Elimination Games

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Abstract

We study two-stage elimination games with four heterogeneous players. In the first stage, the players are allocated into two contests each of which includes two players that compete against each other in an all-pay auction. Then the winners of the two contests interact with each other in the second stage. The outcomes of the interaction in the second stage are given by a general form of the players' payoffs as functions of their types (abilities). Then, we examine the effect of the timing of the competitions in the first stage on the players' expected payoffs and their total effort. In particular, we demonstrate that if the players' payoff functions in the second stage are separable in the players' types then the timing of the play in the first stage of either simultaneous or sequential contests does not affect the players' expected payoff as well as their total effort in the elimination games.

Keywords: Elimination games, all-pay contests, simultaneous contests, sequential contests.

JEL classification: D44, J31, D72, D82

1 Introduction

An elimination tournament is a multi-stage game in which in each stage some of the contestants are removed while the others advance to the next stage until the final stage. The elimination tournament was first studied in the statistical literature. The pioneering paper of David (1959)

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considered the winning probability of the top player in a four-player tournament with a random seeding (see also Glenn 1960 and Searles 1963 for early contributions). Other papers (see, for example, Hwang 1982, Horen and Reizman 1985 and Schwenk 2000, Ariegi and Dimirov 2020, and Ariegi 2022) consider various optimality criteria for choosing seedings. These works assume that for each game among two players, there is a fixed, exogenously given probability that one player will beat his opponent. This probability does not depend on the stage of the tournament in which the particular game takes place nor on the identity of the expected opponent at the next stage. As opposed to the statistical literature, in the economic literature (see, for example, Rosen 1986, Gradstein and Konrad 1999, Kräkel 2004, Groh et al. 2012, Sela 2022, and Cohen et al. 2023) the winning probabilities in each game are endogenous in that they result from equilibrium strategies and are dependent on continuation winning values. Moreover, the winning probabilities depend on the stage of the tournament in which the game takes place as well as on the identity of the future expected opponents.

In elimination tournaments, especially in sports, in each stage there are similar interactions among the players, namely the players compete in the same form of contest in every stage, and the only difference between the stages is the number of competitions and the set of the players that participate. However, in our elimination games, the kind of interaction among the agents in each stage could be completely different. For instance, it could be that the players compete against each other in one stage and later the winners of this stage cooperate with each other in the next stage. Accordingly, we consider two-stage elimination games where the results of the second stage are given by the payoffs of the players as general functions of their types (abilities). Then, given these payoff functions, we analyze the players' strategies in the first stage of the elimination games. By this characterization we capture many kinds of elimination games since in the second stage the players' payoff functions may represent many kinds of interactions. We assume that in the second stage, the players' payoff functions are monotonically increasing in both players' types, namely, each player's expected payoff in the second stage increases in his ability as well as in the other agent's ability.

To illustrate such a scenario, consider some candidates who compete in the first stage to be elected to a political party where the winners come from different competitions in their areas such that the winner of each area wins a position in his party. In the second stage, the power of this party and its chance to win in the election depends on the types of its members such that the higher the types (regarding personality or charisma) of the party's members, the higher is its chance to win. Thus, in the second elimination stage, the success of each member depends on his own type as well as the types of the other members in his party, namely, the members' expected payoffs increase in their own types and the types of the other members in their party.

The realm of sports provides another example. Consider football teams from countries all over Europe who compete against each other to qualify for the Champions League. The income of each team from participating in the Champion League is quite high so it comprises a significant part of their financial budget. Income is also derived from broadcasting rights where it depends on the reputation of the teams. As such, the higher the number of highly reputed teams, the higher will be the income of all the qualified teams in the Champions League. Therefore, not only does the team's own type (reputation) affect its income but also do the other teams' types. Here also, the expected payoff of a team increases in the types of all the other ones that qualify to the final stage of the Champions League.

Last, we can consider athletes in the Olympic Games who compete, for example in races. These athletes compete in an elimination race where in the final the best runners who won in the previous stages compete against each other. One of their goals besides winning medals might be breaking the world record. Thus, the better the runners who qualified for the final are, the higher the chance that the world record, or any other record for that matter, will be broken. Again, the types of runners in the final affect the chance of each of them breaking the world record.

Formally, we study elimination two-stage games with four players. In the first stage, the players are allocated into two all-pay auctions (contests) A and B, each of which includes two players.[1] The players in each contest have different types which represent their abilities,and are commonly known. The player in each contest who exerts the highest effort wins, but all the players bear their costs of their effort. The winners of both contests continue to the second stage, while the losers do not continue and do not get any payoff. The winners of the first stage interact with each other in the second stage, and their payoffs are given as functions of their types where these payoff functions are increasing in the players' types.

The contests in the first stage of our elimination games could be either simultaneous or sequential. Simultaneous and sequential contests are mechanisms and are not just restricted to sports or political contests. In fact, they could be employed also to study resource allocation problems in military and systems defense (Clark and Konrad 2007), or research and development portfolio selection (Clark and Konrad 2008). Whether sequential or simultaneous moves endogenously arise in equilibrium and which is better for stimulating competition are long-standing questions in several fields of economics, particularly contests (see, for example, Baik and Shogren 1992, Morgan 2003, Fu 2006, Konrad and Leininger 2007, Serena 2017 and Mago and Seremeta 2019).

We compare the players' expected payoffs as well as their total effort in the first stage of either simultaneous or sequential contests, and show the relationship between the players' winning probabilities and their expected payoffs. In particular, we provide sufficient and necessary conditions for the players to be indifferent between simultaneous and sequential elimination games. In other words, the players have the same expected payoff in both forms of the contests. Our results provide conditions under which the order of the contests in the first stage of the sequential elimination game does not affect the players' expected payoffs.

[1] For all-pay auctions under complete information see Hillman and Samet (1987), Hillman and Riley (1989), Baye et al. (1996), Che and Gale (1998), Clark and Riis (1998), Siegel (2009), and Sela (2012), and Segev (2020). For all-pay auctions under incomplete information see Amann and Leininger (1996), Krishna and Morgan (1997), Moldovanu and Sela (2001, 2006), Moldovanu et al. (2012), Segev and Sela (2014), Ødegaard and Anderson (2014), Olszewski and Siegel (2016), and Einy et al. (2017).

Our findings indicate that in elimination games, if in the second stage, the expected payoffs of the players are separable functions of their types, then their probabilities of winning in the simultaneous and sequential elimination games are the same, and therefore the players' expected payoffs are the same. The separability of the expected payoffs in the second stage is not a necessary condition for this payoff equivalence. However, without the separability of the players' payoff functions, an equivalence of the simultaneous and sequential elimination games might depend on the allocation of players among the two all-pay auctions. In other words, for some allocations, there is an equivalence while for other allocations there is not. We can conclude that if players have separable functions in elimination games then the timing of play, namely whether simultaneous or sequential, should not affect the players' payoff or their total output.

The rest of the paper is organized as follows: In Section 2 we present our model of elimination games. In Section 3 we analyze the equilibrium of simultaneous elimination games and in Section 4 the equilibrium of sequential elimination games. In Section 5 we compare the players' expected payoffs as well as their total effort for these two games, and in Section 6 we provide some applications of our results to elimination games with specific forms of players' payoff functions. Section 7 concludes. The proofs appear in the appendix.

2 The model

We study an elimination two-stage game with four players. In the first stage, the players are allocated in two all-pay auctions each of which includes two players. The players' types in contest A are a_i, $i = 1, 2$, where $a_1 \geq a_2$, and in contest B they are b_j, $j = 1, 2$ where $b_1 \geq b_2$. The players' types represent their abilities, the higher is the type of a player, the higher is his ability. These types are commonly known. We refer to a player with type a_i, $i = 1, 2$ as player a_i and to player with type b_j, $j = 1, 2$ as player b_j. In the first stage, each player i in contest A exerts an effort x_i, and each player j in contest B exerts an effort y_j, $j = 1, 2$. The player in each contest who exerts the highest effort wins but all the players bear their costs of effort. The winners of both contests continue to the second stage, while the losers do not continue and do not get anything. In the second stage, the winners of the first stage interact with each other, and their payoffs are given as functions of their types. Formally, if player i from contest A and player j from contest B are the winners of the first stage, then player a_i's payoff in the second stage is $g_A(a_i, b_j)$ and player b_j's payoff in this stage is $g_B(a_i, b_j)$, where $g_A, g_B : R^2 \to R^1$ are increasing in both players' types. When the players in both contests simultaneously exert their efforts in the first stage, we refer to this model as the simultaneous elimination game. But, when the players in the first stage first compete in contest A and later in contest B, we refer to this model as the sequential elimination game .

