# On subgame consistency of the Shapley-Shubik power index

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**Abstract**

We present a new axiomatization of the Shapley-Shubik power index based on three axioms. The central axiom – **occasional subgame-consistency on average** (**OSCoA**) – requires the power of a player to coincide with the average of his power in one-player-out subgames, for *just one* game $v$ on any given support (which must be essential for the game). The choice of $v$ may be player-dependent but $v$ must have no veto players. The other two axioms are the standard **Transfer** and **Dummy**. We also formulate some stronger variants of **OSCoA** that do not explicitly require the support of such a game $v$ to be essential.

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## 1 Introduction

Felsenthal and Machover (1998) persuasively argue that, in quantifying a priori voting power of individual voters under a decision rule describable by a simple (voting) game, a distinction should be made between two notions of power: *I-power* and *P-power*. They view I-power as the "voter's potential *influence* over the outcome of divisions of

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the decision-making body: whether proposed bills are adopted or blocked," whereas P-power is the "voter's expected relative share in a fixed *prize* available to the winning coalition under a decision rule."[1] They also argue that, among the two best-known and most popular voting power indices – the Banzhaf power index (henceforth, $\boldsymbol{BPI}$)[2] and the Shapley-Shubik power index (henceforth, $\boldsymbol{SSPI}$)[3] – the former is better suited to measure I-power, and the latter P-power.

The suitability of the $\boldsymbol{BPI}$ to quantify I-power is intuitively evident: it measures the probability that player (voter) $i$'s Yes vote swings the voting outcome from No to Yes when he joins a random set of other Yes voters, under the uniform distribution over such sets.[4] On the other hand, the $\boldsymbol{SSPI}$ seems to be better tailored to measure P-power: it is an import of the Shapley (1953) value, a solution concept defined for cooperative TU games and designed to predict (or suggest) the way in which the worth of the grand coalition in the game is to be shared between the players; in the context of voting games, this solution concept suggests a division of what may be regarded as some tangible "prize of power" (with total worth normalized to 1).

There are some grounds to doubt that the $\boldsymbol{SSPI}$ is always a conceptually adequate measure of voting power, however.[5] Felsenthal and Machover (1998, 2005) highlight several possible shortcomings of the $\boldsymbol{SSPI}$, including the following two. For one, as pointed out already, the $\boldsymbol{SSPI}$'s origin is the Shapley value, which was

[1] The quotations are from Felsenthal and Machover (2005, Section 3).

[2] As is often done in the literature, the term "Banzhaf power index" is used for brevity, although the origin of the $\boldsymbol{BPI}$ lies in multiple works (Penrose (1946), Banzhaf (1965, 1966, 1968), Coleman (1971)).

[3] Introduced in Shapley and Shubik (1954).

[4] Felsenthal and Machover (1998) call this version of $\boldsymbol{BPI}$ "the Banzhaf measure." The uniform distribution over the sets of Yes voters represents a state of complete *a priori* ignorance regarding the voters' preferences, which seems the most appropriate when attempting to quantify *a priori* the influence of individual voters.

[5] In contrast to the $\boldsymbol{BPI}$, even the usual probabilistic interpretation of the $\boldsymbol{SSPI}$ – whereby a player's power is his probability to be pivotal in a random (and uniformly distributed) ordering of all voters, who are gradually persuaded to vote Yes – may be objectionable as a model underlying power measurement. That is, for example, because the voting process may be secret or simultaneous, so pivotality is not well-defined; not everyone may be ultimately persuaded to vote Yes; and, if the voting is indeed observable, players have an incentive to position themselves strategically in an ordering so as to become pivotal. This underscores the need for an axiomatic justification for the $\boldsymbol{SSPI}$, as is repeatedly stressed by Felsenthal and Machover (1998, Section 6).

designed to be a sharing rule of a concrete amount of utility that is collectively attainable by the grand coalition. But, in the context of voting games, a winning coalition of Yes voters may not derive a tangible benefit – or that benefit may be in the form of public rather than private good – or "winning" may be just a label attached to subsets of players that suffice to approve a proposal. In all these cases, measuring P-power may not be feasible.

If the P-power, is for any reason, not relevant or not measurable, can the $\boldsymbol{SSPI}$ at least be viewed as a quantification of the I-power, as is the case with the $\boldsymbol{BPI}$? This takes us to another point raised by Felsenthal and Machover (1998, 2005): the Shapley value is efficient, which means that the $\boldsymbol{SSPI}$ for different players must sum up to 1, making the $\boldsymbol{SSPI}$ *de facto* – and perhaps artificially – normalized. But this calls into question the validity of using $\boldsymbol{SSPI}$ to compare the same player's influence across different games – that is, the $\boldsymbol{SSPI}$'s capacity to capture *absolute*, rather than relative, power of players may be in doubt. Axiomatic derivations of the $\boldsymbol{SSPI}$ (starting with Dubey (1975)) tended to a priori impose efficiency as one of the axioms,[6] thereby not helping to alleviate the above concern.

This work offers a new axiomatization of the $\boldsymbol{SSPI}$ that attempts to stay as distant as possible from explicitly or implicitly assuming efficiency of the power index. Thus, the perceived normalization of the total power to unity will be a strictly ex-post, implied feature of the index. Our central axiom, which we begin to describe next, requires at least an occasional *consistency* between a player's power in a simple game and the attribution of power to this player in subgames, in a way that may reasonably be expected from an I-power index.

Cooperative game theory has devoted a fair amount of attention to consistency of various solution concepts – that is, to their invariance under a reduction of the player set for a particular specification of the game in which only the remaining players are active.[7] Different notions of that residual, or *reduced*, game upon which the

[6] Among the works that characterize the $\boldsymbol{SSPI}$ without the efficiency axiom are Laruelle and Valenciano (2001), Einy and Haimanko (2011) and Chen et al. (2024), which are touched upon later in the introduction.

[7] Sobolev (1975) pioneered this approach, axiomatizing the prenucleolus. Davis and Maschler's (1964) definition of a "reduced" game was later used in Peleg's (1986) axiomatizations of the core and the prekernel. Another notion of reduced game was used by Hart and Mas-Colell (1989) to

consistency requirement is predicated characterize different solution concepts in the TU setting. However, those reduced games share a common feature: in determining the new characteristic function, any coalition drawn from the remaining players makes a utility transfer to (or receives a transfer from) subsets of players outside the reduced game.[8] In particular, the existing reduced game concepts do not seem to be applicable to simple games, which in most applications do not attribute to winning coalitions a tangible divisible utility but merely label coalitions that are sufficient to approve a proposal under some underlying decision rule. In fact, technically "reducing" a simple game according to the usual notions is bound to produce a game that is not even simple, making little sense if a simple game is viewed just as a labeling device.

Our approach to consistency is built around *subgames*, obtained by removing a single player from the game's support.[9] For a *simple* game, considering a subgame obtained by removing one *non-veto* player from the support is meaningful because the resulting subgame remains simple. This subgame represents the conceivable scenario in which one player becomes inactive, and the minimal winning coalitions become (by default) those that could push a proposal through in the original game without the now-inactive player's vote. However, it would be too simplistic to think that a player $i$ retains the same influence after some $j \neq i$ is inactivated. Evidently, $i$'s influence is affected by two opposing forces: $j$'s inactivation may reduce the set of minimal winning coalitions containing $i$ (making $i$ less influential), but the set of minimal winning coalitions excluding $i$ may shrink as well (boosting $i$'s relative influence). These forces, however, do not generally balance each other. At one extreme, *only* the minimal winning coalitions with $i$ are affected by $j$'s inactivation, clearly weakening

characterize the Shapley value.

