TWO-STAGE MATCHING CONTESTS

Aner Sela

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Monaster Center for
Economic Research
Ben-Gurion University of the Negev
P.O. Box 653
Beer Sheva, Israel

Fax: 972-8-6472941
Tel: 972-8-6472286

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Aner Sela†

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Abstract

We study two-sided matching contests with two sets of agents, each of which includes n heterogeneous agents with commonly known types. In the first stage, the agents simultaneously send their costly efforts and then the order of choosing a partner from the other set is determined according to the Tullock contest success function. In the second stage, each agent chooses a partner from the other set, and an agent has a positive revenue if there is a matching in which he chooses a partner from the other set and this partner also chooses him. We analyze the agents' equilibrium efforts in the first stage as well as their choices of partners in the second stage, and demonstrate that if the agents' values, which are functions of the types of the agents who are matched, are either multiplicative or additive, their efforts are not necessarily monotonically increasing in their types.

Keywords: Matching, Tullock contest

JEL classification: D44, J31, D72, D82

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†Department of Economics, Ben-Gurion University of the Negev, 84105 Beer Sheva, Israel, anersela@bgu.ac.il

1 Introduction

In numerous situations the goal of heterogeneous agents from one set is to form matches with agents from another set. In such two-sided matching contests, agents in each set compete against each other to perceive their relative status in their sets for achieving the best match from the other set. For example, one set includes universities who compete against each other and signal their quality by hiring the best faculty, while the other set includes research scientists who compete to publish their works in high-ranking scientific journals and win grants in order to demonstrate their academic standing. Similarly, one set can include universities who invest in hiring the best researchers and teachers as well as in providing the best conditions for the students, while the other set includes student candidates who compete to achieve the highest grades in order to be admitted to the best possible universities.

In this paper we study a two-sided matching model in which 1) both sets of agents are active; 2) there is complete information; 3) the number of agents is finite; and 4) the agents compete in Tullock contests. Formally, we study a two-sided matching model under complete information in which there are two sets of agents, one of n heterogeneous firms and the other of n heterogeneous workers, each of which has commonly known types. In the first stage, the firms compete against each other in order to have priority of choosing a worker, and vice versa, the workers compete against each other in order to have priority of choosing a firm. In order to win the agents simultaneously exert their costly efforts which indicate their willingness to be matched. The firm (workers) to first choose a worker (firm) is determined by the Tullock contest success function which takes into account the efforts of all the firms (workers).[1] Then, the next firm (worker) to choose a worker (firm) is determined by the Tullock contest success function which is based on the efforts of all the firms (workers) excluding the effort of the first winner. This continues until each agent chooses one partner from the other side.

[1] For more information about Tullock contests, see, among others, Tullock (1980), Skaperdas (1996), Szidarovszky and Okuguchi (1997), and Baye and Hoppe (2003).

Once the choice order is determined in both sets, in the second stage, the agents sequentially choose partners from the other side. The choices in the two sets are done simultaneously. The agents have a value function that is monotonically increasing in both types of firms and workers who are matched. We say that there is a matching for agent i iff there is $j \in \{1, ..., n\}$ such that firm i chooses worker j and vice versa, namely, worker j also chooses firm i. In other words, an agent has a positive revenue if he is matched independently of whether or not the other agents are matched as well. We assume that there is no direct interaction between the two sets, the agents do not know the outcomes in the other sets and therefore there is an agency which matches the agents from both sets according to their choices.

A two-sided matching contest in which in both sets the agents compete in a Tullock contest was also studied by Cohen, Rabi and Sela (2020). However, these authors assumed only one stage such that after the agents in both sets exert their efforts they are assortatively matched, namely, the firm which won first place is matched with the worker who won first place, the firm which won second place is matched with the worker who won second place, and so on until all the firms and workers are matched with each other. In our model, we assume that the match between the two sets is not assortatively done, and the agents may have the right to choose their partners. Furthermore, while in Cohen, Rabi and Sela (2020) there is always a matching between the two sets of agents, we assume that there is a matching between a firm and a worker only if they choose each other. We believe that this situation reflects a real-life two-sided matching contest better than an automatic matching without a choice of agents. For example, a student will not be admitted to a university if he thinks that he has no chance of being accepted, and a university will not make an offer to a researcher if there is no chance that this researcher is interested in applying.

We begin with $2x2$ matching contests in which there are two agents on each side, one of which is referred to the high-type firm (worker) and the other one as the low-type firm (worker). We analyze some general properties of the equilibrium efforts and show that, independent of the form of the agent's value function, as a result of the matching, it is always profitable for each agent to choose the agent with the highest type from the other set. This result is not obvious since an agent has a higher expected payoff if he is matched with a higher type, but he also has a higher expected payoff if he chooses a partner who also chooses him with a higher probability.

Then we study $2x2$ matching contests with a multiplicative value function of the agents' types and show that the high-type firm's effort is higher than that of the low-type iff the high-type worker's effort is higher than that of the low-type. In addition, if both sides are symmetric, namely, the high-type firm and worker and the low-type firm and worker are the same, the low-type agents always exert higher efforts than the high-type ones. This result is quite unusual for standard (one-sided) contests, in particular, for a Tullock contest in which the higher the agent's type (value of winning) is, the higher is his equilibrium effort.

We also show that in $2x2$ matching contests with an additive value function of the agents' types, the firms exert the same positive effort and similarly the workers exert the same positive effort. In that case, matching between firms and workers is completely random. The intuition behind this result is that high-type agents are aware that they are in high demand from the other side, and therefore it is of no advantage to exert high costly efforts. The paradox is, however, that the high-type agents from both sides exert relatively low efforts, and therefore they have a higher probability to be matched with low-type agents and vice versa.

We also study nxn matching contests with an additive value function. We first study a $3x3$ matching contest and explicitly characterize the agents' equilibrium efforts. The results indicate that, similarly to the $2x2$ matching contest with an additive value function, there is an equilibrium in which all the firms exert the same effort, and similarly, all the workers exert the same effort. Afterwards, we generalize this result to any nxn matching contest and prove that in equilibrium all the firms exert the same effort as well as do all the workers. Accordingly, we should not expect the behavior of agents in larger matching contests to dramatically differ from their behavior in $2x2$ matching contests.

1.1 Outline

The rest of the paper is organized as follows: In Section 2, we present our matching contest. In Section 3, we analyze the equilibrium of the $2x2$ matching contest, and in Sections 3 and 4 we study the $2x2$ matching contests with multiplicative and additive value functions. In Section 5 we first study the $3x3$ matching contest with additive value functions and then its generalization to nxn matching contests. Section 6 concludes. The Appendix contains some of the proofs.

1.2 Related literature

The study of matching contests in the literature can be classified according to several criterion such as:

1) Matching contests in which only one set of agents is active and the other set is passive (see, for example, Spence 1973, Chao and Wilson 1989, Pesendorfer 1995, and Fernandez and Gali 1999), and, on the other hand, matching contests in which both sets are active (see, for example, Damiano and Li 2007, Hoppe, Moldovanu, and Sela 2009, Hoppe, Moldovanu, and Ozdenoren 2011, and Dizdar, Moldovanu and Szech 2019).

2) Matching contests under complete information (see, for example, Cole, Mailath, and Postlewaite 2001a,b, Bullow and Levin 2006, Peters 2007, and Bhaskar and Hopkins 2016), and, on the other hand, matching contests under incomplete information (see, for example, Spence 1973, Damiano and Li 2007, Hoppe, Moldovanu, and Sela 2009, Hoppe, Moldovanu, and Ozdenoren 2011, and Dizdar, Moldovanu and Szech 2019).

