for $x \ge b$ we take $\alpha_D(x) = \beta_D(x) = 1$. By substituting the solutions of $f(\cdot)$ for $x < b$ into (18) for $x \ge b$ we get

$$\begin{aligned} f(x) &= \lambda e^{-\mu B(x)} \int_0^b e^{\mu B(w)} f(w)dw + \lambda\pi e^{-\mu B(x)} \\ &= \lambda e^{-\mu[B(b)+x-b]} \left[\int_0^a \frac{k_0}{\alpha_D(w)} e^{\lambda A(w)} dw \right. \\ &\quad + \int_a^b \left(\frac{k_1}{\alpha_D(w)} e^{\lambda A(w)} - \frac{f(b)}{\alpha_D(w)} e^{\lambda A(w)} \right. \\ &\quad \left. \left. \times \int_0^w \frac{e^{\mu B(y)-\lambda A(y)}}{\beta_D(y)} dy \right) dw \right] + \lambda\pi e^{-\mu[B(b)+x-b]}, \end{aligned}$$

so that

$$\begin{aligned} k_2 &= \lambda\pi e^{-\mu[B(b)-b]} + \lambda e^{-\mu[B(b)-b]} \left[\int_0^a \frac{\lambda\pi}{\alpha_D(w)} e^{\lambda A(w)} dw \right. \\ &\quad + \int_a^b \left(\frac{k_1}{\alpha_D(w)} e^{\lambda A(w)} - \frac{f(b)}{\alpha_D(w)} e^{\lambda A(w)} \right. \\ &\quad \left. \left. \times \int_0^w \frac{e^{\mu B(y)-\lambda A(y)}}{\beta_D(y)} dy \right) dw \right]. \end{aligned}$$

Now k_0, k_1 and k_2 are expressed in terms of each other and π. The final solution is obtained via the normalizing condition (15).

Example 2. We briefly consider a special choice for $\alpha_D(x)$ and $\beta_D(x)$, in addition to the assumption that the jump sizes are exponential. More precisely, we take

$$\alpha_D(x) = \begin{cases} \alpha_0, & x < \nu, \\ \alpha_1 & \nu \le x < b, \\ 1, & x \ge b, \end{cases} \tag{20}$$

and

$$\beta_D(x) = \begin{cases} \beta_0, & x < \gamma, \\ \beta_1, & \gamma \le x < b, \\ 1, & x \ge b. \end{cases} \tag{21}$$

We assume, without loss of generality, that $0 < \nu < a < \gamma < b$, but a similar analysis can be performed for some other combination of the parameters ν, γ and a. It is natural to assume that $\alpha_1 > \alpha_0$ but this assumption does not have any effect on the analysis. It is now trivial to verify that

$$A(x) = \begin{cases} \frac{x}{\alpha_0}, & x < \nu, \\ \frac{\nu}{\alpha_0} + \frac{x-\nu}{\alpha_1}, & \nu \le x < b, \\ \frac{\nu}{\alpha_0} + \frac{b-\nu}{\alpha_1} + x - b, & x \ge b, \end{cases} \tag{22}$$

and

$$B(x) = \begin{cases} \frac{x}{\beta_0}, & x < \gamma, \\ \frac{\gamma}{\beta_0} + \frac{x-\gamma}{\beta_1}, & \gamma \le x < b, \\ \frac{\gamma}{\beta_0} + \frac{b-\gamma}{\beta_1} + x - b, & x \ge b. \end{cases} \tag{23}$$

Substitution in (19) gives the density $f(\cdot)$.

5. Analysis of the main performance measures

In this section we express the five key performance measures of the cash surplus model in the density $f(\cdot)$ that was determined in the previous section.

Performance measures (i) and (ii): Time to ruin and deficit at ruin

Lemma 1. *Let $f(\cdot)$ be the steady state density of $\mathbf{D_{art}}$ and let ξ be the deficit at ruin of $\mathbf{R}$ (which is also the overflow $D_{art}(\tau) - b$) and define the distribution $H_\xi(x) = \Pr(\xi \le x)$. Then*

(i) $$H_\xi(x) = 1 - \frac{f(x+b)}{f(b)}$$

so that

$$E\xi = \frac{\int_b^\infty f(w)dw}{f(b)},$$

and

(ii) $$E\tau = \frac{1 - \int_b^\infty f(w)dw}{f(b)}.$$

Proof. (i) The function

$$\frac{f(x+b)}{\int_0^\infty f(z+b)dz}, \quad x > 0,$$

is the conditional steady state density of $\mathbf{D_{art}}$ given $\mathbf{D_{art}} > b$. By deleting the time periods in which $\mathbf{D_{art}} \le b$ and gluing together the time periods in which $\mathbf{D_{art}} > b$ we obtain a typical sample path of the *forward recurrence time* of a renewal process whose interrenewal times have the same distribution as ξ. Designate the equilibrium density of ξ by $h_e(\cdot)$. Then by renewal theory (p. 65 of Wolff, 1989)

$$h_e(x) = \frac{1 - H_\xi(x)}{E\xi}.$$

That means that

$$\frac{f(x+b)}{\int_0^\infty f(z+b)dz} = \frac{1 - H_\xi(x)}{E\xi}. \tag{24}$$

Substituting $x = 0$ we get

$$E\xi = \frac{\int_0^\infty f(z+b)dz}{f(b)}. \tag{25}$$

The proof of (i) is complete by substituting (25) into (24).

