Notes on monetary theory for microeconomists

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Abstract: We present a reinterpretation of what is known as the classical dichotomy, and illustrate its power, within the framework of the fiscal theory of the price level, by deriving some insights in a language that, hopefully, is accessible to microeconomists.

1 Introduction

"Money is a fiction invented by the macroeconomists," used to say Nir Dagan in our conversations at the Hebrew University. And he was right. Most of the courses we either took or taught did not have money in them, and none of them explained why money exists. True, the IS-LM model featured a demand for money, but we were never told where it came from. And in any case, it was commonly accepted that this model was tantamount to idolatry. The central model of economics was, and still is, the Walrasian model, the one in which individuals are endowed and ultimately consume peanuts and olives, goods that make our lives worth living, and in which money has, at most, the role of a unit of account. The basic Walrasian model of an exchange economy could be extended to include production, public goods, externalities, intertemporal trade, and even uncertainty, without the need for a medium of exchange. The Walrasian model seemed to reject any attempt to introduce money. The main textbook on microeconomics theory, Mas-Colell, Whinston, and Green (1995), does not mention money, except for a short subsection which deals with

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the overlapping generations model. On the other hand, money and inflation has been a topic of interest for economists since economics became a science. Moreover, economists seemed to be speaking about monetary policy most of the time. To be sure, there were theories that tried to explain the determination of prices and the causes of inflation by means of money, most notably the quantity theory of money. But this theory, however well it predicted inflation, seemed to be a mechanical devise imported from the area of physics. Still, the most prestigious economists seemed to advocate for the quantity theory of money. The inebriating talks by Milton Friedman were, and still are, so convincing that it seems impossible that the quantity theory were unfounded.

My own attempts to come up with an explanation of the existence of money were fruitless. The closer I got was a finite overlapping generations model in which money was transferred from generation to generation and which gave to the last generation the right to some last-period peanuts. But I knew that this was cheating because these last period peanuts, however meagre, made money a real asset when everybody knows that money, if we want it to be money, must be a purely financial asset that does not provide neither utility nor property rights on future goods. Then, I recalled having read a paper by Polterovich (1993) on rationing and queues in which money was used as a means to allocating goods owned by the government. This way of introducing money was also cheating, but at least here money was, arguably, not a pure real asset. Also, this paper showed me that money may have a role in an economy that is not a pure private ownership one. When there is a government that provides services, there might be a role for money. An then, I found Woodford (1995) and the recent book by Cochrane (2021), which exhibit what is known as the fiscal theory of the price level. As its name suggests, this theory is aimed at explaining the determination of the price level in a monetary economy, without apealing to the quantity theory. But for me, the main insight was the idea that money may have value because it can be used to pay taxes. The idea is that if we move slightly away from a pure private ownership economy to one in which the government can tax part of your endowment but at the same time accepts money as a means of payment, people may want to hold money. This genial idea is so simple that it is no surprise that almost nobody saw it. And as an added bonus, this idea also works in finite economies, the workhorse of microeconomists.

Once money is successfully introduced in a simple Walrasian economy, we can study some of the topics that concerned monetary economists in the past. One is the fascinating debate about the "classical dichotomy." According to the classical position, "money is a veil behind which the action of real economic forces is concealed" (Pigou (2021)). Samuelson (1968) tolds us that the classical economists, he himself among them, "thought that real outputs and inputs and price ratios depended essentially in the longest run on real factors, such as tastes, technology, and endowments," and that the absolute level of prices is determined in the money market. As a result, an increase in the quantity of money would cause a proportional increase in all prices. Patinkin (1949), however, argued that there is an inconsistency in the classical position. Roughly, if the real side of the economy is insensitive to the quantity of money, then an increase in this quantity cannot unchain any reaction in the commodity markets that induce a proportional increase in the nominal prices. Patinkin further provided an integration of value and monetary theory whereby all prices, relative and absolute, are jointly determined and the neutrality of money is still preserved.

In these notes I propose a reinterpretation of the classical dichotomy. In a monetary economy, money must have some role. For instance, it may be used as a means to distributing publicly-owned goods, or to pay taxes. When we compare a monetary economy with the one that is obtained from it by erasing all money endowments as well the government-owned goods and taxes, we see that both money and taxes also had a redistributive effect; money endowments provide purchasing power and taxes excise purchasing power. In that sense, the introduction of money has a real effect. But then, given a monetary economy, we can conceive of an associated real economy that preserves the redistributive effect just mentioned, and whose set of feasible allocations coincides with that of the original monetary economy. This is the economy that lies behind the proverbial veil. The Walrasian equilibrium of this real economy is the one that determines the relative prices as well as all produciton plans and consumption bundles. Once the relative prices and all other real variables are determined, monetary prices, and the remaining monetary variables are determined by the quantity equation. We will illustrate the workings of the classical dichotomy under this interpretation by means of two models. The first one consists of a simple economy in which money is used to privatize publicly owned goods. The second, is an instance of the fiscal theory of the price level.

Lest is unclear, let me state from the outset that there is no innovation in this notes. They formulate the classical theory and the modern fiscal theory of the price level within the standard Walrasian model, which is the dominant tool used by economists to understand reality. My main purpose is to formulate them in a way that is digestible for microeconomists, trying to define everything upfront, from primitive variables to solution concepts, and avoiding the excessive use of technical jargon which constitutes, more often than not, an insurmountable barrier to entry.

2 The classical dichotomy

The "classical dichotomy" states that relative prices are determined in the real part of the economy (by the equilibrium conditions of the commodity markets), and that the price level is determined in the monetary sector. An early and eloquent statement of this dichotomy can be found in Cassel (On quantitative thinking in economics, as cited by (Patinkin, 1965, p. 620)):

> Thus the general price-problem is divided into two problems: first a problem of how the *relative* prices are determined; secondly, a problem of how the *general level* of prices is fixed. This separation of the two different sides of the general price-problem ... is so natural, and has such great scientific and educational advantage, that it is hardly possible to do without it.

Cassel considered that the determination of the relative prices is the task of the general economic theory, and the determination of the level of prices is the task of the theory of money. This dichotomy is based on what Leontief calls the "homogeneity postulate" (Leontief (1936)):

> ... the *quantity of any service or any commodity demanded or supplied by a firm or an individual remains unchanged if all the prices upon which it (directly)*

depends increase or decrease exactly in the same proportion. In mathematical terms, this means that all supply and demand functions … are homogeneous functions of zero degree.

Patinkin criticized the classical dichotomy and proposed an integration of monetary and value theory. He pointed out an inconsistency in the classical dichotomy. He essentially showed that if the homogeneity postulate holds, then neither the commodity market nor the money market are able to determine the absolute level of prices. Specifically (Patinkin (1949)),

> [i]f the supply of all goods depends only on relative prices, then, of necessity, the demand for money can depend only on relative prices. Thus absolute prices appear nowhere in the system, and hence obviously cannot be "determined" by it.

The formal argument runs as follows. Consider a monetary economy with L commodities and money. Denote agent h's excess demand for commodity ℓ by $z^h_\ell(p, \bar{m}^h)$ and his demand for money by $m^h(p, \bar{m}^h)$, where $\bar{p} = (p_1, \ldots, p_L)$ is a vector or commodity prices and $\bar{m}^h$ is agent h's initial endowment of money. Assume that p^* is a vector of equilibrium prices. Then, at these prices all markets clear:

$$\sum_h z^h_\ell(p^*, \bar{m}^h) = 0 \qquad \ell = 1, \ldots, L \tag{1a}$$

$$\sum_h m^h(p^*, \bar{m}^h) = \sum_h \bar{m}^h \tag{1b}$$

By the homogeneity postulate, for any positive scalar λ,

$$\sum_h z^h_\ell(\lambda p^*, \bar{m}^h) = 0 \qquad \ell = 1, \ldots, L$$

Namely, all commodity markets continue to clear if all prices are multiplied by λ. Therefore, by Walras's law, the money market clears as well:

$$\sum_h m^h(\lambda p^*, \bar{m}^h) = \sum_h \bar{m}^h$$

which means that λp^* is a vector of equilibrium prices as well; namely absolute prices are indetermined.

Patinkin's integration of monetary and value theory consists of postulating that the excess demand functions are homogenous of degree 0 in $\left(\bar{p}, \bar{m}^h\right)$ and that the demand functions for money are homogeneous of degree one in the same variables. As a result, if the initial endowments of money are multiplied by $\lambda > 0$, the equilibrium conditions will no longer be satisfied (causing what Patinkin calls a real balance effect), but if p^* was an equilibrium vector of prices before the change in money supply, λp^* is an equilibrium vector of prices after the change. This relation between money supply and equilibrium prices is commonly referred to as the neutrality of money.