3) Matching contests in which there is a continuum number of agents (see, for example, Cole, Mailath, and Postlewaite 2001a,b, Peters and Siow 2002, Damiano and Li 2007, Hoppe, Moldovanu, and Ozdenoren 2011 and Bhaskar and Hopkins 2016), and, on the other hand, matching contests in which there is a finite number of agents (see, for example, Hoppe, Moldovanu, and Sela 2009 and Dizdar, Moldovanu and Szech 2019).

4) Matching contests in which the agents compete in the all-pay contest (see, for example,

Hoppe, Moldovanu, and Sela 2009,and, Dizdar, Moldovanu and Szech 2019), or, in Tullock contests (see Cohen, Rabi and Sela 2020), or, in rank-order tournaments (Bhaskar and Hopkins 2016).

In all this related literature there is no a combination, as in our paper, of the following four classifications: 1) both sets of agents are active 2) there is complete information 3) the number of agents is finite, and, 4) the agents compete in Tullock contests.

2 The matching contest

There is a set $M = \{1, 2, ..., n\}$ of n firms and a set $W = \{1, 2, ..., n\}$ of n workers. The firms' types are m_i, where $m_i \geq m_{i+1}$, $i = 1, ..., n-1$, and the workers' types are w_j, where $w_j > w_{j+1}$, $j = 1, ..., n-1$. There is bo connection between the sets, but all these types are commonly known. The matching contest proceeds as follows: In the first stage, each firm $i, i = 1, 2, ..., n$ exerts a costly effort x_i, and each worker $j, j = 1, 2, ..., n$ exerts a costly effort y_j. Efforts are submitted simultaneously. If there is a matching between firm i and worker j after exerting efforts of x_i and y_j correspondingly, the firm's utility is $f(m_i, w_j) - x_i$ and, similarly, the worker's utility is $f(m_i, w_j) - y_j$, where $f : R^2 \to R^1$ is the value function which is monotonically increasing in the types of firms and workers who are matched. If there is not a matching, the utility of an agent is negative and is equal to the cost of this agent's effort.[2] Matching between firms and workers can be done as follows: The firms (workers) exert their efforts and then the order in which they choose partners from the other set is determined according to the method of Clark and Riis (1998) which is as follows: The firm to choose a partner first is determined by the probability success function which takes into account the efforts of all the firms. Formally, firm i, $i = 1, ..., n$ wins the first choice of a partner with probability $\frac{x_i}{\sum_{k=1}^{n} x_k}$, where x_k is firm k's effort, $k = 1, ..., n$. Then, the second choice of a partner is determined by the probability success function which is based on the efforts of all the firms excluding the effort of the first winner. Formally, firm i, $i = 1, ..., n$ wins

[2] This is a straightforward generalization of the standard one-sided Tullock contest. On the existence of equilibrium in Tullock contests see Szidarovszky and Okuguchi (1997) and Einy et al. (2015).

the second choice with probability $\sum_{\substack{k=1 \\ k \neq i}}^{n} \frac{x_k}{\sum_{j=1}^{n} x_j} \frac{x_i}{\sum_{\substack{j=1 \\ j \neq k}}^{n} x_j}$, and so on until every firm chooses one worker. Similarly, the order of the workers' choices for partners is determined by their efforts.

Then, in the second stage, the agents sequentially choose partners from the other sets according to the results of the contests. It is assumed that when an agent chooses a partner he does not know anything about the results in the other set, but only the types of the agents there. Then an agency makes a matching of the agents from both sets according to their choices as follows: Firm i is matched iff there is $j \in \{1, ..., n\}$ such that firm i chooses worker j and vice versa, worker j also chooses firm i. In that case, if an agent is matched, he has an expected payoff that is equal to his value function minus the cost of his effort. On the other hand, if an agent is not matched he has a negative expected payoff that is equal to his cost of effort. We say that a matching contest has a (subgame perfect) equilibrium if every agent chooses an effort and a partner that maximizes his expected utility given the efforts of the other agents as well as the other agents' choices for partners in both sets. Note that this matching contest, only if the two sets have the same number of agents, has always the trivial corner equilibrium in which all the agents do not exert an effort in the first stage. We focus on the analysis of the interior equilibrium in which all the agents are active in the first stage.[3]

3 The 2x2 matching contest

In order to analyze the subgame perfect equilibrium of the two-stage matching contest we begin with the second stage and go backwards to the first one.

[3]The designer has several ways to prevent the existence of the corner equilibrium in which all the agents are passive in the first stage.

3.1 The second stage

Consider a set $M = \{l, h\}$ of two firms and a set $W = \{l, h\}$ of two workers. We call the types m_h and w_h the high-type firm and worker, respectively, and the other types, m_l and w_l the low-type firm and worker, respectively. Suppose that firm $i, i = h, l$ exerts effort x_i and worker $j, j = h, l$ exerts effort y_j. Then, in the second stage, the agents sequentially choose agents from the other set according to the order that was determined in the first stage by their efforts. In the following, we examine the conditions under which a firm which wins the option to choose a worker, chooses the one with the highest type w_h, and similarly, a worker who wins the option to choose a firm, chooses the one with the highest type m_h.

Now suppose that the high-type firm wins the first choice of a partner. Then, if this firm chooses the low-type worker instead of the high-type worker, and all the other agents choose the highest type if they win, the firm can be matched only with the low-type worker. This occurs if the high-type firm wins and the low-type worker wins, or, alternatively, if the high-type firm loses and the low-type worker wins. Then, the expected payoff of the high-type firm is

$$\widetilde{E}_h = f(m_h, w_l)\left[\frac{x_h}{x_h + x_l}\frac{y_l}{y_h + y_l}\right] + f(m_h, w_l)\left[\frac{x_l}{x_h + x_l}\frac{y_l}{y_h + y_l}\right] - x_h$$

where x_h is his optimal effort in this case. On the other hand, if the high-type firm wins and chooses the high-type worker, and also chooses the same effort x_h that was optimal in the previous case, and all the other agents choose the highest type when they win, the expected payoff of the high-type firm is

$$E_h = f(m_h, w_h)\left[\frac{x_h}{x_h + x_l}\frac{y_h}{y_h + y_l}\right] + f(m_h, w_l)\left[\frac{x_l}{x_h + x_l}\frac{y_l}{y_h + y_l}\right] - x_h$$

Then we have

$$\Delta E_h = E_h - \widetilde{E}_h = f(m_h, w_h)\left[\frac{x_h}{x_h + x_l}\frac{y_h}{y_h + y_l}\right] - f(m_h, w_l)\left[\frac{x_h}{x_h + x_l}\frac{y_l}{y_h + y_l}\right]$$

Thus, we obtain that $\Delta E_h \geq 0$ iff

$$\frac{y_h}{y_l} \geq \frac{f(m_h, w_l)}{f(m_h, w_h)}$$

Similarly, if the high-type worker wins, he chooses the high-type firm iff

$$\frac{x_h}{x_l} \geq \frac{f(m_l, w_h)}{f(m_h, w_h)}$$

Thus, we obtain

Lemma 1 *In the second stage of the 2x2 matching contest, when the high-type firm wins to be the first to choose a worker, this firm chooses the high-type worker iff the workers' efforts satisfy* $\frac{y_h}{y_l} \geq \frac{f(m_h, w_l)}{f(m_h, w_h)}$. *Similarly, when the high-type worker wins to be the first to choose a firm, he chooses the high-type firm iff the firms' efforts satisfy* $\frac{x_h}{x_l} \geq \frac{f(m_l, w_h)}{f(m_h, w_h)}$.