(ii) By definition of $\mathbf{D_{art}}$, $E\tau = ET - E\xi$. Thus, it is enough to show that $ET = \frac{1}{f(b)}$. But level b is downcrossed only once during a cycle of $\mathbf{D_{art}}$, and this downcrossing occurs at the end of the cycle T. By the strong Markov property, it is possible to define the cycle as the time between successive downcrossings of level b. Clearly the expected cycle length and the rate of renewals are reciprocal to each other. But by LCT, $f(b)$ is the rate of downcrossings of level b, since $\alpha_D(b) = 1$ and the cash process decreases at rate 1 whenever it is above level b. The proof is complete. □

Performance measure (iii): The dividend until ruin

Lemma 2. *Let R_{div} be the dividend withdrawn until ruin. Then*

$$ER_{div} = \frac{\alpha_D(0+)\left[1 - \int_0^\infty f(x)dx\right]}{f(b)}.$$

Proof. By the normalizing condition (15), $1 - \int_0^\infty f(x)dx$ is the steady state probability π that the dam is empty, which equals the steady state probability of being in a dividend period for the surplus process. The mean total length of all the dividend periods during the cycle T is obtained by multiplication with ET, which equals $\frac{1}{f(b)}$. Finally observe that the net income fraction for the insurance firm during dividend periods is $\alpha_R(b-) = \alpha_D(0+)$. The result follows. □

Performance measures (iv) and (v): The amount of money transferred to/from the reinsurer

Designate the expected amount of money transferred from the firm to the reinsurer until ruin by R_{to} and the expected amount of money transferred from the reinsurer to the firm until ruin by R_{from}. We compute R_{to} with the convention that during dividend periods the proportion $1 - \frac{\alpha_D(0+)}{c}$ of the dividend is transferred to the reinsurer. Similarly, we compute R_{from} with the convention that ruin means bankruptcy, which means that all of the deficit at ruin is covered by the reinsurer.

Lemma 3.

(i) $$R_{to} = \frac{\left[\int_0^b [c - \alpha_D(x)]f(x)dx + \pi[c - \alpha_D(0+)]\right]}{f(b)},$$

(ii) $$R_{from} = \frac{\int_0^b [1 - \beta_D(x)]\alpha_D(x)f(x)dx}{f(b)} + \int_a^b [1 - \beta_D(x)]dx,$$

where the payment of the deficit by the reinsurer is not taken into account in (ii).

Proof. (i) We have

$$\begin{aligned} R_{to} &= E\left[\int_0^T [c - \alpha_D(D_{art}(t))]I_{D_{art}(t)<b}dt\right] \\ &= ET\, E[(c - \alpha_D(D_{art}(\infty)))I_{D_{art}(\infty)<b}] \\ &= ET\left[\int_0^b [c - \alpha_D(x)]f(x)dx + \pi[c - \alpha_D(0+)]\right] \\ &= \frac{\int_0^b [c - \alpha_D(x)]f(x)dx + \pi[c - \alpha_D(0+)]}{f(b)}, \end{aligned} \tag{26}$$

where $D_{art}(\infty)$ is the steady state random variable of $\mathbf{D_{art}}$. The first step of (26) is the definition of R_{to}. The second step is obtained by renewal theory. To explain the third step we note that up to time τ the process $\mathbf{D_{art}} < b$. Finally, in the fourth step we substitute $ET = 1/f(b)$.

(ii) Similarly to (i), we have, with $N(x, dt)$ the number of downcrossings of level x in $[t, t + dt)$:

$$R_{from} = E\left[\int_0^T [1 - \beta_D(D_{art}(t))]N(D_{art}(t), dt)\right]. \tag{27}$$

Let us now take a closer look at the rate of downcrossings of level x. Every cycle of $\mathbf{D_{art}}$ starts at level $D_{art}(0) = a$ and after a single upcrossing of level b (at τ) the cycle T ($T = \tau + D(\tau) - b$) is terminated at level $b+$. Accordingly, by *level crossing theory* for any $x \in (0, a) \cup [b, \infty)$ the long run average number of upcrossings is equal to the long run average number of downcrossings. Here, since $\mathbf{D_{art}}$ is an artificial process, the set of states $\{x : x \in [b, \infty)\}$ is not of relevance, since $\beta_D(x) = 1$ by assumption for $\{x : x \in [b, \infty)\}$. However (see also the proof of Theorem 1), for any $x \in [a, b)$ the number of upcrossings until ruin minus the number of downcrossings until ruin per cycle is equal to 1, which is the number of downcrossings of level $b+$. By renewal theory this means that in $\mathbf{D_{art}}$ for any $x \in [a, b)$ (in terms of long run average) we have

{rate of upcrossings of level x}
 − {rate of downcrossings of level $b+$}
 = {rate of downcrossings of level x}.

Formally, since by definition $\alpha_D(b+) = 1$, we have

{rate of upcrossings of level x}

$$= \begin{cases} \alpha_D(x)f(x), & 0 < x < a, \\ \alpha_D(x)f(x) + f(b+), & a \le x < b, \\ f(x), & x \ge b, \end{cases}$$

where in the region $x \ge b$ we take into account that, by construction of $\mathbf{D_{art}}$, $\alpha_D(x) = 1$. It now follows that

$$\begin{aligned} R_{from} &= E\left[\int_0^T [1 - \beta_D(D_{art}(t))]N(D_{art}(t), dt)\right] \\ &= ET\left[\int_0^b [1 - \beta_D(x)]\alpha_D(x)f(x)dx + f(b)\int_a^b [1 - \beta_D(x)]dx\right] \\ &= \frac{\int_0^b [1 - \beta_D(x)]\alpha_D(x)f(x)dx}{f(b)} + \int_a^b [1 - \beta_D(x)]dx. \quad \square \end{aligned} \tag{28}$$

Remark 1. At first glance, it looks surprising that the second component of R_{from} in Lemma 3 is independent of the expected cycle length $1/f(b)$. However, in every cycle there is exactly one extra downcrossing between a and b, that gives a contribution to R_{from}.

It is natural to assume that at time of ruin the reinsurer pays the deficit by covering the entire claim causing the ruin. Clearly, the reinsurer will be profitable (on the average) until ruin when $R_{to} - R_{from} - E\xi \geq 0$.

5.1. A proportionality result

We derive two results which are formulated in terms of the steady state density $f(\cdot)$. The first result is used as a preliminary for the second, which is a *proportionality result*. In Section 6.1 we consider the special case of exponential interarrival times and exponential gross jumps. Then, the proportionality result becomes a more explicit result.

