Correlation and Entanglement of multipartite Qubit, Qutrit, and n-level Systems

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Quantum entanglement is an information resource; it plays an important role in many protocols for quantum-information processing, including quantum computation, quantum cryptography, teleportation, super-dense coding, and quantum error correction protocols. Moreover, multipartite entanglement offers a means of enhancing interferometric precision beyond the standard quantum limit and is therefore relevant to increasing the precision of atomic clocks by decreasing projection noise in spectroscopy. But it turns out that even entanglement of two two-level (qubit) systems is not well understood! In this case, there is a Peres-Horodecki criterion for determining whether a mixed states are entangled, but the physical meaning of this criterion is not understood. Recently, we have been able to express the criterion in terms of invariant parameters in the density matrix. For the purpose of discussion here, let us consider only qubits. For two uncorrelated qubits, call them A and B, we can write the density matrix as a tensor product, $\rho_{AB} = \rho_A \rho_B$, where the individual qubit density matrices can be written as $\rho_J = \frac{1}{2}(1 + \mathbf{n}_J \cdot \boldsymbol{\sigma}_J)$, where $J = A, B$, the $\boldsymbol{\sigma}_J$ are Pauli matrices for particle J and the Bloch vectors are $\mathbf{n}_J = \langle \boldsymbol{\sigma}_J \rangle = \mathrm{Tr}\, \boldsymbol{\sigma}_J \rho_J$. For two correlated qubits,

$$\rho_{AB} = \frac{1}{4} \left[(1 + \mathbf{n}_A \cdot \boldsymbol{\sigma}_A)(1 + \mathbf{n}_B \cdot \boldsymbol{\sigma}_B) + \boldsymbol{\sigma}_A \cdot \mathbf{C}^{AB} \cdot \boldsymbol{\sigma}_B \right], \tag{1}$$

where the tensor $\mathbf{C}^{AB}$ specifies the qubit correlations,

$$C_{ij}^{AB} \equiv \langle \sigma_{i,A} \sigma_{j,B} \rangle - \langle \sigma_{i,A} \rangle \langle \sigma_{j,B} \rangle = \langle \sigma_{i,A} \sigma_{j,B} \rangle - n_{i,A}\, n_{j,B}. \tag{2}$$

The density matrix ρ_{AB} is a 4×4 Hermitian matrix with trace unity, so 15 parameters are required to parameterize it. The 3 components of $\mathbf{n}_A$, the 3 components of $\mathbf{n}_B$, and the 9 components C_{ij} of the 3×3 matrix $\mathbf{C}$, where we no longer explicitly show subsystem superscripts, are sufficient for this purpose. Our bipartite correlation measure for an n-level and m-level system is based on the $(n^2 - 1) \times (m^2 - 1)$ correlation matrix $\mathbf{C}$:

$$\mathcal{E}_C \equiv \frac{n_<^2}{4(n_<^2 - 1)} \mathrm{Tr}\, \mathbf{C}\mathbf{C}^T = \frac{n_<^2}{4(n_<^2 - 1)} \sum_{i,j} C_{ij} C_{ji}^T, \tag{3}$$

where $n_< = \min(n, m)$. $\mathcal{E}_C = \frac{n_<^2}{4(n_<^2 - 1)} \mathrm{Tr}\, (\rho_{AB} - \rho_A \rho_B)^2$ is a nonnegative real number.

The correlation matrix $\mathbf{C}$ quantifies the correlation and the entanglement of bipartite states. For pure two-qubit states, the number of nonzero singular values (NSVs) of $\mathbf{C}$ is zero for non-entangled states ($\mathbf{C}$ vanishes), and is three for entangled states. For classically-correlated states with two terms in the sum, only one NSV occurs, two NSVs occur for three terms, three NSVs occur for four or more terms, and for entangled (i.e., quantum-correlated) mixed states there are three NSVs. These cases are summarized in Fig. 1. Entangled mixed states can be differentiated from classically-correlated states with 3 NSVs by applying the Peres-Horodecki (PH) partial transposition condition [which corresponds to changing the sign of $n_{y,B}$ and the matrix elements C_{iy}^{AB} that multiply $\sigma_{y,B}$ in (1), and determining whether the resulting ρ is still a genuine density matrix — if it is, the state is classically correlated] to the density matrices with 3 NSVs. The *only* categories that cannot be distinguished without use of the PH condition are the mixed-entangled and the classically correlated states with ≥ 4 terms.