Now, suppose that the low-type firm wins the first choice of a partner. Then, if this firm chooses the low-type worker instead of the high-type worker and all the other agents choose the high type if they win, this firm can be matched only with the low-type worker. This occurs if the low-type firm wins and the high-type worker wins, or, alternatively, if the low-type firm loses and the low-type worker loses. Then, the expected payoff of the low-type firm is

$$\widetilde{E}_l = f(m_l, w_l) \left[\frac{x_l}{x_h + x_l} \frac{y_h}{y_h + y_l} \right] + f(m_l, w_l) \left[\frac{x_h}{x_h + x_l} \frac{y_h}{y_h + y_l} \right] - x_l$$

where x_l is his optimal effort in this case. On the other hand, if the low-type firm chooses the high-type worker, and also chooses the same effort x_l that was optimal in the previous case, and all the other types choose the high type when they win, then the expected payoff of the low-type firm is

$$E_l = f(m_l, w_h) \left[\frac{x_l}{x_h + x_l} \frac{y_l}{y_h + y_l} \right] + f(m_l, w_l) \left[\frac{x_h}{x_h + x_l} \frac{y_h}{y_h + y_l} \right] - x_l$$

Then we have

$$\Delta E_l = E_l - \widetilde{E}_l = f(m_l, w_h) \left[\frac{x_l}{x_h + x_l} \frac{y_l}{y_h + y_l} \right] - f(m_l, w_l) \left[\frac{x_l}{x_h + x_l} \frac{y_h}{y_h + y_l} \right]$$

Thus, we obtain that $\Delta E_l \geq 0$ iff

$$\frac{y_h}{y_l} \leq \frac{f(m_l, w_h)}{f(m_l, w_l)}$$

Similarly, if the low-type worker wins he chooses the high-type firm iff

$$\frac{x_h}{x_l} \leq \frac{f(m_h, w_l)}{f(m_l, w_l)}$$

Hence, we obtain

Lemma 2 *In the second stage of the 2x2 matching contest, when the low-type firm wins to be the first to choose a worker, this firm chooses the high-type worker iff the workers' efforts satisfy* $\frac{y_h}{y_l} \leq \frac{f(m_l, w_h)}{f(m_l, w_l)}$. *Similarly, when the low-type worker wins to be the first to choose a firm, he chooses the high-type firm iff the firms' efforts satisfy* $\frac{x_h}{x_l} \leq \frac{f(m_h, w_l)}{f(m_l, w_l)}$.

If we combine Lemmas (1) and (2) we obtain

Claim 1 *In the second stage of the 2x2 matching contests when a firm wins the option to choose a worker, this firm chooses the worker with the highest type if*

$$\frac{f(m_h, w_l)}{f(m_h, w_h)} \leq \frac{y_h}{y_l} \leq \frac{f(m_l, w_h)}{f(m_l, w_l)} \tag{1}$$

Similarly, when a worker wins the option to choose a firm, he chooses the firm with the highest type if

$$\frac{f(m_l, w_h)}{f(m_h, w_h)} \leq \frac{x_h}{x_l} \leq \frac{f(m_h, w_l)}{f(m_l, w_l)} \tag{2}$$

3.2 The first stage

Suppose that conditions (1) and (2) are satisfied and the agents make their choices according to Claim 1. Then, there are only two possible forms of matching: The first is the positive-assortative form, namely $m_h - w_h$ and $m_l - w_l$, which occurs when both high-type agents win the first choice of a partner. The second is the negative-assortative matching, namely, $m_h - w_l$ and $m_l - w_h$ which occurs when both low-type agents win the first choice of a partner. If a high-type agent from one set and a low-type agent from the other set win no matching occurs. Thus, in the $2x2$ matching contest, when firm $i, i = h, l$ exerts effort x_i and worker $j, j = h, l$ exerts effort y_j, we have:

1) The high-type firm is matched with the high-type worker iff they both win, and the high-type firm is matched with the low-type worker iff this firm loses and the low-type worker wins. Thus, the maximization problem of the high-type firm is

$$\max_{x_h} f(m_h, w_h)\left[\frac{x_h}{x_h+x_l}\frac{y_h}{y_h+y_l}\right] + f(m_h, w_l)\left[\frac{x_l}{x_h+x_l}\frac{y_l}{y_h+y_l}\right] - x_h \tag{3}$$

2) The low-type firm is matched with the high-type worker iff this firm wins and the high-type worker loses, and the low-type firm is matched with the low-type worker iff they both lose. Thus, the maximization problem of the low-type firm is

$$\max_{x_l} f(m_l, w_h)\left[\frac{x_l}{x_h+x_l}\frac{y_l}{y_h+y_l}\right] + f(m_l, w_l)\left[\frac{x_h}{x_h+x_l}\frac{y_h}{y_h+y_l}\right] - x_l \tag{4}$$

3) The high-type worker is matched with the high-type firm iff they both win, and the high-type worker is matched with the low-type firm iff he loses and the low-type firm wins. Thus, the maximization problem of the high-type worker is

$$\max_{y_h} f(m_h, w_h)\left[\frac{y_h}{y_h+y_l}\frac{x_h}{x_h+x_l}\right] + f(m_l, w_h)\left[\frac{y_l}{y_h+y_l}\frac{x_l}{x_h+x_l}\right] - y_h \tag{5}$$

4) The low-type worker is matched with the high-type firm iff he wins and the high-type firm loses, and the low-type worker is matched with the low-type firm iff they both lose. Thus, the maximization problem of the low-type worker is

$$\max_{y_l} f(m_h, w_l)\left[\frac{y_l}{y_h+y_l}\frac{px_l}{x_h+x_l}\right] + f(m_l, w_l)\left[\frac{y_h}{y_h+y_l}\frac{x_h}{x_h+x_l}\right] - y_l \tag{6}$$

The F.O.C. of the maximization problems (3), (4), (5), and (6) are

$$\begin{aligned}
f(m_h, w_h)\frac{x_l}{(x_h+x_l)^2}\frac{y_h}{y_h+y_l} - f(m_h, w_l)\frac{x_l}{(x_h+x_l)^2}\frac{y_l}{y_h+y_l} &= 1 \\
f(m_l, w_h)\frac{x_h}{(x_h+x_l)^2}\frac{y_l}{y_h+y_l} - f(m_l, w_l)\frac{x_h}{(x_h+x_l)^2}\frac{y_h}{y_h+y_l} &= 1 \\
f(m_h, w_h)\frac{y_l}{(y_h+y_l)^2}\frac{x_h}{x_h+x_l} - f(m_l, w_h)\frac{y_l}{(y_h+y_l)^2}\frac{x_l}{x_h+x_l} &= 1 \\
f(m_h, w_l)\frac{y_h}{(y_h+y_l)^2}\frac{x_l}{x_h+x_l} - f(m_l, w_l)\frac{y_h}{(y_h+y_l)^2}\frac{x_h}{x_h+x_l} &= 1
\end{aligned} \tag{7}$$

and the S.O.C. of the maximization problems (3), (4), (5), and (6) are

$$
\begin{aligned}
&-f(m_h,w_h)\frac{2x_l}{(x_h+x_l)^3}\frac{y_h}{y_h+y_l}+f(m_h,w_l)\frac{2x_l}{(x_h+x_l)^3}\frac{y_l}{y_h+y_l}\\
&-f(m_l,w_h)\frac{2x_h}{(x_h+x_l)^3}\frac{y_l}{y_h+y_l}+f(m_l,w_l)\frac{2x_1}{(x_h+x_l)^2}\frac{y_h}{y_h+y_l}\\
&-f(m_1,w_h)\frac{2y_l}{(y_h+y_l)^3}\frac{x_h}{x_h+x_l}+f(m_l,w_h)\frac{2y_l}{(y_h+y_l)^2}\frac{x_l}{x_h+x_l}\\
&-f(m_h,w_l)\frac{2y_h}{(y_h+y_l)^2}\frac{x_l}{x_h+x_l}+f(m_l,w_l)\frac{2y_h}{(y_h+y_l)^2}\frac{x_h}{x_h+x_l}
\end{aligned}
$$

It can be verified that the F.O.C imply that the S.O.C. are satisfied.