3 Simultaneous elimination games

In this section, we assume that in the first stage the players compete simultaneously in all-pay auctions, where in the second stage, the players' payoffs are given by $g_A(a_i, b_j)$ and $g_B(a_i, b_j)$ according to the types of the winners in the first stage, $i = 1, 2$ and $j = 1, 2$. It can be easily verified that there is no pure strategy equilibrium in this stage. Thus, we let $F_{a_i}(x)$ be player i's effort cumulative distribution function, $i = 1, 2$, in contest A, and $F_{b_j}(x)$ be player j's effort cumulative distribution function, $j = 1, 2$, in contest B. Then, in contest A, player a_2's mixed-strategy equilibrium $F_{a_2}(x)$ is given by

$$(g_A(a_1, b_1)p_{b_1} + g_A(a_1, b_2)p_{b_2})F_{a_2}(x) - x = u_{a_1}, \tag{1}$$

where p_{b_j} is the probability that player b_j, $j = 1, 2$, wins in contest B, and u_{a_1} is the expected payoff of player a_1 in contest A. Likewise, in contest A, player a_1's mixed strategy equilibrium is given by

$$(g_A(a_2, b_1)p_{b_1} + g_A(a_2, b_2)p_{b_2})F_{a_1}(x) - x = u_{a_2}, \tag{2}$$

where u_{a_2} is the expected payoff of player a_2 in contest A. By the same arguments as in the analysis of the standard all-pay auction (Hillman and Riley 1989 and Baye et al. 1996), the expected payoff of the weaker player ($a_2 < a_1$) is equal to zero, and therefore, we have

$$u_{a_2} = 0. \tag{3}$$

Then, by (2), player a_1's mixed strategy equilibrium in contest A is

$$F_{a_1}(x) = \frac{x}{g_A(a_2, b_1)p_{b_1} + g_A(a_2, b_2)p_{b_2}}. \tag{4}$$

Note that $F_{a_1}(0) = 0$, $F_{a_1}(g_A(a_2, b_1)p_{b_1} + g_A(a_2, b_2)p_{b_2}) = 1$, and $F_{a_1}(x)$ is strictly increasing on the interval $[0, g_A(a_2, b_1)p_{b_1} + g_A(a_2, b_2)p_{b_2}]$. Thus, the maximal effort of the players in contest A is $g_A(a_2, b_1)p_{b_1} + g_A(a_2, b_2)p_{b_2}$, and if player a_1 exerts this maximal effort he wins for sure. Therefore, player a_1's expected payoff in contest A is

$$\begin{aligned} u_{a_1} &= (g_A(a_1, b_1)p_{b_1} + g_A(a_1, b_2)p_{b_2}) - (g_A(a_2, b_1)p_{b_1} + g_A(a_2, b_2)p_{b_2}) \\ &= (g_A(a_1, b_1) - g_A(a_2, b_1))p_{b_1} + (g_A(a_1, b_2) - g_A(a_2, b_2))p_{b_2}. \end{aligned} \tag{5}$$

By (1) and (5), player a_2's mixed strategy equilibrium in contest A is

$$F_{a_2}(x) = \frac{x + (g_A(a_1, b_1) - g_A(a_2, b_1))p_{b_1} + (g_A(a_1, b_2) - g_A(a_2, b_2))p_{b_2}}{g_A(a_1, b_1)p_{b_1} + g_A(a_1, b_2)p_{b_2}}. \tag{6}$$

Note that $F_{a_2}(0) = \frac{(g_A(a_1,b_1)-g_A(a_2,b_1))p_{b_1}+(g_A(a_1,b_2)-g_A(a_2,b_2))p_{b_2}}{g_A(a_1,b_1)p_{b_1}+g_A(a_1,b_2)p_{b_2}} < 1$, $F_{a_2}(g_A(a_2, b_1)p_{b_1} + g_A(a_2, b_2)p_{b_2}) = 1$, and $F_{a_2}(x)$ is strictly increasing on the interval $[0, g_A(a_2, b_1)p_{b_1} + g_A(a_2, b_2)p_{b_2})]$. Player a_2's probability of winning in contest A is given by

$$\begin{aligned} p_{a_2} &= 1 - p_{a_1} = \int_0^{g_A(a_2,b_1)p_{b_1}+g_A(a_2,b_2)p_{b_2}} (\int_0^y F'_{a_1}(x)dx)F'_{a_2}(y)dy \\ &= \int_0^{g_A(a_2,b_1)p_{b_1}+g_A(a_2,b_2)p_{b_2}} (\int_0^y \frac{1}{g_A(a_2, b_1)p_{b_1} + g_A(a_2, b_2)p_{b_2}}dx)\frac{1}{g_A(a_1, b_1)p_{b_1} + g_A(a_1, b_2)p_{b_2}}dy \\ &= \frac{g_A(a_2, b_1)p_{b_1} + g_A(a_2, b_2)p_{b_2}}{2(g_A(a_1, b_1)p_{b_1} + g_A(a_1, b_2)p_{b_2})}, \end{aligned} \tag{7}$$

where p_{a_i} , $i = 1, 2$ is player a_i's winning probability in contest A. Note that since $a_1 > a_2$ and $\frac{dg_A(a,b)}{da} \geq 0$ we obtain that $p_{a_1} \geq p_{a_2}$.

Similarly, we can characterize player b_1's mixed-strategy in contest B which is

$$F_{b_1}(x) = \frac{x}{g_B(a_1, b_2)p_{a_1} + g_B(a_2, b_2)p_{a_2}}, \tag{8}$$

and his expected payoff is

$$u_{b_1} = (g_B(a_1,b_1) - g_B(a_1, b_2))p_{a_1} + (g_B(a_2, b_1) - g_B(a_2, b_2))p_{a_2}. \tag{9}$$

Player b_2's mixed strategy equilibrium in contest B is

$$F_{b_2}(x) = \frac{x + (g_B(a_1,b_1) - g_B(a_1, b_2))p_{a_1} + (g_B(a_2, b_1) - g_B(a_2, b_2))p_{a_2}}{g_B(a_1, b_1)p_{a_1} + g_B(a_2, b_1)p_{a_2})}, \tag{10}$$

and his expected payoff is

$$u_{b_2} = 0. \tag{11}$$

Player b_2's winning probability in contest B is given by

$$\begin{aligned} p_{b_2} &= 1 - p_{b_1} = \int_0^{g_B(a_1,b_2)p_{a_1}+g_B(a_2,b_2)p_{a_2}} (\int_0^y F'_{b_1}(x)dx)F'_{b_2}(y)dy \\ &= \int_0^{g_B(a_1,b_2)p_{a_1}+g_B(a_2,b_2)p_{a_2}} (\int_0^y \frac{1}{g_B(a_1, b_2)p_{a_1} + g_B(a_2, b_2)p_{a_2}}dx)\frac{1}{g_B(a_1, b_1)p_{a_1} + g_B(a_2, b_1)p_{a_2}}dy \\ &= \frac{g_B(a_1, b_2)p_{a_1} + g_B(a_2, b_2)p_{a_2}}{2(g_B(a_1, b_1)p_{a_1} + g_B(a_2, b_1)p_{a_2})}. \end{aligned} \tag{12}$$

Note that since $b_1 > b_2$ and $\frac{dg_B(a,b)}{db} \geq 0$ we obtain that $p_{b_1} \geq p_{b_2}$.

The players' expected total effort is given by

$$\begin{aligned} TE &= \frac{g_A(a_2, b_1)p_{b_1} + g_A(a_2, b_2)p_{b_2}}{2}(1 + \frac{g_A(a_2, b_1)p_{b_1} + g_A(a_2, b_2)p_{b_2}}{(g_A(a_1, b_1)p_{b_1} + g_A(a_1, b_2)p_{b_2})}) \\ &+ \frac{g_B(a_1, b_2)p_{a_1} + g_B(a_2, b_2)p_{a_2}}{2}(1 + \frac{g_B(a_1, b_2)p_{a_1} + g_B(a_2, b_2)p_{a_2}}{g_B(a_1, b_1)p_{a_1} + g_B(a_2, b_1)p_{a_2}}). \end{aligned} \tag{13}$$

4 Sequential elimination games

In this section, we assume that players sequentially compete in the first stage, first in contest A and later in contest B. We begin with the analysis of contest B and go backwards to that of contest A.

4.1 Contest B

Suppose that player $i, i = 1, 2$, in set A won earlier. Let $F_{b_j}(x)$ be player j's effort cumulative distribution function, $j = 1, 2$, in contest B. Then, player b_2's mixed-strategy equilibrium $F_{b_2}(x)$ in contest B is given by

$$g_B(a_i, b_1)F_{b_2}(x) - x = c_{b_1}, \tag{14}$$

where c_{b_1} is b_1's expected payoff and is equal to

$$c_{b_1} = g_B(a_i, b_1) - g_B(a_i, b_2). \tag{15}$$

Player b_1's mixed strategy equilibrium in contest B is given by

$$g_B(a_i, b_2)F_{b_1}(x) - x = c_{b_2}. \tag{16}$$

where c_{b_2} is b_2's expected payoff and is equal to

$$c_{b_2} = 0. \tag{17}$$

By (16), player b_1's mixed strategy equilibrium in contest B is

$$F_{b_1}(x) = \frac{x}{g_B(a_i, b_1)}, \tag{18}$$

and by (14), player b_2's mixed strategy equilibrium is

$$F_{b_2}(x) = \frac{x + g_B(a_i, b_1) - g_B(a_i, b_2)}{g_B(a_i, b_2)}. \tag{19}$$

The players' winning probabilities in contest B are

$$\begin{aligned} q_{b_1} &= 1 - \frac{g_B(a_i, b_2)}{2g_B(a_i, b_1)} \\ q_{b_2} &= \frac{g_B(a_i, b_2)}{2g_B(a_i, b_1)}. \end{aligned} \tag{20}$$

Note that since $b_1 > b_2$ and $\frac{dg_B(a,b)}{db} \geq 0$ we obtain that $q_{b_1} \geq q_{b_2}$.