3 The classical theory of money; a reinterpretation

In this section we illustrate the reinterpretation of the classical dichotomy outlined above, in the simplest of all contexts, exchange economies. We begin by defining what all economists know.

There are L commodities. An *exchange economy* consists of a set of agents $I = \{1, 2, \ldots, I\}$. Each agent i is characterized by

- A preference relation $\succsim_i$ on $\mathbb{R}^L_+$
- An initial bundle of commodities $\omega^i \in \mathbb{R}^L_+$.

We denote the exchange economy by $\mathcal{EE} = (\succsim_i, \omega^i)_{i \in I}$.

We denote the aggregate bundle of commodities by $\omega = \sum_{i \in I} \omega^i$. An *allocation* is a collection of bundles, $(x^i)_{i \in I}$, one for each agent. An allocation, $(x^i)_{i \in I}$, is *feasible* if $\sum_{i \in I} x^i = \omega$. An allocation, $(x^i)_{i \in I}$ and a vector of prices $p \in \mathbb{R}^L$ is a *competitive equilibrium* if

1. for all $i \in I$, the bundle x^i is affordable at p, namely $px^i \leq p\omega^i$
2. the bundle x^i is at least as good for i as any affordable bundle: $x^i \succsim_i x$ for all $x \in X$ s.t. $px \leq p\omega^i$
3. the allocation is $(x^i)_{i \in I}$ is feasible.

If $(x^i)_{i\in I}$ and $p \in \mathbb{R}^L$ constitute a competitive equilibrium, we say that $(x^i)_{i\in I}$ is a *competitive allocation supported by* p.

The following is a well known fact: If $\langle (x^i)_{i\in I}, p \rangle$ constitutes a competitive equilibrium of $\mathcal{EE}$, so does $\langle (x^i)_{i\in I}, \lambda p \rangle$ for any $\lambda > 0$. This fact is sometimes described as saying that the absolute prices that support a competitive allocation are not determined; only relative prices are.

Exchange economies represent private property economies; each agent owns a bundle of commodities and there is no bundle of commodities that does not belong to an agent. In these economies, there is no role for money, and there is no need for money. We will now turn to economies in which some commodities belong to no specific agent. They belong to the "government." Since the government does not consume commodities, it distributes them by means of a novel monetary mechanism. These economies are akin to the amusement park described by Lucas in his commencement address delivered at the University of Chicago on December 9, 1988. They also resemble the method used by Feeding America to allocate food to food banks (Prendergast (2017, 2022)).

A *tickets economy* consists of a set of agents $I = \{1, 2, \ldots, I\}$, and a government. Each agent i is characterized by

- A preference relation $\succsim_i$ on $\mathbb{R}^L_+$
- An initial bundle of commodities $\omega^i \in \mathbb{R}^L_+$
- An initial amount of money $m^i \geq 0$

The government owns an initial endowment of commodities $\omega^G \in X$.

We denote the economy by $\mathcal{E} = \left((\succsim_i, \omega^i, m^i)_{i\in I}, \omega^G \right)$.

We assume that $\sum_{i\in I} \omega^i + \omega^G \gg 0$ and that $\omega^G \neq 0$. We denote the aggregate bundle of commodities by $\omega = \sum_{i\in I} \omega^i + \omega^G$ and the total amount of money in the economy by $M = \sum_{i\in I} m^i$. As before, an *allocation* is a collection of bundles, $(x^i)_{i\in I}$, one for each agent. An allocation, $(x^i)_{i\in I}$, is *feasible* if $\sum_{i\in I} x^i = \omega$. An allocation, $(x^i)_{i\in I}$ and a vector of prices $p \in \mathbb{R}^L$ is a *tickets equilibrium* of the tickets economy $\mathcal{E}$ if

1. for all $i \in I$, the bundle x^i is affordable at p: $px^i \leq p\omega^i + m^i$ for all $i \in I$

2. the bundle x^i is at least as good for i as any affordable bundle: $x^i \succsim_i x$ for all $x \in X$ s.t $px \leq p\omega^i + m^i$

3. the allocation is $(x^i)_{i\in I}$ is feasible,

4. the amount of money in the economy equals the value of the government goods: $\sum_{i\in I} m^i = p\omega^G$.

If $(x^i)_{i\in I}$ and $p \in \mathbb{R}^L$ constitute a tickets equilibrium, we say that $(x^i)_{i\in I}$ is an *allocation supported by* p.

Note that the tickets economy $\mathcal{E} = \left((\succsim_i, \omega^i, m^i)_{i\in I}, \omega^G\right)$ can be seen as the exchange economy $\mathcal{EE} = (\succsim_i, \omega^i)_{i\in I\cup\{G\}}$ that is obtained from $\mathcal{E}$ by adding one commodity, money, from which no agent in I derives utility, and by adding one agent, the government, that consumes only money. Furthermore, a tickets equilibrium of the tickets economy $\mathcal{E}$ is none other than a competitive equilibrium of the exchange economy $\mathcal{EE}$ in which the price of money is set to be 1.

As opposed to the case of competitive equilibria, nominal prices are determined in a tickets equilibrium; an equiproportional change in nominal prices does not preserve the equilibrium. However, an equiproportional increase in money prices and in the quantity of money does preserve the equilibrium. This is stated in the following observation, the proof of which is left to the reader, and which is nothing but the expression of the neutrality of money in the context of our tickets economies.

Observation 1 (The neutrality of money) *Let* $\mathcal{E} = \left((\succsim_i, \omega^i, m^i)_{i\in I}, \omega^G\right)$ *and* $\mathcal{E}' = \left((\succsim_i, \omega^i, \lambda m^i)_{i\in I}, \omega^G\right)$ *be two economies where* $\lambda > 0$. *If* $(x^i)_{i\in I}$ *is an allocation in* $\mathcal{E}$ *supported by* p, *then* $(x^i)_{i\in I}$ *is an allocation in* $\mathcal{E}'$ *supported by* λp.

Let $\mathcal{E} = \left(I, (\succsim_i, \omega^i, m^i)_{i\in I}, \omega^G\right)$ be a tickets economy. The associated *real economy* is he exchange economy $\mathcal{RE} = \left(\succsim_i, \omega^i + \alpha^i\omega^G\right)_{i\in I}$ where $\alpha^i = \frac{m^i}{M}$. The associated real economy is obtained from the monetary economy by distributing the "common goods" among the individuals in proportion to their initial money holdings. In the tickets economy, money's only purpose is to distribute the commodities owned by the government among the agents. The "real" part of the economy is the exchange economy that is obtained after this distribution has taken place. Note in particular, that $(x^i)_{i\in I}$ is a feasible allocation in the tickets economy $\mathcal{E}$ if an only if it is a feasible allocation in the associated real economy $\mathcal{RE}$.

Since at a tickets equilibrium, the purchasing power of a dollar is one Mth of the social endowment ω^G, it is not surprising that the equilibrium of the tickets economy is also a competitive equilibrium of its associated real economy.

Observation 2 *If $\left\langle (x^i)_{i\in I}, p\right\rangle$ constitutes a tickets equilibrium of the tickets economy $\mathcal{E} = \left(I, (\succcurlyeq_i, \omega^i, m^i)_{i\in I}, \omega^G\right)$, then $\left\langle (x^i)_{i\in I}, p\right\rangle$ is also a competitive equilibirum of the associated real economy $\mathcal{EE} = \left(\succcurlyeq_i, \omega^i + \alpha^i\omega^G\right)_{i\in I}$ where $\alpha^i = \frac{m^i}{M}$.*

The proof is immediate from the definitions and is left to the reader. The following observation is a converse of the above one. It is an expression of the classical dichotomy.

Observation 3 (The classical dichotomy) *Let $\left\langle (x^i)_{i\in I}, p\right\rangle$ be a competitive equilibirum of the exchange economy $\mathcal{EE} = \left(\succcurlyeq_i, \omega^i + \alpha^i\omega^G\right)_{i\in I}$ where $\alpha^i = \frac{m^i}{M}$. Let $V = \frac{p\omega}{p\omega^G}$.Then, $\left\langle (x^i)_{i\in I}, p^*\right\rangle$, is an equilibrium of the the tickets economy $\mathcal{E} = \left((\succcurlyeq_i, \omega^i, m^i)_{i\in I}, \omega^G\right)$, where $p^* = \frac{M}{p\omega^G}p$. Furthermore, the following quantity equation holds:*

$$MV = p^*\omega. \tag{2}$$

Notice that V is the ratio of the value of the aggregate bundle of commodities to the value of publicly-owned goods. Since it is a ratio of nominal values, V depends only on relative prices and therefore is a real variable. It follows from (2) that V is the ratio of the value of the final goods to the quantity of money, and for that reason it is known as the velocity of money. The observation states, that the equilibrium relative prices of the tickets economy $\mathcal{E} = \left((\succcurlyeq_i, \omega^i, m^i)_{i\in I}, \omega^G\right)$ can be calculated as the equilibrium relative prices of the exchange economy $\mathcal{RE} = \left(\succcurlyeq_i, \omega^i + \alpha^i\omega^G\right)_{i\in I}$, and the equilibrium price level can be figured out by means of the quantity equation (2).

