On the Existence of Bayesian Cournot Equilibrium

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Abstract

We show that when firms have asymmetric information about the market demand and theirs costs, a (Bayesian) Cournot equilibrium in pure strategies may not exist, or be unique. In fact, we are able to construct surprisingly simple and robust examples of duopolies with these features. However, we also find sufficient conditions for existence, and for uniqueness, of Cournot equilibrium in a certain class of industries. More general results arise when negative prices are possible.

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1 Introduction

The Cournot model is widely used in studies of imperfectly competitive industries. Its standard version, which is concerned with the case of firms producing a homogeneous good with *complete* information about demand and production costs, has been extensively studied. However, in the past thirty years a fairly big amount of research has been dedicated to questions that arise when the information is *incomplete*, i.e., when there is uncertainty about the market demand and/or the firms’ cost functions, and firms have asymmetric information about them. (See, e.g., Gal-Or [6,7], Raith [13], Sakai [14,15], Shapiro [16], Vives [18,19,20], and Einy et al [3,4].)

In oligopolies with incomplete information, some of the questions that had been addressed concern the *value of information* to a firm (that is, whether and by how much a firm can benefit from receiving additional information), as well as firms’ *incentives to share information*. Naturally, treating these questions requires comparisons of the (pure strategy Bayesian) Cournot equilibrium outcomes in industries that differ with respect to the information endowments of the firms. The scope of these exercises is thus limited to classes of industries for which a Cournot equilibrium exists. Moreover, sharp and general conclusions are hard to obtain unless Cournot equilibrium is also *unique* under various information endowments of the firms.

For a complete information oligopoly, there is an extensive literature concerned with the existence and uniqueness of Cournot equilibrium under various assumptions on the demand and cost functions. A well known and general condition for existence of equilibrium is found in Novshek [12], who generalizes earlier results (of, e.g., Szidarovszky and Yakowitz [17]). More recent developments can be found in Amir [1], where equilibrium existence and uniqueness results are established by making a connection with the theory of supermodular games (Milgrom and Roberts [11]).

The issues of existence of Cournot equilibrium in incomplete information oligopolies have so far been largely bypassed in the literature by making strong assumptions. For instance, Gal-Or [6], Vives [18,19], and Raith [13] assume that the linear market demand is uncertain, but allow the possibility that *negative* prices may arise for large

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1Henceforth we shall use the expression "Cournot equilibrium" to refer to both the pure strategy Cournot equilibria of an oligopoly with complete information, and to the pure strategy Bayesian Cournot equilibria in oligopolies with incomplete information.
outputs, in order not to break the linearity of the demand function.\textsuperscript{2} While negative prices may make sense in some contexts, a model is of a greater appeal if such a possibility is avoided. As we shall see, even if equilibrium prices are positive, the mere possibility of negative prices in some states of nature has strategic implications that are crucial in sustaining equilibrium behavior.\textsuperscript{3}

In other papers (Sakai [14]), incomplete information is assumed only on firms’ linear costs, which again allows to avoid the general problem of equilibrium existence. In a non-linear setting, Einy et al [4] derive conditions under which the value of public information in an oligopoly is either positive or negative, but assume that the firms are symmetrically informed, which allows to reduce the equilibrium existence question to that in a complete information oligopoly. The assumption of symmetry of information, and a reduction to the complete information case that it allows, also stand behind the existence result of Lagerlöf [8]. In Einy et al [3] a categorical approach is used: it is assumed that an equilibrium exists, and then its properties are studied.

In this work we ask whether (and when) a Cournot equilibrium exists, and is unique, in an oligopoly with asymmetric information. Unfortunately, with regard to existence our findings are disappointing: there are examples of \textit{duopolies} with differential information, including one with \textit{linear} inverse demand and cost functions, that possess no Cournot equilibrium (see Examples 1 and 2).\textsuperscript{4} The reason for the non-existence in these examples lies in that, although the inverse demand function is well behaved in all states of nature (it is linear in Example 1, and concave in Example 2, before it reaches zero), the expected payoff functions of firms do not have "nice" properties, such as concavity or submodularity, that would guarantee existence of equilibrium via known theorems. Despite the good behavior of the demand function when it is positive, once it reaches zero for some level of aggregate output and is

\textsuperscript{2}In these papers, linearity of the demand function is instrumental in the proofs of existence and uniqueness of a Cournot equilibrium.

\textsuperscript{3}We are not the first to note the strategic impact of requiring that prices be always non-negative: Malueg and Tsutsui [10] show that this requirement affects the known results on feasibility of information sharing in an ex-ante symmetric linear duopoly. We show that requiring that prices be non-negative affects the \textit{existence} of Cournot equilibrium in oligopolies with asymmetric information.

\textsuperscript{4}The information structure in these examples is very simple too: one firm is better informed than the other.
“truncated,” i.e., forced to stay at zero for all larger outputs, a discontinuity occurs in its derivative. As a result, the expected payoff functions lose properties conducive to equilibrium existence.

In contrast, when prices are allowed to become negative, the good behavior of the demand and cost functions does lead to expected payoff functions that are well behaved (e.g., submodular, or concave in the firm’s strategy), and we obtain existence results – see Theorem 1 – under a variant of Novshek [12] condition. There is a simple reason why these results cannot be used, in general, to deduce equilibrium existence in oligopolies with non-negative prices: ruling out negative prices changes strategic considerations of the firms, sometimes in a very significant way. Specifically, a profile of strategies that is a Cournot equilibrium when negative prices are possible may cease being an equilibrium if the demand is truncated to avoid negative prices: whereas deviations to large outputs may be deterred by the possibility of negative prices, when prices are always non-negative these deviations may be profitable\(^5\) – see Remark 1.

Even though the existence of a Cournot equilibrium with non-negative prices is a more scarce phenomenon, we characterize a class of oligopolies with incomplete information in which a Cournot equilibrium does exist – see Theorem 3 and Corollary 1. The key feature of this class is the existence of certain thresholds of output which no firm will ever desire to exceed, and which guarantee positive prices in every state of nature if firms adhere to them. (Existence of such thresholds is guaranteed if firms’ marginal costs increase sufficiently fast.) This condition rules out the possibility that truncating the demand to avoid negative prices creates profitable deviations that are unprofitable without truncation.

On the front of uniqueness, it turns out that even a simple duopoly may have multiple Cournot equilibria – see Example 4. Nevertheless, we establish that in an oligopoly with two types of firms in which one type has superior information, whenever a Cournot equilibrium exists, it must be unique – see Theorems 2 and 4.

\(^5\)If prices are non-negative it may be profitable for a firm with deficient information to increase output in order to increase revenue in states where demand is high, knowing that in states where demand is low the revenue is bounded from below by zero.
2 Cournot Competition with Asymmetric Information

Consider an industry where a set of firms, $N = \{1, 2, ..., n\}$, compete in the production of a homogeneous good. There is uncertainty about the market demand and the production costs. This uncertainty is described by a finite set $\Omega$ of states of nature, together with a probability measure $\mu$ on $\Omega$, which represents the common prior belief of the firms about the distribution of the realized state. The information of the firms about the state of nature may be asymmetric: the private information of firm $i \in N$ is given by a partition $\Pi^i$ of $\Omega$ into disjoint sets. For any $\omega \in \Omega$, $\Pi^i (\omega)$ denotes the information set of $i$ given $\omega$, that is, the element of $\Pi^i$ that contains $\omega$.\footnote{The assumption that $\Omega$ is finite is not necessary, and is made only to simplify the presentation – See Remark 7.} W.l.o.g., we assume that $\mu$ has full support on $\Omega$, that is, $\mu (\Pi^i (\omega)) > 0$ for every $i \in N$ and $\omega \in \Omega$.

If $q^i (\omega)$ denotes the quantity of the good produced by firm $i$ in state $\omega \in \Omega$, and $Q (\omega) \equiv \sum_{i=1}^{n} q^i (\omega)$ is the aggregate output in $\omega$, then the profit of firm $i$ in $\omega$ is given by

$$h^i (\omega, (q^1 (\omega), ..., q^n (\omega))) = q^i (\omega) P (\omega, Q (\omega)) - c^i (\omega, q^i (\omega)),$$

where $P (\omega, \cdot)$ is the inverse demand function in $\omega$, and $c^i (\omega, \cdot)$ is the cost function of firm $i$ in $\omega$. Some standard assumptions will be made on the inverse demand and cost functions, but we postpone their introduction until Sections 4 and 5. For now, we will merely assume that in every $\omega \in \Omega$ there exists a level of aggregate output $\overline{Q} (\omega)$ such that

$$P (\omega, \overline{Q} (\omega)) = 0 < P (\omega, Q)$$

for every $Q < \overline{Q} (\omega)$. Thus, $\overline{Q} (\omega)$ denotes the horizontal intercept of the inverse demand function, which we refer to as the demand intercept in $\omega$.