[8]Such transfers can be seen as arising from bargaining between a given (sub)coalition of the remaining players and (sub)coalitions of outside players over the extent of intergroup cooperation and the corresponding compensation based on some solution concept (which is given *ex ante* or determined *ex post*). See, e.g., Hart and Mas-Colell (1989, Section 4) for a survey of some prominent reduced game notions, and Pérez and Sun (2021) for an axiomatic treatment of reduced game mappings.

[9]Unlike the standard notion of a subgame (see, e.g., Definition 3.2.2 in Peleg and Sudhölter (2007)), our approach *retains in the player set* the player removed from the game's support, thereby turning him into a null player. In viewing subgames this way, we follow Béal et al. (2016), who refer to this procedure as the *nullification* of a player.

$i$; and at another extreme, $j$'s switch-off removes *only* minimal winning coalitions without $i$, thereby strengthening $i$.

Although removing a player may significantly affect the decision structure in the game, it turns out that if a simple game $v$ has no veto players (and hence inactivation of *any* single player leaves the game simple), then according to the $\boldsymbol{SSPI}$ the power of any given player $i$ in $v$ equals the *average* of $i$'s power in all *one-player-out* subgames $v_{-j}$ (obtained by removing a single player $j$ from $v$'s support).[10] This fact is a simple corollary of a known recursive formula for the Shapley value in general TU games, which expresses any player's value as the average of his values in one-player-out subgames and his marginal contribution to the grand coalition.[11] In the context of simple games without veto players, however, this representation takes a whole new meaning: any player $i$'s power, although generally not consistent with the power that the $\boldsymbol{SSPI}$ attributes to him in a particular subgame, is **subgame-consistent on average (**henceforth, **SCoA)**, being equal to the average of $i$'s power in all one-player-out subgames $v_{-j}$.

The property of **SCoA** appears to be quite appealing in the context of a general I-power index: the knowledge of winning coalitions in the subgames $v_{-j}$ for all $j$ fully accounts for the decision structure in the original game $v$, and so **SCoA** seems adequate as a minimal[12] consistency requirement. Alternatively, **SCoA** can be viewed as a simple *robustness* feature of a power index in relation to a hypothetical scenario in which one (and only one) player randomly drops out of the game $v$'s support (by becoming inactive, i.e., abstaining from the vote), with equal probability for any player in the support to be chosen for inactivity. **SCoA** then means that the power measured in $v$ is equal – in expectation – to the power measured in the one-player-out subgame created by a randomly inactive player.

We elevate the property of **SCoA** to the status of an *axiom* for general power indices, but only need its weak form. Specifically, a power index is said to be **occasionally subgame-consistent on average (**henceforth, **OSCoA)** if, for each

[10]This property of the $\boldsymbol{SSPI}$ has been partially observed, in a much more limited context of specific weighted majority games, by Gafni et al. (2021, Lemma 3.1).

[11]This formula goes back (at least) to Maschler and Owen (1989) and Hart and Mas-Colell (1989).

[12]As noted earlier, full consistency of a power index with subgame power is unachievable, making consistency *on average* the next best desirable property.

subset $T$ of players, the power of a player in $T$ is the average of his power in one-player-out subgames for *at least one* (but not every!) game $v$. Such a game $v$ is required to have $T$ as its essential support[13] and to be without veto players, but the choice of the game $v$ may be player-dependent. Our main result is that **OSCoA** – supplemented by the standard **Transfer (T)** and **Dummy (D)** axioms[14] – uniquely characterizes a power index as the $\boldsymbol{SSPI}$.

In addition to avoiding the usual efficiency axiom, our axiomatization of the $\boldsymbol{SSPI}$ does not involve an explicit symmetry or anonymity assumption.[15] In particular, the **OSCoA** axiom does not even require the "occasional" game $v$ for which the subgame consistency holds on average to be the same for every player in the support $T$. There are also no restrictions on how the choice of $v$ depends $T$ (although the averaging principle behind our consistency notion does contain some equal-treatment aspect, in that the one-player-out subgames are given the same weight in averages).

It is worth noting that previous axiomatizations of the $\boldsymbol{SSPI}$ that avoided the efficiency axioms tended to explicitly impose symmetry or some explicit symmetry-related features. The recent work of Chen et al. (2024) employs – like us – just three axioms in its characterization of the $\boldsymbol{SSPI}$: two of them, **T** and **D,** are the same as in our setting, and the third, cross-invariance, requires all games that are *symmetric* within the same support to be treated *identically*.[16] In the $\boldsymbol{SSPI}$ characterizations of Laruelle and Valenciano (2001) and Einy and Haimanko (2011), symmetry is an explicit axiom alongside two types of "gain-loss" assumptions that replace efficiency.[17]

[13]The assumption that the support is essential is extensively discussed in Section 4.1, and its necessity is demonstrated in Remark 7. Some sufficient conditions for games used in **OSCoA** to have an essential support are provided in Remarks 4 and 5, and these are incorporated into the axiomatization of the $\boldsymbol{SSPI}$ in Corollary 1.

[14]These axioms are due to Dubey (1975) and Dubey and Shapley (1979).

[15]Since we do not *a priori* assume symmetry or non-negativity of the power index, it is not a *semivalue* in the sense of Einy (1987) and Dubey et al. (1981). Consequently, the characterization results for semivalues provided in these works cannot be applied in our setting.

[16]Condition in a somewhat similar spirit is used in an older axiomatization of the $\boldsymbol{SSPI}$, due to Blair and McLean (1990), in the context of modeling the players' preferences over simple games. In that work, however, that condition is supplemented by the assumption of symmetry in each player's evaluation of his position in unanimity games.

[17]In Laruelle and Valenciano (2001), the central efficiency-replacing axiom is the **total gain-loss balance**. It requires that, following the deletion of a minimal winning coalition from the game's

This paper is organized as follows. Section 2 recalls the basic definitions pertaining to games, simple games, and power indices. Section 3 introduces and motivates the concept of subgame consistency on average. Section 4 states the axioms, and Section 5 presents the main results: Theorem 1 provides our axiomatization of the $\boldsymbol{SSPI}$, and Corollary 1 incorporates into it some strengthened versions of **OSCoA** that do not involve the essential support assumption. Section 6 concludes, and the Appendix contains most of the proofs.