Proof. Let $\left\langle (x^i)_{i\in I}, p\right\rangle$ be a competitive equilibirum of the exchange economy $\mathcal{RE} = \left(\succcurlyeq_i, \omega^i + \alpha^i\omega^G\right)_{i\in I}$ where $\alpha^i = \frac{m^i}{M}$. Then, $(x^i)_{i\in I}$ is feasible and for all $i \in I$, the bundle x^i is affordable at p and it is at least as good as any affordable bundle. Since $(x^i)_{i\in I}$ is feasible in $\mathcal{RE}$ as well, we only need to show that for all $i \in I$, x^i is affordable at p^* and that $p^*\omega^G = M$. For all i,

$$\begin{aligned} px^i \leq p\left(\omega^i + \alpha^i\omega^G\right) &\Leftrightarrow \frac{M}{p\omega^G}px^i \leq \frac{M}{p\omega^G}p\left(\omega^i + \frac{m^i}{M}\omega^G\right) \\ &\Leftrightarrow p^*x^i \leq p^*\omega^i + m^i \end{aligned}$$

namely, bundle x^i is affordable at p^*. Also,

$$p^*\omega^G = \frac{M}{p\omega^G}p\omega^G = M$$

namely the quantity of money is the market value of the government goods. Finally, the quantity equation (2) holds. Indeed,

$$MV = M\frac{p\omega}{p\omega^G} = p^*\omega.$$

■

4 The classical dichotomy and the fiscal theory of the price level

In this section we illustrate the reinterpretation of the classical dichotomy within the framework of the fiscal theory of the price level. See Woodford (1995) and Cochrane (2021). In this model, there are no government-owned goods, but the the government has taxation power. We use a model with infinitely many periods in which individuals' preferences do not depend on money. For them, money is a means for transfering purchasing power from one period to the next. This, by itself does not infuse value to money. However, since the government accepts money as a means of payment, government-issued money will have value.

We proceed as in the previous section. We first define a real economy, with no money. We later define a monetary economy in which there is a government that issues money and imposes taxes on the population. Finally, we define a real economy associated with the monetary economy and show that their equilibrium real variables coincide, and that the monetary variables of the latter are easily determined by the quantity equation.

The formal details of the model are as follows. Time is discrete: $1, 2, \ldots$. There is a countable number of commodities: peanuts in period t, for $t = 1, 2, \ldots$. A bundle of commodities is a list $\overline{c} = (c_1, c_2, \ldots)$ of non-negative quantities of peanuts, one for each period. A *real economy* consists of a set of agents $H = \{1, 2, \ldots, H\}$. Each agent h is characterized by

- his homothetic, monotonic and concave utility function $U^h(\overline{c}) = \sum_{t=1}^{\infty} \beta^{t-1} u^h(c_t)$
- his initial endowment of commodities $\overline{\omega}^h = (\omega_1^h, \omega_2^h, \ldots)$, where $\omega_t^h \in (0, \omega_{\max})$.

We summarize the real economy by $\mathcal{RE} = \left(U^h, \overline{\omega}^h\right)_{h \in H}$.

The definition of a competitive equilibrium given in Section 3 is valid here as well. For the reader's convenience, we restate it.

An *allocation* is a collection of bundles, $\left(\overline{c}^h\right)_{h \in H}$, one for each agent. An allocation, $\left(\overline{c}^h\right)_{h \in H}$, is *feasible* if

$$\sum_{h \in H} \overline{c}^h = \sum_{h \in H} \overline{\omega}^h.$$

An allocation, $\left(\overline{c}^h\right)_{h \in H}$ and a vector of prices $\bar{p} = (p_1, p_2 \ldots)$ is a *competitive equilibrium* if

1. for all $h \in H$, the bundle $\overline{c}^h$ is *affordable* at $\bar{p}$, namely $\bar{p}\overline{c}^h \leq \bar{p}\overline{\omega}^h$.
2. the bundle $\overline{c}^h$ is at least as tasty for h as any affordable bundle: $U^h(\overline{c}^h) \geq U^h(\overline{c})$ for all $\overline{c}$ such that $\bar{p}\overline{c} \leq \bar{p}\overline{\omega}^h$
3. the allocation is $\left(\overline{c}^h\right)_{h \in H}$ is feasible.

4.1 Characterization of competitive equilibria

We denote the aggregate endowment by $\sum_{h \in H} \bar{\omega}^h = \bar{\omega} = (\omega_1, \omega_2 \ldots)$. For simplicity we assume that all individuals have the same preferences, $u^h = u$ and that u is differentiable. We further assume that the sequence $\frac{u'(\omega_{t+1})}{u'(\omega_t)}$ is bounded. Given that preferences are identical and homothetic, we have that the efficient allocations and, in particular, the competitive allocation satisfy

$$\frac{c_{t+1}^h}{c_t^h} = \frac{\omega_{t+1}}{\omega_t} \qquad t = 1, 2, \ldots, \ h \in H \tag{3}$$

The competitive equilibrium prices, $(p_1, p_2, \ldots)$, therefore, satisfy

$$\frac{p_{t+1}}{p_t} = \beta \frac{u'(\omega_{t+1})}{u'(\omega_t)} \qquad t = 1, 2, \ldots$$

We denote the value $\beta \frac{u'(\omega_{t+1})}{u'(\omega_t)}$ by δ_t. It is the subjective discount factor from $t+1$ to t, of the individuals when they consume a positive proportion of the aggregate endowment. The price ratio $\frac{p_{t+1}}{p_t}$ is the real discount rate from $t+1$ to t, or equivalently, the reciprocal of the gross interest rate. It is the amount of peanuts in period t that needs to be paid in order to get one peanut in $t+1$. With this notation we have that in equilibrium

$$\frac{p_{t+1}}{p_t} = \delta_t \qquad t = 1, 2, \ldots$$

Let's further denote by

$$\delta_{t_0 t_1} = \prod_{\tau = t_0}^{t_1 - 1} \delta_\tau$$

the subjective discount rate from period t_0 to period $t_1 (\geq t_0)$, where we adopt the convention that for any function g, $\prod_{\tau=t}^{t-1} g(\tau) = 1$. Then, since by monotonicty of preferences $\bar{p}\bar{c} = \bar{p}\bar{\omega}^h$, it follows from (3) that the proportion of the aggregate bundle that each individual consumes is the proportion of the value of his initial bundle out of the value of the aggregate bundle:

$$\begin{aligned} \bar{c}^h &= \frac{\sum_{t=1}^{\infty} p_t \omega_t^h}{\sum_{t=1}^{\infty} p_t \omega_t} \bar{\omega} = \frac{\sum_{t=1}^{\infty} \left(p_1 \prod_{\tau=1}^{t-1} \frac{p_{\tau+1}}{p_\tau} \right) \omega_t^h}{\sum_{t=1}^{\infty} \left(p_1 \prod_{\tau=1}^{t-1} \frac{p_{\tau+1}}{p_\tau} \right) \omega_t} \bar{\omega} \\ &= \frac{\sum_{t=1}^{\infty} \delta_{1t} \omega_t^h}{\sum_{t=1}^{\infty} \delta_{1t} \omega_t} \bar{\omega}. \end{aligned} \tag{4}$$

Note that since $\frac{u'(\omega_{t+1})}{u'(\omega_t)}$ is bounded and $\beta \in (0,1)$, we have that $\lim_{t\to\infty} \delta_{1t} = 0$. Since ω^h is also bounded and non-negative, we obtain that $\sum_{t=1}^{\infty} \delta_{1t}\omega_t^h < \infty$ and $\sum_{t=1}^{\infty} \delta_{1t}\omega_t < \infty$, which implies that $\overline{c}^h$ is well defined. We summarize the above discussion in the following observation.