(1) Let J be the number of dividend periods until ruin. Then

$$EJ = \frac{1 - \int_0^\infty f(x)dx}{\lambda f(b)}. \tag{29}$$

To see this note that, with R_{div} the dividend withdrawn until ruin: $ER_{div} = \alpha_D(0+)EJEI$. Clearly, for the mean dividend period we have $EI = 1/\lambda$. Now the result follows by Lemma 2.

(2) Let $\theta(x; b)$ be the probability that level 0 is reached before level b is upcrossed by $\mathbf{D_{art}}$ when the starting point is $0 < x < b$, and let

$$\Theta(b) = \int_0^\infty \theta(x; b)dG(B(x)).$$

Then $1 - \theta(a; b)$ can be interpreted as the probability that no dividend period occurred until ruin. $\Theta(b)$ is the conditional probability given that $\mathbf{D_{art}}$ has just left 0, it will return to 0 again before upcrossing b; i.e., given that the dividend period is terminated there will be another dividend period before ruin. Thus, $1-\Theta(b)$ is the probability given that the process has just left the dividend period, there will be no dividends until ruin. Then the next proportionality result holds:

$$\frac{\theta(a; b)}{1 - \Theta(b)} = \frac{1 - \int_0^\infty f(x)dx}{\lambda f(b)}. \tag{30}$$

To see this recall that by the strong Markov property

$$\Pr(J = n) = \begin{cases} 1 - \theta(a; b), & n = 0, \\ \theta(a; b)(1 - \Theta(b))\Theta(b)^{n-1}, & n \geq 1. \end{cases}$$

Then,

$$EJ = \frac{\theta(a; b)}{1 - \Theta(b)}.$$

Now compare with (29) and the result follows.

6. The dual model

By the dual model we mean a risk model with generally distributed claim inter-arrival times and exponentially distributed claim amounts. Such models have already been discussed in the actuarial literature see, e.g. Borovkov and Dickson (2008) and Frostig et al. (2012). In most queueing and dam models in which the arrival process is not a Poisson process an exact analysis seems very intricate if not impossible. Indeed, a balance equation of type (9) still holds (see Cohen, 1977), but it does not provide a solution of the steady state density, since the steady state law and the law just before jumps are different. PASTA does not hold because the arrival process is not Poisson. As a result, the balance equation comprises *two unknowns* which are the steady state density and the limiting density of the state just before jumps (in the language of queueing these are the densities of the *work* and that of the *waiting time*). However, there is a special case that is tractable although the arrival process is not a Poisson process. This is the case in which the claims arrive according to a renewal process (with interarrival times having distribution $C(\cdot)$, say) and the gross negative jump sizes in the surplus $\mathbf{R_{mod}}$ are $\exp(\mu)$ distributed. Clearly, under this assumption the surplus cash process is not a Markov process. To solve the steady state density we use an approach based on duality between $M/G/1$ type dams and $G/M/1$ type dams. Other variants of this approach have been introduced in Adan et al. (2005), Perry and Stadje (2003) and Perry et al. (2013).

The non-Markov model $\mathbf{R_{mod}}$

Between negative jumps the original risk process $\mathbf{R}$ increases at input rate $\alpha_R(x)$ $(0 < x < b)$ until level b is reached. After reaching level b the risk process $\mathbf{R}$ stays at level b until the next negative jump and so forth. We now define a modified risk process $\mathbf{R_{mod}} = \{R_{mod}(t) : t \geq 0\}$ such that below level b the modified process $\mathbf{R_{mod}}$ is a probabilistic replication of $\mathbf{R}$, but when $\mathbf{R}$ reaches level b (if reached) the process $\mathbf{R_{mod}}$ still continues to increase at rate 1.

Then, the time of the first negative jump in $\mathbf{R}$ (at the end of I) is also the time of a negative jump in $\mathbf{R_{mod}}$, but the latter jump equals I plus an $\exp(\mu)$ distributed amount—and from that time $\mathbf{R}$ and $\mathbf{R_{mod}}$ are again stochastically equal until the moment (if it occurs) that level b is reached again and so forth. This means that every negative jump in $\mathbf{R_{mod}}$ that starts from any level $x > b$ is also a downcrossing of level b.

The gross negative jumps in $\mathbf{R_{mod}}$ are i.i.d., $\exp(\mu)$, random variables, but the net jumps are neither independent nor identically distributed; cf. (6).

Formally, we define the stopping time L_0 for both $\mathbf{R}$ and $\mathbf{R_{mod}}$ such that $L_0 = inf\{t : R(t) = b\} = inf\{t : R_{mod}(t) = b\}$. Thus $\{R(t) : t \leq L_0\} \stackrel{D}{=} \{R_{mod}(t) : t \leq L_0\}$. The time of a negative jump from level b in $\mathbf{R}$ is also the time of a negative jump in $\mathbf{R_{mod}}$, but while the starting points are different the end points of the jumps are stochastically the same. From that time on the probabilistic replication of $\mathbf{R_{mod}}$ from $\mathbf{R}$ is renewed (until reaching level b again, if this event occurs). For every increment of time t such that $R(L_0 + t) = b$ we have $R_{\mathbf{mod}}(L_0 + t) = b + t$. Next, let $L_1 = inf\{t > L_0 : R(t) < b\}$. We then define $\mathbf{R_{mod}}$ such that $R_{mod}(L_1+) = R(L_1+)$. In words, the probabilistic replication of $\mathbf{R_{mod}}$ from $\mathbf{R}$ is renewed at L_1. Note that every time of a negative jump in $\mathbf{R}$ is also the time of a negative jump in $\mathbf{R_{mod}}$, but a negative jump that occurs whenever $\mathbf{R_{mod}} > b$ automatically downcrosses level b and the gross undershoot below b in $\mathbf{R_{mod}}$ is also $\exp(\mu)$ distributed. Finally, we assume that $\mathbf{R_{mod}}$ is a regenerative process whose first cycle was described above. That is, after downcrossing level 0 (time of ruin) a new cycle starts from level $b - a$.