# 2 Preliminaries

## 2.1 TU games and simple games

Let $N = \{1, 2, ..., n\}$, $n \geq 2$, be the *player set*, which will be fixed throughout. Denote the collection of all *coalitions* (subsets of $N$) by $2^N$, and the empty coalition by $\varnothing$. Then a TU game on $N$ (or simply a *game*) is a map $v : 2^N \to \mathbb{R}$ with $v(\varnothing) = 0$. The space of all games is denoted by $\mathcal{G}$. A coalition $T \subseteq N$ is called a *carrier* of $v \in \mathcal{G}$ if $v(S) = v(S \cap T)$ for any $S \subseteq N$; $\boldsymbol{T}(v)$ will stand for the *support* of $v$, defined as its minimal carrier. Given $v \in \mathcal{G}$ and $i \in \boldsymbol{T}(v)$, we denote by $v_{-i}$ the *one-player-out* subgame of $v$ obtained by "removing" $i$ from $v$'s support, i.e., letting $v_{-i}(S) = v(S \backslash \{i\})$ for any $S \subseteq N$. Player $i$ is called a *dummy* in $v$ if $v(S \cup \{i\}) = v(S) + v(\{i\})$ for every $S \subseteq N \backslash \{i\}$; if, in addition, $v(\{i\}) = 0$ then $i$ is a *null player*. Note that $i \in \boldsymbol{T}(v)$ becomes a null player in the game $v_{-i}$.[18]

The concept of a simple (voting) game is embedded in the framework of TU games. The domain $\mathcal{SG} \subset \mathcal{G}$ of *simple games* on player (voter) set $N$ consists of all $v \in \mathcal{G}$ such that

(i) $v(S) \in \{0, 1\}$ for all $S \subset N$;

(ii) $v(N) = 1$;

(iii) $v$ is *monotonic*, i.e., if $S \subset T$ then $v(S) \leq v(T)$.

support, the total loss in power for the players in that coalition equals the total gain in power for the players in its complement. In Einy and Haimanko (2011), efficiency is replaced by the **gain-loss** axiom, which requires that a gain in power for one player entails *some* (not necessarily equivalent) loss in power for another player when the game changes.

[18] Using the terminology of Béal et al. (2016), the one-player-out subgame $v_{-i}$ is obtained from $v$ by *nullifying* player $i$.

A coalition $S$ – which may now be interpreted as the set of players who cast a Yes vote – is said to be *winning* in $v \in \mathcal{SG}$ if $v(S) = 1$, and *losing* otherwise. The set of simple games will be denoted by $\mathcal{SG}$. For any $v \in \mathcal{SG}$, denote by $\mathbb{W}_{\min}(v)$ the set of its minimal winning coalitions, and by $\mathbb{W}(v)$ the set of all winning coalitions that are contained in $\boldsymbol{T}(v)$. Given a non-empty set $T \subseteq N$, denote by $u_T \in \mathcal{SG}$ the *unanimity game* with support $T$ (i.e., $\mathbb{W}_{\min}(u_T) = \{T\}$).

We call $i \in N$ a *veto player* in a game $v \in \mathcal{SG}$ if $v(N\backslash\{i\}) = 0$, i.e., no coalition can win without $i$. Denote by $\mathcal{SG}_{no-veto}$ the subset of $\mathcal{SG}$ consisting of simple games in which all players are non-veto. Given $v \in \mathcal{SG}_{no-veto}$ and $i \in \boldsymbol{T}(v)$, note that the grand coalition $N$ remains winning in the one-player-out subgame $v_{-i}$, i.e. $v_{-i}(N) = 1$, and hence the removal of $i$ does not affect the simple game's status: $v_{-i} \in \mathcal{SG}$.

Finally, for any $v, w \in \mathcal{SG}$ define $v \vee w, v \wedge w \in \mathcal{SG}$ by:

$$(v \vee w)(S) = \max\{v(S), w(S)\},$$

$$(v \wedge w)(S) = \min\{v(S), w(S)\}$$

for all $S \subseteq N$. (It is evident that $\mathcal{SG}$ is closed under operations $\vee, \wedge$.) Thus, a coalition is winning in $v \vee w$ if and only if it is winning in at least one of $v$ or $w$, and it is winning in $v \wedge w$ if and only if it is winning in both $v$ and $w$. Note that any game $v \in \mathcal{SG}$ can be represented as

$$v = u_{T_1} \vee u_{T_2} \vee ... \vee u_{T_k},$$

where $T_1, T_2, ..., T_k \subseteq \boldsymbol{T}(v)$ is a list of the elements of $\mathbb{W}_{\min}(v)$.

## 2.2 Power indices

A *power index* is a mapping $\phi : \mathcal{SG} \to \mathbb{R}^n$. For each $i \in N$ and $v \in \mathcal{SG}$, the $i^{\text{th}}$ coordinate of $\phi(v) \in \mathbb{R}^n$, $\phi(v)(i)$, is interpreted as the voting power of player $i$ in the game $v$. The Banzhaf power index (henceforth $\boldsymbol{BPI}$) and the Shapley-Shubik power index (henceforth $\boldsymbol{SSPI}$) are among the best known power indices. The $\boldsymbol{BPI}$ is given for each $v \in \mathcal{SG}$ and $i \in N$ by

$$\boldsymbol{BPI}(v)(i) = \sum_{S \subseteq N\backslash\{i\}} \frac{1}{2^{n-1}} [v(S \cup \{i\}) - v(S)]. \tag{1}$$

Here, for each $i \in N$, $\boldsymbol{BPI}(v)(i)$ is the probability that adding player $i$'s Yes vote to a random set (coalition) of all other Yes voters, drawn w.r.t. the uniform distribution over the subsets of $N\backslash\{i\}$, swings the voting outcome (namely, that the random coalition $S \subseteq N\backslash\{i\}$ switches from losing to winning when joined by $i$).

The $\boldsymbol{SSPI}$ is given for each $v \in \mathcal{SG}$ and $i \in N$ by

$$\boldsymbol{SSPI}(v)(i) = \sum_{S\subseteq N\backslash\{i\}} \frac{|S|!\,(n-|S|-1)!}{n!}\left[v(S\cup\{i\}) - v(S)\right]. \tag{2}$$

Thus, the formula for $\boldsymbol{SSPI}$ is a modification of (1), wherein the original coefficient $\frac{1}{2^{n-1}}$ in each summand – the probability of the coalition of all other Yes voters being $S$ under the uniform distribution – is replaced by $\frac{|S|!(n-|S|-1)!}{n!}$. For each $i \in N$, $\boldsymbol{SSPI}(v)(i)$ may be thought of as arising from a simple model of sequential voting, in which all players sequentially join the Yes vote according to a random and uniformly distributed ordering of $N$. Then, $\boldsymbol{SSPI}(v)(i)$ in (2) is the probability that player $i$ is pivotal in a random ordering of $N$ – namely, the chance that the coalition of players preceding $i$ in joining the Yes vote is losing, but becomes winning once $i$ is counted.

# 3 Subgame consistency on average

The Shapley (1953) value $\boldsymbol{SH}$ on the space $\mathcal{G}$ of all games is given by the same formula – (2) – as the $\boldsymbol{SSPI}$ but for a general $v \in \mathcal{G}$, and among its distinctive characteristics is the efficiency property, namely that $\sum_{i\in N}\boldsymbol{SH}(v)(i) = v(N)$ holds for any $v \in \mathcal{G}$. The $\boldsymbol{SSPI}$ inherits this property, satisfying the equality

$$\sum_{i\in N}\boldsymbol{SSPI}(v)(i) = 1 \tag{3}$$

for each $v \in \mathcal{SG}$. The postulate of efficiency in axiomatic characterizations of the $\boldsymbol{SSPI}$ (and the perceived implication of "normalization" associated with (3)) is, in part, what casts the $\boldsymbol{SSPI}$ more as a measure of P-power – pertaining to the allocation of the tangible spoils enjoyed by the winning coalition – and less as a measure of I-power, which concerns the influence of voters on the voting outcome.