The players' total effort in the second stage is

$$TE_2(a_i) = \frac{g_B(a_i, b_2)}{2}\left(1 + \frac{g_B(a_i, b_2)}{g_B(a_i, b_1)}\right),$$

where $a_i, i = 1, 2$, is the winner in contest A.

4.2 Contest A

Let $F_{a_i}(x)$ be player a_i's effort cumulative distribution function, $i = 1, 2$, in contest A. Then, player a_2's mixed-strategy equilibrium $F_{a_2}(x)$ in set A is given by

$$(g_A(a_1, b_1)q_{b_1} + g_A(a_1, b_2)q_{b_2})F_{a_2}(x) - x = c_{a_1}, \tag{21}$$

where c_{a_1} is player a_1's expected payoff and is equal to

$$\begin{aligned} c_{a_1} &= (g_A(a_1, b_1)q_{b_1} + g_A(a_1, b_2)q_{b_2}) \\ &\quad -(g_A(a_2, b_1)q_{b_1} + g_A(a_2, b_1)q_{b_2}). \end{aligned} \tag{22}$$

Player a_1's mixed strategy equilibrium in contest A is given by

$$(g_A(a_2, b_1)q_{b_1} + g_A(a_2, b_1)q_{b_2})F_{a_1}(x) - x = c_{a_2}. \tag{23}$$

where c_{a_2} is player a_2's expected payoff and is equal to

$$c_{a_2} = 0.$$

By (23), player a_1's mixed strategy equilibrium in contest A is

$$F_{a_1}(x) = \frac{x}{g_A(a_2, b_1)q_{b_1} + g_A(a_2, b_1)q_{b_2}}, \tag{24}$$

and by (21), player a_2's mixed strategy equilibrium is

$$F_{a_2}(x) = \frac{x + (g_A(a_1, b_1)q_{b_1} + g_A(a_1, b_2)q_{b_2}) - (g_A(a_2, b_1)q_{b_1} + g_A(a_2, b_1)q_{b_2})}{g_A(a_1, b_1)q_{b_1} + g_A(a_1, b_2)q_{b_2}}. \tag{25}$$

The players' winning probabilities in contest A are

$$\begin{aligned} q_{a_1} &= 1 - q_{a_2} \\ q_{a_2} &= \frac{g_A(a_2, b_1)q_{b_1} + g_A(a_2, b_2)q_{b_2}}{2(g_A(a_1, b_1)q_{b_1} + g_A(a_1, b_2)q_{b_2})}. \end{aligned} \tag{26}$$

Note that since $a_1 > a_2$ and $\frac{dg_A(a,b)}{da} \geq 0$ we obtain that $q_{a_1} > q_{a_2}$.

By the above analysis, we obtain that the players' expected payoffs in contest A are

$$\begin{aligned} c_{a_1} &= (g_A(a_1,b_1)-g_A(a_2,b_1))q_{b_1}+(g_A(a_1,b_2)-g_A(a_2,b_2))q_{b_2} \\ c_{a_2} &= 0, \end{aligned} \tag{27}$$

and in contest B they are

$$\begin{aligned} c_{b_1} &= q_{a_1}(g_B(a_1,b_1)-g_B(a_1,b_2))+q_{a_2}(g_B(a_2,b_1)-g_B(a_2,b_2)) \\ c_{b_2} &= 0. \end{aligned} \tag{28}$$

The players' expected total effort in contest A is

$$\begin{aligned} TE &= TE_1+q_{a_1}TE_2(a_1)+q_{a_2}TE_2(a_2) \\ &= \frac{g_A(a_2,b_1)q_{b_1}+g_A(a_2,b_2)q_{b_2}}{2}\left(1+\frac{g_A(a_2,b_1)q_{b_1}+g_A(a_2,b_2)q_{b_2}}{g_A(a_1,b_1)q_{b_1}+g_A(a_1,b_2)q_{b_2}}\right) \\ &\quad +q_{a_1}\frac{g_B(a_1,b_2)}{2}\left(1+\frac{g_B(a_1,b_2)}{g_B(a_1,b_1)}\right) \\ &\quad +q_{a_2}\frac{g_B(a_2,b_2)}{2}\left(1+\frac{g_B(a_2,b_2)}{g_B(a_2,b_1)}\right). \end{aligned} \tag{29}$$

5 A comparison of elimination games

In this section, we compare the players' expected payoffs as well as their expected total effort in the simultaneous and sequential elimination games. We show that the players' winning probabilities play a key role in this comparison. We first indicate some relations of the players' winning probabilities.

Proposition 1 *If the winning probabilities of the players in the second contest (contest B) of the sequential elimination game are equal to the winning probabilities of these players in the simultaneous elimination game, then the winning probabilities of the players in the first contest (contest A) of the sequential elimination game are also equal to the winning probabilities of these players in the simultaneous elimination game. That is,* $p_{b_i}=q_{b_i}$ *implies that* $p_{a_i}=q_{a_i}$*,* $i=1,2$*.*

By Proposition 1, it is possible that the players in the first contest (contest A) of the sequential elimination game have the same winning probabilities as in the simultaneous elimination game, but this is not true for the players in the second contest (contest B) of the sequential elimination game who might have different winning probabilities than in the simultaneous elimination game. This result will have a meaningful effect on the comparison of the players 'expected payoffs in the elimination games as follows:

Proposition 2 *If*

$$g_A(a_1, b_1) - g_A(a_2, b_1) > g_A(a_1, b_2) - g_A(a_2, b_2),$$

then player a_1's expected payoff in the simultaneous elimination game is larger than in the sequential elimination game iff his winning probability in the simultaneous elimination game is larger than in the sequential elimination game, that is, $p_{b_1} > q_{b_1}$. Player a_2's expected payoff is zero in both simultaneous and sequential elimination games.

Similarly, if

$$g_B(a_1, b_1) - g_B(a_1, b_2) > g_B(a_2, b_1) - g_B(a_2, b_2),$$

then player b_1's expected payoffs in the simultaneous elimination game is larger than in the sequential elimination game iff his winning probability in the simultaneous elimination game is larger than in the sequential elimination game, that is, $p_{a_1} > q_{a_1}$. Player b_2's expected payoff is zero in both simultaneous and sequential elimination games.

In particular, if $p_{b_1} = q_{b_1}$, namely, the players' winning probabilities in contest B are the same in the simultaneous and sequential elimination games, then they have the same expected payoffs.

By Proposition 2 we can conclude that

Conclusion 1 *The players have the same expected payoffs in both contests A and B of the sequential elimination game if in contest B the players have the same winning probabilities in the simultaneous and sequential elimination games, that is, if $p_{b_1} = q_{b_1}$.*

A relevant question is now when and under which conditions the players' winning probabilities are the same in the sequential and simultaneous elimination games. In the next section we examine this issue.

6 Applications

Assume that the players' payoff functions in the second stage are multiplicatively separable as follows:

$$\begin{aligned} g_A(a,b) &= \psi_A(a)\varphi_A(b) \qquad (30) \\ g_B(a,b) &= \varphi_B(a)\psi_B(b). \end{aligned}$$

Then, by Proposition 2, we have

Proposition 3 *Assume that in the second stage of the simultaneous and sequential elimination games players have separable payoff functions as given by (30). Then, if $\psi'_A(a) \geq 0$ and $\psi'_B(b) \geq 0$, all the players have the same probabilities of winning in both simultaneous and sequential elimination games. Thus, they also have the same expected payoffs in these games. In addition, the players' expected total effort is the same.*

By Proposition 3, we can conclude that

Conclusion 2 *Assume that in the second stage of the sequential elimination game the players have separable payoff functions as given by (30). Then, if $\psi'_A(a) \geq 0$ and $\psi'_B(b) \geq 0$, the order of contests A and B does not affect the players' expected payoffs.*

Now, assume first that the players' payoff functions in the second stage are additively separable as follows:

$$\begin{aligned} g_A(a,b) &= \psi_A(a)+\varphi_A(b) \qquad (31) \\ g_B(a,b) &= \varphi_B(a)+\psi_B(b). \end{aligned}$$

In that case, we show that the players who play in contest A have the same probabilities of winning in the simultaneous and sequential games. On the other hand, the players who play in contest B do not necessarily have the same probabilities of winning. Nevertheless, by Proposition 2 we have

Proposition 4 *Assume that in the second stage of the simultaneous and sequential elimination games players have separable payoff functions as given by (31). Then, if* $\psi'_A(a) \geq 0$ *and* $\psi'_B(b) \geq 0$*, all the players have the same expected payoffs in the sequential and and simultaneous elimination games. However, the players' expected total effort is not the same.*

Again, by Proposition 4, we can conclude that

Conclusion 3 *Assume that in the second stage of the sequential elimination game players have separable payoff functions as given by (30). Then, if* $\psi'_A(a) \geq 0$ *and* $\psi'_B(b) \geq 0$*, the order of the contests* A *and* B *does not affect the players' expected payoffs.*

In the second stage of the elimination game when there is an interaction between the two winners of the first stage, sometimes the stronger player (the player with the higher type) "pulls up" the weaker one such that the players have payoff functions in the second stage as follows:

$$g_A(a,b) = g_B(a,b) = \max\{\psi(a), \varphi(b)\}. \tag{32}$$

In that case, the allocation of the players in the two contests A and B has a crucial effect on the outcome as we show in the following result.