The steady state analysis of $\mathbf{R_{mod}}$ is based on a certain duality argument between non-Markov risk processes with exponential negative jumps and Markov dam processes with positive jumps. This argument was developed in Perry and Stadje (2003). In fact, the risk model introduced in Perry and Stadje (2003) is only a special case of $\mathbf{R_{mod}}$, because the negative jumps there are i.i.d. random variables, while the negative jumps of $\mathbf{R_{mod}}$ here are not i.i.d. random variables, since they are state dependent. It should be noted that despite the model here is more general than that of Perry and Stadje (2003) the idea behind the methodology is the same, since the construction of the Markovian dual dam process $\mathbf{D_{dual}} = \{D_{\mathbf{dual}}(t) : t \geq 0\}$ from $\mathbf{R_{mod}}$ is due to sample paths.

The construction of $\mathbf{D_{dual}}$ from $\mathbf{R_{mod}}$ is carried out in two phases; cf. Fig. 2. First, we construct the artificial mountain process $\mathbf{M} = \{M(t) : t \geq 0\}$ from $\mathbf{R_{mod}}$. As explained in the proof of Theorem 1 we replace a negative jump that starts at level x and ends at level w by a piece of trajectory that decreases continuously at rate $\beta_R(\cdot)$.

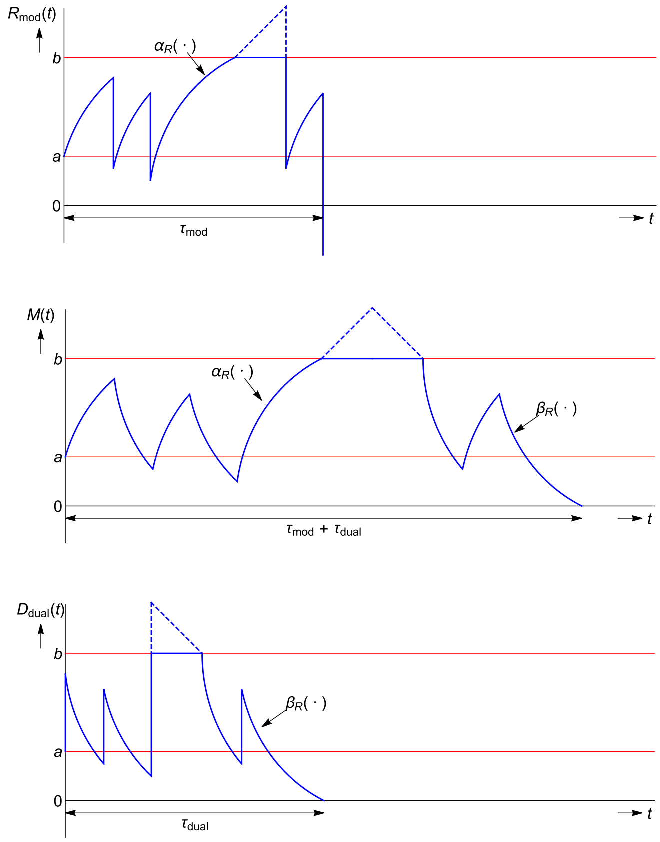

Fig. 2. The construction of $\mathbf{D_{dual}}$ from $\mathbf{R_{mod}}$, via $\mathbf{M}$.

Let the OFF periods be the periods where the process $M(t)$ decreases, and the ON periods be the periods where it increases. That is, the state of $\mathbf{M}$ at the end of each OFF period is stochastically equal to the state of $\mathbf{R_{mod}}$ immediately after the corresponding negative jump. As a result, the mountain $\mathbf{M}$ is a process whose continuous sample path alternates between independent ON and OFF periods, where the ON periods are i.i.d. random variables with generic distribution $G(\cdot)$ and the OFF periods are independent and $\exp(\mu)$ distributed. Also, when the content level is equal to $x > 0$ then during the ON periods the rate of upward slope in $\mathbf{M}$ is $\alpha_R(x)$ and during OFF periods the rate of downward slope in $\mathbf{M}$ is $\beta_R(x)$. In the second phase we construct the dual dam process $\mathbf{D_{dual}}$ from $\mathbf{M}$ by deleting the ON periods and gluing together the OFF periods. $\mathbf{D_{dual}}$ is a Markov process, since the positive jumps arrive according to a Poisson process of rate λ, but note that the positive jumps of $\mathbf{D_{dual}}$ are state dependent (cf. Fig. 2) in the sense that if the jump starts at level x the probability that the jump will be greater than y is equal to $1 - G(\int_x^{x+y} \frac{1}{\alpha_R(w)} dw)$ (note that $\alpha_R(x) = 1$ for all $x \geq b$).

We designate the cycle of $\mathbf{M}$ by $\tau_{mod} + \tau_{dual}$, where τ_{mod} is the cycle (time to ruin) of $\mathbf{R_{mod}}$ and τ_{dual} is the cycle of $\mathbf{D_{dual}}$. The next lemma relates the law of $\mathbf{R_{mod}}$ and that of $\mathbf{D_{dual}}$ where, for the sake of simplicity, we assume that the starting point of $\mathbf{R_{mod}}$ is a (instead of $b - a$).