To distance ourselves from the notion of P-power, and, ultimately, to put the $\boldsymbol{SSPI}$ in a different light, instead of efficiency we focus on some aspects of consistency in measuring voting power in a game and its one-player-out subgames, which is something that I-power may in principle exhibit. Consider a power index $\phi$, a game $v \in \mathcal{SG}_{no-veto}$ with support $T := \boldsymbol{T}(v)$, and $i \in T$. When a player $j \neq i$ is removed from $v$'s support,[19] there is little reason to expect $i$'s influence in the ensuing subgame $v_{-j}$ to remain the same as in the original $v$, i.e., to have

$$\phi(v)(i) = \phi(v_{-j})(i). \tag{4}$$

That is because – as argued in the introduction – $j$'s removal from the support potentially changes the decision structure in the game, conceivably adding to, or subtracting from, the influence held by each of the remaining players in $v$. However, when the entire collection $\{v_{-j}\}_{j \in T}$ of one-player-out subgames of $v$ is taken into account, the full complexity of $\mathbb{W}(v)$ is brought back, and one may view as desirable a generalized version of (4), whereby $i$'s power in one-player-out subgames of $v$ coincides *on average* with his power in $v$ :

$$\phi(v)(i) = \frac{1}{|T|} \sum_{j \in T} \phi(v_{-j})(i). \tag{5}$$

According to (5), the $\phi$-measured power in $v$ is *consistent* – on average – with the power that $\phi$ quantifies in one-player-out subgames of $v$. For an alternative interpretation, consider a hypothetical scenario in which one (and only one) player is randomly dropped out of the game $v$'s support $T$ (i.e., the player becomes inactive, abstaining from vote), with equal probability for any player in $T$ to be chosen for inactivity. Then (5) means that the $\phi$-based power measurement is – in expectation – *robust* to the introduction of random inactivity of one player: the $\phi$-measured power in $v$ coincides with its expectation in a random one-player-out subgame of $v$ that is created in the above scenario.

**Definition 1**. A power index $\phi$ has the property of *subgame consistency on average* (henceforth, **SCoA**) if (5) holds for every $v \in \mathcal{SG}_{no-veto}$ with support $T = \boldsymbol{T}(v)$ and every $i \in T$.

In what follows we verify that $\boldsymbol{SSPI}$ has the **SCoA** property, but $\boldsymbol{BPI}$ does not.

**Proposition 1**. The $\boldsymbol{SSPI}$ has the **SCoA** property.

[19] Since $v \in \mathcal{SG}_{no-veto}$, $|T| \geq 2$ and so there exists $j \in T$ who is not $i$.

**Proof**. See the Appendix.

**Remark 1**. The $\boldsymbol{BPI}$ does *not* have the **SCoA** property even in three-player simple games (for two-player games, it coincides with the $\boldsymbol{SSPI}$ which has the **SCoA** property). When $n = 3$, consider for example the game $v = u_{\{1,2\}} \vee u_{\{2,3\}} \vee u_{\{1,3\}}$. For this $v \in \mathcal{SG}_{no-veto}$, we have $v_{-1} = u_{\{2,3\}}$, $v_{-2} = u_{\{1,3\}}$, and $v_{-3} = u_{\{1,2\}}$, and thus

$$\begin{aligned}\boldsymbol{BPI}(v) &= \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \\ &\neq \frac{1}{3}\left[\left(0, \frac{1}{2}, \frac{1}{2}\right) + \left(\frac{1}{2}, 0, \frac{1}{2}\right) + \left(\frac{1}{2}, \frac{1}{2}, 0\right)\right] \\ &= \frac{1}{3}\left[\boldsymbol{BPI}(v_{-1}) + \boldsymbol{BPI}(v_{-2}) + \boldsymbol{BPI}(v_{-3})\right].\end{aligned}$$

In fact, a stronger claim is valid, as can be easily checked:

$$\boldsymbol{BPI}(v)(i) \neq \frac{1}{3}\left[\boldsymbol{BPI}(v_{-1})(i) + \boldsymbol{BPI}(v_{-2})(i) + \boldsymbol{BPI}(v_{-3})(i)\right]$$

for *every* $v \in \mathcal{SG}_{no-veto}$ with $\boldsymbol{T}(v) = N = \{1, 2, 3\}$ and *every* $i \in N$.

**Remark 2**. The restriction made in Definition 1 to checking (5) in the **SCoA** property only for games without veto players – i.e., games in the domain $\mathcal{SG}_{no-veto}$ – is not merely due to the fact that a one-player-out subgame $v_{-i}$ becomes the null game (and thus lies outside the domain $\mathcal{SG}$) if $i$ is a veto player in the game $v \in \mathcal{SG}$. The more substantial reason is that the $\boldsymbol{SSPI}$, a power index that we aim to characterize, fails to satisfy (5) for games $v \in \mathcal{SG} \backslash \mathcal{SG}_{no-veto}$. This is already evident[20] in the simplest example: for the unanimity game $v = u_{\{1,2\}} \in \mathcal{SG} \backslash \mathcal{SG}_{no-veto}$ with two veto players, $v_{-1} = v_{-2}$ is the null game,[21] and thus $\mathbf{SPI}(v) = (\frac{1}{2}, \frac{1}{2}) \neq (0, 0) = \frac{1}{2}\left[\boldsymbol{SH}(v_{-1}) + \boldsymbol{SH}(v_{-2})\right]$.

# 4 Axioms

Even if a power index $\phi$ is known to have the **SCoA** property and its values are given for games with one-player support, it still cannot be uniquely determined on

[20] In fact, the failure of (5) for games $v \in \mathcal{SG} \backslash \mathcal{SG}_{no-veto}$ is apparent from the proof of Proposition 1 because, for a veto player $i$, (12) in that proof would not imply (5).

[21] Because $v_{-1}, v_{-2} \notin \mathcal{SG}$, in the equality that follows we apply the $\boldsymbol{SH}$ and not the $\boldsymbol{SSPI}$ to the subgames.

the entire $\mathcal{SG}$ by a recursive use of (5), simply because (5) does not apply to games with veto players. In order to uniquely characterize the $\boldsymbol{SSPI}$, we will need to appeal to the standard transfer and dummy axioms, and then supplement them with a weak version of the **SCoA** property.

## 4.1 Transfer and Dummy

The following two axioms are standard, and have been commonly assumed in axiomatizations of the $\boldsymbol{SSPI}$ and $\boldsymbol{BPI}$ starting with Dubey (1975) and Dubey and Shapley (1979).

**Axiom I: Transfer (T).** $\phi(v \vee w) + \phi(v \wedge w) = \phi(v) + \phi(w)$ for all $v, w \in \mathcal{SG}$.

As was noted, e.g., in Dubey et al. (2005), **T** can be restated in an equivalent but conceptually clearer form, amounting to a requirement that the change in power following an addition of winning coalitions to the game depends on just that added set.[22]

**Axiom II: Dummy (D).** If $v \in \mathcal{SG}$ and $i$ is a dummy player in $v$, then $\phi(v)(i) = v(\{i\})$.

**D** may be viewed as introducing a scale of measurement. If $i$ is dummy in $v \in \mathcal{SG}$, then he is either a null player (when $v(\{i\}) = 0$), who never swings the outcome, and it is convenient to label his influence as 0; or a *dictator* (when $v(\{i\}) = 1$), who always determines the outcome single-handedly, and it is convenient to label that level of influence as 1.