A strategy for firm $i$ is a function $q^i : \Omega \to \mathbb{R}_+$ that specifies its output in every state of nature, subject to measurability with respect to $i$’s private information (i.e., $q^i$ is constant on every information set of firm $i$). The set of strategies of firm $i$ will be denoted by $B (\Omega, \Pi^i)$. Given a strategy profile $q = (q^1, ..., q^n) \in \prod_{j=1}^{n} B (\Omega, \Pi^j)$
the expected profit of firm $i$ is:

$$H^i(q) = E \left[ h^i(\cdot, (q^1(\cdot), ..., q^1(\cdot))) \right] = E \left[ q^i(\cdot) P(\cdot, Q(\cdot)) - c^i(\cdot, q^i(\cdot)) \right].$$

A strategy profile $q_\star \in \prod_{j=1}^n B(\Omega, \Pi^j)$ is a Cournot equilibrium if no firm finds it profitable to unilaterally deviate to another strategy, i.e., if for every $i \in N$ and $q^i \in B(\Omega, \Pi^i)$

$$H^i(q_\star) \geq H^i(q_\star \mid q^i),$$

(1)

where $(q_\star \mid q^i)$ stands for profile of strategies which is identical to $q_\star$ in all but the $i$th strategy, which is replaced by $q^i$. This is equivalent to requiring

$$E(\hat{h}^i(\cdot, q_\star(\cdot)) \mid \Pi^i(\omega)) \geq E(\hat{h}^i(\cdot, (q_\star \mid q^i)(\cdot)) \mid \Pi^i(\omega))$$

for every $\omega \in \Omega$, where $E(g(\cdot) \mid A)$ stands for the expectation of a random variable $g$ conditional on event $A$.

## 3 A Cournot Equilibrium May Not Exist: Examples

Consider a linear duopoly where firms’ marginal costs are positive and the market demand is non-increasing. In this setting, a Cournot equilibrium exists when firms have complete information. Surprisingly, when firms’ have asymmetric information existence of a Cournot equilibrium cannot be guaranteed even in this simple setting, as Example 1 below illustrates.

Equilibrium non-existence in Example 1 is driven by the difference in firms’ information about the demand intercept $\tilde{Q}$. We shall see in Section 5 (Example 3) that an incomplete information linear duopoly does possess a Cournot equilibrium if $\tilde{Q}$ is known to both firms (and thus, in particular, if the demand intercept is the same in all states of nature).

**Example 1.** Consider the following linear duopoly with asymmetric information. The set of states of nature $\Omega$ consists of just two states, $\omega_1$ and $\omega_2$. The probability of $\omega_1$ is $\frac{1}{4}$, and the probability of $\omega_2$ is $\frac{3}{4}$. Firm 1 is informed about the realized state
of nature, while firm 2 has *no information* about it; i.e., \( \Pi^1 = \{\omega_1, \omega_2\} \) and \( \Pi^2 = \{\Omega\} \). The inverse demand function is

\[
P(\omega, Q) = \max \{1 - b(\omega)Q, 0\},
\]

where \( b(\omega_1) = \frac{1}{4} \) and \( b(\omega_2) = 1 \). Thus, both \( P(\omega_1, \cdot) \) and \( P(\omega_2, \cdot) \) are linear till they reach zero, at which point they are truncated and set to be equal to zero. This ensures that the prices are always non-negative: in the state \( \omega \) the inverse demand function \( P \) is positive on \( [0, \bar{Q}(\omega)] \), and is zero for \( Q \geq \bar{Q}(\omega) \), where \( \bar{Q}(\omega_1) = 4 \) and \( \bar{Q}(\omega_2) = 1 \). The marginal costs of firm 1 are \( c^1(\omega_1) = 2 \) and \( c^1(\omega_2) = \frac{1}{100} \). Firm 2 has a constant marginal cost \( c^2 = \frac{1}{100} \) in both states of nature.

We show that *no Cournot equilibrium exists* in this industry. Note that the marginal revenue of firm 1 in \( \omega_1 \) is always below its marginal cost; hence maximizing profits entails that firm 1 produces zero in this state. Thus, in looking for an equilibrium we restrict attention to those strategies of firm 1, \( q^1 \in B(\Omega, \Pi^1) \), that prescribe producing zero in \( \omega_1 \); i.e., \( q^1 \) can be identified with a scalar \( x \equiv q^1(\omega_2) \in \mathbb{R}_+ \). Also, since firm 2 does not know the realized state, a strategy of firm 2, \( q^2 \in B(\Omega, \Pi^2) \), must specify the same output in both states of nature; i.e., \( q^2 \) can be identified with a scalar \( y \equiv q^2(\omega_1) = q^2(\omega_2) \in \mathbb{R}_+ \). Accordingly, the strategies of firms 1 and 2 will be regarded as scalars \( x, y \in \mathbb{R}_+ \).

Note that for any \( (x, y) \in \mathbb{R}_+^2 \) the expected profit of firm 1 is

\[
H^1(x, y) = \begin{cases} 
\frac{3}{4} \left[ (1 - x - y) x - \frac{x}{100} \right] & \text{if } x + y < 1, \\
\frac{3}{4} \left[ -\frac{x}{100} \right] & \text{if } x + y \geq 1.
\end{cases}
\]

That is, if the aggregate output \( x + y \) is below \( \bar{Q}(\omega_2) = 1 \), then firm 1’s revenue in \( \omega_2 \) is positive, and this results in the expected profit of \( \frac{3}{4} \left[ (1 - x - y) x - \frac{x}{100} \right] \). If \( x + y \geq \bar{Q}(\omega_2) = 1 \), then the revenue in \( \omega_2 \) is zero and firm 1 incurs losses equal to its costs.

The expected profit function of firm 2 has a more complicated form, since this firm is equally active in both states of nature, unlike firm 1 which does not produce in \( \omega_1 \). For \( (x, y) \in \mathbb{R}_+^2 \), the expected profit of firm 2 is

\[
H^2(x, y) = \begin{cases} 
\hat{H}^2(x, y) & \text{if } x + y < 1, \\
\hat{H}_2(y) & \text{if } 1 \leq x + y \text{ and } y \leq 4, \\
-\frac{y}{100} & \text{if } y > 4.
\end{cases}
\]
where

\[ H_2^2(x, y) = \frac{1}{4}(1 - \frac{y}{4})y + \frac{3}{4}(1 - x - y) y - \frac{y}{100}, \tag{6} \]

and

\[ \hat{H}^2(y) = \frac{1}{4}(1 - \frac{y}{4})y - \frac{y}{100}. \]

Note that \( H_2^2 \) equals to the expected profit of firm 2 when the inverse demand is \( P_i(\omega_i, Q) = 1 - b(\omega_i)Q \) for \( i = 1, 2 \); i.e., when negative prices are possible. When the inverse demand is the truncated function (3), however, prices are non-negative, and the revenue is not less than with possibly negative prices; i.e., \( H_2^2(x, y) \leq H^2(x, y) \) for every \((x, y) \in \mathbb{R}_+^2\). Furthermore, \( H^2 = H^2 \) whenever

\[ x + y < \hat{Q}(\omega_2) = 1, \]

since this guarantees that the inverse demand function (3) is positive in both states of nature. When

\[ \hat{Q}(\omega_2) = 1 \leq x + y \text{ and } y \leq 4 = \hat{Q}(\omega_1), \]

the expected profit of firm 2 becomes \( \hat{H}^2 \), which the expected revenue of firm 2 in \( \omega_1 \) minus its costs. Finally, when

\[ y > \hat{Q}(\omega_1) = 4, \]

the revenue of firm 2 is zero in both states of nature, and the firm incurs losses equal to its costs.