Lemma 4. *Let $f_{\mathbf{R_{mod}}}(\cdot)$ and $f_{\mathbf{D_{dual}}}(\cdot)$ be the steady state densities of $\mathbf{R_{mod}}$ and $\mathbf{D_{dual}}$, respectively. Then for any $x > 0$,*

$$f_{\mathbf{R_{mod}}}(x) = \begin{cases} \frac{E\tau_{dual}}{E\tau_{mod}} \left[\frac{\beta_R(x)}{\alpha_R(x)} f_{\mathbf{D_{dual}}}(x) - \frac{\beta_R(0)}{\alpha_R(x)} f_{\mathbf{D_{dual}}}(0) \right], & 0 < x < a, \\ \frac{E\tau_{dual}}{E\tau_{mod}} \frac{\beta_R(x)}{\alpha_R(x)} f_{\mathbf{D_{dual}}}(x), & x \geq a. \end{cases}$$

Proof. (i) $x \geq a$. Let $U_{\mathbf{R_{mod}}}(x)$ and $D_{\mathbf{R_{mod}}}(x)$ be the number of upcrossings and the number of downcrossings of level x during the cycle τ_{mod} of $\mathbf{R_{mod}}$, respectively. Similarly, let $U_{\mathbf{D_{dual}}}(x)$ and $D_{\mathbf{D_{dual}}}(x)$ be the number of upcrossings and the number of downcrossings of level x during the cycle τ_{dual} of $\mathbf{D_{dual}}$. $\mathbf{R_{mod}}$ and $\mathbf{D_{dual}}$ are regenerative processes; thus, for every $x \geq a$ every upcrossing in $\mathbf{R_{mod}}$ [$\mathbf{D_{dual}}$] is compensated by a downcrossing and thus $U_{\mathbf{R_{mod}}}(x) = D_{\mathbf{R_{mod}}}(x)$ and $U_{\mathbf{D_{dual}}}(x) = D_{\mathbf{D_{dual}}}(x)$.

By the construction of $\mathbf{D_{dual}}$ from $\mathbf{R_{mod}}$ (via $\mathbf{M}$) the probability law of the number of upcrossings in $\mathbf{R_{mod}}$, of any level $x \geq a$, is equal to the probability law of the number of upcrossings in $\mathbf{D_{dual}}$, which means that the expected number of upcrossings in one cycle of $\mathbf{R_{mod}}$ is equal to the expected number of upcrossings in one cycle of $\mathbf{D_{dual}}$. Thus

$$EU_{\mathbf{R_{mod}}}(x) = ED_{\mathbf{R_{mod}}}(x) = EU_{\mathbf{D_{dual}}}(x) = ED_{\mathbf{D_{dual}}}(x). \quad (31)$$

By renewal theory $EU_{\mathbf{R_{mod}}}(x)/E\tau_{mod}$ and $EU_{\mathbf{D_{dual}}}(x)/E\tau_{dual}$ are interpreted as the long-run rates of upcrossings in $\mathbf{R_{mod}}$ and in $\mathbf{D_{dual}}$, respectively. By LCT (see Cohen, 1977)

$$EU_{\mathbf{R_{mod}}}(x)/E\tau_{mod} = \alpha_R(x) f_{\mathbf{R_{mod}}}(x) \quad (32)$$

and

$$EU_{\mathbf{D_{dual}}}(x)/E\tau_{dual} = \beta_R(x) f_{\mathbf{D_{dual}}}(x). \quad (33)$$

Now substitute (32) and (33) into (31) to obtain

$$E\tau_{mod} \cdot \alpha_R(x) f_{\mathbf{R_{mod}}}(x) = E\tau_{dual} \cdot \beta_R(x) f_{\mathbf{D_{dual}}}(x). \quad (34)$$

(ii) $0 \leq x < a$. Consider the sample path of a cycle of the mountain $\mathbf{M}$. On the one hand, for every $x \in [0, a)$ the number of downcrossings is 1 larger than the number of upcrossings. Thus, by construction of $\mathbf{D_{dual}}$ this means that the number of downcrossings of level x in $\mathbf{D_{dual}}$ is 1 larger than the number of upcrossings in $\mathbf{R_{mod}}$. On the other hand, level 0 is downcrossed only once (at the end of each cycle) in both $\mathbf{D_{dual}}$ and $\mathbf{R_{mod}}$. Thus, with probability 1

$$U_{\mathbf{R_{mod}}}(0) = D_{\mathbf{D_{dual}}}(0) = 1. \quad (35)$$

Thus

$$U_{\mathbf{R_{mod}}}(x) + U_{\mathbf{R_{mod}}}(0) = D_{\mathbf{R_{mod}}}(x) \stackrel{\mathcal{D}}{=} D_{\mathbf{D_{dual}}}(x), \quad (36)$$

where the second step of (36) is obtained by the construction of $\mathbf{D_{dual}}$ from $\mathbf{R_{mod}}$. Take expectations in (36) and substitute (35) to obtain

$$EU_{\mathbf{R_{mod}}}(x) = ED_{\mathbf{D_{dual}}}(x) - ED_{\mathbf{D_{dual}}}(0). \quad (37)$$

Now apply (32) and (33) to (37) in order to obtain

$$E\tau_{mod} \cdot \alpha_R(x) f_{\mathbf{R_{mod}}}(x) = E\tau_{dual} \cdot \beta_R(x) f_{\mathbf{D_{dual}}}(x) - E\tau_{dual} \cdot \beta_R(0) f_{\mathbf{D_{dual}}}(0).$$

The proof is complete. □

It follows by Lemma 4 that the steady state law of $\mathbf{R_{mod}}$ can be expressed in terms of the steady state law of $\mathbf{D_{dual}}$. Before calculating the density $f_{\mathbf{D_{dual}}}(\cdot)$, we introduce

$$A_R(x) = \int_0^x \frac{1}{\alpha_R(y)} dy,$$
$$B_R(x) = \int_0^x \frac{1}{\beta_R(y)} dy.$$

The Markov model

The process $\mathbf{D_{dual}}$ is a Markov process. The times between positive jumps are $\exp(\mu)$ distributed and the gross positive jumps have distribution $C(\cdot)$, so that the probability that a net positive jump starting at w will upcross x is $1 - C(A_R(x) - A_R(w))$. In order to ease the notation we introduce the constant (unknown yet) $\zeta := \beta_R(0)f_{\mathbf{D_{dual}}}(0)$.