## 4.2 The main axiom

Our main axiom on a power index is weaker than the **SCoA** requirement, in that it imposes the latter only on *some* games. However, a technical condition is involved: these games must have an *essential support*. While simple to state, this condition lacks an obvious intuitive interpretation, but we will show later (in Remarks 4, 5, and

[22]Formally, the following is required: if $v \geq v' \in \mathcal{SG}$ and $u \geq u' \in \mathcal{SG}$ are such that $v - v' = u - u'$, then $\phi(v') - \phi(v) = \phi(u') - \phi(u)$.

Corollary 1) that it can be replaced by more natural (although stronger) assumptions, and we will discuss its necessity in Remark 7.

We recall from Shapley (1953, Lemma 3) that each $v \in \mathcal{G}$ is representable as a linear combination of unanimity games,

$$v = \sum_{\varnothing \neq T \subseteq \boldsymbol{T}(v)} c_T(v) u_T \tag{6}$$

for a uniquely determined set $\{c_T(v)\}_{\varnothing \neq T \subseteq \boldsymbol{T}(v)}$ of coefficients, given by

$$c_T(v) = \sum_{S \subseteq T} (-1)^{|T|-|S|} v(S) \tag{7}$$

for each non-empty $T \subseteq \boldsymbol{T}(v)$.[23]

**Definition 2**. We say that $v \in \mathcal{SG}$ has an *essential support* if $c_{\boldsymbol{T}(v)}(v) \neq 0$.[24]

**Remark 3**. Not every $v \in \mathcal{SG}$ (or even $v \in \mathcal{SG}_{no-veto}$) has an essential support. For example, if $N = \{1, 2, 3, 4\}$, then $v = u_{\{1,2\}} \vee u_{\{2,3\}} \vee u_{\{3,4\}} \in \mathcal{SG}_{no-veto}$ has full support $\boldsymbol{T}(v) = N$, yet this support is not essential because $c_{\boldsymbol{N}}(v) = 0$ as easily seen from (7).

**Remark 4**. Despite the observation in Remark 3, the class of games in $\mathcal{SG}$ (or even in $\mathcal{SG}_{no-veto}$) that have an essential support is quite broad. Indeed:

(i) whenever $|\mathbb{W}(v)|$ is an **odd** number, $v \in \mathcal{SG}$ has an essential support;

(ii) for any $T \subseteq N$, $|T| \geq 2$, and any $|T| + 1 \leq m \leq 2^{|T|} - 1$ (including all **odd** values of such $m$), there exists $v \in \mathcal{SG}_{no-veto}$ with $\boldsymbol{T}(v) = T$ and $|\mathbb{W}(v)| = m$;

(iii) if the support $T$ of $v \in \mathcal{SG}_{no-veto}$ is not essential, then there exists a game $v' \in \mathcal{SG}_{no-veto}$ with $T$ as its essential support that is obtained from $v$ by adding a single winning coalition.

(See the Appendix for the proof.)

[23] In fact, $c_T(v)$ is defined by (7) for any non-empty $T \subseteq N$, but it is easy to see that $c_T(v) = 0$ whenever $T \nsubseteq \boldsymbol{T}(v)$.

[24] In using this term we follow, e.g., Besner (2022), who refers to any coalition $T$ with $c_T(v) \neq 0$ as *essential* in the game $v$.

**Remark 5**. Any game $v \in \mathcal{SG}$ that is *symmetric* within its support $\boldsymbol{T}(v)$ (i.e., $v(S)$ depends only on $|S \cap \boldsymbol{T}(v)|$) has an essential support. (See the Appendix for the proof.)

**Axiom III: Occasional subgame consistency on average (OSCoA).** A power index $\phi$ is *occasionally subgame-consistent on average* (henceforth, **OSCoA**) if, given any $T \subseteq N, |T| \geq 2$, and any $i \in T$, there exists a game $\overline{v} \in \mathcal{SG}_{no-veto}$ with *essential* support $\boldsymbol{T}(\overline{v}) = T$ for which

$$\phi(\overline{v})(i) = \frac{1}{|T|} \sum_{j \in T} \phi(\overline{v}_{-j})(i). \tag{8}$$

As its name suggests, **OSCoA** requires the **SCoA** property to hold only occasionally: for any possible support set $T$, the consistency requirement in (8) needs to materialize only $|T|$ times (once for every player in $T$), and the choice of a game $\overline{v}$ for which (8) holds may depend on $i \in T$. Naturally, the **OSCoA** property is weaker than **SCoA** (notice that, by Remarks 4 or 5, for any given $T \subseteq N, |T| \geq 2$, there *exists* a game $v \in \mathcal{SG}_{no-veto}$ for which $T$ is the *essential* support, and so any $\phi$ with the **SCoA** property satisfies **OSCoA).**[25] Our main result in the next section, Theorem 1, shows that going in the opposite direction is possible if the axioms from Section 4.1 are assumed as well: any $\phi$ satisfying **OSCoA** in conjunction with **T** and **D** has the **SCoA** property *a posteriori* because it must coincide with the $\boldsymbol{SSPI}$.

# 5 The results

Our main result below axiomatizes the $\boldsymbol{SSPI}$.

**Theorem 1**. There exists one, and only one, power index satisfying **T, D** and **OSCoA**, and it is the $\boldsymbol{SSPI}$.

**Proof.** The fact that the $\boldsymbol{SSPI}$ satisfies **T** and **D** is well known (see, e.g., Dubey (1975)). By Proposition 1, the $\boldsymbol{SSPI}$ has the **SCoA** property, and thus also satisfies the weaker **OSCoA**.

[25] Clearly, **OSCoA** is *strictly* weaker than **SCoA** when $n \geq 3$: a power index obtained from the $\boldsymbol{SSPI}$ by arbitrarily changing the latter for just *one* game $v \in \mathcal{SG}_{no-veto}$ with $\boldsymbol{T}(\overline{v}) = N$ satisfies **OSCoA** and **D** but neither **SCoA** nor **T**.

To prove uniqueness, consider a power index $\phi$ that is subject to **T, D** and **OSCoA.** We start with a lemma on the connection between $\phi(v)$ for a given $v \in \mathcal{SG}$ and $\phi$'s values for unanimity games, proved in the Appendix.

**Lemma 1**. For any $v \in \mathcal{SG}$,

$$\phi(v) = \sum_{\varnothing \neq L \subseteq \{1,2,\ldots,k\}} (-1)^{|L|+1} \phi\left(u_{\cup_{l \in L} T_l}\right), \tag{9}$$

where $T_1, T_2, ..., T_k \subseteq \boldsymbol{T}(v)$ is a list of the elements of $\mathbb{W}_{\min}(v)$. Furthermore, the coefficient of $u_{\boldsymbol{T}(v)}$ in $v$'s expansion in (6) is given by

$$c_{\boldsymbol{T}(v)}(v) = \sum_{\varnothing \neq L \subseteq \{1,2,\ldots,k\}, \cup_{l \in L} T_l = \boldsymbol{T}(v)} (-1)^{|L|+1} \tag{10}$$

(namely, $c_{\boldsymbol{T}(v)}(v)$ is the total weight given to $\phi\left(u_{\boldsymbol{T}(v)}\right)$ in the summation in (9)).