Proposition 5 *Assume that in the second stage of the simultaneous and sequential elimination games players have payoff functions as given by (32).*

1) When the two strongest players (the players with the highest types) are allocated in the same contest in the first stage, for example, when $a_1 > a_2 > b_1 > b_2$*, then the players have the same winning probabilities, and therefore the same expected payoffs in the simultaneous and sequential elimination games. In addition, the players' expected total efforts are the same.*

2) When the two strongest players (the players with the highest types) are allocated in different contests, for example, when $a_1 > b_1 > b_2 > a_2$, then the players do not necessarily have the same winning probabilities and not the same expected payoffs in the simultaneous and sequential elimination games. Likewise, the players' expected total efforts are not necessarily the same.

The results are similar when in the interaction between the two winners of the first stage, the weaker player "pulls down" the stronger player such that both players have payoff functions in the second stage as follows:

$$g_A(a,b) = g_B(a,b) = \min\{\psi(a), \varphi(b)\}. \tag{33}$$

In that case, we have

Proposition 6 *Assume that in the second stage of the simultaneous and sequential elimination games players have payoff functions as given by (33).*

1) When the two strongest players (the players with the highest types) are allocated in the same contest, for example, when $a_1 > a_2 > b_1 > b_2$, then the players have the same winning probabilities and therefore the same expected payoffs in the simultaneous and sequential elimination games. In addition, the players' total efforts are the same.

2) When the two strongest players (the players with the highest types) are allocated in different contests, for example, when $a_1 > b_1 > b_2 > a_2$, then the players do not necessarily have the same probabilities of winning and not the same expected payoffs in the simultaneous and sequential elimination games. Likewise, the players' total efforts are not necessarily the same.

By Propositions 5 and 6, we can conclude that

Conclusion 4 *If in the second stage of the sequential elimination game players have payoff functions as given by (32) or by (32), then, the order of the contests A and B does not affect the players' expected payoffs.*

7 Conclusion

We study two-stage elimination games with four players where in the first stage the players are allocated into two all-pay auctions, each of which includes two players. The winners of the two contests interact with each other in the second stage. The results of this interaction are given by a general form of these players' payoff functions where we assume that they are monotonically increasing in both players' types.

We show that if the players' payoff functions are either multiplicatively separable or additively separable then the timing of play in the first stage, namely, whether the two contests are played simultaneously or sequentially, does not affect the players' expected payoff as well as their total effort. Furthermore, the timing of play in the first stage of the sequential elimination game does not affect the players' expected payoffs nor their total effort.

We also show that the equivalence of the players' winning probabilities implies the equivalence of the players' expected payoffs. Furthermore, the equivalence of the players' winning probabilities in one contest of the sequential (the second one) and the simultaneous elimination games is sufficient to imply the equivalence of all the players' expected payoffs. It turns out that even if players have different winning probabilities in the simultaneous and sequential games, they still might have the same expected payoffs.

We consider elimination games with four players and two contests in the first stage, each of which includes two players. Our analysis can be trivially generalized to the case with more than two players in each contest, since then it is well known that only the two players with the highest types will be active while all the others will not exert any effort in the first stage and therefore will not participate in the second stage as well. On the other hand, a possible less simpler generalization of our model could be with more than two contests, each of which includes two players, and then all the winners of the contests in the first stage participate in the second one.

8 Appendix

8.1 Proof of Proposition 1

By (7), the players' winning probabilities in contest A of the simultaneous elimination game are

$$\begin{aligned} p_{a_2} &= \frac{g_A(a_2, b_1)p_{b_1} + g_A(a_2, b_2)p_{b_2}}{2(g_A(a_1, b_1)p_{b_1} + g_A(a_1, b_2)p_{b_2})} \\ p_{a_1} &= 1 - p_{a_2}, \end{aligned}$$

and by (26), the players' winning probabilities in contest A of the simultaneous elimination game are

$$\begin{aligned} q_{a_2} &= \frac{g_A(a_2, b_1)q_{b_1} + g_A(a_2, b_2)q_{b_2}}{2(g_A(a_1, b_1)q_{b_1} + g_A(a_1, b_2)q_{b_2})} \\ q_{a_1} &= 1 - q_{a_2}. \end{aligned}$$

By a comparison of these winning probabilities, it can be easily verified that $p_{b_1} = q_{b_1}$ implies that $p_{a_2} = q_{a_2}$. It can be verified that $p_{a_2} = q_{a_2}$ does not imply that $p_{a_1} = q_{a_1}$.

8.2 Proof of Proposition 2

The players' expected payoffs in the simultaneous elimination game which are given by (3), (5), (9), and (11), are

$$\begin{aligned} u_{a_1} &= (g_A(a_1, b_1) - g_A(a_2, b_1))p_{b_1} + (g_A(a_1, b_2) - g_A(a_2, b_2))p_{b_2} \\ u_{a_2} &= 0, \end{aligned}$$

and

$$\begin{aligned} u_{b_1} &= (g_B(a_1, b_1) - g_B(a_1, b_2))p_{a_1} + (g_B(a_2, b_1) - g_B(a_2, b_2))p_{a_2} \\ u_{b_2} &= 0. \end{aligned}$$

On the other hand, the players' expected payoffs in the sequential elimination game which are given by (15), (17), (27), and (28), are

$$
\begin{aligned}
c_{a_1} &= (g_A(a_1,b_1) - g_A(a_2,b_1))q_{b_1} + (g_A(a_1,b_2) - g_A(a_2,b_2))q_{b_2} \\
c_{a_2} &= 0,
\end{aligned}
$$

and

$$
\begin{aligned}
c_{b_1} &= (g_B(a_1,b_1) - g_B(a_1,b_2))q_{a_1} + (g_B(a_2,b_1) - g_B(a_2,b_2))q_{a_2} \\
c_{b_2} &= 0.
\end{aligned}
$$

The difference between players a_1's and b_1's expected payoffs in the simultaneous and sequential elimination games are:

$$
\begin{aligned}
u_{a_1} - c_{a_1} &= (g_A(a_1,b_1) - g_A(a_2,b_1))(p_{b_1} - q_{b_1}) + \\
&\quad (g_A(a_1,b_2) - g_A(a_2,b_2))(p_{b_2} - q_{b_2}),
\end{aligned}
$$

and

$$
\begin{aligned}
u_{b_1} - c_{b_1} &= (g_B(a_1,b_1) - g_B(a_1,b_2))(p_{a_1} - q_{a_1}) + \\
&\quad (g_B(a_2,b_1) - g_B(a_2,b_2))(p_{a_2} - q_{a_2}).
\end{aligned}
$$

Since $(p_{b_1} - q_{b_1}) = -(p_{b_2} - q_{b_2})$, we obtain that if $g_A(a_1,b_1) - g_A(a_2,b_1) > g_A(a_1,b_2) - g_A(a_2,b_2)$, then, player a_1's expected payoff in the simultaneous elimination game is larger than in the sequential elimination game iff $p_{b_1} > q_{b_1}$. Similarly, since $(p_{a_1} - q_{a_1}) = -(p_{a_2} - q_{a_2})$, we obtain that if $g_B(a_1,b_1) - g_B(a_1,b_2) > g_B(a_2,b_1) - g_B(a_2,b_2)$, then, player a_1's expected payoff in the simultaneous elimination game is larger than in the sequential elimination game iff $p_{b_1} > q_{b_1}$.

In particular if $p_{b_i} = q_{b_i}, i = 1,2$ then $u_{a_i} = c_{a_i}, i = 1,2$. By Proposition 1, $p_{b_i} = q_{b_i}$ implies that $p_{a_i} = q_{a_i}, i = 1,2$ which imply that $u_{b_i} = c_{b_i}, i = 1,2$. Thus, the players' expected payoffs are the same in the simultaneous and sequential elimination games.