Observation 4 *The real economy $\mathcal{RE} = (U, \overline{\omega}^h)$ has a unique competitive allocation $\bar{c}$, which is defined by (4). The supporting competitive prices are unique up to multiplication by a positive constant and satisfy $\frac{p_{t+1}}{p_t} = \delta_t$ for $t = 1, 2, \ldots$.*

4.2 A monetary pre-economy

We will now describe economies in which there is a government that issues and redeems debt, and raises taxes. Time is discrete: $1, 2, \ldots$. As before, there is a countable number of commodities: peanuts in period t, for $t = 1, 2, \ldots$. There are H individuals, each one characterized by

- his homothetic, monotonic and concave utility function $U^h(\bar{c}) = \sum_{t=1}^{\infty} \beta^{t-1} u^h(c_t)$
- his initial endowment of commodities $\bar{\omega}^h = (\omega_1^h, \omega_2^h, \ldots,)$, where $\omega_t^h \in (0, \omega_{\max})$
- his initial endowment of bonds b_1^h.

These bonds are denominated in period 1 dollars and can be used to pay for commodities and to pay taxes. They can equivalently be thought of as cash.

There is a government that collects taxes from each individual h whose real values are given by $\bar{s}^h = \left(s_1^h, s_2^h, \ldots\right)$. We assume that $s_t^h < \omega_t^h$ for $t = 1, 2, \ldots$.

We denote the pre-economy by $\mathcal{PE} = \left\langle \left(U^h, \bar{\omega}^h, b_1^h, \bar{s}^h\right)_{h\in H} \right\rangle$. We call it pre-economy because the policy variables to be used by the monetary authority are unspecified. As before, we assume that all the individuals have the same differentiable utility functions, $u^h = u$, and that the sequence $\frac{u'(\omega_{t+1})}{u'(\omega_t)}$ is bounded.

We denote the initial quantity of money by $B_1 = \sum_{h=1}^{H} b_1^h$, and the aggregate real tax revenues at $t = 1, 2, \ldots$, by $s_t = \sum_{h=1}^{H} s_t^h$. We assume that $s_t > 0$ for almost all $t \in \{1, 2, \ldots\}$.

An *allocation* consists of a bundle $\bar{c}^h = (c_1^h, c_2^h \ldots) \geq 0$ and a portfolio $\bar{b}^h = (b_1^h, b_2^h, \ldots)$ for each individual, and an issuance of bonds $\bar{B} = (B_1, B_2, \ldots) \geq 0$ by the monetary authority. Note that a portfolio is allowed to have negative entries. B_t is the amount of dollars that the government promises to pay in period t to the bond holders. An allocation $\langle(\bar{c}^h, \bar{b}^h), \bar{B}\rangle$ is *feasible*, if $\sum_{h=1}^H c_t^h = \omega_t$ and $\sum_{h=1}^H b_t^h = B_t$ for $t = 1, 2, \ldots$. Namely, if at every period, the amount of peanuts consumed equals the amount of peanuts in the economy, and the amount of bonds in the individuals' portfolios equals the amount issued by the monetary authority.

Given a vector of spot peanut prices, $(P_1, P_2 \ldots)$, the taxes that individual h is required to pay are $T_t^h = P_t s_t^h$, and the total tax revenue in period t is given by $T_t = \sum_{h=1}^H T_t^h = \sum_{h=1}^H P_t s_t^h$, for $t = 1, 2, \ldots$.

Money that is not used to pay taxes is transformed into a bond that promises to pay one dollar the next period at a rate of Q_t dollars in t per dollar in $t+1$. That is, Q_t is the nominal discount rate: the amount of dollars in period t that needs to be paid in order to get one dollar in $t+1$. Given a vector of spot peanut prices, $(P_1, P_2 \ldots)$, and of discount rates $(Q_1, Q_2, \ldots)$, bundle $(c_1, c_2, \ldots)$ and portfolio $(b_1, b_2, \ldots)$ are *(jointly) affordable* for individual h if $b_1 = b_1^h$ and

$$\begin{aligned} P_t c_t + Q_t b_{t+1} &\leq P_t \omega_t^h + b_t - T_t^h \qquad t = 1, 2, \ldots \\ \lim_{t\to\infty} \inf Q_{1t+1} b_{t+1} &\geq 0 \end{aligned}$$

That is, bundle $(c_1, c_2, \ldots)$ and portfolio $(b_1, b_2, \ldots)$ are (jointly) affordable for individual h if for every period, the amount of money spent on consumption and on bonds, does not exceed the value of his net income plus the fruits of his savings, and if the present value of his savings is not negative in the long run. The individual may have debts (negative savings) in some, and even all periods. But he is not allowed to have a sequence of debts whose present values are bounded away from 0.[1]

Given a vector of discount rates $(Q_1, Q_2, \ldots)$, a vector of bond issuances $(B_1, B_2, \ldots)$

[1]It can be shown that this restriction can be replaced by the following borrowing limits: $b_{t+1} \leq \sum_{\tau=t+1}^{\infty} Q_{t+\tau}(P_\tau \omega_\tau^h - T_\tau^h)$. See Santos and Woodford (1997) and Aiyagari (1994) for a discussion and derivation of similar borrowing constraints.

and of tax revenues $(T_1, T_2, \ldots)$ we say that *the government's budget is balanced* if

$$B_t = T_t + Q_t B_{t+1} \qquad t = 1, 2, \ldots$$

That is, current debt is financed either by taxes or by new debt. We now define an equilibrium of the pre-economy $\mathcal{PE}$.

A *monetary equilibrium* of $\mathcal{PE}$ consists of a vector of spot prices $\bar{P} = (P_1, P_2, \ldots)$, nominal discount rates $\bar{Q} = (Q_1, Q_2, \ldots)$, consumption bundles $\bar{c}^{h*} = (c_1^{h*}, c_2^{h*}, \ldots)_{h \in H}$, portfolios $\bar{b}^{h*} = \left(b_1^{h*}, b_2^{h*}, \ldots\right)_{h \in H}$, and a bond issuance $\bar{B} = (B_1, B_2, \ldots)$ such that

1. For each individual h, the bundle $\bar{c}^{h*}$ and portfolio $\bar{b}^{h*}$ are jointly affordable given $\bar{P}$ and $\bar{Q}$;

2. $U^h(\bar{c}^*) \geq U^h(\bar{c})$ for all affordable consumption bundle-protfolio pairs $\left(\bar{c}, \bar{b}\right)$;

3. The government's budget is balanced;

4. The allocation $\left(\left(\bar{c}^{h*}, \bar{b}^{h*}\right)_{h \in H}, \bar{B}\right)$ is feasible.

The problem with the concept of equilibrium of a pre-economy, is that there are too many endogenous variables, and as a result, it is indeterminate. In other words, there is a plethora of equilibria. On the other hand, this affords the monetary authority the ability to select the equilibrium by appropriately choosing some of endogenous variables and setting their value. The variables chosen by the monetary authority will typically be either the nominal discount rates or the debt levels, or some combination thereoff. The choice of these policy variables and the setting of their values by the monetary authority is referred to as *monetary policy.* On the other hand, the real tax revenues $(\bar{s}^h)_{h \in H}$, are exogenous variables in the pre-economy, and the role of fiscal policy is to set their values.

We have said that the pre-economy has a plethora of equilibria. Nevertheless, some interesting things can be said about all of them. But before that, we digress and comment on the relation of this model and Patinkin's integration of value and monetary theories.

5 The fiscal theory and Patinkin's integration of monetary and value theories

Patinkin's integration of monetary and value theories consisted in building a model in which the demand for commodities are homogeneous of degree 0 in prices and money endowments, and the demand for money is homogeneous of degree one in the same variables. Patinkin's apprach consisted in introducing money in the utility function. Specifically, he assumed that individuals' preferences depend not only on the commodity bundle they consume but also on their real balances, namely the purchasing power of the money they hold. In this way, nominal prices are determined by the interaction of the real and monetary sides of the economy, and at the same time, the neutrality of money holds –doubling the quantity of money results in doubling the nominal prices.