By identifying firms’ strategies with scalars \( x, y \) we have in effect converted the incomplete information duopoly into a complete information game, with payoff functions given by (4) and (5). In this game, the reaction function of firm 1 is

\[ R^1(y) = \max \left\{ \frac{1}{2} \left( \frac{99}{100} - y \right), 0 \right\}. \tag{7} \]

The reaction function of firm 2 is

\[ R^2(x) = \begin{cases} R^2_-(x) & \text{if } x \leq \bar{x}, \\ \frac{48}{25} & \text{if } x \geq \bar{x}, \end{cases} \]

where

\[ R^2_-(x) = \max \left\{ \frac{198}{325} - \frac{6}{13}x, 0 \right\}, \tag{8} \]

and \( \bar{x} = \frac{33}{25} - \frac{8}{25} \sqrt{13} \approx 0.16622 \). (Here \( \bar{x} \) is the smallest solution of the quadratic equation \( H^2_-(x, R^2_-(x)) = \hat{H}^2 \left( \frac{48}{25} \right) \).)
The function $R^2_2(x)$ in (8) is simply the reaction of firm 2 under the assumption of possibly negative prices (that is, with $H^2_2$ given in (6) as firm 2’s expected profit function, instead of the more complicated $H^2$). From the expression (5) for $H^2$ it is clear that, in setting its best response to $x \leq 1$, firm 2 must first find $y_1 \in [0, 1 - x]$ that maximizes its payoff according to $H^2$, then find $y_2 \in [1 - x, 4]$ that maximizes $\hat{H}^2$, and then choose the best between $y_1$ and $y_2$. As follows from the definition of $R^2_2$, $y_1 = \min \{ R^2_2(x), 1 - x \}$. And $y_2 = \frac{48}{25}$ independently of $x$, since $\frac{48}{25}$ is the unique maximizer of $\hat{H}^2$.

We can now see the source of the composite nature of $R^2$, and of its sharp discontinuity at $\bar{x}$. For $x < \bar{x}$,

$$H^2(x, y_1) = H^2_2(x, R^2_2(x)) > \hat{H}^2 \left( \frac{48}{25} \right) = H^2(x, y_2),$$

and thus $y_1$ maximizes $H^2(x, y)$ and $R^2(x) = R^2_2(x)$. For $x > \bar{x}$,

$$H^2(x, y_1) \leq H^2_2(x, R^2_2(x)) < \hat{H}^2 \left( \frac{48}{25} \right) = H^2(x, y_2),$$

and thus $y_2$ maximizes $H^2(x, y)$ and $R^2(x) = \frac{48}{25}$. Since $R^2_2(\bar{x}) < \frac{48}{25}$, $R^2$ jumps upwards at $\bar{x}$ – see Figure 1 below. Plotting $R^1$ and $R^2$ together clearly shows that the graphs of these functions do not cross, and therefore a Cournot equilibrium does not exist. 

$\blacksquare$
Figure 1: Figure 1
Remark 1. (Possibly negative prices versus always non-negative prices.)

Note that in the scenario where negative prices are possible (i.e., when $H^2$ is firm 2’s expected profit, and accordingly $R^2$ is its reaction function) the graphs of $R^1$ and $R^2$ cross at the unique point $x_*=\frac{99}{200}, y_*=\frac{99}{200}$, which is the Cournot equilibrium in this case. It is interesting to observe the role of negative prices in sustaining equilibrium behavior in this case: the fear of negative prices arising in $\omega_2$, and of the ensuing negative revenue in this state, makes deviations of firm 2 to high outputs $y$ (i.e., such that $x+y>1$) unprofitable. When prices are always non-negative, however, revenue in $\omega_2$ is bounded below by zero, making such deviations profitable – specifically, $y=\frac{48}{25}$ is a profitable deviation from $R^2(x)$ when $x>\bar{x}$. This explains the tendency of $R^2$ to be above $R^2$ seen in Figure 1, and stresses the strategic impact of truncating the inverse demand to avoid negative prices. Note also that the expected profit of firm 2 when prices may be negative, $H^2$, is concave in strategies of firm 2. However, firm 2’s expected profit when prices are always non-negative, $H^2$, is not concave due to its composite nature (which is the result of truncating the inverse demand (3), as was explained following formula (5)). Thus, when the inverse demand function is truncated to avoid negative prices, the firms’ expected profit functions tend to lose properties conducive to equilibrium existence. This will be seen in full generality in Sections 4 and 5. ■

Remark 2. (Non-scalar strategies of firm 1.) In Example 1 it was assumed that the marginal cost of production of firm 1 in state $\omega_1$ is 2, which implies that firm 1 produces zero in this state. This simplifies the presentation, since the equilibrium strategy of firm 1 is identified with a scalar representing its output in state $\omega_2$, but is not necessary to obtain an example where a Cournot equilibrium does not exist. We only need the quantity produced by firm 1 in $\omega_1$ to be sufficiently small. For instance, in the above duopoly a Cournot equilibrium does not exist even when the marginal costs of firm 1 in $\omega_1$ are very small for $q \in [0, 0.01]$, but exceed 2 for $q \in [0.02, \infty)$.■

Remark 3. (Nearly complete information.) A Cournot equilibrium may fail to exist even in an industry with nearly complete information. Indeed, consider a linear duopoly where firms know the inverse demand function and their costs with a probability near one, but with a small probability $\varepsilon>0$ firms’ information about the state of demand and costs is as in Example 1. More precisely, let us assume that
\( \omega_1, \omega_2 \in \Omega, \mu(\{\omega_1\}) = \frac{1}{2} \varepsilon \) and \( \mu(\{\omega_2\}) = \frac{3}{2} \varepsilon \). Both firms know when \( \{\omega_1, \omega_2\} \) and each \( \omega \in \Omega \setminus \{\omega_1, \omega_2\} \) occur, but firm 1 can distinguish between \( \omega_1 \) and \( \omega_2 \) while firm 2 cannot. On \( \{\omega_1, \omega_2\} \), let the inverse demand function and the cost functions be the same as in Example 1. Then an argument similar to the one used above shows that no Cournot equilibrium exists in this industry. ■

**Remark 4.** (A linear duopoly where the inverse demand has a constant slope.) In Example 1 the maximum price is 1 in both states of nature, while the slope of the demand (and the demand intercept \( \bar{Q}(\omega_1) \)) are variable. However, a simple modification yields an example of an industry in which in all states of nature the inverse demand function has a constant slope of -1 when it is positive (but the maximum price is variable), and in which a Cournot equilibrium does not exists either. Simply multiply the function \( P(\omega_1, \cdot) \) by 4 in state \( \omega_1 \), to obtain a new \( P(\omega_1, Q) = \max\{4 - Q, 0\} \) with slope -1 on \([0, 4] \). To offset the four-fold increase of \( P(\omega_1, \cdot) \), divide the probability of \( \omega_1 \) by 4 (and then rescale the vector \((\frac{1}{4} \mu(\{\omega_1\}), \mu(\{\omega_2\})) = (\frac{1}{16}, \frac{3}{4}) \) to obtain new probabilities \((\frac{1}{16}, \frac{13}{16}) \) for the two states of nature). Equilibrium non-existence can be established by similar arguments. ■

**Example 2.** (Cournot equilibrium non-existence in a duopoly where the inverse demand is piecewise-linear and concave.) Consider the duopoly described in Example 1 with just one change: let the demand in \( \omega_1 \) be given by\(^7\)

\[
P(\omega_1, Q) = \begin{cases} 
1, & \text{if } Q \leq \frac{99}{100}, \\
100(1 - Q) & \text{if } \frac{99}{100} < Q \leq 1, \\
0 & \text{if } Q > 1.
\end{cases}
\]

Note that in this example the demand intercept is constant, \( \bar{Q}(\omega_1) = \bar{Q}(\omega_2) = 1 \), and thus known to both firms. Using our previous notations, for any \((x, y) \in \mathbb{R}_+^2 \) the expected profit of firm 1 is given by (4) above, while the expected profit of firm 2 is

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\(^7\)The function \( P(\omega_1, \cdot) \) can be made smooth in a small neighborhood of \( Q = 0.99 \), and strictly decreasing on \([0, 1]\) by adding to it a function \( \varepsilon(Q) \equiv 0.001(1 - Q) \). This change would make the presentation more cumbersome without essentially affecting the qualitative arguments.
Thus, the reaction function of firm 1 is given by (7), and of firm 2 by

\[
R^2(x) = \begin{cases} 
\frac{33}{50} - \frac{1}{2} x & \text{if } x \leq \bar{x}, \\
\frac{99}{100} & \text{if } x \geq \bar{x}.
\end{cases}
\]

where \(\bar{x} = \frac{33}{50} - \frac{9}{25} \sqrt{22} \approx 0.1943\). (Here \(\bar{x}\) is the smallest solution of the quadratic equation \(\frac{1}{4}y_1 + \frac{3}{4}(1-x-y_1)y_1 - \frac{94}{100} = \frac{1}{4}y_2 - \frac{99}{100}\), where \(y_1 = \frac{33}{50} - \frac{1}{2}x\) and \(y_2 = \frac{99}{100}\).)

It is easy to see that the reaction functions do not cross, just as in Figure 1, and therefore a Cournot equilibrium does not exists in this industry. ■

## 4 Cournot Equilibrium when Negative Prices are Possible

In this section we study conditions for existence and uniqueness of Cournot equilibrium, assuming that prices may become negative in some states of nature if firms’ outputs are sufficiently large. The possibility of negative prices, that may be meaningful in certain contexts, plays an important role in guaranteeing equilibrium existence as we mentioned in Remark 1. In addition, (the proofs of) the results presented now will be instrumental in establishing conditions for existence of a Cournot equilibrium when prices are restricted to be non-negative, which is the topic of the next section.