The next theorem is a balance LCT equation for the steady state density of $\mathbf{D_{dual}}$.

Theorem 2. *Let $C(\cdot)$ be the distribution of the gross jump sizes in $\mathbf{D_{dual}}$. Then, the balance equation for the steady state density $f_{\mathbf{D_{dual}}}(\cdot)$ is given by:*

$$\beta_R(x)f_{\mathbf{D_{dual}}}(x) = \begin{cases} \mu\int_0^x [1-C(A_R(x)-A_R(w))] \times f_{\mathbf{D_{dual}}}(w)dw + \zeta, & 0<x<a,\\ \mu\int_0^x [1-C(A_R(x)-A_R(w))] \times f_{\mathbf{D_{dual}}}(w)dw + \zeta[1-C(A_R(x)-A_R(a))], & a\le x<b,\\ \mu\int_0^b [1-C(A_R(x)-A_R(w))] \times f_{\mathbf{D_{dual}}}(w)dw + \zeta[1-C(A_R(x)-A_R(a))], & x\ge b. \end{cases} \tag{38}$$

Proof. The proof is very similar to the proof of Theorem 1, where λ is replaced by μ and G by C. The other difference is the rate of upcrossing level x starting at 0. Notice that here:

(i) For $0 < x < a$, ζ is the rate of upcrossings of level x after reaching 0 since at the beginning of each cycle (or at the end of the previous cycle) the process jumps to a.

(ii) For $a \le x < b$ or $x \ge b$ the rate of upcrossing level x from 0 is $\zeta[1-C(A_R(x)-A_R(w))]$. ζ is the rate of downcrossing 0 at the end of each cycle (or going to a at the beginning of the next cycle) and then the probability of upcrossing x is $1-C(A_R(x)-A_R(w))$. □

The solution to $f_{\mathbf{D_{dual}}}(x)$ in (38) is similar to the solution of $f(x)$ in (10). We use the notation:

$$Q(x,w) := Q_1(x,w) := \frac{\mu[1-C(A_R(x)-A_R(w))]}{\beta_R(x)}.$$

Also, define

$$Q_{n+1}(x,w) = \int_w^x Q_1(x,y)Q_n(y,w)dy.$$

We get in (38)

$$f_{\mathbf{D_{dual}}}(x) = \begin{cases} \frac{\zeta}{\beta_R(x)} + \int_0^x Q(x,w) \times f_{\mathbf{D_{dual}}}(w)dw, & 0<x<a,\\ \frac{\zeta}{\mu}Q(x,a) + \int_0^x Q(x,w) \times f_{\mathbf{D_{dual}}}(w)dw, & a\le x<b,\\ \frac{\zeta}{\mu}Q(x,a) + \int_0^b Q(x,w) \times f_{\mathbf{D_{dual}}}(w)dw, & x\ge b. \end{cases} \tag{39}$$

Solving for $f_{\mathbf{D_{dual}}}(\cdot)$ in $0 < x < a$ we get

$$f_{\mathbf{D_{dual}}}(x) = \frac{\zeta}{\beta_R(x)} + \sum_{n=1}^{\infty}\int_0^x \zeta\frac{Q_n(x,w)}{\beta_R(w)}dw = \frac{\zeta}{\beta_R(x)} + \zeta\sum_{n=1}^{\infty} V_n(x), \tag{40}$$

where $V_n(x) := \int_0^x \frac{Q_n(x,w)}{\beta_R(w)}dw$.

To solve for $f_{\mathbf{D_{dual}}}(\cdot)$ in $a \le x < b$ we introduce

$$m(x) := \frac{\zeta}{\mu}Q(x,a) + \int_0^a Q(x,w)f_{\mathbf{D_{dual}}}(w)dw,$$

so that, for $a \le x < b$:

$$f_{\mathbf{D_{dual}}}(x) = m(x) + \int_a^x Q(x,w)f_{\mathbf{D_{dual}}}(w)dw.$$

Iteration now yields:

$$f_{\mathbf{D_{dual}}}(x) = m(x) + \sum_{n=1}^{\infty}\int_a^x Q_n(x,w)m(w)dw. \tag{41}$$

In the region $x \ge b$ we simply substitute the solutions (40) and (41) into (39). From the first equation of (39) it follows that $\zeta = \beta_R(0)f_{\mathbf{D_{dual}}}(0)$. Finally we obtain $f_{\mathbf{D_{dual}}}(0)$ by using the normalizing condition, as ζ appears linearly in each of the $f_{\mathbf{D_{dual}}}(x)$ expressions on $(0,a)$, $[a,b)$ and $[b,\infty)$. Lemma 4 finally gives $f_{\mathbf{R_{mod}}}(x)$.

Now we are in a position to find the five performance measures for the risk model with renewal arrivals and exponentially distributed gross claim sizes.

Performance measure (i): Time to ruin

To find the mean time to ruin $E\tau_{mod}$, we observe that

$$E\tau_{mod} = [\text{rate of downcrossings of level 0 by } \mathbf{R_{mod}}]^{-1}, \tag{42}$$

so that we need to find the rate of downcrossings of level 0 by $\mathbf{R_{mod}}$. To this end we designate $H(\cdot)$ as the limiting distribution of the state in $\mathbf{R_{mod}}$ just before a negative jump. Clearly, $H(\cdot)$ is an absolutely continuous distribution for all $0 < x < b$, but it has an atom at level b. Let $h(\cdot)$ be the density with respect to $H(\cdot)$. Then, with λ denoting one divided by the mean interval between two negative jumps of the renewal process of claim arrivals, the rate of downcrossings of level 0 by $\mathbf{R_{mod}}$ is equal to

$$\lambda\int_0^b e^{-\mu B_R(x)}h(x)dx + \alpha_R(b-)f_{\mathbf{R_{mod}}}(b-)e^{-\mu B_R(b)}. \tag{43}$$

As in the proof of Theorem 1, $e^{-\mu B_R(x)}$ is the probability that a negative jump starting at state x is bigger than x, thus starting at x ruin occurs. The first component corresponds to jumps in $\mathbf{R_{mod}}$ from some level $x \in (0,b)$. The second component is the rate of downcrossings that start from (above) level b. The downcrossing rate of level b also equals the upcrossing rate of level b, which is $\alpha_R(b-)f_{\mathbf{R_{mod}}}(b-)$. It has to be multiplied by $e^{-\mu B_R(b)}$ because it has to go all the way through 0.