In the model presented in the previous section, money is not an argument in the individual's utility functions. Nevertheless, individuals do demand money. They do so to transfer purchasing power from one period to the other. One may wonder if the resulting system of excess demands shares the homogeneity properties required for Patinkin's integration. The answer is affirmative. To see this, note that if bundle $(c_1, c_2, \ldots)$ and portfolio $(b_1, b_2, \ldots)$ are jointly affordable for individual h given spot prices $(P_1, P_2, \ldots)$, discount rates $(Q_1, Q_2, \ldots)$, and initial holdings b_1^h, then for all $\lambda > 0$, bundle $(c_1, c_2, \ldots)$ and portfolio $\lambda(b_1, b_2, \ldots)$ are jointly affordable for individual h given, prices $\lambda(P_1, P_2, \ldots)$, discount rates $(Q_1, Q_2, \ldots)$, and initial holdings λb_1^h. Formally,

$$P_t c_t + Q_t b_{t+1} \leq P_t \omega_2^h + b_t - P_t s_t^h \quad \Leftrightarrow \quad \lambda P_t c_t + Q_t \lambda b_{t+1} \leq \lambda P_t \omega_2^h + \lambda b_t - \lambda P_t s_t^h \qquad t = 1, 2, \ldots$$
$$\text{and} \ \lim_{t\to\infty} \inf Q_{1t+1} b_{t+1} \geq 0 \quad \Leftrightarrow \quad \lim_{t\to\infty} \inf Q_{1t+1} \lambda b_{t+1} \geq 0$$

Consequently, if bundle $(c_1, c_2, \ldots)$ and portfolio $(b_1, b_2, \ldots)$ are utility maximizing for individual h given prices $(P_1, P_2, \ldots)$, discount rates $(Q_1, Q_2, \ldots)$ and initial holdings b_1^h, then bundle $(c_1, c_2, \ldots)$ and portfolio $\lambda(b_1, b_2, \ldots)$ are utility maximizing for individual h given , $\lambda(P_1, P_2, \ldots)$, $(Q_1, Q_2, \ldots)$ and λb_1^h. This means that the commodity demand functions are homogeneous of degree 0 in prices and initial money holdings, and that the money demand functions are linearly homogeneous in the same variables. As a result, in this model, the individual demands satisfy the homogeneity properties required by Patinkin, and therefore the neutrality of money is expected to hold, as shall soon be verified.

6 The real economy associated with the pre-economy

As we have mentioned before, the equilibrium of the pre-economy is indeterminate becasue there are too many endogenous variables. However, the government can steer the economy toward one of them by an appropriate choice of policy variables. This will be the topic of next section. But, no matter what policy instruments the monetary authority decides to use, the initial public debt will still be B_1, whose payment will be fully financed by raising taxes. Consequently, the real value of the public debt will, in any equilibrium, be equal to the real value of the stream of taxes collected $(s_1, s_2, \ldots)$. A bond holder, therefore, is essentially entitled to a share of this stream. Specifically, it is as if individual h, who holds b_1^h bonds, owns a proportion b_1^h/B_1 of the stream $(s_1, s_2, \ldots)$. Additionally, invidual h has a stream of tax liabilities of $\left(s_1^h, s_2^h, \ldots\right)$. As a result, the endowment of individual h is augmented by $\frac{b_1^h}{B_1}(s_1, s_2, \ldots) - \left(s_1^h, s_2^h, \ldots\right)$. For this reason, no matter what the exogenous and the endogenous variables of the economy are, the associated real economy is the one in which individual h's initial endowment is $(\omega_1^h, \omega_2^h \ldots) + \frac{b_1^h}{B_1}(s_1, s_2, \ldots) - \left(s_1^h, s_2^h, \ldots\right)$. Formally, given a pre-economy $\mathcal{PE} = \left\langle \left(U^h, \bar{\omega}^h, b_1^h, \bar{s}^h\right)_{h\in H} \right\rangle$, *the real economy associated* with it is

$$\mathcal{RE} = \left\langle \left(U^h, \bar{\omega} - s^h + \frac{b_1^h}{B_1} s\right)_{h\in H} \right\rangle.$$

Note that, since $\sum_{h\in H}\left(s^h + \frac{b_1^h}{B_1} s\right) = -s + s = 0$, the sets of feasible allocations of $\mathcal{PE}$ and of its associated real economy $\mathcal{RE}$ are the same. We define the following two magnitudes associated with the above real economy, which will be useful later. One is the present value of the stream of the economy's resources or lifetime real GDP for short

$$Y = \sum_{t=1}^{\infty} \delta_{1t}\omega_t \tag{5}$$

and the second is the ratio of the lifetime real GDP and the present value of the stream of real tax revenues

$$V = \frac{\sum_{t=1}^{\infty} \delta_{1t}\omega_t}{\sum_{t=1}^{\infty} \delta_{1t} s_t}. \tag{6}$$

We are now ready to state the following proposition, which is the expression of the classical dichotomy within the fiscal theory.

Proposition 1 [The classical dichotomy] Let $\mathcal{PE} = \left\langle \left(U^h, \bar{\omega}^h, b_1^h, \bar{s}^h\right)_{h \in H} \right\rangle$ be a pre-economy, and let $\mathcal{RE}$ be its associated real economy. If $\left\langle \bar{P}, \bar{Q}, \left(\bar{c}^{h*}, \bar{b}^{h*}\right)_{h \in H}, \bar{B} \right\rangle$ is a monetary equilibrium of $\mathcal{PE}$, then $\langle (\bar{c}^{h*})_{h \in H}, \bar{p} \rangle$, constitutes a competitive equilibrium of $\mathcal{RE}$, where $\bar{p} = (p_1, p_2 \ldots)$ satisfies

$$\frac{p_{t+1}}{p_t} = \frac{Q_t P_{t+1}}{P_t} \qquad t = 1, 2, \ldots.$$

Furthermore, the following quantity equation holds:

$$P_1 Y = B_1 V \tag{7}$$

Proof. See appendix. ■

Since the equilibrium of any real economy is essentially unique, we obtain as a corollary of Proposition 1 that the equilibrium relative prices of the pre-economy are determined in the real economy associated with it, and the equilibrium price level can be calculated by the quantity equation.

As established in Observation 4, the competitive equilibrium prices, $(p_1, p_2, \ldots)$, of $\mathcal{RE}$ satisfy

$$\frac{p_{t+1}}{p_t} = \delta_t \qquad t = 1, 2, \ldots$$

Note that these prices are present value prices, not spot prices. That is, when $p_1 = 1$, their units are peanuts in period 1 per unit of peanuts in period t. By the classical dichotomy, the equilibrium relative prices in the monetary economies and in the associated real economy are the same. In particular, the price of peanuts at $t + 1$ in terms of peanuts at t is

$$\frac{Q_t P_{t+1}}{P_t} = \delta_t \qquad t = 1, 2, \ldots \tag{8}$$

This is the expression of the Fisher equation in this model: the gross real interest equals the product of the gross interest rate and the gross rate of inflation.

The following insight follows immediately from (8).

Insight 1 Inflation is always and everywhere a monetary phenomenom. Fiscal policy does not affect inflation. Inflation is determined exclusively by the monetary policy. Specifically, inflation in period $t+1$ is determined by the nominal interest rate in period t.

As established in Observation 4, the competitive bundles are given by

$$\overline{c}^h = \frac{\sum_{\tau=1}^{\infty} \delta_{1\tau} \left(\omega_\tau^h + \left(\frac{b_1^h}{B_1} - \frac{s_\tau^h}{s_\tau} \right) s_\tau \right)}{\sum_{\tau=1}^{\infty} \delta_{1\tau} \omega_\tau} \bar{\omega}, \qquad h \in H$$

which leads to the following.

Insight 2 Monetary policy does not affect the equilibrium consumption bundles, which are determined in the real economy. Fiscal policy, on the other hand, affects the equilibrium allocation since, unless bond and tax liabilities shares are equal, it induces a redistribution of income.

7 Monetary economies

In order to transform a pre-economy into a monetary economy, we need to define which of the variables are exogenous. The exogenous variables will be called policy variables. We have already assumed that the real tax revenues $\left(\bar{s}^h\right)_{h \in H}$ are exogenous variables. Therefore, the candidates for policy variables are the nominal discount rates $(Q_1, Q_2, \ldots)$ and the nominal debt levels $(B_2, \ldots)$. However, not all of them can simultaneously be exogenous, for otherwise the system will be inconsistent. And not all of them can be endogenous, because otherwise the system would be indeterminate. One possibility, then, would be letting the nominal discount rates be the policy variables and allowing the debt levels to be determined endogenously. Another possibility would be letting the debt levels be the policy variables, and allowing the discount rates to be determined in equilibrium. A third possibility is any mixture of them. Here, we will restrict attention to the first two alternatives, interest pegging and debt management.

Interest pegging and fiscal policy Under this regime, the monetary authority sets the nominal interest rates and lets the individuals decide how much money to lend. The economy can be summarized by

$$\mathcal{E} = \left\langle \left(U^h, (\omega_1^h, \omega_2^h, \ldots), b_1^h, \left(s_1^h, s_2^h, \ldots\right)\right)_{h \in H}, (Q_1, Q_2, \ldots) \right\rangle$$

and the main endogenous variables will be the spot peanut prices $(P_1, P_2, \ldots)$ and debt levels $(B_2, \ldots)$. The remaining endogenous variables, e.g. the tax revenues, can be calculated easily once prices are known.