8.3 Proof of Proposition 3

Assume that players in both elimination games have the payoff functions $g_A(a,b) = \psi_A(a)\varphi_A(b)$ and $g_B(a,b) = \psi_B(a)\varphi_B(b)$. Then, by (7), in contest A of the simultaneous elimination game, the players' winning probabilities are

$$\begin{aligned}
p_{a_2} &= \frac{g_A(a_2,b_1)p_{b_1} + g_A(a_2,b_2)p_{b_2}}{2(g_A(a_1,b_1)p_{b_1} + g_A(a_1,b_2)p_{b_2})} \\
&= \frac{\psi_A(a_2)\varphi_A(b_1)q_{b_1} + \psi_A(a_2)\varphi_A(b_2)q_{b_2}}{2(\psi_A(a_1)\varphi_A(b_1)q_{b_1} + \psi_A(a_1)\varphi_A(b_2)q_{b_2})} \\
&= \frac{\psi_A(a_2)}{2\psi_A(a_1)} \\
p_{a_1} &= 1 - \frac{\psi_A(a_2)}{2\psi_A(a_1)},
\end{aligned}$$

and, by (12), the players' winning probabilities in contest B are

$$\begin{aligned}
p_{b_2} &= \frac{g_B(a_1,b_2)p_{a_1} + g_B(a_2,b_2)p_{a_2}}{2(g_B(a_1,b_1)p_{a_1} + g_B(a_2,b_1)p_{a_2})} \\
&= \frac{\varphi_B(a_1)\psi_B(b_2)p_{a_1} + \varphi_B(a_2)\psi_B(b_2)p_{a_2}}{2(\varphi_B(a_1)\psi_B(b_1)p_{a_1}q_{a_1} + \varphi_B(a_2)\psi_B(b_1)p_{a_1}q_{a_2})} \\
&= \frac{\psi_B(b_2)}{2\psi_B(b_1)} \\
p_{b_1} &= 1 - \frac{\psi_B(b_2)}{2\psi_B(b_1)}.
\end{aligned}$$

Likewise, by (26), in contest A of the sequential elimination game, the players' winning probabilities are

$$\begin{aligned}
q_{a_2} &= \frac{g_A(a_2,b_1)q_{b_1} + g_A(a_2,b_2)q_{b_2}}{2(g_A(a_1,b_1)q_{b_1} + g_A(a_1,b_2)q_{b_2})} \\
&= \frac{\psi_A(a_2)\varphi_A(b_1)q_{b_1} + \psi_A(a_2)\varphi_A(b_2)q_{b_2}}{2(\psi_A(a_1)\varphi_A(b_1)q_{b_1} + \psi_A(a_1)\varphi_A(b_2)q_{b_2}} = \frac{\psi_A(a_2)}{2\psi_A(a_1)} \\
q_{a_1} &= 1 - \frac{\psi_A(a_2)}{2\psi_A(a_1)}.
\end{aligned}$$

and by (20), in the sequential elimination game, the players' probabilities of winning in contest B are

$$\begin{aligned}
q_{b_2} &= \frac{g_B(a_i,b_2)}{2g_B(a_i,b_1)} = \frac{\varphi_B(a_i)\psi_B(b_2)}{2\varphi_B(a_i)\psi_B(b_1)} = \frac{\psi_B(b_2)}{2\psi_B(b_1)} \\
q_{b_1} &= 1 - \frac{\psi_B(b_2)}{2\psi_B(b_1)},
\end{aligned}$$

We can see that

$$\begin{aligned} p_{a_i} &= q_{a_i}, i = 1, 2 \\ p_{b_i} &= q_{b_i}, i = 1, 2 \end{aligned}$$

Thus, according to Proposition 2, the players' expected payoffs are the same in the sequential and simultaneous elimination games.

Now, by (13), the players' expected total effort in the simultaneous elimination game is

$$\begin{aligned} TE_{sim} &= \frac{g_A(a_2,b_1)p_{b_1} + g_A(a_2,b_2)p_{b_2}}{2}(1 + \frac{g_A(a_2,b_1)p_{b_1} + g_A(a_2,b_2)p_{b_2}}{(g_A(a_1,b_1)p_{b_1} + g_A(a_1,b_2)p_{b_2})}) \\ &+ \frac{g_B(a_1,b_2)p_{a_1} + g_B(a_2,b_2)p_{a_2}}{2}(1 + \frac{g_B(a_1,b_2)p_{a_1} + g_B(a_2,b_2)p_{a_2}}{g_B(a_1,b_1)p_{a_1} + g_B(a_2,b_1)p_{a_2}}) \\ &= \frac{a_2(b_1 p_{b_1} + b_2 p_{b_2})}{2}(1 + \frac{a_2}{a_1}) + \frac{b_2(a_1 p_{a_1} + a_2 p_{a_2})}{2}(1 + \frac{b_2}{b_1}) \\ &= p_{a_1}\frac{b_2 a_1}{2}(1 + \frac{b_2}{b_1}) + p_{a_2}\frac{b_2 a_2}{2}(1 + \frac{b_2}{b_1}) + p_{b_1}\frac{b_1 a_2}{2}(1 + \frac{a_2}{a_1}) + p_{b_2}\frac{b_2 a_2}{2}(1 + \frac{a_2}{a_1}). \end{aligned}$$

And, by (29), the players' expected total effort in the sequential elimination game is

$$\begin{aligned} TE_{\text{seq}} &= TE_1 + q_{a_1}TE_2(a_1) + q_{a_2}TE_2(a_2) \\ &= \frac{g_A(a_2,b_1)q_{b_1} + g_A(a_2,b_2)q_{b_2}}{2}(1 + \frac{g_A(a_2,b_1)q_{b_1} + g_A(a_2,b_1)q_{b_2}}{g_A(a_1,b_1)q_{b_1} + g_A(a_1,b_2)q_{b_2}}) \\ &+ q_{a_1}\frac{g_B(a_1,b_2)}{2}(1 + \frac{g_B(a_1,b_2)}{g_B(a_1,b_1)}) + q_{a_2}\frac{g_B(a_2,b_2)}{2}(1 + \frac{g_B(a_2,b_2)}{g_B(a_2,b_1)}) \\ &= \frac{a_2(b_1 q_{b_1} + b_2 q_{b_2})}{2}(1 + \frac{a_2}{a_1}) + q_{a_1}\frac{a_1 b_2}{2}(1 + \frac{b_2}{b_1}) + q_{a_2}\frac{a_2 b_2}{2}(1 + \frac{b_2}{b_1}) \\ &= q_{a_1}\frac{a_1 b_2}{2}(1 + \frac{b_2}{b_1}) + q_{a_2}\frac{a_2 b_2}{2}(1 + \frac{b_2}{b_1}) + q_{b_1}\frac{b_1 a_2}{2}(1 + \frac{a_2}{a_1}) + q_{b_2}\frac{b_2 a_2}{2}(1 + \frac{a_2}{a_1}). \end{aligned}$$

We can see that if the probabilities in the simultaneous and sequential elimination games are the same, then the expected total effort is the same in these games. namely, if $q_{a_i} = p_{a_i}, q_{b_i} = p_{b_i}, i = 1, 2$, then $TE_{sim} = TE_{\text{seq}}$.

8.4 Proof of Proposition 4

Assume that the players in both elimination games have the payoff functions $g_A(a,b) = \psi_A(a)\varphi_A(b)$ and $g_B(a,b) = \psi_B(a)\varphi_B(b)$. Then, by (7), in contest A of the simultaneous elimination game, the players' winning probabilities are

$$\begin{aligned}
p_{a_2} &= \frac{g_A(a_2,b_1)p_{b_1}+g_A(a_2,b_2)p_{b_2}}{2(g_A(a_1,b_1)p_{b_1}+g_A(a_1,b_2)p_{b_2})} \\
&= \frac{\psi_A(a_2)+\varphi_A(b_1)p_{b_1}+\varphi_A(b_2)p_{b_2}}{2(\psi_A(a_1)+\varphi_A(b_1)p_{b_1}+\varphi_A(b_2)p_{b_2})} \\
p_{a_1} &= 1-p_{a_2},
\end{aligned}$$

and, by (12), the players' winning probabilities in contest B are

$$\begin{aligned}
p_{b_2} &= \frac{g_B(a_1,b_2)p_{a_1}+g_B(a_2,b_2)p_{a_2}}{2(g_B(a_1,b_1)p_{a_1}+g_B(a_2,b_1)p_{a_2})} \\
&= \frac{\psi_B(b_2)+\varphi_A(a_1)p_{a_1}+\varphi_A(a_2)p_{a_2}}{2(\psi_B(b_1)+\varphi_A(a_1)p_{a_1}+\varphi_A(a_2)p_{a_2})}.
\end{aligned}$$