Now let us determine the density $h(\cdot)$. From the duality between $\mathbf{R_{mod}}$ and $\mathbf{D_{dual}}$ the level just before a negative jump in $\mathbf{R_{mod}}$ is stochastically equal to the sum of the level just before a positive jump and the net jump size in $\mathbf{D_{dual}}$ (see Fig. 2). In the dual process $\mathbf{D}_{dual}$ the claim arrival process is Poisson thus the PASTA law can be applied. This means that

$$h(x) = \int_0^x f_{\mathbf{D_{dual}}}(w)d_wC(A_R(x)-A_R(w)). \tag{44}$$

We thus have, combining Eqs. (42)–(44):

$$E\tau_{mod} = 1\Big/\left[\lambda\int_0^b e^{-\mu B_R(x)}\int_0^x f_{\mathbf{D_{dual}}}(w)d_wC(A_R(x) - A_R(w))dx + \alpha_R(b-)f_{\mathbf{R_{mod}}}(b-)e^{-\mu B_R(b)}\right]. \tag{45}$$

Performance measure (ii): Deficit at ruin

Clearly, by the memoryless property the deficit is $\exp(\mu)$ distributed, since it is the undershoot below level 0 in $\mathbf{R_{mod}}$.

Performance measure (iii): The dividend until ruin

By the duality between $\mathbf{R_{mod}}$ and $\mathbf{D_{dual}}$, the length of the dividend period I in $\mathbf{R_{mod}}$ is stochastically equal to the size of the overshoot in $\mathbf{D_{dual}}$. Note that since $\mathbf{D_{dual}}$ downcrosses 0 only once per cycle, at rate $\alpha_D(0)f_{\mathbf{D_{dual}}}(0)$, the expected cycle length $E[\tau_{dual}]$ in $\mathbf{D_{dual}}$ is $\frac{1}{\alpha_D(0)f_{\mathbf{D_{dual}}}(0)}$. Thus:

$$EI = E[\tau_{dual}]P(\mathbf{D_{dual}} > b) = \frac{\int_b^\infty f_{\mathbf{D_{dual}}}(w)dw}{\alpha_D(0)f_{\mathbf{D_{dual}}}(0)}.$$

Performance measures (iv) and (v): The amount of money transferred to/from the reinsurer

In the proof of the next lemma we use similar arguments as in that of Lemma 3. However, there are several changes.

Lemma 5.

$$\text{(i)}\quad R_{to} = E\tau_{mod}\left[\int_0^b [c-\alpha_R(x)]f_{\mathbf{R_{mod}}}(x)dx + [c-\alpha_R(b-)]\right]\int_b^\infty f_{\mathbf{R_{mod}}}(x)dx,$$

$$\text{(ii)}\quad R_{from} = E\tau_{mod}\int_0^b [1-\beta_R(x)]f_{\mathbf{D_{dual}}}(x)dx = E\tau_{mod}\left[\int_0^b [1-\beta_R(x)]\alpha_R(x)f_{\mathbf{R_{mod}}}(x)dx\right] + \int_0^a [1-\beta_R(x)]dx,$$

where $E\tau_{mod}$ is given in (45).

Proof. (i)

$$\begin{aligned} R_{to} &= E\left[\int_0^{\tau_{mod}} [c-\alpha_R(R_{mod}(t))]dt\right] \\ &= E\tau_{mod}E[c-\alpha_R(R_{mod}(\infty))] \\ &= E\tau_{mod}\left[\int_0^b [c-\alpha_R(x)]f_{\mathbf{R_{mod}}}(x)dx + [c-\alpha_R(b-)] \times \int_b^\infty f_{\mathbf{R_{mod}}}(x)dx\right]. \end{aligned}$$

(ii)

$$R_{from} = E\tau_{mod}\left[\int_0^b [1-\beta_R(x)] \times (\text{rate of downcrossings of level } x \text{ by } \mathbf{R_{mod}})dx\right].$$

That rate of downcrossings equals the rate of downcrossings in the dual process, and hence

$$R_{from} = E\tau_{mod}\left[\int_0^b [1-\beta_R(x)]\beta_R(x)f_{\mathbf{D_{dual}}}(x)dx\right]. \tag{46}$$

Using Lemma 4 we can also write this as

$$R_{from} = E\tau_{mod}\left[\int_0^b [1-\beta_R(x)]\beta_R(x)f_{\mathbf{D_{dual}}}(x)dx + \beta_R(0)f_{\mathbf{D_{dual}}}(0)\int_0^a [1-\beta_R(x)]f_{\mathbf{D_{dual}}}(x)dx\right]. \tag{47}$$

The last part of (ii) now follows by observing that $E\tau_{mod} = (\beta_R(0)f_{\mathbf{D_{dual}}}(0))^{-1}$. □

6.1. Poisson arrivals: explicit result for $\theta(a; b)$

We now consider the special case in which the times between successive claims *and* the gross claim sizes are exp(λ) distributed and exp(μ) distributed, respectively. In this special case the probability $\theta(a; b)$ (cf. Section 5.1) can be computed explicitly. The computation is carried out with regard to the *dual dam* process $\mathbf{D_{dual}}$ and it is based on cycle maximum analysis. Define M as the *cycle maximum* of $\mathbf{D_{dual}}$—with the restriction that if the cycle maximum exceeds b, we put M equal to b. By the duality concept M is also the maximal value of the surplus $\mathbf{R_{mod}}$ until ruin.