We now show by induction on $m = 1, 2, ..., n$ that

$$\phi(v) = \boldsymbol{SSPI}(v) \tag{11}$$

for any $v \in \mathcal{SG}$ with $|\boldsymbol{T}(v)| = m$. For $m = 1$, the equality (11) follows from **D** because all players in $v \in \mathcal{SG}$ with $|\boldsymbol{T}(v)| = 1$ are dummies. Now assume that $m \geq 2$ and that (11) has been established for all $v \in \mathcal{SG}$ with $|\boldsymbol{T}(v)| < m$. The next lemma, proved in the Appendix, shows that (11) holds for any unanimity game with support of size $m$ :

**Lemma 2.** Under the induction hypothesis, for any $T \subseteq N$, $|T| = m$ $(\geq 2)$, we have $\phi(u_T) = \boldsymbol{SSPI}(u_T)$.

By the induction hypothesis and Lemma 2, we now know that (11) holds for all unanimity games $u_T$ for which $|T| \leq m$. Thus, if $v \in \mathcal{SG}$ is *any* game with $|\boldsymbol{T}(v)| = m$, by applying (9) in Lemma 1 to both $\phi$ and the $\boldsymbol{SSPI}$ we obtain (11) for such a $v$ as well, which finishes the induction step. We conclude that (11) holds for any $v \in \mathcal{SG}$, meaning that $\phi = \boldsymbol{SSPI}$. ■

**Remark 6.** The axioms **T, D** and **OSCoA** are independent. Indeed, the dictatorial index $\phi^d$ (given by $\phi^d(v)(i) = v(\{i\})$ for every $v \in \mathcal{SG}$ and $i \in N$) satisfies **T** and **D** but not **OSCoA.**[26] The index that is identically zero satisfies **T** and **OSCoA**

[26] By Remark 1, the $\boldsymbol{BPI}$ also satisfies **T** and **D** but not **OSCoA,** though only when $n \geq 3$.

but not **D.** Finally, an index that coincides with the $\boldsymbol{SSPI}$ on all games with the exception of all those $v \in \mathcal{SG}\backslash\mathcal{SG}_{no-veto}$ for which $\boldsymbol{T}(v) = N$, and is the zero vector for all games in the latter category, satisfies **D** and **OSCoA** but not **T**.

**Remark 7**. Our statement of **OSCoA** includes a requirement that the games $\overline{v}$ on which condition (8) is imposed must have an essential support. This requirement cannot be dropped when $n \geq 4$ if one desires the $\boldsymbol{SSPI}$ to be uniquely characterized by **T, D** and **OSCoA** as in Theorem 1.[27] (See the Appendix for the proof.)

The assumption that the games used in **OSCoA** have an essential support can be replaced by some stronger conditions that do not explicitly involve essentiality, while still yielding the same axiomatization of the $\boldsymbol{SSPI}$ as in Theorem 1. This is summarized in the following corollary:

**Corollary 1**. A power index is the $\boldsymbol{SSPI}$ if and only if it satisfies **T, D,** and, given any $T \subseteq N$, $|T| \geq 2$, and any $i \in T$, at least one of the following two conditions is fulfilled:

(**a**) there exists $\overline{v} \neq u_T$ in $\mathcal{SG}$ with $\boldsymbol{T}(\overline{v}) = T$, which is symmetric within $T$ and for which (8) holds;

(**b**) there exists $\overline{v} \in \mathcal{SG}_{no-veto}$ with $\boldsymbol{T}(\overline{v}) = T$ and an odd $|\mathbb{W}(\overline{v})|$, for which (8) holds.

**Proof.** The $\boldsymbol{SSPI}$ satisfies the conditions stated in the corollary because it has the **SCoA** property by Proposition 1, implying *both* (**a**) and (**b**). Conversely, assume that a power index adheres to the stated conditions, and consider any $T \subseteq N$, $|T| \geq 2$, and $i \in T$. If (**a**) is satisfied, then $T$ is the essential support of $\overline{v}$ by Remark 5. The game $\overline{v}$ cannot have veto players (otherwise, by the symmetry of $\overline{v}$ within $T$, every player in $T$ would be a veto player, meaning that $\overline{v} = u_T$ and contradicting the assumption in (**a**)). That is, $\overline{v} \in \mathcal{SG}_{no-veto}$. And, if (**b**) is satisfied, $\overline{v} \in \mathcal{SG}_{no-veto}$ has $T$ as its essential support by Remark 4. Thus, in either of the two cases, there exists $\overline{v} \in \mathcal{SG}_{no-veto}$ with the essential support $T$ that satisfies (8). Accordingly, the power index satisfies **OSCoA**, and together with **T** and **D** this implies – by Theorem 1 – that the index is the $\boldsymbol{SSPI}$.$\blacksquare$

[27] When $n = 2$ or $n = 3$, *all* simple games have an essential support, and so the requirement of having an essential support is automatically fulfilled by any $\overline{v}$.

# 6 Concluding remarks

Our axioms avoid explicit imposition of symmetry (or anonymity) assumptions, although some measure of symmetry is inevitably implicit in **D** and **OSCoA**. Indeed, it follows from **D** that null players are treated symmetrically. Also, the arithmetic average taken in (8) within the statement of **OSCoA** assigns the same weight to the one-player-out subgames for all players. What counters the symmetry is that the games $\overline{v}$ for which (8) holds may be different for different players, and the relation between the given support $T$ and the game $\overline{v}$ that it essentially supports does not need to be covariant under player permutations. Note also that when Corollary 1(i) requires (8) to hold for a symmetric game with a given support in the $\boldsymbol{SSPI}$ characterization, the choice of such a game is support-dependent and may be player-dependent[28] – and thus, symmetry is in no way explicitly spelled out even with this requirement.

We would like to stress again that, in **OSCoA,** the requirement that the games used for (8) must have an essential support is imposed out of necessity, not by choice: Remark 7 explains why the essential support assumption cannot be dropped without forfeiting the uniqueness of a power index. It may be of interest to explore the embedding of our finite games into the setting with an infinite universe of players, which could make the recursive aspect of consistency in **OSCoA** more potent, possibly allowing for a weakening or removal of the essential support assumption.[29]

If, instead of **OSCoA,** we assume that a power index *a priori* satisfies its stronger version – specifically, the **SCoA** in Definition 1 that imposes (8) for *all* games $\overline{v} \in \mathcal{SG}_{no-veto}$ supported on a given $T$ – then the essentiality of support becomes moot, and Theorem 1 yields the same axiomatization of the $\boldsymbol{SSPI}$. Note, however, that replacing **OSCoA** with a stronger **SCoA** does not eliminate the need for either **T**

[28] In particular, the assumption in Corollary 1(i) does not imply the cross-invariance axiom of Chen et al. (2024).