Likewise, by (26), in contest A of the sequential elimination game the players' winning probabilities are

$$\begin{aligned}
q_{a_2} &= \frac{g_A(a_2,b_1)q_{b_1}+g_A(a_2,b_2)q_{b_2}}{2(g_A(a_1,b_1)q_{b_1}+g_A(a_1,b_2)q_{b_2})} \\
&= \frac{\psi_A(a_2)+\varphi_A(b_1)q_{b_1}+\varphi_A(b_2)q_{b_2}}{2(\psi_A(a_1)+\varphi_A(b_1)q_{b_1}+\varphi_A(b_2)q_{b_2})} \\
q_{a_1} &= 1-q_{a_2},
\end{aligned}$$

and by (20), the players' probabilities of winning in contest B are

$$\begin{aligned}
q_{b_2} &= q_{a_1}\frac{g_B(a_1,b_2)}{2g_B(a_1,b_1)}+q_{a_2}\frac{g_B(a_2,b_2)}{2g_B(a_2,b_1)} \\
&= q_{a_1}\frac{\varphi_B(a_1)+\psi_B(b_2)}{2(\varphi_B(a_1)+\psi_B(b_1))}+q_{a_2}\frac{\varphi_B(a_2)+\psi_B(b_2)}{2(\varphi_B(a_2)+\psi_B(b_1))} \\
q_{b_1} &= 1-q_{b_2}.
\end{aligned}$$

We can see that in contest A, the players have the same probabilities of winning in the simultaneous and sequential elimination games. Thus, by Proposition 2, in contest B, the players' expected payoffs in the simultaneous and sequential elimination games are the same, but they are not necessarily the same in contest A. However, in contest A of the simultaneous elimination game, by (5)and (3), the players' expected payoffs are

$$\begin{aligned}
u_{a_1} &= (g_A(a_1,b_1) - g_A(a_2,b_1))p_{b_1} + (g_A(a_1,b_2) - g_A(a_2,b_2))p_{b_2} \\
&= (\psi_A(a_1) + \varphi_A(b_1) - \psi_A(a_2) - \varphi_A(b_1))p_{b_1} + (\psi_A(a_1) + \varphi_A(b_2) - \psi_A(a_2) + \varphi_A(b_2))p_{b_2} \\
&= (\psi_A(a_1) - \psi_A(a_2))(p_{b_1} + p_{b_2}) = \psi_A(a_1) - \psi_A(a_2) \\
u_{a_2} &= 0.
\end{aligned}$$

On the other hand, in the sequential elimination game, by (27), the players' expected payoffs in contest A are

$$\begin{aligned}
u_{a_1} &= c_{a_1} = (g_A(a_1,b_1) - g_A(a_2,b_1))q_{b_1} + (g_A(a_1,b_2) - g_A(a_2,b_2))q_{b_2} \\
&= (\psi_A(a_1) - \psi_A(a_2))(q_{b_1} + q_{b_2}) = \psi_A(a_1) - \psi_A(a_2) \\
u_{a_2} &= 0.
\end{aligned}$$

Hence, we obtain that the players in contest A also have the same expected payoffs in the simultaneous and sequential elimination games.

Now, we compare the total effort in these elimination games. In the simultaneous game, by (13), the players' expected total effort is

$$\begin{aligned}
TE_{sim} &= \frac{g_A(a_2,b_1)p_{b_1} + g_A(a_2,b_2)p_{b_2}}{2}\left(1 + \frac{g_A(a_2,b_1)p_{b_1} + g_A(a_2,b_2)p_{b_2}}{(g_A(a_1,b_1)p_{b_1} + g_A(a_1,b_2)p_{b_2})}\right) \\
&\quad + \frac{g_B(a_1,b_2)p_{a_1} + g_B(a_2,b_2)p_{a_2}}{2}\left(1 + \frac{g_B(a_1,b_2)p_{a_1} + g_B(a_2,b_2)p_{a_2}}{g_B(a_1,b_1)p_{a_1} + g_B(a_2,b_1)p_{a_2}}\right) \\
&= \frac{a_2 + b_1p_{b_1} + b_2p_{b_2}}{2}\left(1 + \frac{a_2 + b_1p_{b_1} + b_2p_{b_2}}{a_1 + b_1p_{b_1} + b_2p_{b_2}}\right) \\
&\quad + \frac{b_2 + a_1p_{a_1} + a_2p_{a_2}}{2}\left(1 + \frac{b_2 + a_1p_{a_1} + a_2p_{a_2}}{b_1 + a_1p_{a_1} + a_2p_{a_2}}\right).
\end{aligned}$$

In the sequential game, by (29), the players' expected total effort is

$$\begin{aligned}
TE_{\text{seq}} &= TE_1 + q_{a_1}TE_2(a_1) + q_{a_2}TE_2(a_2) \\
&= \frac{g_A(a_2,b_1)q_{b_1} + g_A(a_2,b_2)q_{b_2}}{2}(1 + \frac{g_A(a_2,b_1)q_{b_1} + g_A(a_2,b_1)q_{b_2}}{g_A(a_1,b_1)q_{b_1} + g_A(a_1,b_2)q_{b_2}}) \\
&\quad + q_{a_1}\frac{g_B(a_1,b_2)}{2}(1 + \frac{g_B(a_1,b_2)}{g_B(a_1,b_1)}) \\
&\quad + q_{a_2}\frac{g_B(a_2,b_2)}{2}(1 + \frac{g_B(a_2,b_2)}{g_B(a_2,b_1)}) \\
&= \frac{a_2 + b_1q_{b_1} + b_2q_{b_2}}{2}(1 + \frac{a_2 + b_1q_{b_1} + b_2q_{b_2}}{a_2 + b_1q_{b_1} + b_2q_{b_2}}) \\
&\quad + q_{a_1}\frac{a_1+b_2}{2}(1 + \frac{a_1+b_2}{a_1+b_1}) + q_{a_2}\frac{a_2+b_2}{2}(1 + \frac{a_2+b_2}{a_2+b_1}).
\end{aligned}$$

Assume that $a_1 = a_2$ and $b_1 = b_2$. Then $p_{a_i} = q_{a_i} = 0.5$, $p_{b_i} = q_{b_i} = 0.5$. In that case, the expected total effort in the simultaneous game is

$$\begin{aligned}
TE_{sim} &= 0.5\frac{b_2a_1}{2}(1 + \frac{b_2}{b_1}) + 0.5\frac{b_2a_2}{2}(1 + \frac{b_2}{b_1}) + 0.5\frac{b_1a_2}{2}(1 + \frac{a_2}{a_1}) + 0.5\frac{b_2a_2}{2}(1 + \frac{a_2}{a_1}) \\
&= \frac{1}{4a_1b_1}(a_1b_2 + a_2b_1)(a_1 + a_2)(b_1 + b_2) \\
&= \frac{1}{4ab}(ab + ab)(a + a)(b + b) = 2ab,
\end{aligned}$$

while the total effort in the sequential game is

$$\begin{aligned}
TE_{\text{seq}} &= \frac{a + b(0.5) + b(0.5)}{2}(1 + \frac{a + b(0.5) + b(0.5)}{a + b(0.5) + b(0.5)}) + (0.5)\frac{a+b}{2}(1 + \frac{a+b}{a+b}) + (0.5)\frac{a+b}{2}(1 + \frac{a+b}{a+b}) \\
&= a + b.
\end{aligned}$$

Thus, the expected total effort is not necessarily the same in the simultaneous and sequential elimination games.

9 Proof of Proposition 5

Assume that players in both elimination games have the payoff functions $g_A(a,b) = g_B(a,b) = \max\{\psi(a), \varphi(b)\}$.

1) If $a_1 > a_2 > b_1 > b_2$, then, by (7) and (12), the players' probabilities of winning in the simultaneous elimination game are

$$\begin{aligned} p_{a_2} &= \frac{g_A(a_2,b_1)p_{b_1} + g_A(a_2,b_2)p_{b_2}}{2(g_A(a_1,b_1)p_{b_1} + g_A(a_1,b_2)p_{b_2})} = \frac{a_2}{2a_1} \\ p_{a_1} &= 1 - p_{a_2}, \end{aligned}$$

and

$$\begin{aligned} p_{b_2} &= \frac{g_B(a_1,b_2)p_{a_1} + g_B(a_2,b_2)p_{a_2}}{2(g_B(a_1,b_1)p_{a_1} + g_B(a_2,b_1)p_{a_2})} \\ &= \frac{a_1 p_{a_1} + a_2 p_{a_2}}{2(a_1 p_{a_1} + a_2 p_{a_2})} = \frac{1}{2} \\ p_{b_1} &= \frac{1}{2}. \end{aligned}$$

while, by (26) and (20), the players' probabilities of winning in the sequential elimination game are

$$\begin{aligned} q_{a_2} &= \frac{g_A(a_2,b_1)q_{b_1} + g_A(a_2,b_2)q_{b_2}}{2(g_A(a_1,b_1)q_{b_1} + g_A(a_1,b_2)q_{b_2}} = \frac{a_2}{2a_1} \\ q_{a_1} &= 1 - q_{a_2}, \end{aligned}$$

and

$$\begin{aligned} q_{b_2} &= \frac{g_B(a_i,b_2)}{2g_B(a_i,b_1)} = \frac{a_i}{2a_i} = \frac{1}{2} \\ q_{b_1} &= \frac{1}{2}. \end{aligned}$$