The following four conditions will be assumed on the inverse demand and the cost functions:

(i) For every \(\omega \in \Omega\), \(P(\omega, \cdot)\) is non-increasing.

(ii) For every \(\omega \in \Omega\) and \(i \in N\), \(c^i(\omega, \cdot)\) is strictly increasing and continuous.

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(iii) There exists \( Z > 0 \) such that \( P(\omega, Z) \leq 0 \) for every \( \omega \in \Omega \).

(iv) For every \( \omega \in \Omega \), \( P(\omega, \cdot) \) is twice continuously differentiable and satisfies

\[
Q P''(\omega, Q) + P'(\omega, Q) \leq 0
\]

for every \( Q \in \mathbb{R}_+ \). (At \( Q = 0 \) we have in mind the right-side derivatives of \( P \) and \( P' \)).

Condition (iv), which is borrowed from Novshek [12], is satisfied, e.g., by all concave and twice continuously differentiable inverse demand functions.\(^9\) Condition (iii) allows the possibility of negative prices in some states of nature if the total output is sufficiently large. This condition (in conjunction with the rest, particularly (iv)) has far reaching implications for the existence of Cournot equilibrium, as Theorem 1 below demonstrates. If, instead of assuming (iii) and (iv), we truncate the inverse demand when it reaches zero in order to guarantee that prices are non-negative, and impose condition (iv) only the range of outputs where it is positive (which will be the framework of the next section), then a Cournot equilibrium may fail to exist even in a linear duopoly as shown in Example 1. However, we have also seen in Remark 1 that if the demand in Example 1 is not truncated and thus negative prices may arise, then equilibrium existence is restored because the firms’ incentives change dramatically.

Theorem 1 is a general result on Cournot equilibrium existence when negative prices are possible:

**Theorem 1.** Consider an oligopoly that satisfies (i) – (iv). If either \( n = 2 \), or \( c_i'(\omega, \cdot) \) is convex for every \( i \in N \) and \( \omega \in \Omega \), then there exists a Cournot equilibrium.

---

\(^8\)Condition (iii) is implied by (i) in conjunction with our assumption on the existence of the demand intercept \( \bar{Q} \), made in Section 2. Indeed, one can simply take \( Z \equiv \max_{\omega \in \Omega} \bar{Q}(\omega) \). However, we chose to state (iii) as a separate condition since its modified version will be conveniently used in the next section. Moreover, condition (iii) can replace our assumption on the existence of \( \bar{Q} \) for the duration of this section: our results here make no use of \( \bar{Q} \) or any of its properties.

\(^9\)Assuming that each \( P(\omega, \cdot) \) is log-concave, a condition used by Amir [1] to prove existence of Cournot equilibrium in a duopoly when information is complete, does not appear to be a viable alternative to assuming condition (iv). As we note in the proof of Theorem 1, (9) implies certain properties of the state-dependent revenue functions (decreasing differences, concavity), that are inherited by the expected revenue function. This would not have been the case for log-concave \( P(\omega, \cdot) \).
It is the nice behavior of the expected payoff functions that stands behind the existence result in Theorem 1. Indeed, its proof reveals that if firms’ cost functions are convex, then the expected profit function of each firm is concave in its strategy; and if there are just two firms, then the profit function of each firm has increasing differences in its strategy, i.e., is submodular – see Section 6.1 in the Appendix. Both these conditions imply existence of a Cournot equilibrium via known results. When prices are constrained to be non-negative, however, the differentiability condition in (iv) breaks up abruptly at the point $\bar{Q}$ where the price becomes zero. As a result, the expected profit functions lose some of these nice properties and this, as illustrated by Examples 1 and 2, may cause non-existence of Cournot equilibrium.

The following theorem establishes uniqueness of Cournot equilibrium in an oligopoly in which there are two types of firms, one of which possesses superior information (i.e., has a finer information partition). Before stating the theorem, we introduce the following strengthened versions of (i) and (ii):

(i’) For every $\omega \in \Omega$, $P(\omega, \cdot)$ is strictly decreasing.

(ii’) For every $\omega \in \Omega$ and $i \in N$, $c^i(\omega, \cdot)$ is strictly increasing, twice continuously differentiable and convex.

**Theorem 2.** Consider an oligopoly satisfying (i’), (ii’), (iii), and (iv). Suppose further that the set $N$ of firms can be partitioned into two disjoint sets, $K$ and $M$, such that $1 \in K$, $2 \in M$, and such that $\Pi^i = \Pi^1$, $c^i = c^1$ for every $i \in K$, $\Pi^j = \Pi^2$, $c^j = c^2$ for every $j \in M$, and $\Pi^1$ is finer than $\Pi^2$. Then there exists a unique Cournot equilibrium.

**Proof.** See Section 6.2 of the Appendix. ■

5 Cournot Equilibrium when Prices are Non-Negative

We have shown in the previous section that when Novshek condition is satisfied even in the region of negative prices, existence of a Cournot equilibrium can be established
under quite general conditions. When negative prices are ruled out (by truncating the 
inverse demand function when it reaches zero, thereby breaking its differentiability), 
existence of a Cournot equilibrium cannot be guaranteed as seen in Examples 1 and 
2. In this section we discuss this latter case and present positive results that apply to 
a certain class of oligopolies.

In order to rule out negative prices, we will consider the following assumption on 
the inverse demand function, which replaces condition (iii) of Section 4:

(iii’) For every \( \omega \in \Omega \), \( P(\omega, Q) = 0 \) if \( Q \geq \overline{Q}(\omega) \). (Thus, the inverse demand is 
fixed at zero beyond the demand intercept \( \overline{Q}(\omega) \).

Since the Novshek condition (iv) will now be used in conjunction with condition 
(iii’), it must be restated in the following form, that requires (9) to hold only below 
the intercept \( \overline{Q} \):

(iv’ ) For every \( \omega \in \Omega \), \( P(\omega, \cdot) \) is twice continuously differentiable and satisfies 
\(QP''(\omega, Q) + P'(\omega, Q) \leq 0 \) on \([0, \overline{Q}(\omega)]\).\(^\text{10}\)

Theorem 3 below assumes that there exists a profile of state-dependent thresholds 
of output, \( \overline{q} \in \prod_{i=1}^{n} B(\Omega, \Pi^i) \), such that for every \( \omega \in \Omega \)
\[
\sum_{i=1}^{n} \overline{q}^i(\omega) \leq \overline{Q}(\omega), \tag{10}
\]
which no firm will never desire to exceed; i.e., such that for each firm \( i \) the strate-
gies \( q^i \in B(\Omega, \Pi^i) \) satisfying \( q^i(\omega) > \overline{q}^i(\omega) \) in some \( \omega \in \Omega \) are weakly dominated. 
(Existence of such thresholds is guaranteed if marginal costs of firms increase suffi-
ciently fast.) When this condition holds then a Cournot equilibrium exists under the 
assumptions analogous to those of Theorem 1.

**Theorem 3.** Consider an oligopoly satisfying (i), (ii), (iii’), and (iv’). Suppose 
further that there exists \( \overline{q} \in \prod_{i=1}^{n} B(\Omega, \Pi^i) \) satisfying (10), and such that for every 
strategy profile \( q \in \prod_{i=1}^{n} B(\Omega, \Pi^i) \) and every \( i \in N \)
\[
H^i(q) \leq H^i(q \mid \min(q^i, \overline{q}^i)). \tag{11}
\]
\(^{10}\text{At the endpoints of the interval this refers to the continuity of one-sided derivatives.}\)
If either \( n = 2 \), or \( c^i(\omega, \cdot) \) is convex for every \( i \in N \) and \( \omega \in \Omega \), then there exists a Cournot equilibrium.

**Proof.** See Section 6.3 of the Appendix.

Intuitively, the inequality (11) in Theorem 3 means that no firm wants to produce too much, since firm \( i \) will not reduce its expected profit by reducing its output to the level \( q^i \). And when firms produce below the thresholds \( (\overline{q}^i)^n_{i=1} \), prices are positive, and deviations are evaluated in the domain where the inverse demand function is twice continuously differentiable and obeys the inequality in (iv'). This allows us to establish existence of a Cournot equilibrium using arguments analogous to those of Section 4, where the Novshek condition holds on the entire \( \mathbb{R}_+ \).