If we divide the LHS of the first two equations of (7) by each other, and also divide both RHS of these equations by each other, we obtain that

$$\frac{x_l(f(m_h,w_h)y_h - f(m_h,w_l)y_l)}{x_h(f(m_2,w_h)y_l - f(m_l,w_l)y_h)} = 1 \tag{8}$$

Similarly, if we divide both LHS of the last two equations of (7) by each other, and divide the RHS of these equations by each other, we obtain

$$\frac{y_l(f(m_h,w_h)x_h - f(m_l,w_h)x_l)}{y_h(f(m_h,w_l)x_l - f(m_l,w_l)x_h)} = 1 \tag{9}$$

By (8), we obtain that $f(m_h,w_h)y_h - f(m_h,w_l)y_l > 0$ iff $f(m_l,w_h)y_l - f(m_l,w_l)y_h > 0$, and by (9) that $f(m_h,w_h)x_h - f(m_l,w_h)x_l > 0$ iff $f(m_h,w_l)x_l - f(m_l,w_l)x_h$. Thus, we have upper and lower bounds on the ratio of the equilibrium efforts as follows: The workers' equilibrium efforts satisfy

$$\frac{f(m_l,w_h)}{f(m_l,w_l)} > \frac{y_h}{y_l} > \frac{f(m_h,w_l)}{f(m_h,w_h)} \tag{10}$$

and the firms' equilibrium efforts satisfy

$$\frac{f(m_h,w_l)}{f(m_l,w_l)} > \frac{x_h}{x_l} > \frac{f(m_l,w_h)}{f(m_h,w_h)} \tag{11}$$

Since the inequalities (10) and (11) are exactly identical to (1) and (2), respectively, we obtain that

Proposition 1 *In a 2x2 matching contest the agents' equilibrium efforts in the first stage are given by (7) and in the second stage each agent who wins in the first stage chooses the agent with the highest type from the other set.*

Note that by (8), if $x_h > x_l$, then $f(m_l, w_h)y_l - f(m_l, w_l)y_h < f(m_h, w_h)y_h - f(m_h, w_l)y_l$, and similarly, by (9), if $y_h > y_l$, then $f(m_h, w_l)x_l - f(m_l, w_l)x_h < f(m_h, w_h)x_h - f(m_l, w_h)x_l$. Thus, we have a sufficient condition for the high-type agents to exert higher efforts than the low-type ones.

Lemma 3 *In the 2x2 matching contest, if* $\frac{f(m_l,w_h)+f(m_h,w_l)}{f(m_h,w_h)+f(m_l,w_l)} > 1$, *then the high-type firm's effort* x_h *is larger than the low-type firm's effort* x_l *iff the high-type worker's effort* y_h *is larger than the low-type worker's effort* y_l.

In the following sections, we will see that the condition in Lemma 3 is not necessarily satisfied and, in particular, the agents' effort are not necessarily monotonically increasing in their types.

4 The 2x2 matching contest with a multiplicative value function

We assume now that the agents' value function is multiplicative, namely, $f(m_i, w_j) = m_i w_j$, $i = k, h$, $j = k, h$.[4] By (7), the agents' equilibrium efforts satisfy:

$$\begin{aligned} v_h w_h \frac{x_l}{(x_h+x_l)^2}\frac{y_h}{y_h+y_l} - v_h w_l \frac{x_l}{(x_h+x_l)^2}\frac{y_l}{y_h+y_l} &= 1 \\ v_l w_h \frac{x_h}{(x_h+x_l)^2}\frac{y_l}{y_h+y_l} - v_l w_l \frac{x_h}{(x_h+x_l)^2}\frac{y_h}{y_h+y_l} &= 1 \\ w_h v_h \frac{y_l}{(y_h+y_l)^2}\frac{x_h}{x_h+x_l} - w_h v_l \frac{y_l}{(y_h+y_l)^2}\frac{x_l}{x_h+x_l} &= 1 \\ w_l v_h \frac{y_h}{(y_h+y_l)^2}\frac{x_l}{x_h+x_l} - w_l v_l \frac{y_h}{(y_h+y_l)^2}\frac{x_h}{x_h+x_l} &= 1 \end{aligned} \tag{12}$$

Furthermore, by (10) and (11), the agents' equilibrium efforts also satisfy:

$$\begin{aligned} \frac{y_h}{y_l} &> \frac{w_l}{w_h} \\ \frac{x_h}{x_l} &> \frac{m_l}{m_h} \end{aligned}$$

[4]Note that our results in this section can be immediately extended to value functions of the form $f(m_i, w_j) = \delta(m_i)\rho(w_j)$, where δ and ρ are strictly increasing and differentiable.

Note that Lemma 3 does not hold in this case since $\frac{f(m_l,w_h)+f(m_h,w_l)}{f(m_h,w_h)+f(m_l,w_l)} = \frac{m_l w_h + m_h w_l}{m_h w_h + m_l w_l} < 1$. However, if we divide the LHS of the first two equations of (12) by each other, and also divide the RHS of these equations by each other, we obtain that

$$\frac{x_l(v_h w_h y_h - v_h w_l y_l)}{x_h(v_l w_h y_l - v_l w_l y_h)} = 1$$

Thus, if $x_l > x_h$ then

$$v_h w_h y_h - v_h w_l y_l < v_l w_h y_l - v_l w_l y_h$$

or, alternatively,

$$y_h(v_h w_h + v_l w_l) < y_l(w_h y_l + v_h w_l)$$

Since $v_h w_h + v_l w_l > w_h y_l + v_h w_l$, we obtain that $y_l > y_h$. Similarly, we obtain that if $y_l > y_h$, then $x_l > x_h$. Thus, although the condition in Lemma 3 does not hold, we have

Claim 2 *In the 2x2 matching contest with a multiplicative value function, the high-type firm's effort is higher than that of the low-type iff the high-type worker's effort is higher than that of the low-type.*

Assume now that the matching contest is symmetric, namely, $v_h = w_h$ and $v_l = w_l$. By (12), the agents' equilibrium efforts satisfy

$$\begin{aligned} v_h^2 \frac{x_l x_h}{(x_h+x_l)^3} - v_h v_l \frac{x_l^2}{(x_h+x_l)^3} &= 1 \\ v_l v_h \frac{x_h x_l}{(x_h+x_l)^3} - v_l^2 \frac{x_h^2}{(x_h+x_l)^3} &= 1 \end{aligned} \tag{13}$$

where x_h is the symmetric effort of the high-type agents (both firm and worker) and x_l is the symmetric effort of the low-type agents. If we divide the LHS of the two equations of (13) by each other, and also the RHS of these equations by each other, we obtain that

$$v_l^2 x_h^2 + v_h(v_h - v_l)x_l x_h - v_h v_l x_l^2 = 0$$

Thus, the high-type agents' effort is given by

$$x_h = \frac{-v_h(v_h - v_l)x_l + x_l\sqrt{v_h^2(v_h - v_l)^2 + 4v_h v_l^3}}{2v_l^2}$$

We can see that $x_h \leq x_l$ iff

$$(2v_l^2 + v_h(v_h - v_l))^2 \geq v_h^2(v_h - v_l)^2 + 4v_h v_l^3$$

Since the last inequality is always satisfied, we have

Proposition 2 *In the symmetric* $2x2$ *matching contest with a multiplicative value function, the low-type agents always exert an effort that is higher than or equal to that of the high-type ones.*

The above result is not straightforward since in one-sided contests, usually the higher the value of an agent is, the higher is his equilibrium effort. In other words, high-type agents usually exert higher efforts in an equilibrium than low-type agents. Although the low-type agents' effort is higher than that of the high-type agents, namely, $x_l > x_h$, by (3) and (4), the difference between the agents' expected utilities satisfy

$$u_h - u_l = (v_h^2 - v_l^2)\frac{x_h^2}{x_h + x_l} + (x_l - x_h) > 0$$

Thus, in the symmetric $2x2$ matching contest with a multiplicative value function, the high-type agents' expected payoff is higher than that of the low-type agents.