Since $p_{a_i} = q_{a_i}$ and $p_{b_i} = q_{b_i}, i = 1, 2$, by Proposition 2, we obtain that the players' expected payoffs are the same in the simultaneous and sequential elimination games

Now, by (13), the expected total effort in the simultaneous elimination game is

$$\begin{aligned} TE_{sim} &= \frac{g_A(a_2,b_1)p_{b_1} + g_A(a_2,b_2)p_{b_2}}{2}(1 + \frac{g_A(a_2,b_1)p_{b_1} + g_A(a_2,b_2)p_{b_2}}{(g_A(a_1,b_1)p_{b_1} + g_A(a_1,b_2)p_{b_2})}) \\ &+ \frac{g_B(a_1,b_2)p_{a_1} + g_B(a_2,b_2)p_{a_2}}{2}(1 + \frac{g_B(a_1,b_2)p_{a_1} + g_B(a_2,b_2)p_{a_2}}{g_B(a_1,b_1)p_{a_1} + g_B(a_2,b_1)p_{a_2}}) \\ &= \frac{a_2}{2}(1 + \frac{a_2}{a_1}) + a_1 p_{a_1} + a_2 p_{a_2}, \end{aligned}$$

and, by (29), the expected total effort in the sequential elimination game is

$$
\begin{aligned}
TE_{\text{seq}} &= TE_1 + q_{a_1} TE_2(a_1) + q_{a_2} TE_2(a_2) = \\
&\frac{g_A(a_2,b_1)q_{b_1} + g_A(a_2,b_2)q_{b_2}}{2}\left(1 + \frac{g_A(a_2,b_1)q_{b_1} + g_A(a_2,b_2)q_{b_2}}{g_A(a_1,b_1)q_{b_1} + g_A(a_1,b_2)q_{b_2}}\right) + \\
&q_{a_1}\frac{g_B(a_1,b_2)}{2}\left(1 + \frac{g_B(a_1,b_2)}{g_B(a_1,b_1)}\right) + q_{a_2}\frac{g_B(a_2,b_2)}{2}\left(1 + \frac{g_B(a_2,b_2)}{g_B(a_2,b_1)}\right) \\
&= \frac{a_2}{2}\left(1 + \frac{a_2}{a_1}\right) + a_1 q_{a_1} + a_2 q_{a_2}.
\end{aligned}
$$

Thus, if $a_1 > a_2 > b_1 > b_2$, the total effort is the same in the simultaneous and sequential elimination games.

2) If $a_1 > b_1 > b_2 > a_2$, then, by (7) and (12), the players' winning probabilities in the simultaneous elimination game are

$$
\begin{aligned}
p_{a_2} &= \frac{g_A(a_2,b_1)p_{b_1} + g_A(a_2,b_2)p_{b_2}}{2(g_A(a_1,b_1)p_{b_1} + g_A(a_1,b_2)p_{b_2})} = \frac{b_1 p_{b_1} + b_2 p_{b_2}}{2a_1} \\
p_{a_1} &= 1 - p_{a_2},
\end{aligned}
$$

and

$$
\begin{aligned}
p_{b_2} &= \frac{g_B(a_1,b_2)p_{a_1} + g_B(a_2,b_2)p_{a_2}}{2(g_B(a_1,b_1)p_{a_1} + g_B(a_2,b_1)p_{a_2})} = \frac{a_1 p_{a_1} + b_2 p_{a_2}}{2(a_1 p_{a_1} + b_1 p_{a_2})} \\
p_{b_1} &= 1 - p_{b_2}.
\end{aligned}
$$

Then we obtain that

$$
\begin{aligned}
p_{b_2} &= \frac{a_1\left(1 - \frac{b_1(1-p_{b_2}) + b_2 p_{b_2}}{2a_1}\right) + b_2 \frac{b_1(1-p_{b_2}) + b_2 p_{b_2}}{2a_1}}{2\left(a_1\left(1 - \frac{b_1(1-p_{b_2}) + b_2 p_{b_2}}{2a_1}\right) + b_1 \frac{b_1(1-p_{b_2}) + b_2 p_{b_2}}{2a_1}\right)} \\
&= \frac{2a_1^2 + b_1^2 - a_1 b_1}{b_1^2 - a_1 b_1 + a_1 b_2 - b_1 b_2}.
\end{aligned}
$$

On the other hand, by (26) and (20), the players' winning probabilities in the sequential elimination game are

$$
\begin{aligned}
q_{a_2} &= \frac{g_A(a_2,b_1)q_{b_1} + g_A(a_2,b_2)q_{b_2}}{2(g_A(a_1,b_1)q_{b_1} + g_A(a_1,b_2)q_{b_2})} = \frac{b_1 q_{b_1} + b_2 q_{b_2}}{2a_1} \\
q_{a_1} &= 1 - q_{a_2},
\end{aligned}
$$

and

$$
\begin{aligned}
q_{b_2} &= \frac{g_B(a_i,b_2)}{2g_B(a_i,b_1)} = \frac{a_1}{2a_1} = \frac{1}{2} \text{ if } a_i = a_1 \\
q_{b_2} &= \frac{g_B(a_i,b_2)}{2g_B(a_i,b_1)} = \frac{b_2}{2b_1} \text{if } a_i = a_2 \\
q_{b_1} &= 1 - q_{b_2}.
\end{aligned}
$$

Then, we obtain that

$$
\begin{aligned}
q_{b_2} &= \frac{1}{2}q_{a_1} + \frac{b_2}{2b_1}q_{a_2} = \frac{1}{2}(1 - \frac{b_1q_{b_1} + b_2q_{b_2}}{2a_1}) + \frac{b_2}{2b_1}\frac{b_1q_{b_1} + b_2q_{b_2}}{2a_1} \\
&= \frac{1}{4a_1b_1}\left(b_2^2q_{b_2} - b_1^2q_{b_1} + 2a_1b_1 + b_1b_2q_{b_1} - b_1b_2q_{b_2}\right) \\
&= \frac{1}{4a_1b_1}\left(b_2^2q_{b_2} - b_1^2(1-q_{b_2}) + 2a_1b_1 + b_1b_2(1-q_{b_2}) - b_1b_2q_{b_2}\right) \\
&= -\frac{-b_1^2 + 2a_1b_1 + b_1b_2}{b_1^2 + b_2^2 - 4a_1b_1 - 2b_1b_2}
\end{aligned}
$$

When we compare the winning probabilities in contest B when $b_1 = b_2$ in the simultaneous and sequential elimination games we obtain that

$$
\begin{aligned}
p_{b_2} - q_{b_2} &= \frac{2a_1^2 + b^2 - a_1b}{b^2 - a_1b + a_1b - b_1b} - (-\frac{-b^2 + 2a_1b + bb}{b^2 + b^2 - 4a_1b - 2bb}) \\
&= \frac{1}{2b(b - b_1)}\left(b^2 - 2ba_1 + b_1b + 4a_1^2\right)
\end{aligned}
$$

A similar comparison holds for the winning probabilities in contest A. Thus, if $a_1 > b_1 > b_2 > a_2$, the winning probabilities are not necessarily the same in the simultaneous and sequential elimination games. In that case, it can be verified that the total effort is also not the same.

9.1 Proof of Proposition 6

Assume that players in both simultaneous and sequential elimination games have the payoff functions $g_A(a,b) = g_B(a,b) = \max\{\psi(a), \varphi(b)\}$.

1) If $a_1 > a_2 > b_1 > b_2$, then, by (7) and (12), the players' winning probabilities in the

simultaneous elimination game are

$$\begin{aligned} p_{a_2} &= \frac{g_A(a_2,b_1)p_{b_1}+g_A(a_2,b_2)p_{b_2}}{2(g_A(a_1,b_1)p_{b_1}+g_A(a_1,b_2)p_{b_2})} = \frac{b_1p_{b_1}+b_2p_{b_2}}{2(b_1p_{b_1}+b_2p_{b_2})} = \frac{1}{2} \\ p_{a_1} &= \frac{1}{2}, \end{aligned}$$

and

$$\begin{aligned} p_{b_2} &= \frac{g_B(a_1,b_2)p_{a_1}+g_B(a_2,b_2)p_{a_2}}{2(g_B(a_1,b_1)p_{a_1}+g_B(a_2,b_1)p_{a_2})} = \frac{b_2p_{a_1}+b_2p_{a_2}}{2(b_1p_{a_1}+b_1p_{a_2})} = \frac{b_2}{2b_1} \\ p_{b_1} &= 1-p_{b_1}, \end{aligned}$$

while, by (26) and (20), the players' winning probabilities in the sequential elimination game are

$$\begin{aligned} q_{a_2} &= \frac{g_A(a_2,b_1)q_{b_1}+g_A(a_2,b_2)q_{b_2}}{2(g_A(a_1,b_1)q_{b_1}+g_A(a_1,b_2)q_{b_2}} = \frac{b_1q_{b_1}+b_2q_{b_2}}{2(b_1q_{b_1}+b_2q_{b_2})} = \frac{1}{2} \\ q_{a_1} &= \frac{1}{2}, \end{aligned}$$

and

$$\begin{aligned} q_{b_2} &= \frac{g_B(a_i,b_2)}{2g_B(a_i,b_1)} = \frac{b_2}{2b_1} \\ q_{b_1} &= 1-q_{b_2}. \end{aligned}$$

Since $p_{a_i} = q_{a_i}$ and $p_{b_i} = q_{b_i}, i = 1,2$, then by Proposition 2, we obtain that the players' expected payoffs in the simultaneous and sequential elimination games are the same.