**Remark 5.** A close look into the proof of Theorem 3 (see the Appendix) reveals that, in fact, a condition weaker than (11) would suffice to guarantee existence of a Cournot equilibrium. Specifically, (11) could be replaced by the following:

Given a profile strategies \( q \in \prod_{i=1}^n B(\Omega, \Pi^i) \), for every firm \( i \) there exists a strategy\(^{11}\) \( r^i \leq \overline{q}^i \) such that

\[
H^i(q) \leq H^i(q \mid r^i). \tag{12}
\]

Thus, condition (12) is different from (11) in that \( \min(q^i, \overline{q}^i) \) is replaced by a strategy \( r^i \leq \overline{q}^i \): firm \( i \) would prefer some strategy \( r^i \) over \( q^i \notin [0, \overline{q}^i] \), and not necessarily the strategy \( \min(q^i, \overline{q}^i) \) that simply reduces output to the level \( \overline{q}^i \) whenever it exceeds \( \overline{q}^i \).

Remark 5 leads to the following corollary that establishes existence of a Cournot equilibrium if there are thresholds \( (\overline{q}^i)^n_{i=1} \) satisfying (10) such that the expected the monopoly profit of any firm \( i \) under any strategy exceeding \( \overline{q}^i \) is non-positive, given \( i \)'s information.

**Corollary 1.** Consider an oligopoly that satisfies (i), (ii), (iii'), and (iv'), and assume that \( c^i(\cdot, 0) \equiv 0 \) for every \( i \in N \). Suppose further that there exists \( \overline{q} \in \mathbb{R}_+ \).

\(^{11}\)Here and henceforth, we use the notation \( h \leq g \) (for \( h, g : \Omega \to \mathbb{R}_+ \)) if and only if \( h(\omega) \leq g(\omega) \) for every \( \omega \in \Omega \).
\[ \prod_{i=1}^{n} B(\Omega, \Pi^i) \] such that (10) holds, and
\[
E\left( h^i(\cdot, (q^i, 0^{-i})) \mid \Pi^i(\omega) \right) \leq 0
\] (13)
for every \( i \in N \), every \( \omega \in \Omega \) and every strategy \( q^i \) that exceeds \( \bar{q}^i \) on \( \Pi^i(\omega) \). (Here \( 0^{-i} \) stands for the profile of strategies of all firms but \( i \) according to which every firm produces zero in every state of nature.) If either \( n = 2 \), or \( c^i(\omega, \cdot) \) is convex for every \( i \in N \) and \( \omega \in \Omega \), then there exists a Cournot equilibrium.

**Proof.** Given a profile of strategies \( q \), consider the strategy \( r^i \) of \( i \) which is equal to 0 on the \( \Pi^i \)-measurable set \( A = \{ \omega \mid q^i(\omega) > \bar{q}^i(\omega) \} \), and to \( q^i \) on \( A^c \). If \( \omega \in A \),
\[
E\left( h^i(\cdot, q(\cdot)) \mid \Pi^i(\omega) \right) \leq E\left( h^i(\cdot, (q^i, 0^{-i})) \mid \Pi^i(\omega) \right) \leq 0 = E\left( h^i(\cdot, q(\cdot) \mid r^i(\cdot)) \mid \Pi^i(\omega) \right)
\]
as follows from conditions (i), (13), and the assumption of zero fixed costs. And if \( \omega \in A^c \), then \( q^i = r^i \) on \( \Pi^i(\omega) \), and thus for every \( \omega \in \Omega \)
\[
E\left( h^i(\cdot, q(\cdot)) \mid \Pi^i(\omega) \right) \leq E\left( h^i(\cdot, q(\cdot) \mid r^i(\cdot)) \mid \Pi^i(\omega) \right).
\] (14)
By taking the expectation over \( \omega \) in (14), we obtain (12). Existence of a Cournot equilibrium then follows by Remark 5.

The following theorem is a counterpart of Theorem 2 when prices are restricted to non-negative. It establishes conditions that guarantee that, when a Cournot equilibrium exists, it is also unique. Let us first restate (i') in the form appropriate in the current setting of non-negative prices:

(i'') For every \( \omega \in \Omega \), \( P(\omega, \cdot) \) is strictly decreasing on \([0, \bar{Q}(\omega)]\).

**Theorem 4.** Consider an oligopoly satisfying (i''), (ii'), (iii'), and (iv'). Suppose further that the set \( N \) of firms can be partitioned into two disjoint sets, \( K \) and \( M \), such that \( 1 \in K \), \( 2 \in M \), and such that \( \Pi^i = \Pi^1 \), \( c^i = c^1 \) for every \( i \in K \), \( \Pi^j = \Pi^2 \), \( c^j = c^2 \) for every \( j \in M \), and \( \Pi^1 \) is finer than \( \Pi^2 \). Assume also that the demand
intercept \( \bar{Q} \) is strictly positive and measurable with respect to \( \Pi^2. \)
If a Cournot equilibrium exists (e.g., under conditions of either Theorem 3 or Remark 5), then it is unique.

**Proof.** See Section 6.4 of the Appendix. \( \blacksquare \)

Example 3 below is an application of Theorems 3 and 4 to a duopoly with linear demand. In particular, we show that a Cournot equilibrium exists provided the demand intercept \( \bar{Q} \) is known to both firms.

**Example 3 (Linear Demand and Complete Information on the \( Q \)-intercept).** Suppose that \( n = 2. \) Let \( \alpha, \beta : \Omega \rightarrow \mathbb{R}_{++} \) be strictly positive functions, and assume moreover that \( \beta \in B(\Omega, \Pi^1) \cap B(\Omega, \Pi^2), \) where \( \Pi^1 \) and \( \Pi^2 \) are information endowments of the duopolists. Suppose that for any \( \omega \in \Omega, \)

\[
P(\omega, Q) = \max \{ \alpha(\omega) (\beta(\omega) - Q), 0 \},
\]

and that the two cost functions satisfy (ii'). Here \( \bar{Q} = \beta. \) Since \( \beta \) is both \( \Pi^1 \)- and \( \Pi^2 \)-measurable, both firms know the demand intercept \( \bar{Q} \) in every state of nature. This is a crucial difference with Example 1, where \( \bar{Q} \) was not measurable with respect to the information partition of firm 2, and a Cournot equilibrium does not exist. As we shall see, the measurability \( \bar{Q} = \beta \) with respect to both partitions leads to a different conclusion.

We prove that there is a Cournot equilibrium. Let \( \bar{q}^1 = \bar{q}^2 \equiv \bar{\frac{1}{2}} \beta \in B(\Omega, \Pi^1) \cap B(\Omega, \Pi^2). \) Clearly \( (\bar{q}^1, \bar{q}^2) \) satisfies (10) in Theorem 3. We show next that condition (11) of that theorem also holds.

Let \( q = (q^1, q^2) \in B(\Omega, \Pi^1) \times B(\Omega, \Pi^2). \) If \( q^i \in [0, \bar{q}^i] \) for every \( i \) then (11) is trivial, and so suppose that \( q^i \notin [0, \bar{q}^i] \) for some firm \( i. \) We claim that then

\[
E \left( h^i(\cdot, q(\cdot)) \mid \Pi^i(\omega) \right) \leq E \left( h^i(\cdot, q(\cdot)) \mid \bar{q}^i(\cdot) \right) \mid \Pi^i(\omega) \right)
\]

\[\text{(15)}\]

\[^{12}\text{This condition did not appear in the statement of Theorem 2. It is needed only when prices are restricted to be non-negative. Indeed, without } \bar{Q}\text{'s measurability with respect to both fields, there are counterexamples to uniqueness even if all firms have the same information, see Lagerlöf [9].}\]
for every $\omega \in \Omega$. Assume first that $q^i(\omega) > \overline{q}^i(\omega)$. Note that for every $x \geq \overline{q}^i(\omega) = \frac{1}{2} \beta(\omega)$, the following holds: either
\[
\frac{\partial}{\partial x} \left[ xP(\omega, x + y) - c^i(\omega, x) \right] \leq \alpha(\omega) (\beta(\omega) - 2x) - \frac{\partial}{\partial x} c^i(\omega, x) < 0,
\]
if $x + y < \beta(\omega)$, or
\[
\frac{\partial}{\partial x} \left[ xP(\omega, x + y) - c^i(\omega, x) \right] = -\frac{\partial}{\partial x} c^i(\omega, x) < 0,
\]
if $x + y \geq \beta(\omega)$. Accordingly,
\[
h^i(\omega, q(\omega)) < h^i(\omega, q(\omega) | \overline{q}^i(\omega)).
\]
Since $q^i(\omega) > \overline{q}^i(\omega)$ implies the same inequality for every $\omega' \in \Pi^i(\omega)$, (15) follows.
When $q^i(\omega) \leq \overline{q}^i(\omega)$, $\min (q^i(\cdot), \overline{q}^i(\cdot)) = q^i(\cdot)$ and thus (15) holds trivially. By taking the expectation over $\omega$ in (15), we obtain (11). We conclude that in a duopoly with linear prices (11) holds for every $q = (q^1, q^2)$ and every $i = 1, 2$.