5 The 2x2 matching contest with an additive value function

We next assume that the agents' value function is additive, namely, $f(m_i, w_j) = m_i + w_j$, $i = l, h2$, $j = l, h$.[5] By (7), the agents' equilibrium efforts satisfy:

$$\begin{aligned}
(v_h + w_h)\frac{x_h}{(x_h + x_l)^2}\frac{y_h}{y_h + y_l} - (v_h + w_l)\frac{x_l}{(x_h + x_l)^2}\frac{y_l}{y_h + y_l} &= 1 \\
(v_l + w_h)\frac{x_h}{(x_h + x_l)^2}\frac{y_l}{y_h + yl} - (v_l + w_l)\frac{x_h}{(x_h + x_l)^2}\frac{y_h}{y_h + y_l} &= 1 \\
(w_h + v_h)\frac{y_l}{(y_h + y_l)^2}\frac{x_h}{x_h + x_l} - (w_h + v_l)\frac{y_l}{(y_h + y_l)^2}\frac{x_l}{x_h + x_l} &= 1 \\
(w_l + v_h)\frac{y_h}{(y_h + y_l)^2}\frac{x_l}{x_h + x_l} - (w_l + v_l)\frac{y_h}{(y_h + y_l)^2}\frac{x_1}{x_h + x_l} &= 1
\end{aligned} \tag{14}$$

By Lemma 3, since $\frac{f(m_l,w_h)+f(m_h,w_l)}{f(m_h,w_h)+f(m_l,w_l)} = \frac{m_l+w_h+m_h+w_l}{m_h+w_h+m_l+w_l} = 1$, we have

Claim 3 *In the 2x2 matching contest with an additive value function the high-type firm's effort is higher than that of the low-type firm iff the high-type worker's effort is higher than that of the low-type worker.*

Assume now that the matching contest is symmetric, namely, $v_h = w_h$ and $v_l = w_l$. By (14), the agents' equilibrium efforts satisfy

$$\begin{aligned}
(2v_h)\frac{x_l x_h}{(x_h + x_l)^3} - (v_h + v_l)\frac{x_l^2}{(x_h + x_l)^3} &= 1 \\
(v_l + v_h)\frac{x_h x_l}{(x_h + x_l)^3} - (2v_l)\frac{x_h^2}{(x_h + x_l)^3} &= 1
\end{aligned} \tag{15}$$

where x_h is the symmetric effort of the high-type agents (both firm and worker) and x_l is the symmetric effort of the low-type agents. If we divide the LHS of the two equations of (15) by each other, and also divide the RHS of these equations by each other, we obtain that

$$2v_l x_h^2 + (v_h - v_l)x_l x_h - (v_h + v_l)x_l^2 = 0$$

Thus the symmetric effort of the high-type agents is given by

[5] Note that our results in this section can be immediately extended to value functions having the form $f(m_i, w_j) = \delta(m_i) + \rho(w_j)$, where δ and ρ are strictly increasing and differentiable.

$$x_h = \frac{-(v_h - v_l)x_l + x_l\sqrt{(v_h - v_l)^2 + 8(v_h + v_l)v_l}}{4v_l}$$

This implies that in the symmetric 2x2 matching contest with an additive value function the equilibrium efforts are

$$x_h = x_l = \frac{v_h - v_l}{8}$$

Therefore, both the low-type and the high-type agents exert the same effort. Similarly, the solution of equations (14) yields the equilibrium efforts of the asymmetric $2x2$ matching contests as follows:

Proposition 3 *In the asymmetric $2x2$ matching contest with an additive output function the equilibrium efforts are*

$$\begin{aligned} x_h &= x_l = \frac{w_h - w_l}{8} \\ y_h &= y_l = \frac{v_h - v_l}{8} \end{aligned}$$

and the agents' expected payoffs are

$$\begin{aligned} u_{m_h} &= \frac{4v_h + w_h + 3w_l}{8} \\ u_{m_l} &= \frac{4v_l + w_h + 3w_l}{8} \\ u_{w_h} &= \frac{4w_h + v_h + 3v_l}{8} \\ u_{w_l} &= \frac{4w_l + v_h + 3v_l}{8} \end{aligned}$$

By Proposition 3, we can see that the high-type agents have higher expected utilities than the low-type agents, namely, $u_{m_h} \geq u_{m_l}$ and $u_{w_h} > u_{wl}$. We can also see that if the expected utility of the high-type firm is higher than the expected utility of the high-type worker, then the expected utility of the low-type firm is lower than the expected utility of the low-type worker, namely, $u_{m_h} \geq u_{w_h}$ iff $u_{w_l} \geq u_{m_l}$.

6 The nxn matching contest with an additive value function

Consider first the $3x3$ matching contest. There is a set $M = \{1, 2, 3\}$ of three firms and a set $W = \{1, 2, 3\}$ of three workers. The firms' types are m_1, m_2, m_3, $m_1 \geq m_2 \geq m_3$, and the workers' types are w_1, w_2, w_3, $w_1 \geq w_2 \geq w_3$. The agents' value function is additive, namely, $f(m_i, w_j) = m_i + w_j$, $i = 1, 2, 3$, $j = 1, 2, 3$.

We assume that in the second stage when a firm gets the option to choose a worker it chooses the one with the highest type that has not as yet been chosen by another firm, and similarly, when an worker gets the option to choose a firm, he chooses the one with the highest type that has not as yet been chosen by another worker. These strategies (choices) of the agents will be verified as equilibrium strategies given the agents' efforts in the first stage.

Proposition 4 *In an equilibrium of the* $3x3$ *matching contest with an additive value function, all the firms exert the same effort*

$$x = \frac{8w_1 + 2w_2 - 10w_3}{108}$$

and all the workers exert the same effort

$$y = \frac{8m_1 + 2m_2 - 10m_3}{108}$$

Proof. See Appendix. ■

Note that the assumption about the agents' strategies in the second stage holds by the equilibrium efforts given by Proposition 4, since if all the agents in each set exert the same effort in the first stage, then the best choice for each agent in the second stage is to choose the agent from the other set with the highest type such that the order of the choices is correlated with the strength of the types.

Now, consider a set $M = \{1, 2, ..., n\}$ of n firms and a set $W = \{1, 2, ..., n\}$ of n workers where $n > 3$. The firms' types are $m_1, ..., m_n$, $m_i \geq m_{i+1}, i = 1, ..., n-1$, and the workers' types are $w_1, ..., w_n$, $w_j \geq w_{j+1}, j = 1, 2, ..., n-1$. Then, similarly to Proposition 4, we obtain

Proposition 5 *In an equilibrium of the nxn matching contest with an additive value function, all the firms exert the same effort and all the workers exert the same effort.*

Proof. *See Appendix.* ■

Similarly to the case of the $3x3$ matching contest, in the nxn matching contest where $n > 3$, in the second stage, an agent chooses the agent with the highest type from the other set that was not chosen earlier such that in each set the highest type is chosen first, the second highest type is chosen second, and so on until the lowest type is chosen last.