[29] Such an embedding is an important feature of the finite TU games framework in Dubey et al. (1981) and of the simple games framework in Einy (1987). However, these works consider *semivalues*, which in the simple games context are power indices that, in addition to **T** and **D**, are *a priori* assumed to satisfy the symmetry (anonymity) axiom and to be non-negative. Since we do not make these latter two assumptions, the semivalue characterization results provided in these works are not directly applicable in our setting.

or **D**. This is because the indices described in Remark 6, which satisfy **OSCoA** and only one of the axioms **T** and **D,** in fact have the (full) **SCoA** property**.** But, just as mentioned in the previous paragraph, considering an infinite universe of players – in which **(O)SCoA** would have a stronger recursive aspect – might allow for some weakening of the other two axioms.

# 7 Appendix

## 7.1 Proof of Proposition 1

The following recursive formula[30] for the $\boldsymbol{SSPI}$ (and, more generally, the Shapley value $\boldsymbol{SH}$) is due to Maschler and Owen (1989) and Hart and Mas-Colell (1989): for every $v \in \mathcal{SG}$ and $i \in \boldsymbol{T}(v)$,

$$\boldsymbol{SSPI}\,(v)\,(i) = \frac{1}{|\boldsymbol{T}(v)|} \sum_{j \in \boldsymbol{T}(v) \setminus \{i\}} \boldsymbol{SSPI}\,(v_{-j})\,(i) + \frac{1}{|\boldsymbol{T}(v)|} \left[ v(\boldsymbol{T}(v)) - v(\boldsymbol{T}(v) \setminus \{i\}) \right]. \tag{12}$$

If $v \in \mathcal{SG}_{no-veto}$ then no $i$ is a veto player, and thus the second term of the RHS of (12) is zero. Also, since $i$ is obviously a null player in $v_{-i}$, $\boldsymbol{SSPI}\,(v_{-i})\,(i) = 0$ by (2), and hence one may take the sum over *all* $j \in \boldsymbol{T}(v)$ in the first term of the RHS of (12). It therefore follows from (12) that $\phi = \boldsymbol{SSPI}$ satisfies (5) for every $v \in \mathcal{SG}_{no-veto}$ and every $i \in T = \boldsymbol{T}(v)$.

## 7.2 Proof of the claims in Remark 4

(i) Clearly,

$$c_{\boldsymbol{T}(v)}(v) \bmod 2 \overset{\text{by (7)}}{=} \left( \sum_{S \subseteq \boldsymbol{T}(v)} (-1)^{|\boldsymbol{T}(v)| - |S|}\, v(S) \right) \bmod 2$$

$$= \left( \sum_{S \subseteq \boldsymbol{T}(v)} v(S) \right) \bmod 2 = |\mathbb{W}(v)| \bmod 2 \neq 0$$

and hence $c_{\boldsymbol{T}(v)}(v) \neq 0$.

[30] For its explicit statement see, e.g., Pérez-Castrillo and Wettstein (2001, p. 282).

(ii) Order *all* non-empty coalitions $T_1 = T, T_2, ..., T_{2^{|T|}-1}$ contained in $T$ in such a way that $T_k \supseteq T_{k+1}$ for every $1 \leq k < 2^{|T|} - 1$. Then, given any $|T| + 1 \leq m \leq 2^{|T|} - 1$, it is easy to verify that for $v_m := u_{T_1} \vee u_{T_2} \vee ... \vee u_{T_m}$, we have $\boldsymbol{T}(v_m) = T$ and $\mathbb{W}(v_m) = \{T_1, T_2, ..., T_m\}$, hence $|\mathbb{W}(v_m)| = m$.

(iii) Notice that $|\mathbb{W}(v)|$ is an *even* number, since otherwise $T = \boldsymbol{T}(v)$ would have been essential by (i). Thus $|\mathbb{W}(v)| < 2^{|T|} - 1$, and so $\mathbb{W}(v)$ does *not* contain at least one non-empty subset of $T$. Let $1 \leq m < |T|$ be the highest number for which $\mathbb{W}(v)$ does *not* contain *at least one* subset of $T$ of size $m$, and pick some $T' \notin \mathbb{W}(v)$ with $|T'| = m$. Then $v' := v \vee u_{T'} \in \mathcal{SG}_{no-veto}$ is a game with $\boldsymbol{T}(v') = T$ that is obtained from $v$ by adding just one winning coalition: $\mathbb{W}(v') = \mathbb{W}(v) \cup \{T'\}$. Accordingly, $|\mathbb{W}(v')| = |\mathbb{W}(v)| + 1$. This is an odd number, which implies by (i) that $v'$ has an essential support.

## 7.3 Proof of the claim in Remark 5

A game $v$ described in the remark is necessarily a $q$-majority game[31] on $\boldsymbol{T}(v)$ for some integer $1 \leq q \leq |\boldsymbol{T}(v)|$, and so, by (7),

$$\begin{aligned} c_{\boldsymbol{T}(v)}(v) &= \sum_{S \subseteq \boldsymbol{T}(v)} (-1)^{|\boldsymbol{T}(v)| - |S|} v(S) = \sum_{s=q}^{|\boldsymbol{T}(v)|} (-1)^{|\boldsymbol{T}(v)| - s} \binom{|\boldsymbol{T}(v)|}{s} \\ &= \sum_{s=q}^{|\boldsymbol{T}(v)|} (-1)^{|\boldsymbol{T}(v)| - s} \binom{|\boldsymbol{T}(v)|}{|\boldsymbol{T}(v)| - s} = \sum_{k=0}^{|\boldsymbol{T}(v)| - q} (-1)^k \binom{|\boldsymbol{T}(v)|}{k} \\ &= (-1)^{|\boldsymbol{T}(v)| - q} \binom{|\boldsymbol{T}(v)| - 1}{|\boldsymbol{T}(v)| - q} \neq 0, \end{aligned}$$

where the last equality is based on a known formula.[32]

## 7.4 Proof of Lemma 1

It is easy to see (by induction on $k$) that

$$v = u_{T_1} \vee u_{T_2} \vee ... \vee u_{T_k} = \sum_{\varnothing \neq L \subseteq \{1,2,...,k\}} (-1)^{|L|+1} \bigwedge_{l \in L} u_{T_l}. \tag{13}$$

[31] That is, $v(S) = 1$ if and only if $|S \cap \boldsymbol{T}(v)| \geq q$.

[32] The equality used here, $\sum_{k=0}^{m} (-1)^k \binom{r}{k} = (-1)^m \binom{r-1}{m}$ for $m \leq r - 1$, appears, e.g., as (5.16) in Graham et al (1989, p. 165). It can also be easily established by induction on $m$.

Since for any non-empty $L \subseteq \{1, 2, ..., k\}$,

$$\bigwedge_{l \in L} u_{T_l} = u_{\cup_{l \in L} T_l}, \tag{14}$$

the preceding equality also yields

$$v = \sum_{\varnothing \neq L \subseteq \{1,2,\ldots,k\}} (-1)^{|L|+1} u_{\cup_{l \in L} T_l}. \tag{15}$$

Using the assumption that $\phi$ satisfies **T,** it can be shown by induction on $k$ that $\phi(v)$ can be represented as follows,[33] in full analogy with (13):

$$\phi(v) = \sum_{\varnothing \neq L \subseteq \{1,2,\ldots,k\}} (-1)^{|L|+1} \phi\left(\bigwedge_{l \in L} u_{T_l}\right).$$

This yields (9) via (14).