Now, we compare the expected total effort in these games. In the simultaneous elimination game, by (13), the players' expected total effort is

$$\begin{aligned} TE_{sim} &= \frac{g_A(a_2,b_1)p_{b_1}+g_A(a_2,b_2)p_{b_2}}{2}(1+\frac{g_A(a_2,b_1)p_{b_1}+g_A(a_2,b_2)p_{b_2}}{(g_A(a_1,b_1)p_{b_1}+g_A(a_1,b_2)p_{b_2})}) \\ &\quad +\frac{g_B(a_1,b_2)p_{a_1}+g_B(a_2,b_2)p_{a_2}}{2}(1+\frac{g_B(a_1,b_2)p_{a_1}+g_B(a_2,b_2)p_{a_2}}{g_B(a_1,b_1)p_{a_1}+g_B(a_2,b_1)p_{a_2}}) \\ &= \frac{b_1p_{b_1}+b_2p_{b_2}}{2}2+\frac{b_2}{2}(1+\frac{b_2}{b_1}), \end{aligned}$$

and, by (29), the expected total effort in the sequential elimination game is

$$\begin{aligned} TE &= TE_1 + q_{a_1} TE_2(a_1) + q_{a_2} TE_2(a_2) = \\ & \frac{g_A(a_2,b_1)q_{b_1} + g_A(a_2,b_2)q_{b_2}}{2}\left(1 + \frac{g_A(a_2,b_1)q_{b_1} + g_A(a_2,b_2)q_{b_2}}{g_A(a_1,b_1)q_{b_1} + g_A(a_1,b_2)q_{b_2}}\right) \\ & + + q_{a_1}\frac{g_B(a_1,b_2)}{2}\left(1 + \frac{g_B(a_1,b_2)}{g_B(a_1,b_1)}\right) + q_{a_2}\frac{g_B(a_2,b_2)}{2}\left(1 + \frac{g_B(a_2,b_2)}{g_B(a_2,b_1)}\right) \\ &= \frac{b_1 q_{b_1} + b_2 q_{b_2}}{2} 2 + q_{a_1}\frac{b_2}{2}\left(1 + \frac{b_2}{b_1}\right) + q_{a_2}\frac{b_2}{2}\left(1 + \frac{b_2}{b_1}\right). \end{aligned}$$

Thus, if $a_1 > a_2 > b_1 > b_2$, the total effort is the same in the simultaneous and sequential elimination games.

2) If $a_1 > b_1 > b_2 > a_2$, then, by (7) and (12), the players' winning probabilities in the simultaneous elimination game are

$$\begin{aligned} p_{a_2} &= \frac{g_A(a_2,b_1)p_{b_1} + g_A(a_2,b_2)p_{b_2}}{2(g_A(a_1,b_1)p_{b_1} + g_A(a_1,b_2)p_{b_2})} \\ &= \frac{a_2}{2(b_1 p_{b_1} + b_2 p_{b_2})} \\ p_{a_1} &= 1 - p_{a_2}, \end{aligned}$$

and

$$\begin{aligned} p_{b_2} &= \frac{g_B(a_1,b_2)p_{a_1} + g_B(a_2,b_2)p_{a_2}}{2(g_B(a_1,b_1)p_{a_1} + g_B(a_2,b_1)p_{a_2})} \\ &= \frac{b_2 p_{a_1} + a_2 p_{a_2}}{2(b_1 p_{a_1} + a_2 p_{a_2})} \\ p_{b_1} &= 1 - p_{b_2}. \end{aligned}$$

Then we obtain that

$$\begin{aligned} p_{b_2} &= \frac{b_2 p_{a_1} + a_2 p_{a_2}}{2(b_1 p_{a_1} + a_2 p_{a_2})} = \frac{b_2\left(1 - \frac{a_2}{2(b_1 p_{b_1} + b_2 p_{b_2})}\right) + a_2 \frac{a_2}{2(b_1 p_{b_1} + b_2 p_{b_2})}}{2\left(b_1\left(1 - \frac{a_2}{2(b_1 p_{b_1} + b_2 p_{b_2})}\right) + a_2 \frac{a_2}{2(b_1 p_{b_1} + b_2 p_{b_2})}\right)} \\ &= \frac{a_2^2 - a_2 b_2 + 2b_1 b_2}{2a_2^2 + 4b_1^2 - 2b_2^2 - 2a_2 b_1 + 2b_1 b_2}. \end{aligned} \tag{34}$$

while, by (26) and (20), the players' winning probabilities in the sequential elimination game are

$$\begin{aligned} q_{a_2} &= \frac{g_A(a_2,b_1)q_{b_1} + g_A(a_2,b_2)q_{b_2}}{2(g_A(a_1,b_1)q_{b_1} + g_A(a_1,b_2)q_{b_2}} = \frac{a_2}{2(b_1 q_{b_1} + b_2 q_{b_2})} \\ q_{a_1} &= 1 - q_{a_2}, \end{aligned}$$

and,

$$\begin{aligned} q_{b_2} &= \frac{g_B(a_i,b_2)}{2g_B(a_i,b_1)} = \left\{ \begin{array}{c} \frac{b_2}{2b_1} \text{ if } a_i = a_1 \\ \frac{1}{2} \text{ if } a_i = a_2 \end{array} \right\} \frac{b_2}{2b_1} \text{ if } a_i = a_1 \\ q_{b_1} &= 1 - q_{b_2}. \end{aligned}$$

Then we obtain that

$$\begin{aligned} q_{b_2} &= \frac{b_2}{2b_1} q_{a_1} + \frac{1}{2} q_{a_2} = \frac{b_2}{2b_1}(1 - \frac{a_2}{2(b_1 q_{b_1} + b_2 q_{b_2})}) + \frac{1}{2}\frac{a_2}{2(b_1 q_{b_1} + b_2 q_{b_2})} \\ &= \frac{b_2}{2b_1}(1 - \frac{a_2}{2(b_1(1-q_{b_2}) + b_2 q_{b_2})}) + \frac{1}{2}\frac{a_2}{2(b_1(1-q_{b_2}) + b_2 q_{b_2})} \\ &= \frac{a_2 b_1 - a_2 b_2 + 2b_1 b_2}{4b_1^2 - 2b_2^2 + 2b_1 b_2}. \end{aligned} \tag{35}$$

By (34) and (35) we have that

$$\begin{aligned} p_{b_2} - q_{b_2} &= \frac{a_2^2 - a_2 b_2 + 2b_1 b_2}{2a_2^2 + 4b_1^2 - 2b_2^2 - 2a_2 b_1 + 2b_1 b_2} - \frac{a_2 b_1 - a_2 b_2 + 2b_1 b_2}{4b_1^2 - 2b_2^2 + 2b_1 b_2} \\ &= \frac{1}{2} a_2 (a_2 - b_1) \frac{b_1 - b_2}{2b_1^2 + b_1 b_2 - b_2^2} \frac{2b_1 - a_2 + b_2}{a_2^2 - a_2 b_1 + 2b_1^2 + b_1 b_2 - b_2^2} \end{aligned}$$

Thus, the players' winning probabilities in both games are not necessarily the same. Furthermore, since we have that

$$(g_A(a_1,b_1) - g_A(a_2,b_1)) = b_1 - a_2 > (g_A(a_1,b_2) - g_A(a_2,b_2)) = b_2 - a_2$$

by Proposition 2, player a_1's expected payoff in the simultaneous elimination game is larger than in the sequential elimination game iff $p_{b_1} > q_{b_1}$ or, alternatively, iff $p_{b_2} < q_{b_2}$.

Similarly, since we have that

$$(g_B(a_1,b_1) - g_B(a_1,b_2)) = b_1 - b_2 > (g_B(a_2,b_1) - g_B(a_2,b_2)) = a_2 - a_2 = 0$$

player b_1' expected payoffs in the simultaneous elimination game is larger than in the sequential elimination game iff $p_{a_1} > q_{a_1}$, or, alternatively, iff $p_{a_2} < q_{a_2}$. In that case, it can be verified that the expected total effort is not the same in the simultaneous and sequential elimination games either.

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