7 Concluding remarks

In standard (one-sided) Tullock contests with either one or several stages, the agents' efforts are monotonically increasing, namely, an agent with a higher type (value of winning) exerts a higher effort in equilibrium than an agent with a lower one. However, we demonstrated that in two-sided matching contests this situation no longer holds such that in equilibrium the agents' efforts might be monotonically decreasing in the agents' types or the same for all the agents, independent of their types. While there is a huge literature on one-sided Tullock contests, little has been said on two-sided (Tullock) contests because they are much more computationally complex. We mainly focused on the $2x2$ matching contests, but our findings indicate that the agents' behavior will be similar in matching contests with any number of agents. We assumed that an agent has a revenue only if he is matched such that he chooses an agent from the other side and this agent chooses him as well. Obviously, in a two-sided matching contest, agents could be matched in different ways, and then the results could be completely different (see Cohen, Rabi, and Sela 2020).

8 Appendix

8.1 Proof of Proposition 4

Firm 1 is matched with worker 1 iff they both win the first choice. The firm is matched with worker 2 iff this firm wins the second choice and worker 2 wins the first choice, and the firm is matched with worker 3 iff this firm wins the third choice and worker 3 wins the first choice. Thus, if the firms' efforts are $x_i, i = 1, 2, 3$, and the workers' efforts are $y_j, j = 1, 2, 3$, the maximization problem of firm 1 in the $3x3$ matching contest is

$$
\begin{aligned}
&\max_{x_1} f(m_1, w_1)\frac{x_1}{x_1 + x_2 + x_3}\frac{y_1}{y_1 + y_2 + y_3} \\
&+f(m_1, w_2)\left[\begin{array}{c} \frac{x_2}{x_1+x_2+x_3}\frac{x_1}{x_1+x_3}\frac{y_2}{y_1+y_2+y_3} \\ +\frac{x_3}{x_1+x_2+x_3}\frac{x_1}{x_1+x_2}\frac{y_2}{y_1+y_2+y_3} \end{array}\right] \\
&+f(m_1, w_3)\left[\begin{array}{c} \frac{x_3}{x_1+x_2+x_3}\frac{x_2}{x_1+x_2}\frac{y_3}{y_1+y_2+y_3} \\ +\frac{x_2}{x_1+x_2+x_3}\frac{x_3}{x_1+x_3}\frac{y_3}{y_1+y_2+y_3} \end{array}\right] \\
&-x_1
\end{aligned}
$$

The F.O.C. is

$$
\begin{aligned}
& f(m_1,w_1)\left[\frac{x_2+x_3}{(x_1+x_2+x_3)^2}\frac{y_1}{y_1+y_2+y_3}\right] \\
& +f(m_1,w_2)\begin{bmatrix} -\tfrac{x_2}{(x_1+x_2+x_3)^2}\tfrac{y_2}{y_1+y_2+y_3}\tfrac{x_1}{x_1+x_3} \\ +\tfrac{x_3}{(x_1+x_3)^2}\tfrac{x_2}{x_1+x_2+x_3}\tfrac{y_2}{y_1+y_2+y_3} \\ -\tfrac{x_3}{(x_1+x_2+x_3)^2}\tfrac{y_2}{y_1+y_2+y_3}\tfrac{x_1}{x_1+x_2} \\ +\tfrac{x_2}{(x_1+x_2)^2}\tfrac{x_3}{x_1+x_2+x_3}\tfrac{y_2}{y_1+y_2+y_3} \end{bmatrix} \\
& +f(m_1,w_3)\begin{bmatrix} -\tfrac{x_3}{(x_1+x_2+x_3)^2}\tfrac{y_3}{y_1+y_2+y_3}\tfrac{x_2}{x_1+x_2} \\ -\tfrac{x_2}{(x_1+x_2)^2}\tfrac{x_3}{x_1+x_2+x_3}\tfrac{y_3}{y_1+y_2+y_3} \\ -\tfrac{x_2}{(x_1+x_2+x_3)^2}\tfrac{y_3}{y_1+y_2+y_3}\tfrac{x_3}{x_1+x_3} \\ -\tfrac{x_3}{(x_1+x_3)^2}\tfrac{x_2}{x_1+x_2+x_3}\tfrac{y_3}{y_1+y_2+y_3} \end{bmatrix} \\
= \quad & 1
\end{aligned}
\tag{16}
$$

Firm 2 is matched with worker 1 iff this firm wins the second choice and worker 1 wins the first choice. The firm is matched with worker 2 iff they both win the second choice, and this firm is matched with worker 3 iff the firm wins the third choice and worker 3 wins the second choice. Thus, the maximization problem of firm 2 is

$$
\begin{aligned}
& \max_{x_2} f(m_2,w_1)\begin{bmatrix} \tfrac{x_2}{x_1+x_2+x_3}\tfrac{y_2}{y_1+y_2+y_3}\tfrac{y_1}{y_1+y_3} \\ +\tfrac{x_2}{x_1+x+x_3}\tfrac{y_3}{y_1+y_2+y_3}\tfrac{y_1}{y_1+y_2} \end{bmatrix} \\
& +f(m_2,w_2)\begin{bmatrix} \tfrac{x_1}{x_1+x_2+x_3}\tfrac{y_1}{y_1+y_2+y_3}\tfrac{x_2}{x_2+x_3}\tfrac{y_2}{y_2+y_3} \\ +\tfrac{x_3}{x_1+x_2+x_3}\tfrac{y_3}{y_1+y_2+y_3}\tfrac{x_2}{x_1+x_2}\tfrac{y_2}{y_1+y_2} \end{bmatrix} \\
& +f(m_2,w_3)\begin{bmatrix} \tfrac{x_1}{x_1+x_2+x_3}\tfrac{y_1}{y_1+y_2+y_3}\tfrac{x_3}{x_2+x_3}\tfrac{y_3}{y_2+y_3} \\ +\tfrac{x_3}{x_1+x_2+x_3}\tfrac{y_2}{y_1+y_2+y_3}\tfrac{x_1}{x_1+x_2}\tfrac{y_3}{y_1+y_3} \end{bmatrix} \\
& -x_2
\end{aligned}
$$