But (10) and (11) imply, by Theorem 3, that the duopoly has a Cournot equilibrium. Now assume in addition that $\Pi^1$ is finer than $\Pi^2$. Then, by Theorem 4, this duopoly’s Cournot equilibrium is unique.

**Example 4 (Non-Uniqueness of Cournot Equilibrium when no Firm Has Superior Information).** Consider a duopoly in which $\Omega$ consists of three states, $\omega_1$, $\omega_2$, and $\omega_3$; each one is chosen by nature with equal probability. Firms’ information partitions are $\Pi^1 = \{\{\omega_1, \omega_2\}, \{\omega_3\}\}$, and $\Pi^2 = \{\{\omega_1, \omega_3\}, \{\omega_2\}\}$; i.e., firm 1 cannot distinguish between $\omega_1$ and $\omega_2$, and firm 2 cannot distinguish between $\omega_1$ and $\omega_3$. In all states of nature firms face the same quadratic inverse demand function
\[
P(Q) = \max\{1 - Q^2, 0\}.
\]
Thus, firms know the inverse demand in every state of nature; they are, however, asymmetrically informed about their costs.$^{13}$ Firm 1 has a constant marginal cost of $\frac{1}{100}$ in states $\omega_1$ and $\omega_2$, while its marginal cost is 2 in $\omega_3$. Firm 2 has a constant marginal cost of $\frac{1}{100}$ in states $\omega_1$ and $\omega_3$, while its marginal cost is 2 in $\omega_2$.

$Lagerlöf$ [9] provides an example of equilibrium non-uniqueness with symmetrically informed firms but with incomplete information on the inverse demand. This example shows that knowing the inverse demand does not guarantee uniqueness either.
Since in \( \omega_3 \) the marginal revenue of firm 1 is always below its marginal cost, firm 1 produces zero in this state in any best response. Similarly, firm 2 produces zero in \( \omega_2 \) in any best response. It follows that each firm \( i \)'s strategy \( q^i \) can, without loss of generality, be identified with a scalar: \( q^1 \) can be viewed as the quantity \( x \) produced by firm 1 in state \( \omega_1 \) (and thus also in \( \omega_2 \)), and \( q^2 \) as the quantity \( y \) produced by firm 2 in state \( \omega_1 \) (and thus also in \( \omega_3 \)).

We claim that both 

\[
q_* = (x_*, y_*) = \left( \frac{3}{10} \sqrt{2}, \frac{3}{10} \sqrt{2} \right) \approx (0.42426, 0.42426)
\]

and 

\[
q^{**} = (x^{**}, y^{**}) = \left( \frac{7}{30} \sqrt{6}, \frac{7}{30} \sqrt{6} \right) \approx (0.57155, 0.57155)
\]

are Cournot equilibria.

Let us show first that \( q_* \) is a Cournot equilibrium. For \( y \in [0, 1 - x_*] \) the expected profit of firm 2,

\[
H^2(x_*, y) = \frac{1}{3} h^2 (\omega_1, (x_*, y)) + \frac{1}{3} h^2 (\omega_2, (x_*, 0)) + \frac{1}{3} h^2 (\omega_3, (0, y)) = \frac{1}{3} y \left( 1 - \left( \frac{3}{10} \sqrt{2} + y \right)^2 \right) + \frac{1}{3} y (1 - y^2) - \frac{2}{3} \frac{y}{100},
\]

has a (unique) maximum on \([0, 1 - x_*] \) at \( y = y_* = \frac{3}{10} \sqrt{2} \). Thus firm 2 has no incentive to deviate from \( y_* \) to another strategy in \([0, 1 - x_*] \). Now, for \( y \in [1 - x_*, 1] \),

\[
H^2(x_*, y) = \frac{1}{3} y \left( 1 - y^2 \right) - \frac{2}{3} \frac{y}{100}.
\]

The maximum of \( \frac{1}{3} y \left( 1 - y^2 \right) - \frac{2}{3} \frac{y}{100} \) on \([1 - x_*, 1] \) is attained at \( y = \frac{7}{30} \sqrt{6} \approx 0.57155 \). This maximum is equal to \( \frac{343}{6750} \sqrt{6} \approx 0.12447 \), and therefore firm 2 has no incentive to deviate from \( y_* \) (that gives it a payoff \( H^2(x_*, y_*) \approx 0.15274 \)) to a strategy in \([1 - x_*, 1] \). Since producing more than 1 would yield a negative expected profit, we have shown that firm 2 will not deviate unilaterally from \( q_* \). By symmetry, the same holds for firm 1, and thus \( q_* \) is indeed a Cournot equilibrium.

We show next that \( q^{**} \) is a Cournot equilibrium. For \( y \in [1 - x^{**}, 1] \) the expected profit of firm 2,

\[
H^2(x^{**}, y) = \frac{1}{3} y (1 - y^2) - \frac{2}{3} \frac{y}{100},
\]
reaches the maximum value of \(\frac{343 \times 6750}{60} \approx 124.47\) at \(y = y_{ss} = \frac{7}{30}\sqrt{6}\). Thus, firm 2 has no incentive to deviate from \(y_{ss}\) to another strategy in \([1 - x_{ss}, 1]\). For \(y \in [0, 1 - x_{ss}]\), the expected profit of firm 2

\[
H^2(x_{ss}, y) = \frac{1}{3}y \left(1 - \left(\frac{7}{30}\sqrt{6} + y\right)^2\right) + \frac{1}{3}y \left(1 - y^2\right) - \frac{2}{3} \frac{y}{100},
\]

reaches the maximum value of \(\approx 0.11798\) at \(y \approx 0.36792\). Hence firm 2 has no incentive to deviate from \(y_{ss}\) to a strategy in \([0, 1 - x_{ss}]\). Since producing more than 1 would yield negative expected profit, this shows that firm 2 will not deviate unilaterally from \(q_{ss}\). By symmetry, the same holds for firm 1, and thus \(q_{ss}\) is another Cournot equilibrium of the duopoly. ■

**Remark 6 (Existence of Cournot Equilibrium in Mixed Strategies).** If the inverse demand function is continuous in every state, under conditions (ii) and (iii') existence of a Cournot equilibrium in mixed strategies follows from Nash equilibrium existence theorem. Indeed, since under (ii) and (iii') producing more than \(\max_{\omega \in \Omega} Q(\omega)\) with positive probability is never a best response, we can assume, w.l.o.g., that the set of (behavioral) mixed strategies of each firm \(i\) is \(S^i = M([0, \max Q])\), where \(M([0, \max Q])\) is the set of all probability distributions on the interval \([0, \max Q]\). The set \(M([0, \max Q])\) is compact in the weak topology on measures (see Billingsley [2]), and hence so is the product set \(S^i\). Since the inverse demand function and the cost functions are continuous as follows from our assumptions, the expected profit function of each firm is continuous in the product (weak) topology on \(\prod_{i=1}^{n} S^i\). The expected profit function \(H^i\) of firm \(i\) is also concave in its own mixed strategies, \(s^i \in S^i\). Consequently, existence of a Cournot equilibrium follows from Nash equilibrium existence theorem. ■

**Remark 7 (Infinitely Many States of Nature).** Throught this paper we maintained the assumption that the set of states of nature \(\Omega\) is finite. However, this assumption is by no means necessary, and was made only to simplify the presentation. In Einy et al [5], a discussion paper on which this article is based, the uncertainty is represented by a probability space \((\Omega, F, \mu)\), where \(\Omega\) is a (possibly infinite) set of states of nature, \(F\) is a \(\sigma\)-field of subsets of \(\Omega\), and \(\mu\) is a common prior. Firm \(i\)’s information is described by a \(\sigma\)-subfield \(F^i\) of \(F\), which is not necessarily generated
by a partition of $\Omega$. The results on existence and uniqueness of Cournot equilibrium, Theorems 1–4 of this paper, remain valid in this more general context. Their proofs follow very closely those presented here, but some additional assumptions are made, which are not needed when $\Omega$ is finite. In particular, it is assumed that the demand intercept $Q$ is bounded, and that the state-dependent inverse demand function, cost functions, and their first and second order derivatives, are bounded uniformly in $\omega$ on some sufficiently big interval $[0, M]$. ■
References


6 Appendix: The Proofs

Denote by $B(\Omega)$ the set of all non-negative real-valued functions on $\Omega$. The following definition of a partial order on $B(\Omega)$ will be needed in the sequel: if $g, h \in B(\Omega)$, $g \geq h$ (respectively, $g > h$) if and only if $g(\omega) \geq h(\omega)$ (respectively, $g(\omega) > h(\omega)$) for every $\omega \in \Omega$. Similarly, we will say that $g \geq h$ (respectively, $g > h$) on $A \subset \Omega$ if and only if $g(\omega) \geq h(\omega)$ (respectively, $g(\omega) > h(\omega)$) for every $\omega \in A$.