Finally, notice that (15) reduces into a representation of $v$ as a linear combination of unanimity games, with the coefficient of $u_{\boldsymbol{T}(v)}$ being

$$\sum_{\varnothing \neq L \subseteq \{1,2,\ldots,k\}, \cup_{l \in L} T_l = \boldsymbol{T}(v)} (-1)^{|L|+1}.$$

But, as stated between equations (6) and (7), this coefficient is uniquely determined and is equal to $c_{\boldsymbol{T}(v)}(v)$, which establishes (10).

## 7.5 Proof of Lemma 2

**Proof of Lemma 2**. Fix any $T \subseteq N$ with $|T| = m$ $(\geq 2)$, and consider some $i \in T$. By **OSCoA,** there exists $\overline{v} \in \mathcal{SG}_{no-veto}$ with *essential* support $T = \boldsymbol{T}(\overline{v})$ for which (8) holds. But, since $|\boldsymbol{T}(\overline{v}_{-j})| < |\boldsymbol{T}(\overline{v})| = |T| = m$ for every $j \in T$, by the induction hypothesis the RHS of (8) is equal to $\frac{1}{|T|} \sum_{j \in T} \boldsymbol{SSPI}(\overline{v}_{-j})(i)$, which in turn equals $\boldsymbol{SSPI}(\overline{v})(i)$ by Proposition 1. Since the LHS of (8) consists of $\phi(\overline{v})(i)$, we conclude that

$$\phi(\overline{v})(i) = \boldsymbol{SSPI}(\overline{v})(i). \tag{16}$$

Now, for the game $\overline{v}$ above, write $\overline{v} = u_{T_1} \vee u_{T_2} \vee ... \vee u_{T_k}$ for a list $T_1, T_2, ..., T_k \subset T$ of the elements of $\mathbb{W}_{\min}(\overline{v})$. By applying (9) of Lemma 1 to both $\phi$ and the $\boldsymbol{SSPI}$ and using (16), we obtain

$$\sum_{\varnothing \neq L \subseteq \{1,2,\ldots,k\}} (-1)^{|L|+1} \phi\left(u_{\cup_{l \in L} T_l}\right)(i) = \sum_{\varnothing \neq L \subseteq \{1,2,\ldots,k\}} (-1)^{|L|+1} \boldsymbol{SSPI}\left(u_{\cup_{l \in L} T_l}\right)(i). \tag{17}$$

[33] This representation is also a corollary of Lemma 2.3 of Einy (1987).

By the induction hypothesis, $\phi\left(u_{\cup_{l\in L}}\right) = \boldsymbol{SSPI}\left(u_{\cup_{l\in L}T_l}\right)$ whenever $\cup_{l\in L}T_l \neq T$, and it thus follows from (17) that

$$\left(\sum_{\varnothing\neq L\subseteq\{1,2,\ldots,k\},\cup_{l\in L}T_l=T}(-1)^{|L|+1}\right)\phi\left(u_T\right)(i) \tag{18}$$

$$= \left(\sum_{\varnothing\neq L\subseteq\{1,2,\ldots,k\},\cup_{l\in L}T_l=T}(-1)^{|L|+1}\right)\boldsymbol{SSPI}\left(u_T\right)(i). \tag{19}$$

By (10) in Lemma 1, (18)-(19) can be restated as

$$c_T(\overline{v})\phi\left(u_T\right)(i) = c_T(\overline{v})\boldsymbol{SSPI}\left(u_T\right)(i).$$

However, recall that $\overline{v}$ was chosen by **OSCoA** in such a way that its support $T = \boldsymbol{T}(\overline{v})$ is essential, and so $c_T(\overline{v}) \neq 0$, implying that $\phi\left(u_T\right)(i) = \boldsymbol{SSPI}\left(u_T\right)(i)$. Since the above argument can be made for any $i \in T$, and then for any $T \subseteq N$ of size $m$, the lemma is established.

## 7.6 Proof of the claim in Remark 7

Let $n = 4$, and construct a power index $\phi$ as follows: let $\phi\left(u_T\right) = \boldsymbol{SSPI}\left(u_T\right)$ whenever $|T| \leq 3$, and let $\phi\left(u_N\right)$ be the zero vector; then extend $\phi$ onto $\mathcal{SG}$ by the equation

$$\phi\left(v\right) = \sum_{\varnothing\neq T\subseteq\boldsymbol{T}(v)} c_T(v)\phi\left(u_T\right) \tag{20}$$

for every $v \in \mathcal{SG}$, using the coefficients $\{c_T(v)\}_{\varnothing\neq T\subseteq\boldsymbol{T}(v)}$ defined in (7). Using an argument in Einy (1987, bottom of p. 186),[34] it can be readily seen that $\phi$ is well-defined on $\mathcal{SG}$ and satisfies **T**. By (2) and (6), we also have

$$\boldsymbol{SSPI}\left(v\right) = \sum_{\varnothing\neq T\subseteq\boldsymbol{T}(v)} c_T(v)\boldsymbol{SSPI}\left(u_T\right) \tag{21}$$

for any $v \in \mathcal{SG}$, and so $\phi\left(v\right) = \boldsymbol{SSPI}\left(v\right)$ whenever $|\boldsymbol{T}(v)| \leq 3$. It follows in particular that $\phi$ satisfies **D,** and that (8) holds for *any* $\overline{v} \in \mathcal{SG}_{no-veto}$ (with or without an essential support) and all players $i \in \boldsymbol{T}(\overline{v})$ whenever $|\boldsymbol{T}(\overline{v})| \leq 3$ (because that is so for the $\boldsymbol{SSPI}$). Now consider $\overline{\overline{v}} = u_{\{1,2\}} \vee u_{\{2,3\}} \vee u_{\{3,4\}} \in \mathcal{SG}_{no-veto}$. We already

[34] Specifically, by applying (20) to every $v \in \mathcal{G}$, $\phi$ can be extended into a well-defined *linear* operator on $\mathcal{G}$, and hence its restriction to $\mathcal{SG}$ is well-defined and satisfies **T**.

noted in Remark 3 that $\overline{\overline{v}}$'s support $\boldsymbol{T}(\overline{\overline{v}}) = N$ is not essential, i.e., $c_{\boldsymbol{T}(\overline{\overline{v}})}(\overline{\overline{v}}) = 0$. This, together with (20) and (21) taken for $v = \overline{\overline{v}}$, implies that $\phi\left(\overline{\overline{v}}\right) = \boldsymbol{SSPI}\left(\overline{\overline{v}}\right)$ because $\phi$ coincides with $\boldsymbol{SSPI}$ for all simple games (and in particular all unanimity games) with support of size 3 and below. So $\phi$ coincides with $\boldsymbol{SSPI}$ on $\overline{\overline{v}}$ as well, and therefore (8) holds for $T = N$ and $\overline{\overline{v}}$ that is *non-essentially* supported on $N$ (and for all players $i \in N$). We conclude that $\phi$ is an index that is different from $\boldsymbol{SSPI}$ (since $\phi\left(u_N\right) \neq \boldsymbol{SSPI}\left(u_N\right)$ by construction), but it would satisfy all the conditions of Theorem 1 if we were to drop the requirement in **OSCoA** that all $\overline{v}$ used for (8) must have an essential support.

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