The F.O.C. is

$$
\begin{aligned}
& f(m_2,w_1)\begin{bmatrix} \frac{x_1+x_3}{(x_1+x_2+x_3)^2}\frac{y_2}{y_1+y_2+y_3}\frac{y_1}{y_1+y_3} \\ +\frac{x_1+x_3}{(x_1+x_2+x_3)^2}\frac{y_3}{y_1+y_2+y_3}\frac{y_1}{y_1+y_2} \end{bmatrix} \\
& +f(m_2,w_2)\begin{bmatrix} -\frac{x_1}{(x_1+x_2+x_3)^2}\frac{y_1}{y_1+y_2+y_3}\frac{x_2}{x_2+x_3}\frac{y_2}{y_2+y_3} \\ +\frac{x_3}{(x_2+x_3)^2}\frac{x_1}{x_1+x_2+x_3}\frac{y_1}{y_1+y_2+y_3}\frac{y_2}{y_2+y_3} \\ -\frac{x_3}{(x_1+x_2+x_3)^2}\frac{y_3}{y_1+y_2+y_3}\frac{x_2}{x_1+x_2}\frac{y_2}{y_1+y_2} \\ +\frac{x_1}{(x_1+x_2)^2}\frac{x_3}{x_1+x_2+x_3}\frac{y_3}{y_1+y_2+y_3}\frac{y_2}{y_1+y_2} \end{bmatrix} \\
& +f(m_2,w_3)\begin{bmatrix} -\frac{x_1}{(x_1+x_2+x_3)^2}\frac{y_1}{y_1+y_2+y_3}\frac{x_3}{x_2+x_3}\frac{y_3}{y_2+y_3} \\ -\frac{x_3}{(x_2+x_3)^2}\frac{x_1}{x_1+x_2+x_3}\frac{y_1}{y_1+y_2+y_3}\frac{y_3}{y_2+y_3} \\ -\frac{x_2}{(x_1+x_2+x_3)^2}\frac{y_2}{y_1+y_2+y_3}\frac{x_1}{x_1+x_2}\frac{y_3}{y_1+y_3} \\ -\frac{x_1}{(x_1+x_2)^2}\frac{x_3}{x_1+x_2+x_3}\frac{y_2}{y_1+y_2+y_3}\frac{y_3}{y_1+y_3} \end{bmatrix} \\
= \quad & 1
\end{aligned} \tag{17}
$$

Firm 3 is matched with worker 1 iff the firm wins the third choice and worker 1 wins the first choice. The firm is matched with worker 2 iff this firm wins the second choice and worker 2 wins the third choice, and the firm is matched with worker 3 iff they both win the third choice. Thus, the maximization problem of firm 3 is

$$
\begin{aligned}
& \max_{x_3} f(m_3,w_1)\begin{bmatrix} \frac{x_3}{x_1+x_2+x_3}\frac{y_2}{y_1+y_2+y_3}\frac{y_3}{y_1+y_3} \\ +\frac{x_3}{x_1+x_2+x_3}\frac{y_3}{y_1+y_2+y_3}\frac{y_2}{y_1+y_2} \end{bmatrix} \\
& +f(m_3,w_2)\begin{bmatrix} \frac{x_1}{x_1+x_2+x_3}\frac{y_1}{y_1+y_2+y_3}\frac{x_3}{x_2+x_3}\frac{y_3}{y_2+y_3} \\ +\frac{x_2}{x_1+x_2+x_3}\frac{y_3}{y_1+y_2+y_3}\frac{x_3}{x_1+x_3}\frac{y_1}{y_1+y_2} \end{bmatrix} \\
& +f(m_3,w_3)\begin{bmatrix} \frac{x_2}{x_1+x_2+x_3}\frac{y_2}{y_1+y_2+y_3}\frac{x_1}{x_1+x_3}\frac{y_1}{y_1+y_3} \\ +\frac{x_1}{x_1+x_2+x_3}\frac{y_1}{y_1+y_2+y_3}\frac{x_2}{x_2+x_3}\frac{y_2}{y_2+y_3} \end{bmatrix} \\
& -x_3
\end{aligned}
$$

The F.O.C. is

$$
\begin{aligned}
& f(m_3,w_1)\begin{bmatrix} \frac{x_1+x_2}{(x_1+x_2+x_3)^2}\frac{y_2}{y_1+y_2+y_3}\frac{y_3}{y_1+y_3} \\ +\frac{x_1+x_2}{(x_1+x_2+x_3)^2}\frac{y_3}{y_1+y_2+y_3}\frac{y_2}{y_1+y_2} \end{bmatrix} \\
& +f(m_3,w_2)\begin{bmatrix} -\frac{x_1}{(x_1+x_2+x_3)^2}\frac{y_1}{y_1+y_2+y_3}\frac{x_3}{x_2+x_3}\frac{y_3}{y_2+y_3} \\ +\frac{x_2}{(x_2+x_3)^2}\frac{x_1}{x_1+x_2+x_3}\frac{y_1}{y_1+y_2+y_3}\frac{y_3}{y_2+y_3} \\ -\frac{x_2}{(x_1+x_2+x_3)^2}\frac{y_3}{y_1+y_2+y_3}\frac{x_3}{x_1+x_3}\frac{y_1}{y_1+y_2} \\ +\frac{x_1}{(x_1+x_3)^2}\frac{x_2}{x_1+x_2+x_3}\frac{y_3}{y_1+y_2+y_3}\frac{y_1}{y_1+y_2} \end{bmatrix} \\
& +f(m_3,w_3)\begin{bmatrix} -\frac{x_2}{(x_1+x_2+x_3)^2}\frac{y_2}{y_1+y_2+y_3}\frac{x_1}{x_1+x_3}\frac{y_1}{y_1+y_3} \\ -\frac{x_1}{(x_1+x_3)^2}\frac{x_2}{x_1+x_2+x_3}\frac{y_2}{y_1+y_2+y_3}\frac{y_1}{y_1+y_3} \\ -\frac{x_1}{(x_1+x_2+x_3)^2}\frac{y_1}{y_1+y_2+y_3}\frac{x_2}{x_2+x_3}\frac{y_2}{y_2+y_3} \\ -\frac{x_2}{(x_2+x_3)^2}\frac{x_1}{x_1+x_2+x_3}\frac{y_1}{y_1+y_2+y_3}\frac{y_2}{y_2+y_3} \end{bmatrix} \\
= \; & 1
\end{aligned}
\tag{18}
$$

Consider first the symmetric case $m_i = w_i, i = 1,2,3$ which yields $x_i = y_i, i = 1,2,3$. Then, by equations (16), (17), and (18), we have

$$
\begin{aligned}
& f(m_1,w_1)\frac{(x_2+x_3)x_1}{(x_1+x_2+x_3)^3} \\
& +f(m_1,w_2)(\frac{-x_2^2x_1}{(x_1+x_2+x_3)^3(x_1+x_3)}+\frac{x_2^2x_3}{(x_1+x_2+x_3)^2(x_1+x_3)^2} \\
& -\frac{-x_3x_1x_2}{(x_1+x_2+x_3)^3(x_1+x_2)}+\frac{x_2^2x_3}{(x_1+x_2+x_3)^2(x_1+x_2)^2}) \\
& +f(m_1,w_3)(\frac{-x_2x_3^2}{(x_1+x_2+x_3)^3(x_1+x_2)}-\frac{x_2x_3^2}{(x_1+x_2+x_3)^2(x_1+x_2)^3} \\
& -\frac{-x_3^2x_2}{(x_1+x_2+x_3)^3(x_1+x_3)}-\frac{x_3^2x_2}{(x_1+x_2+x_3)^2(x_1+x_3)^2}) \\
= \; & 1
\end{aligned}
\tag{19}
$$

and

$$\begin{aligned}
& f(m_2,w_1)(\frac{(x_1+x_3)x_1x_2}{(x_1+x_2+x_3)^3(x_1+x_3)}+\frac{(x_1+x_3)x_3x_1}{(x_1+x_2+x_3)^3(x_1+x_2)}) \\
& +f(m_2,w_2)(\frac{-x_2^2x_1^2}{(x_1+x_2+x_3)^3(x_2+x_3)^2}+\frac{x_1^2x_2x_3}{(x_1+x_2+x_3)^2(x_2+x_3)^3} \\
& -\frac{-x_3^2x_2^2}{(x_1+x_2+x_3)^3(x_1+x_2)^2}+\frac{x_3^2x_1x_2}{(x_1+x_2+x_3)^2(x_1+x_2)^3}) \\
& +f(m_2,w_3)(\frac{-x_1^2x_3^2}{(x_1+x_2+x_3)^3(x_2+x_3)^2}-\frac{x_1^2x_3^2}{(x_1+x_2+x_3)^2(x_2+x_3)^3} \\
& -\frac{-x_2^2x_1x_3}{(x_1+x_2+x_3)^3(x_1+x_2)(x_1+x_3)}-\frac{x_3^2x_2x_1}{(x_1+x_2+x_3)^2(x_1+x_2)^2(x_1+x_3)}) \\
= \; & 1
\end{aligned} \tag{20}$$