6.1 Proof of Theorem 1

6.1.1 Part I: The Case of $n = 2$

Suppose that $n = 2$. We will show first that for each $\omega \in \Omega$ the profit function $h_1^1(\cdot) \equiv h_1^1(\omega, \cdot)$ of firm 1 has decreasing differences in the first coordinate, that is, if $x_1 \geq x_2 \geq 0$ and $y_1 \geq y_2 \geq 0$, then

$$[h_\omega^1(x_1, y_2) - h_\omega^1(x_2, y_2)] - [h_\omega^1(x_1, y_1) - h_\omega^1(x_2, y_1)] \geq 0,$$

i.e.,

$$[x_1 P(\omega, x_1 + y_2) - x_2 P(\omega, x_2 + y_2)] - [x_1 P(\omega, x_1 + y_1) - x_2 P(\omega, x_2 + y_1)] \geq 0.$$  (16)

Since $P(\omega, \cdot)$ is continuously differentiable, this condition is equivalent to

$$\frac{\partial}{\partial y_2} [x_1 P(\omega, x_1 + y_2) - x_2 P(\omega, x_2 + y_2)] \leq 0,$$

or

$$x_1 P'(\omega, x_1 + y_2) - x_2 P'(\omega, x_2 + y_2) \leq 0,$$

for every $x_1 \geq x_2 \geq 0$ and $y_2 \geq 0$. This condition, in turn, is equivalent (since $P'(\omega, \cdot)$ is also continuously differentiable by (iv)) to

$$\frac{\partial}{\partial x_2} [x_2 P'(\omega, x_2 + y_2)] \leq 0,$$

or

$$x_2 P''(\omega, x_2 + y_2) + P'(\omega, x_2 + y_2) \leq 0,$$  (17)

for every $x_2 \geq 0$ and $y_2 \geq 0$. However, (17) is implied by conditions (i) and (iv) on $P$. 

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From (16) it follows that the expected profit function $H^1$ of firm 1 also has decreasing differences in the first coordinate: for every $(q^1, q^2), (\bar{q}^1, \bar{q}^2) \in B(\Omega, \Pi^1) \times B(\Omega, \Pi^2)$ such that $q^1 \geq \bar{q}^1$, $q^2 \geq \bar{q}^2$, 

$$[H^1(q^1, q^2) - H^1(\bar{q}^1, \bar{q}^2)] - [H^1(q^1, \bar{q}^2) - H^1(\bar{q}^1, \bar{q}^2)] \geq 0.$$ 

Similarly, the expected payoff function $H^2$ of firm 2 has decreasing differences in the second coordinate.

With the partial order $\geq$ on $B(\Omega, \Pi^1)$ and the pointwise convergence topology on it, for every $g, h \in B(\Omega, \Pi^1)$ with $g \geq h$ the interval $[g, h] \subset B(\Omega, \Pi^i)$ is a compact lattice. Now denote the constant function on $\Omega$ which is fixed at the level $Z$ (respectively, 0) by the same symbol, $Z$ (respectively, 0), and let strategy profiles of the firms be restricted to $S^1 \times S^2 \equiv [0, Z] \times [0, Z]$, a product of compact lattices. Since the state-dependent inverse demand and cost functions are continuous (by conditions (ii) and (iv)), then each function $H^i$ is continuous on $S^1 \times S^2$ in both coordinates.

Now reverse the order in $S^2$, i.e., replace the order “$\geq$” with “$\geq$” according to which $g \geq^! h$ if and only if $h \geq g$. Then both $H^1$ and $H^2$ exhibit increasing, rather than decreasing, differences. The reversal of order has no effect on continuity of $H^1$ and $H^2$. Since both $S^1$ and $S^2$ are compact lattices, Theorem 5 of [11] implies that there exists a Cournot equilibrium when strategy profiles of the firms are restricted to be in $S^1 \times S^2$.

Denote one such equilibrium by $(q^*_1, q^*_2)$. If $(q^1, q^2) \in B(\Omega, \Pi^1) \times B(\Omega, \Pi^2)$, notice that $H^1(q^1, q^*_2) \leq H^1(\text{min}(q^1, Z), q^*_2)$ and $H^2(q^*_1, q^2) \leq H^2(q^*_1, \text{min}(q^2, Z))$ as follows from the definition of $Z$ in (iii) and from conditions (i) and (ii). Therefore

$$H^1(q^*_1, q^*_2) \geq H^1(\text{min}(q^1, Z), q^*_2) \geq H^1(q^1, q^*_2)$$

and

$$H^2(q^*_1, q^*_2) \geq H^2(\text{min}(q^2, Z)) \geq H^2(q^*_1, q^2),$$

since $(q^*_1, q^*_2)$ is a Cournot equilibrium when the strategy profiles of the firms are restricted to $S^1 \times S^2$. But these inequalities show that $(q^*_1, q^*_2)$ is actually a Cournot equilibrium without any restrictions on strategies.  

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One more thing needs to be verified before applying this theorem, namely that $H^1$ is supermodular in $q^1$ for fixed $q^2$, i.e., for every $q^1, q^1' \in S^1$ and $q^2 \in S^2$, \(H^1(q^1, q^2) + H^1(q^1', q^2) \leq H^1(\text{max}(q^1, q^1'), q^2) + H^1(\text{min}(q^1, q^1'), q^2)\), and similarly for $H^2$. However, it can be easily checked that this inequality actually holds as equality.
6.1.2 Part II: The Case of Convex Cost Functions

Suppose that $c^i(\omega, \cdot)$ is convex for every $i \in N$ and $\omega \in \Omega$. The proof is a direct consequence of the Nash existence theorem.

First, for each $\omega \in \Omega$, $h^i_\omega(\cdot) \equiv h^i(\omega, \cdot)$ is concave in strategies of firm $i$. Indeed, the second derivative of $q^i(\omega)P(\omega, Q(\omega))$ with respect to $q^i(\omega)$ is equal to $q^i(\omega)P''(\omega, Q(\omega)) + 2P'(\omega, Q(\omega))$, which is non-positive as follows from (i) and (iv). Thus, $q^i(\omega)P(\omega, Q(\omega))$ is concave in $q^i(\omega)$, and from convexity of $c^i(\omega, \cdot)$ it follows that $h^i_\omega(q(\omega)) = q^i(\omega)P(\omega, Q(\omega)) - c^i(\omega, q^i(\omega))$ is also concave in $q^i(\omega)$.

The expected profit function $H^i$ clearly inherits concavity in $q^i$ from $h^i_\omega$.

Second, following notations of Part I, restrict the strategy set of each firm $i$ to the compact $S^i = [0, Z]$. As in Part I, $H^i$ is continuous in all coordinates simultaneously on the compact cube $[0, Z]^N$. Thus, all ingredients for the existence of Nash equilibrium are in place, with the above restriction of strategies. However, the restricted equilibrium is an equilibrium in the unrestricted oligopoly as well, which can be shown again exactly as in Part I.

6.2 Proof of Theorem 2

Since all conditions of Theorem 1 are satisfied, the oligopoly has at least one Cournot equilibrium. We will show that it is unique.

Let $q_*$ be a Cournot equilibrium. The strategies in $q_*$ are clearly bounded by $Z$ (beyond which all prices are non-positive by (iii)). Now pick a firm $i$. Since

$$E\left( q^i(\cdot) P\left( \cdot, \sum_{j \neq i} q^j_*(\cdot) + q^i(\cdot) \right) - c^i(\cdot, q^i(\cdot)) \mid \Pi^i(\omega) \right)$$

is maximized (and in particular locally maximized) at $q^i = q^i_*$ for every $\omega \in \Omega$, the Kuhn-Tucker conditions are satisfied:

$$E\left( q^i_*(\cdot) P'(\cdot, Q_*(\cdot)) + P(\cdot, Q_*(\cdot)) - (c^i)'(\cdot, q^i_*(\cdot)) \mid \Pi^i(\omega) \right) = 0$$

for every $\omega$ in which $q^i_* > 0$, and

$$E\left( q^i_*(\cdot) P'(\cdot, Q_*(\cdot)) + P(\cdot, Q_*(\cdot)) - (c^i)'(\cdot, q^i_*(\cdot)) \mid \Pi^i(\omega) \right) \leq 0$$

for every $\omega$ in which $q^i_* = 0$. 