Debt management and fiscal policy Under this regime, the monetary authority issues the debt levels for each period and lets the market determine the interest rates. The economy can be summarized by

$$\mathcal{E} = \left\langle \left(U^h, (\omega_1^h, \omega_2^h, \ldots), b_1^h, \left(s_1^h, s_2^h, \ldots\right)\right)_{h \in H}, (B_2, \ldots, B_L) \right\rangle$$

and the main endogenous variables will be the spot peanut prices $(P_1, P_2, \ldots)$, and nominal discount rates $(Q_1, Q_2, \ldots)$.

We now define a monetary equilibrium under the above two policy regimes.

Let $\mathcal{E}_Q = \left\langle \left(U^h, (\omega_1^h, \omega_2^h, \ldots), b_1^h, \left(s_1^h, s_2^h, \ldots\right)\right)_{h \in H}, (Q_1, Q_2, \ldots) \right\rangle$ be an economy under interest the regime of pegging and fiscal policy.

A *monetary equilibrium* of $\mathcal{E}_Q$ is a monetary equilibrium of the pre-economy $\mathcal{PE} = \left\langle \left(U^h, (\omega_1^h, \omega_2^h, \ldots), b_1^h, \left(s_1^h, s_2^h, \ldots\right)\right)_{h \in H} \right\rangle$ in which the discount rates are $(Q_1, Q_2, \ldots)$.

Similarly, let $\mathcal{E}_B = \left\langle \left(U^h, (\omega_1^h, \omega_2^h, \ldots), b_1^h, \left(s_1^h, s_2^h, \ldots\right)\right)_{h \in H}, (B_2, \ldots) \right\rangle$ be an economy under debt management and tax policy. A *monetary equilibrium* of $\mathcal{E}_B$ is a monetary equilibrium of the pre-economy $\mathcal{PE} = \left\langle \left(U^h, (\omega_1^h, \ldots, \omega_L^h), b_1^h, \left(s_1^h, \ldots, s_L^h\right)\right)_{h \in H} \right\rangle$ in which the debt levels are $(B_2, \ldots)$.

Note that the equilibrium conditions of the economies under the two different policy regimes are similar, but not the same. While, under interest pegging policy, the discount rates are exogenous policy variables and the debts levels are determined endogenously, under debt management, the nature of these variables is interchanged: discount rates are endogenous and debt levels exogenous.

We now turn to the calculation of the equilibrium values of the nominal variables. Before that, let's denote the discount rate between periods t and 1 by $Q_{1t} = \left(\prod_{\tau=1}^{t-1} Q_\tau\right)$.[2] It is the amount of period-1 dollars needed in order to get one dollar in period t.

Proposition 2 Let $\mathcal{E}_Q = \left\langle \left(U^h, \bar{\omega}^h, b_1^h, \bar{s}^h\right)_{h\in H}, (Q_1, Q_2, \ldots)\right\rangle$ be an economy under interest pegging and tax policy. The equilibrium values of the endogenous monetary variables are

$$P_t = \frac{B_1}{Q_{1t}} \frac{\delta_{1t}}{\sum_{\tau=1}^{\infty} \delta_{1\tau} s_\tau} \qquad t = 1, 2, \ldots \tag{9}$$

$$B_t = \frac{B_1}{Q_{1t}} \frac{\sum_{\tau=t}^{\infty} \delta_{1\tau} s_\tau}{\sum_{\tau=1}^{\infty} \delta_{1\tau} s_\tau} \qquad t = 2, 3, \ldots \tag{10}$$

Proof. See appendix. ■

Note that $\frac{B_1}{Q_{1t}}$ is the value of the initial debt in period t dollars and that $\frac{\sum_{\tau=1}^{\infty} \delta_{1\tau} s_\tau}{\delta_{1t}}$ is the value of the total tax liabilites evaluated in period t peanuts. Therefore, according to the above proposition, the spot price of peanuts at t equals the value of the initial debt in period t dollars divided by the value of the total tax liabilites evaluated in period t peanuts. Also, note that $\frac{\sum_{\tau=t}^{\infty} \delta_{1\tau} s_\tau}{\sum_{\tau=1}^{\infty} \delta_{1\tau} s_\tau}$ is the proportion of the total tax liabilities that are yet to be paid at period t. Therefore, according to the above proposition, the outstanding debt in period t equals this proportion of the initial debt in period t dollars.

Proposition 2 says all and everything that this model can say when the policy variables are the interest rates and the real tax revenues. In particular, from equaton (9) we can obtain the following insights.

Insight 3 The neutrality of money holds.

[2]Recall that under the adopted convention, $Q_{1,1} = 1$.

Insight 4 Although inflation is a monetary phenomenom, the level of prices is affected both by monetary and fiscal policies.

The next proposition characterizes the equilibrium of the economies under the regime of debt management and fiscal policy.

Proposition 3 Let $\mathcal{E}_B = \left\langle \left(U^h, \bar{\omega}^h, b_1^h, \bar{s}^h\right)_{h\in H}, (B_2, \ldots, B_L)\right\rangle$ be an economy under debt management and fiscal policy. The equilibrium values of the endogenous monetary variables are

$$P_t = \frac{B_t}{\sum_{\tau=t}^{\infty} \delta_{t\tau} s_\tau} \qquad t = 1, 2, \ldots \tag{11}$$

$$Q_t = \frac{B_t}{B_{t+1}} \frac{\sum_{\tau=t+1}^{\infty} \delta_{t\tau} s_\tau}{\sum_{\tau=t}^{\infty} \delta_{t\tau} s_\tau} \qquad t = 1, 2, \ldots \tag{12}$$

Proof. See appendix. ■

The above proposition states that the spot price of peanuts at t is such as to make the real value of the outstanding debt at t equal to the future tax revenues, both measured in period t peanuts. This is, perhaps, the main insight of the fiscal theory of the price level.

Insight 5 The neutrality of money holds. This follows from (11).

Note that gross inflation is given by

$$\frac{P_{t+1}}{P_t} = \delta_t \frac{B_{t+1}}{B_t} \frac{\sum_{\tau=t}^{\infty} \delta_{t\tau} s_\tau}{\sum_{\tau=t+1}^{\infty} \delta_{t\tau} s_\tau} \qquad t = 1, 2, \ldots \tag{13}$$

That is, gross inflation is proportional to the growth of money supply and inversely proportional to the ratio of the present value of the real tax revenues from next period on, to the present value of the real tax revenues from this period on. Equations (12–13) give us the following:

Insight 6 A percentage change in the quantity of money induces the same percentage change in the gross inflation rate P_{t+1}/P_t and the same percentage change in the gross nominal interest rate $1/Q_t$.

The two implications recorded in the above insight are considered as two central implications of the quantity theory of money. (See Lucas (1980)). Note that they hold even if the equilibrium is not stationary.

Finally, an insight concerning a policy instrument commonly used in certain countries.

Insight 7 If the government repudiates part of the debt from B_t to B_t', it reduces period t inflation and increases period $t+1$ inflation. This follows from equation (13).

The alert reader will notice the similarity of this result to the main insight of Sargent and Wallace (1981).