and

$$\begin{aligned}
& f(m_3,w_1)(\frac{(x_1+x_2)x_2x_3}{(x_1+x_2+x_3)^3(x_1+x_3)}+\frac{(x_1+x_2)x_2x_3}{(x_1+x_2+x_3)^3(x_1+x_2)}) \\
& +f(m_3,w_2)(\frac{-x_3^2x_1^2}{(x_1+x_2+x_3)^3(x_2+x_3)^2}+\frac{x_1^2x_2x_3}{(x_1+x_2+x_3)^2(x_2+x_3)^3} \\
& -\frac{x_3^2x_1x_2}{(x_1+x_2+x_3)^3(x_1+x_2)(x_1+x_3)}+\frac{x_1^2x_2x_3}{(x_1+x_2+x_3)^2(x_1+x_3)^2(x_1+x_2)}) \\
& +f(m_3,w_3)(\frac{-x_1^2x_2^2}{(x_1+x_2+x_3)^3(x_1+x_3)^2}-\frac{x_1^2x_2^2}{(x_1+x_2+x_3)^2(x_1+x_3)^3} \\
& -\frac{x_1^2x_2^2}{(x_1+x_2+x_3)^3(x_2+x_3)^2}-\frac{x_1^2x_2^2}{(x_1+x_2+x_3)^2(x_2+x_3)^3}) \\
= \; & 1
\end{aligned} \tag{21}$$

We conjecture that the solution satisfies $x = x_1 = x_2 = x_3$ which will be verified in the following. By equations (19), (20), and (21), we have

$$\begin{aligned}
(2m_1)\frac{2}{27x}+(m_1+m_2)(\frac{-2}{54x}+\frac{2}{36x})+(m_1+m_3)(\frac{-2}{54x}-\frac{2}{36x}) &= 1 \\
(m_2+m_1)\frac{4}{54x}+(m_2+m_2)(\frac{-2}{108x}+\frac{2}{72x})+(m_2+m_3)(\frac{-2}{108x}-\frac{2}{72x}) &= 1 \\
(m_3+m_1)\frac{4}{108x}+(m_3+m_2)(\frac{-2}{108x}+\frac{2}{72x})+(m_3+m_3)(\frac{-2}{108x}-\frac{2}{72x}) &= 1
\end{aligned}$$

or

$$
\begin{aligned}
(m_1+m_1)\frac{16}{216x}+(m_1+m_2)(\frac{-8}{216x}+\frac{12}{216x})+(m_1+m_3)(\frac{-8}{216x}-\frac{12}{216x}) &= 1\\
(m_2+m_1)\frac{16}{216x}+(m_2+m_2)(\frac{-8}{216x}+\frac{12}{216x})+(m_2+m_3)(\frac{-8}{216x}-\frac{12}{216x}) &= 1\\
(m_3+m_1)\frac{16}{216x}+(m_3+m_2)(\frac{-8}{216x}+\frac{12}{216x})+(m_3+m_3)(\frac{-8}{216x}-\frac{12}{216x}) &= 1
\end{aligned}
$$

Rearranging gives us

$$
\begin{aligned}
16m_1+4m_2-20m_3 &= 216x\\
16m_1+4m_2-20m_3 &= 216x\\
16m_1+4m_2-20m_3 &= 216x
\end{aligned}
$$

Thus, when there is symmetry between both sets of firms and workers we obtain that all the agents exert the same effort $x=\frac{8m_1+2m_2-10m_3}{108}=\frac{8w_1+2w_2-10w_3}{108}$.

Now, assume asymmetry between both sets of firms and workers. In that case, we conjecture that the equilibrium efforts satisfy $x=x_1=x_2=x_3$ and $y=y_1=y_2=y_3$ which will be verified in the following. By equations (16), (17), and (18), we have

$$
\begin{aligned}
(m_1+w_1)\frac{8}{108x}+(m_1+w_2)(\frac{-4}{108x}+\frac{6}{108x})+(m_1+w_3)(\frac{-4}{108x}-\frac{6}{108x}) &= 1\\
(m_2+w_1)\frac{8}{108x}+(m_2+w_2)(\frac{-4}{108x}+\frac{6}{108x})+(m_2+w_3)(\frac{-4}{108x}-\frac{6}{108x}) &= 1\\
(m_3+w_1)\frac{8}{108x}+(m_3+w_2)(\frac{-4}{108x}+\frac{6}{108x})+(m_3+w_3)(\frac{-4}{108x}-\frac{6}{108x}) &= 1
\end{aligned}
$$

Thus, we obtain that all the firms exert the same effort

$$x=\frac{8w_1+2w_2-10w_3}{108}$$

Similarly, it can be verified that all the workers exert the same effort

$$y=\frac{8m_1+2m_2-10m_3}{108}$$

It can be also verified that all the agents have positive expected payoffs in the above equilibrium.

8.2 Proof of Proposition 5

We assume that the efforts of the workers are the same and then prove that the efforts of the firms are the same and vice versa. In the nxn matching contest when agents have an additive value function, the equilibrium effort of firm $i, i = 1, ..., n$ is given by

$$x_i = \arg\max_{\widetilde{x}_i} \sum_{j=1}^{n} (m_i + w_j) P(\text{firm } i \text{ wins the } j\text{-th choice }) P(\text{worker } j \text{ wins the } i\text{-th choice}) \quad (22)$$

where $P(\text{firm } i \text{ wins the } j\text{-th choice })$ denotes the probability that firm i wins the j-th choice, and $P(\text{worker } j \text{ wins the } i\text{-th choice})$ denotes the probability that worker j wins the i-th choice. Since by our assumption the efforts of the workers are the same, for all $j = 2, ..., n$,

$$P(\text{worker } 1 \text{ wins the } i\text{-th choice}) = P(\text{worker } j \text{ wins the } i\text{-th choice})$$

Thus, by (22) we have

$$x_i = \arg\max_{\widetilde{x}_i} P(\text{worker } 1 \text{ wins the } i\text{-th choice}) \sum_{j=1}^{n} (m_i + w_j) P(\text{firm } i \text{ wins the } j\text{-th choice })$$

which is equivalent to

$$x_i = \arg\max_{\widetilde{x}_i} \sum_{j=1}^{n} (m_i + w_j) P(\text{firm } i \text{ wins the } j\text{-th choice}) \quad (23)$$

Since

$$\sum_{j=1}^{n} P(\text{firm } i \text{ wins the } j\text{-th choice }) = 1$$

by (23) we have

$$x_i = \arg\max_{\widetilde{x}_i} \sum_{j=1}^{n} w_j P(\text{ firm } i \text{ wins the } j\text{-th choice }) \quad (24)$$

Since (24) is the same for all $1 \leq i \leq n$, a symmetric solution where $x_i = x_{i+1}, i = 1, ..., n-1$ is feasible. In the same way we can show that if all the firms exert the same effort then all the workers exert the same effort as well. And, since all the firms exert the same effort, by the expected revenue of each firm given by (22), and its equilibrium effort given by (24), it can be verified that all the firms have positive expected payoffs. The same argument holds for the workers.

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