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Note that for each $\omega \in \Omega$ the function

$$F(q, Q) = qP'(\omega, Q) + P(\omega, Q) - (c')'(\omega, q)$$

is decreasing in $q$ and non-increasing in $Q$ when $q \leq Q$. Indeed, $\frac{\partial F}{\partial q} = P'(\omega, Q) - (c')''(\omega, q) < 0$ since $P$ is decreasing and $c$ is convex by (i') and (ii'), and $\frac{\partial F}{\partial Q} = qP''(\omega, Q) + P'(\omega, Q) \leq 0$ as follows from (i') and (iv). Now suppose that $q_*$ and $q_{**}$ are two Cournot equilibria. That $F$ is decreasing in $q$ and non-increasing in $Q$ implies that one cannot have

$$(q_*^i, Q_*) < (q_{**}^i, Q_{**}) \text{ or } (q_*^i, Q_*) > (q_{**}^i, Q_{**})$$

(inequality in both coordinates and strict inequality in the first coordinate) on any atom $\Pi^i(\omega)$ of $\Pi^i$. This is because otherwise conditions (19) and (20) would not hold simultaneously for $\max((q_*^i, Q_*) , (q_{**}^i, Q_{**}))$. To summarize, any firm’s equilibrium strategy and the aggregate output in equilibrium cannot move in the same direction:

$$(q_*^i, Q_*) \not< (q_{**}^i, Q_{**}) \text{ and } (q_*^i, Q_*) \not> (q_{**}^i, Q_{**})$$

(21)

on any element of $\Pi^i$.

We will next show that every Cournot equilibrium satisfies the equal treatment property, i.e., that strategies of firms of the same type are equal. Indeed, if $q_*$ is a Cournot equilibrium, and $q_*^i \neq q_*^j$ where $i$ and $j$ are firms of the same type, then consider an $n$-tuple $q_{**}$ obtained from $q_*$ by interchanging $i$ and $j$. Clearly, $q_{**}$ is also a Cournot equilibrium. However, if $\Pi^i(\omega) \in \Pi^i$ is a set on which w.l.o.g. $q_*^i > q_*^j = q_{**}^i$, then the obvious fact that $Q_* = Q_{**}$ leads to contradiction with (21). Thus, the equal treatment property holds in any Cournot equilibrium.

Now suppose that $q_*$ and $q_{**}$ are Cournot equilibria in the oligopoly. We will show that they coincide. Due to the equal treatment property, $Q_*(\omega) = |K| q_*^1(\omega) + |M| q_*^2(\omega)$, and it will suffice to establish that $q_*^i = q_{**}^i$ for $i = 1, 2$. If $q_*^2$ and $q_{**}^2$ are not equal everywhere, then there exists $\omega \in \Omega$ such that w.l.o.g.

$$q_*^2 > q_{**}^2 \text{ on } \Pi^2(\omega).$$

(22)

Consequently,

$$q_*^1 \leq q_{**}^1 \text{ on } \Pi^2(\omega).$$

(23)
Indeed, if (23) does not hold, there is \( \omega' \in \Pi^2(\omega) \) with \( q_i^1 > q_{**}^1 \) on \( \Pi^1(\omega') \). But \( \Pi^1(\omega') \subseteq \Pi^2(\omega) \) since the information partition of 1 is finer than that of 2. Thus, from (22), also \( Q_* > Q_{**} \) on \( \Pi^1(\omega') \), contradicting (21).

We now claim that

\[
Q_* \geq Q_{**} \text{ on } \Pi^2(\omega). \tag{24}
\]

Indeed, both \( Q_* \) and \( Q_{**} \) are measurable with respect to the information partition of the more informed firm 1, and thus, if (24) does not hold, there is \( \omega'' \in \Pi^2(\omega) \) with

\[
Q_* < Q_{**} \text{ on } \Pi^1(\omega''). \tag{25}
\]

Strict inequality in (23) on \( \Pi^1(\omega'') \subseteq \Pi^2(\omega) \) together with (25) would contradict (21), and thus \( q_i^1 = q_{**}^1 \) on \( \Pi^1(\omega'') \). But then \( Q_* > Q_{**} \) on \( \Pi^1(\omega'') \) because of (22), contrary to the choice of \( \omega'' \). Thus (24) must hold.

But now (22) and (24) contradict (21). Thus, strategies \( q_i^2 \) and \( q_{**}^2 \) must coincide almost everywhere. Now, if \( q_i^1 \) differs from \( q_{**}^1 \) on \( \Pi^1(\omega) \) for some \( \omega \in \Omega \), and w.l.o.g. \( q_i^1 > q_{**}^1 \) on \( \Pi^1(\omega) \), then \( Q_* > Q_{**} \) on \( \Pi^1(\omega) \) since \( q_i^2 = q_{**}^2 \), contradicting (21) again.

We conclude that \( q_i^1 = q_{**}^1 \) as well.

\[ \blacksquare \]

### 6.3 Proof of Theorem 3

First, restrict strategy sets of each firm \( i \) to be \( S^i = [0, \overline{q}^i] \). Note that for every strategy profile \( q \in S^1 \times \ldots \times S^n \), \( Q \leq \sum_{i=1}^n \overline{q}^i \leq \overline{Q} \). Hence, strategy profiles in \( S^1 \times \ldots \times S^n \) have exactly the same properties as if condition (iv') held on \( \mathbb{R}_+ \) (i.e., as if (iv') had the original form (iv)). Thus, just as in the proof of Theorem 1 (replacing \( S^i = [0, Z] \) with \( S^i = [0, \overline{q}^i] \)), there is a Cournot equilibrium \( q_* \in S^1 \times \ldots \times S^n \) in the oligopoly, provided all unilateral deviations of \( i \) considered in (1) are in \( S^i \).

To show that \( q_* \) is a Cournot equilibrium in the unrestricted oligopoly as well, we now prove that unilateral deviations of \( i \) to strategies outside \( S^i \) are not profitable. Indeed, if \( q^i \) is \( i \)th strategy which is not in \( S^i \), then by (11)

\[
H^i(q_* | q^i) \leq H^i(q_* | \min(q^i, \overline{q}^i)),
\]

and by (1)

\[
H^i(q_* | \min(q^i, \overline{q}^i)) \leq H^i(q_*),
\]

since \( \min(q^i, \overline{q}^i) \in S^i \). This proves via (1) that \( q_* \) is indeed a Cournot equilibrium of the oligopoly without restriction on strategies. \[ \blacksquare \]
6.4 Proof of Theorem 4

Note that if \( q^* \) is a Cournot equilibrium, then

\[ Q^* < \bar{Q}. \tag{26} \]

Indeed, if this is not case, consider an \( \omega \in \Omega \) such that \( Q^* \geq \bar{Q} \) on \( \Pi^1(\omega) \) (such an \( \omega \) exists since both \( Q^* \) and \( \bar{Q} \) are measurable with respect to the finest of all information partitions, \( \Pi^1 \)). If there exists a firm \( i \in K \) with \( q^*_i > 0 \) on \( \Pi^1(\omega) \), then \( i \) would benefit by switching its output to zero on \( \Pi^1(\omega) \) and saving its costs, contradicting (2). And if for all \( i \in K \) \( q^*_i = 0 \) on \( \Pi^1(\omega) \), then \( \sum_{j \in M} q^*_j = Q^* \geq \bar{Q} \) on \( \Pi^1(\omega) \). But since both \( \sum_{j \in M} q^*_j \) and \( \bar{Q} \) are measurable with respect to \( \Pi^2 \), there exists an \( \omega' \in \Omega \) such that \( \Pi^1(\omega) \subset \Pi^2(\omega') \) and \( \sum_{j \in M} q^*_j \geq \bar{Q} (> 0) \) on \( \Pi^2(\omega') \). Accordingly, there exists a firm \( j \in M \) with \( q^*_j > 0 \) on \( \Pi^2(\omega') \), and just as before this means that \( j \) has a profitable deviation from \( q^* \) on \( \Pi^2(\omega') \), contradicting (2). We conclude that (26) holds.

But if \( q^* \) and \( q^{**} \) are two Cournot equilibria, it follows from (26) that \( q^* \), \( q^{**} \), and all strategy profiles close to them\(^{15}\) have exactly the same properties as if the differentiability condition in (iv') held for all \( Q \geq 0 \) (i.e., as if (iv') had the original form (iv)). We can therefore show that \( q^* \) and \( q^{**} \) coincide, just as in the proof of Theorem 2, using the first-order conditions derived from maximization of (18). 

\(^{15}\)What we have in mind are strategy profiles that constitute, at each state of nature, small unilateral deviations from \( q^* \) or \( q^{**} \).