In order to better understand the present-day lack of physical intuition regarding entanglement, it is enlightening to consider the Werner two-qubit density matrix composed of a sum of a singlet state and the maximally mixed state, $\rho^W = p|\Psi^-\rangle\langle\Psi^-| + \frac{1-p}{4}\mathbf{1}$, where $|\Psi^-\rangle$ is

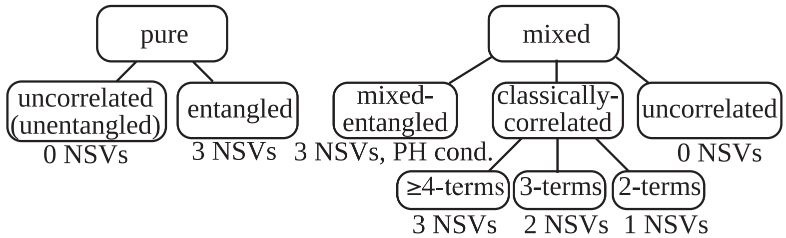

Figure 1: Classification of two-qubit states. Categories can be experimentally distinguished by measuring $\mathbf{n}_A$, $\mathbf{n}_B$, and using Bell measurements to determine the $\mathbf{C}$ matrix.

the singlet, or the more general Werner two-qubit density matrix, $\rho^{GW} = p\,|\psi^-\rangle\langle\psi^-| + \frac{1-p}{4}\mathbf{1}$, where $|\psi^-\rangle = (2\cosh(2\theta))^{-1/2}\,(e^{-\theta}|\uparrow\downarrow\rangle - e^{\theta}|\downarrow\uparrow\rangle)$. ρ^{GW} reduces to ρ^W for $\theta = 0$. For ρ^{GW}, $\mathbf{n}_A = -\mathbf{n}_B = p\tanh(2\theta)\,\hat{\mathbf{z}}$, and

$$\mathbf{C}^{GW} = -p\begin{pmatrix} \operatorname{sech}(2\theta) & 0 & 0 \\ 0 & \operatorname{sech}(2\theta) & 0 \\ 0 & 0 & 1-p+p\operatorname{sech}^2(2\theta) \end{pmatrix}. \tag{4}$$

The PH entanglement criterion shows that this state is entangled if $p[(1+2\operatorname{sech}(2\theta)] \geq 1$. Figure 2 plots the PH criterion limit and the correlation measure, $\mathcal{E}_C(p,\theta) = \sum_i d_i^2 = 1 - p + (2p^2 + p)\operatorname{sech}^2(2\theta)$, for the generalized Werner state. Note that the PH criterion is not obtainable from $\mathbf{C}$ alone, but can be obtained using the invariant parameters $\xi \equiv \sum_i d_i - \frac{\mathbf{n}_A\cdot\mathbf{C}\cdot\mathbf{n}_B}{\mathbf{n}_A\cdot\mathbf{n}_B}$ and $\mathbf{n}_A\cdot\mathbf{n}_B$. More explicitly, $p[1+2\operatorname{sech}(2\theta)] = -\xi + \sqrt{\xi^2/4 - \mathbf{n}_A\cdot\mathbf{n}_B}$, so the PH condition reads

$$-\frac{\xi}{2} + \frac{-\xi + \sqrt{\xi^2 - 4\,\mathbf{n}_A\cdot\mathbf{n}_B}}{2} \geq 1\ , \tag{5}$$

which can be written as the condition: the largest root of the quadratic equation, $(x+\xi/2)^2 + \xi(x+\xi/2) + \mathbf{n}_A\cdot\mathbf{n}_B = 0$, is greater than unity. Thus, mixed state entanglement is determined not only by $\mathbf{C}$ but by additional invariant characteristics of the density matrix, i.e., invariant characteristics composed of the parameters $\mathbf{C}$, $\mathbf{n}_A$ and $\mathbf{n}_B$ used to form the density matrix (whereas the correlation is determined only in terms of $\mathbf{C}$).

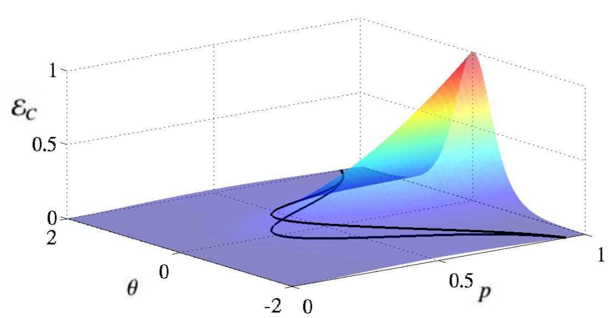

Figure 2: $\mathcal{E}_C(p,\theta)$ versus p and θ for the generalized Werner density matrix ρ^{GW}, and the Peres-Horodecki entanglement criterion limit, $p[1+2\operatorname{sech}(2\theta)] = 1$, drawn on the p-θ plane and projected onto the $\mathcal{E}_C$ surface.