The approach is based on the idea that:

$$\theta(a; b) = \Pr(M < b \mid D_{dual}(0) = a). \tag{48}$$

The analysis of the latter probability is an important issue since $1-\theta(a; b)$ is the probability that no dividend is paid until ruin.

The next theorem is applied to the process $\mathbf{D_{dual}}$ and it is similar in spirit to Theorem 1 in Boxma and Perry (2009). Recall that $\theta(x; y)$ can be interpreted as the probability to reach level 0 before upcrossing level y when the starting point is x for all $a < x < y \le b$ and let $r_M(x)$ be the hazard rate function of M at x. In the theorem, we express $r_M(x)$ into $\theta(x; x)$. Thereafter we determine $\theta(x; x)$, thus also obtaining $r_M(x)$ and hence $\Pr(M < x)$. Finally we use (48) to obtain $\theta(a; b)$.

Theorem 3. *For $a \le x \le b$,*

$$r_M(x) = \frac{\lambda}{\alpha_D(x)}\theta(x; x).$$

Proof. By assumption, the times between negative jumps in $\mathbf{R}$ are exp(λ) distributed and the gross negative jumps are exp(μ) distributed. It follows by the duality argument that the times between successive positive jumps in $\mathbf{D_{dual}}$ are exp(μ) distributed and the gross jump sizes are exp(λ) distributed. Thus, by the lack of memory property of the gross jumps (in $\mathbf{D_{dual}}$) the hazard rate at $[x, x+dx)$ is $\lambda dx/\alpha_D(x)$. Since the latter argument holds for every $x \le b$, regardless of the *history* of $\mathbf{D_{dual}}$ suppose that x, for any arbitrary $a \le x \le b$, is a record value. This means that $M \in [x, x+dx)$ if and only if the latter record value at x is the last record value in the wet (i.e., non-zero) period of $\mathbf{D_{dual}}$ and the probability of the latter event is $\theta(x; x)$. By the strong Markov property, we find $r_M(x)$ by taking the product of $\frac{\lambda}{\alpha_D(x)}$ and $\theta(x; x)$. □

To compute $\theta(x; x)$ note that, due to the fact that $\mathbf{D_{dual}}$ is a Markov process, we have for all $a \le x \le b$ the equation

$$\theta(x; x+dx) = \left[1 - \frac{\mu dx}{\beta_D(x)}\right]\left[\theta(x; x) + (1-\theta(x; x)) \times \frac{\lambda dx}{\alpha_D(x)}\theta(x; x)\right] + o(dx). \tag{49}$$

To understand the right hand side of (49) note that the paths with arrivals in $[0, \frac{dx}{\beta_D(x)})$ do not provide a contribution to $\mathbf{D_{dual}}$, since they have probability μdx and will upcross level $x + dx$ unless the further event of service termination in $[0, \frac{dx}{\alpha_D(x)})$ (having probability $\frac{\lambda dx}{\alpha_D(x)}$) occurs. The term $\theta(x; x)$ in (49) then corresponds to paths which downcross level x and do not upcross it again. The term $(1-\theta(x; x))\frac{\lambda dx}{\alpha_D(x)}\theta(x; x)$ corresponds to paths which downcross level x and upcross again before hitting level 0, with a jump terminating at $u \in [x, x+dx]$ where the value of u does not matter, since $\theta(x; u) = \frac{\lambda dx}{\alpha_D(x)}\theta(x; x) + o(dx)$. Hence, from (49) we get

$$\theta'(x; x) = \frac{\lambda\theta(x; x)}{\alpha_D(x)} - \frac{\mu\theta(x; x)}{\beta_D(x)} - \frac{\lambda\theta^2(x; x)}{\alpha_D(x)}. \tag{50}$$

To simplify (50) substitute for $x \geq a$:

$\eta(x) = 1/\theta(x; x)$.

We get after some elementary algebra

$$\eta'(x) + \eta(x)\left(\frac{\lambda}{\alpha_D(x)} - \frac{\mu}{\beta_D(x)}\right) = \frac{\lambda}{\alpha_D(x)}. \tag{51}$$

Recall the definitions of $B(x)$ and $A(x)$, cf. (8) and (16). By multiplying both sides of (51) by $e^{\lambda A(x)-\mu B(x)}$ we get

$$e^{\lambda A(x)-\mu B(x)}\left[\eta'(x) + \eta(x)\left(\frac{\lambda}{\alpha_D(x)} - \frac{\mu}{\beta_D(x)}\right)\right] = e^{\lambda A(x)-\mu B(x)}\frac{\lambda}{\alpha_D(x)}. \tag{52}$$

Solving for $\eta(x)$ in (52) we get

$\eta(x) = L(x)e^{-\lambda A(x)+\mu B(x)} + c_0 e^{-\lambda A(x)+\mu B(x)}$,

where $L(x) = \int_0^x e^{\lambda A(y)-\mu B(y)} \frac{\lambda}{\alpha_D(y)} dy$ and c_0 is a constant. Obviously, $\eta(0) = 1$ so that $c_0 = 1$. We thus get

$$\eta(x) = [L(x) + 1]e^{-\lambda A(x)+\mu B(x)}. \tag{53}$$

Now substituting (53) into Theorem 3 we obtain

$$r_M(x) = \frac{\lambda e^{\lambda A(x)-\mu B(x)}}{\alpha_D(x)[L(x)+1]},$$

so that

$$\Pr(M \leq x) = \begin{cases} 0, & x < a, \\ 1 - e^{-\int_a^x r_M(y)dy}, & a \leq x < b, \end{cases} \tag{54}$$

$$\Pr(M = b) = e^{-\int_a^b r_M(y)dy}. \tag{55}$$

Finally, one can conclude from (54) and (48) that

$$\theta(a; b) = 1 - e^{-\int_a^b \frac{\lambda e^{\lambda A(x)-\mu B(x)}}{\alpha_D(x)[L(x)+1]} dx}.$$

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