And this is all I have to say about that. (Forrest Gump)

8 Appendix

Proof of Proposition 1: Let $\mathcal{PE} = \left\langle \bar{P}, \bar{Q}, \left(\bar{c}^{h*}, \bar{b}^{h*}\right)_{h\in H}, \bar{B} \right\rangle$ be an equilibrium of the pre-economy and let $\bar{p} = (p_1, p_2, \ldots)$ be a vector of accounting prices such that

$$\frac{p_{t+1}}{p_t} = \frac{Q_t P_{t+1}}{P_t} \qquad t = 1, 2, \ldots. \tag{14}$$

Since $\mathcal{PE}$ and $\mathcal{RE}$ share the same set of feasible allocations, $\left(\bar{c}^{h*}\right)_{h\in H}$ is a feasible allocation of $\mathcal{RE}$. Therefore, to show that $\left(\left(\bar{c}^{h*}\right)_{h\in H}, \bar{p}\right)$ is a competitive equilibrium of the real economy asociated with $\mathcal{PE}$, it is enough to show that for every individual $h \in H$, $\bar{c}^{h*}$ is utility maximizing, given $\bar{p} = (p_1, p_2, \ldots)$. Since the equilibrium prices of $\mathcal{RE}$ are determined up to multiplication by a positive constant, we can assume without loss of generality that $p_1 = P_1$. Then, iterating equation (14),

$$p_t = \left(\prod_{\tau=1}^{t-1} Q_\tau\right) P_t = Q_{1t} P_t \qquad t = 1, 2, \ldots. \tag{15}$$

Before we turn to the proof, we need some preliminary results. Consider agent h's maximization problem in the monetary economy:

$$\max \sum_{t=1}^{\infty} \beta^{t-1} u(c_t) \tag{16}$$
$$\text{s.t. } P_t c_t + Q_t b_{t+1} \leq P_t \left(\omega_t^h - s_t^h\right) + b_t \qquad t = 1, 2, ...$$
$$c_t \geq 0 \qquad t = 1, 2, ...$$
$$b_1 = b_1^h$$
$$\lim_{t\to\infty} \inf Q_{1t+1} b_{t+1} \geq 0$$

and consider the following auxiliary maximization problem

$$\max \sum_{t=0}^{\infty} \beta^{t-1} u(c_t) \tag{17}$$
$$\text{s.t. } \sum_{t=0}^{\infty} p_t c_t \leq \sum_{t=0}^{\infty} p_t \left(\omega_t^h - s_t^h\right) + b_1^h$$
$$c_t \geq 0 \quad t = 1, 2, ...$$

We will show that the above two problems are equivalent, and further, that problem (17) is equivalent to the maximization problem solved by agent h in the real economy $\mathcal{RE}$. As a result, we'll obtain that, given that $\left(\bar{c}^{h*}, \bar{b}^{h*}\right)$ is a solution to (16), $\bar{c}^{h*}$ is utility maximizing for h given $\bar{p}$ in $\mathcal{RE}$. Denote by $B(\bar{P}, \bar{Q})$ the set of feasible consumption paths for problem (16). That is

$$B(\bar{P}, \bar{Q}) = \left\{\bar{c} \geq 0 : \exists \bar{b} \text{ s.t. } P_t c_t + Q_t b_{t+1} \leq P_t \left(\omega_t^h - s_t^h\right) + b_t \text{ for } t \geq 1, \ \lim_{t\to\infty} \inf Q_{1t+1} b_{t+1} \geq 0\right\}$$

Denote by $B(\bar{p})$ the set of feasible consumption paths for problem (17). That is

$$B(\bar{p}) = \left\{\bar{c} \geq 0 : \sum_{t=0}^{\infty} p_t c_t \leq \sum_{t=0}^{\infty} p_t \left(\omega_t^h - s_t^h\right) + b_1^h\right\}$$

Claim 5 $B(\bar{P}, \bar{Q}) = B(\bar{p})$.

Proof. $B(\bar{P},\bar{Q}) \subset B(\bar{p})$: Let $\bar{c} \in B(\bar{P},\bar{Q})$. Then,

$$P_t c_t + Q_t b_{t+1} \leq P_t \left(\omega_t^h - s_t^h\right) + b_t \qquad t = 1, 2, \ldots \tag{18}$$

Multipying both sides by Q_{1t} and adding up the first T terms,

$$\sum_{t=1}^{T} Q_{1t} P_t c_t + \sum_{t=1}^{T} Q_{1t} Q_t b_{t+1} \leq \sum_{t=1}^{T} Q_{1t} P_t \left(\omega_t^h - s_t^h\right) + \sum_{t=1}^{T} Q_{1t} b_t$$

Noting that $Q_{1t} Q_t = Q_{1t+1}$, and canceling the terms $\sum_{t=2}^{T} Q_{1t} b_t$, we have that

$$\sum_{t=1}^{T} Q_{1t} P_t c_t + Q_{1T+1} b_{T+1} \leq \sum_{t=1}^{T} Q_{1t} P_t \left(\omega_t^h - s_t^h\right) + b_1$$

By (15), and since $p_1 = P_1$,

$$\sum_{t=1}^{T} p_t c_t + Q_{1T+1} b_{T+1} \leq \sum_{t=1}^{T} p_t \left(\omega_t^h - s_t^h\right) + b_1$$

Since $b_1 = b_1^h$ and $\lim_{t\to\infty} \inf Q_{1t+1} b_{t+1} \geq 0$, we have that

$$\sum_{t=1}^{\infty} p_t c_t \leq \sum_{t=1}^{\infty} p_t \left(\omega_t^h - s_t^h\right) + b_1^h$$

which means that $\bar{c} \in B(p)$.

$B(\bar{p}) \subset B(\bar{P},\bar{Q})$: Assume now that $\bar{c} \in B(p)$, and define

$$\begin{aligned} b_1 &= b_1^h \\ b_{t+1} &= \frac{P_t \left(\left(\omega_t^h - s_t^h\right) - c_t\right) + b_t}{Q_t} \qquad t = 1, 2, \ldots \end{aligned}$$

We then have that $P_t c_t + Q_t b_{t+1} = P_t \left(\omega_t^h - s_t^h\right) + b_t \quad t = 1, 2, \ldots$. In order to show that $\lim_{T\to\infty} \inf Q_{1T+1} b_{T+1} \geq 0$, multiply both sides by Q_{1t} and add up the first T terms to obtain,

$$\sum_{t=1}^{T} Q_{1t}P_t c_t + \sum_{t=1}^{T} Q_{1t+1}b_{t+1} = \sum_{t=1}^{T} Q_{1t}P_t\left(\omega_t^h - s_t^h\right) + \sum_{t=1}^{T} Q_{1t}b_t$$

Substracitng $\sum_{t=2}^{T} Q_{1t}b_t$ from bot sides and noting that $b_1 = b_1^h$,

$$\sum_{t=1}^{T} Q_{1t}P_t c_t + Q_{1T+1}b_{T+1} = \sum_{t=1}^{T} Q_{1t}P_t\left(\omega_t^h - s_t^h\right) + b_1^h$$

Since $Q_{1t}P_t = p_t$,

$$\sum_{t=1}^{T} p_t c_t + Q_{1T+1}b_{T+1} = \sum_{t=1}^{T} p_t\left(\omega_t^h - s_t^h\right) + b_1^h$$

As a result,

$$Q_{1T+1}b_{T+1} = \sum_{t=1}^{T} p_t\left(\omega_t^h - s_t^h\right) + b_1^h - \sum_{t=1}^{T} p_t c_t$$

and since $\bar{c} \in B(\bar{p})$,

$$\lim_{T\to\infty} Q_{1T+1}b_{T+1} = \sum_{t=1}^{\infty} p_t\left(\omega_t^h - s_t^h\right) + b_1^h - \sum_{t=1}^{\infty} p_t c_t \geq 0$$

This, along with (18), means that $\bar{c} \in B(\bar{P}, \bar{Q})$. ■

We conclude that if $\bar{c} = \left\{c_1^h, c_2^h, \ldots\right\}$ is an optimal consumption path for problem (16) if and only if it is an optimal consumption path for problem (17).

Claim 6 *Let $\left\{\left(c_1^h, b_1^h\right), \left(c_2^h, b_2^h\right), \ldots\right\}$ be a solution to individual h's maximization problem in the monetary economy. Then* $\lim_{T\to\infty} Q_{1T+1}b_{T+1}^h = 0$.

Proof. By monotonicity of u, we must have that

$$P_t c_t^h + Q_t b_{t+1}^h = P_t\left(\omega_t^h - s_t^h\right) + b_t^h \qquad t = 1, 2, ...$$

By Claim 5, $\left\{c_1^h, c_2^h, \ldots\right\} \in B(\bar{p})$. Using the same argument we have used in the second part of the proof ot Claim 5, $\lim_{T\to\infty} Q_{1T+1}b_{T+1} = \sum_{t=1}^{\infty} p_t\left(\omega_t^h - s_t^h\right) + b_1^h - \sum_{t=1}^{\infty} p_t c_t = 0$.

■

Since in equilibrium $B_T = \sum_{h \in H} b_T^{h*}$, as a corollary of the above claim we obtain that $\lim_{T \to \infty} Q_{1T+1} B_{T+1} = 0$. Moreover,

Claim 7 $B_1 = \sum_{t=1}^{\infty} p_t s_t$.

Proof. By budget balance, $B_t = T_t + Q_t B_{t+1} \qquad t = 1, 2, \ldots$. Iterating, and recalling that $T_t = P_t s_t$,

$$B_1 = \sum_{t=1}^{T} \left(\prod_{\tau=1}^{t-1} Q_\tau \right) P_t s_t + \left(\prod_{\tau=1}^{T} Q_\tau \right) B_{T+1} \qquad T = 1, 2, \ldots$$

Since $\left\langle \left(\overline{c}^{h*}, \bar{b}^{h*} \right)_{h \in H}, \bar{B} \right\rangle$ is a feasible allocation, we have that $B_{T+1} = b_{T+1}^{*}$ for all T, and therefore,

$$B_1 = \sum_{t=1}^{T} \left(\prod_{\tau=1}^{t-1} Q_\tau \right) P_t s_t + Q_{1T+1} b_{T+1}^{*} \qquad T = 1, 2, \ldots \tag{19}$$

Since $\lim_{T \to \infty} Q_{1T+1} b_{T+1}^{h*} = 0$,

$$\begin{aligned} B_1 &= \sum_{t=1}^{\infty} Q_{1t} P_t s_t \\ &= \sum_{t=1}^{\infty} p_t s_t \end{aligned} \tag{20}$$

where the last equality follows from (15). ■

It follows from the above claim that $b_1^h = \sum_{t=1}^{\infty} \frac{b_1^h}{B_1} p_t s_t$ and therefore we can write the budget constraint in problem (17) as

$$\sum_{t=0}^{\infty} p_t c_t \leq \sum_{t=0}^{\infty} p_t \left(\omega_t^h - s_t^h \right) + b_1^h = \sum_{t=0}^{\infty} p_t \left(\omega_t^h - s_t^h + \frac{b_1^h}{B_1} s_t \right)$$

We conclude that problem (17) is exactly the maximization problem faced by h in the real economy. And since $\left(\overline{c}^{h*}, \bar{b}^{h*} \right)$ solves the maximization problem of individual h in the monetary economy, $\overline{c}^{h*}$ solves the maximization problem of individual h in the associated real economy $\mathcal{RE}$.

We now show that the quantity equation holds. By (20)

$$\begin{aligned} B_1 &= \sum_{t=1}^{\infty} p_t s_t \\ &= P_1 \sum_{t=1}^{\infty} \prod_{\tau=1}^{t-1} \frac{p_{\tau+1}}{p_\tau} s_t \qquad \text{since } p_t = p_1 \prod_{\tau=1}^{t-1} \frac{p_{\tau+1}}{p_\tau} \text{and } p_1 = P_1. \end{aligned}$$

Since $\frac{p_{\tau+1}}{p_\tau}$ are the equilibrium price ratios of $\mathcal{RE}$,

$$B_1 = P_1 \sum_{t=1}^{\infty} \prod_{\tau=1}^{t-1} \delta_t s_t = P_1 \sum_{t=1}^{\infty} \delta_{1t} s_t$$

Therefore,

$$\begin{aligned} B_1 V &= P_1 \sum_{t=1}^{\infty} \delta_{1t} s_t \frac{\sum_{t=1}^{\infty} \delta_{1t} \omega_t}{\sum_{t=1}^{\infty} \delta_{1t} s_t} \\ &= P_1 Y. \qquad \blacksquare \end{aligned}$$

Proof of Proposition 2: By the quantity equation (7), $P_1 = \frac{B_1}{\sum_{t=1}^{\infty} \delta_{1t} s_t}$. It follows from the Fisher equation (8) that $P_t = \frac{\delta_{1t} P_1}{Q_{1t}}$ for $t = 1, 2, \ldots$, which together with the previous equality implies

$$P_t = \frac{\delta_{1t}}{Q_{1t}} \frac{B_1}{\sum_{\tau=1}^{\infty} \delta_{1\tau} s_\tau} \qquad t = 1, 2, \ldots$$

This shows (9). Finally, we need to compute the public debts. It follows from budget balance that

$$B_{t+1} = \frac{B_t - T_t}{Q_t} = \frac{B_t - P_t s_t}{Q_t} \qquad t = 1, 2, \ldots$$

Therefore,

$$\begin{aligned} B_2 &= \frac{B_1 - \frac{B_1}{\sum_{t=1}^{\infty} \delta_{1t} s_t} s_1}{Q_1} \\ &= \frac{B_1 \sum_{t=2}^{\infty} \delta_{1t} s_t}{Q_1 \sum_{t=1}^{\infty} \delta_{1t} s_t} \end{aligned}$$

namely (10) holds for $t = 2$. Now, assuming that (10) holds for some $t < L-1$,

$$\begin{aligned} B_{t+1} &= \frac{B_t - T_t}{Q_t} = \frac{B_t - P_t s_t}{Q_t} \\ &= \frac{1}{Q_t}\left(\frac{B_1 \sum_{\tau=t}^{\infty} \delta_{1\tau} s_\tau}{Q_{1t} \sum_{\tau=1}^{\infty} \delta_{1\tau} s_\tau} - \frac{B_1 \delta_{1t} s_t}{Q_{1t} \sum_{\tau=1}^{\infty} \delta_{1\tau} s_\tau}\right) \\ &= \frac{B_1 \sum_{\tau=t+1}^{\infty} \delta_{1\tau} s_\tau}{Q_{1t+1} \sum_{\tau=1}^{\infty} \delta_{1\tau} s_\tau} \end{aligned}$$

We have shown by induction that (10) holds for $t = 2, \ldots$.

Proof of Proposition 3: The proof is by induction. By the quantity equation (7),

$$P_1 = \frac{B_1}{\sum_{t=1}^{\infty} \delta_{1t} s_t}$$

That is, (11) holds for $t = 1$. By budget balance,

$$Q_t = \frac{B_t - T_t}{B_{t+1}} = \frac{B_t - P_t s_t}{B_{t+1}} \qquad t = 1, 2, \ldots$$

In particular,

$$\begin{aligned} Q_1 &= \frac{B_1 - \frac{B_1}{\sum_{t=1}^{\infty} \delta_{1t} s_t} s_1}{B_2} = \frac{B_1}{B_2} \frac{\sum_{t=1}^{\infty} \delta_{1t} s_t - s_1}{\sum_{t=1}^{\infty} \delta_{1t} s_t} \\ &= \frac{B_1}{B_2} \frac{\sum_{t=2}^{\infty} \delta_{1t} s_t}{\sum_{t=1}^{\infty} \delta_{1t} s_t} \end{aligned}$$

which means that for $t = 1$, (12) holds as well. Assume now that (11-12) hold for some $t < L$. Then,

$$\begin{aligned} P_{t+1} &= \frac{\delta_t P_t}{Q_t} \\ &= \frac{\delta_t \frac{B_t}{\sum_{\tau=t}^{\infty} \delta_{t\tau} s_\tau}}{\frac{B_t}{B_{t+1}} \frac{\sum_{\tau=t+1}^{\infty} \delta_{1\tau} s_\tau}{\sum_{\tau=t}^{\infty} \delta_{1\tau} s_\tau}} && \text{by the induction hypothesis} \\ &= \frac{\delta_t \frac{B_t}{\sum_{\tau=t}^{\infty} \delta_{t\tau} s_\tau}}{\frac{B_t}{B_{t+1}} \frac{\sum_{\tau=t+1}^{\infty} \delta_{t\tau} s_\tau}{\sum_{\tau=t}^{\infty} \delta_{t\tau} s_\tau}} && \text{since } \delta_{1\tau} = \delta_{1t}\delta_{t\tau} \\ &= \frac{B_{t+1}}{\sum_{\tau=t+1}^{\infty} \delta_{t+1\tau} s_\tau} && \text{since } \frac{\delta_{t\tau}}{\delta_t} = \delta_{t+1\tau} \end{aligned}$$

which means that (11) holds for $t+1$ as well. Further,

$$\begin{aligned} Q_{t+1} &= \frac{B_{t+1} - T_{t+1}}{B_{t+2}} = \frac{B_{t+1} - P_{t+1} s_{t+1}}{B_{t+2}} \\ &= \frac{B_{t+1} - \frac{B_{t+1}}{\sum_{\tau=t+1}^{\infty} \delta_{t+1\tau} s_{\tau}} s_{t+1}}{B_{t+2}} \\ &= \frac{B_{t+1}}{B_{t+2}} \frac{\sum_{\tau=t+1}^{\infty} \delta_{t+1\tau} s_{\tau} - s_{t+1}}{\sum_{\tau=t+1}^{\infty} \delta_{t+1\tau} s_{\tau}} \\ &= \frac{B_{t+1}}{B_{t+2}} \frac{\sum_{\tau=t+2}^{\infty} \delta_{t+1\tau} s_{\tau}}{\sum_{\tau=t+1}^{\infty} \delta_{t+1\tau} s_{\tau}} \end{aligned}$$

which means that (12) holds for $t+1$ as well. We conclude, then, that (11-12) holds for all $t = 1, 2, \ldots$. ■

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