# EFFORT ALLOCATIONS IN ELIMINATION TOURNAMENTS

Aner Sela

Discussion Paper No. 21-03

August 2021

Monaster Center for
Economic Research
Ben-Gurion University of the Negev
P.O. Box 653
Beer Sheva, Israel

Fax: 972-8-6472941
Tel: 972-8-6472286

# Effort Allocations in Elimination Tournaments

Aner Sela*

August 11, 2021

**Abstract**

We study elimination tournaments with $n$ stages and $2^n$ symmetric players. The players have heterogeneous effort budgets that decrease within the stages proportionally to the efforts allocated in the previous stages such that for each effort unit that a player allocates, he loses $\alpha$ (the fatigue parameter) units of effort from his budget. We show that if the fatigue parameter $\alpha$ is larger than $\frac{1}{n}$, the players equally allocate their efforts over all the first $n-1$ stages, and only in the final stage, they exert a lower effort.

**JEL Classification Numbers** D72, D82, D44

**Keywords** Game theory, elimination tournaments, knockout tournaments.

## 1 Introduction

In each stage of an elimination contest some of the players are removed while others advance to the next stage until the final stage in which usually one player wins a prize. In this paper, we focus on elimination tournaments in which teams or individual players play pair-wise matches, and the winner advances to the next round while the loser is eliminated from the competition. Many sportive events are organized as elimination tournaments including the ATP tennis tournaments; professional playoffs in US-basketball, football, baseball and hockey; NCAA college basketball; the FIFA (soccer) world-championship playoffs; the UEFA Champions League; Olympic disciplines such as fencing, boxing, and wrestling; and top level bridge and chess tournaments.

*Department of Economics, Ben-Gurion University of the Negev, Beer–Sheva 84105, Israel. Email: anersela@bgu.ac.il

The elimination tournament was first studied in the statistical literature. The pioneering paper of David (1959) considered the winning probability of the top player in a four-player tournament with a random seeding (see also Glenn 1960 and Searles 1963 for early contributions). Most works in this literature suggest formulas for computing overall probabilities by which various players will win the tournament (see Horen and Reizman 1985 who consider general, fixed win probabilities and analyze tournaments with four and eight players) while others (see, for example, Hwang 1982, Horen and Reizman 1985, Schwenk 2000, Ryvkin 2010, and Karpov 2016) consider various optimality criteria for choosing seedings. These works assume that for each game among players $i$ and $j$ there is a fixed, exogenously given probability that $i$ beats $j$. This probability does not depend on the stage of the tournament in which the particular game takes place nor on the identity of the expected opponent at the next stage. As opposed to the statistical literature, in the economic literature, the winning probabilities in each game become endogenous in that they result from equilibrium strategies and are dependent on continuation values of winning. Moreover, the win probabilities depend on the stage of the tournament in which the game takes place as well as on the identity of the future expected opponents (see, Gradstein and Konrad 1999, Groh et al. 2012, Stracke et al. 2014, Krakel 2014, and Netanel and Sela 2017).

Here we concentrate on the effort allocation in elimination tournaments. In contrast to the common economic models, we assume that players do not have effort cost functions, but instead, each player has an effort budget by which he decides about his effort level in each match. The players compete according to the contest success function of the Tullock contest (see Tullock 1980). In each stage part of a player's effort is completely diminished while part is recycled. In other words, the players have heterogeneous effort budgets that decrease within the stages proportionally to the effort allocated in the previous stages, such that for each effort unit that a player allocates in any stage, he loses $0 < \alpha \leq 1$ units of effort from his budget.

We consider $n$-stage elimination tournaments with $2^n$ symmetric players. We show that if the fatigue parameter $\alpha$ is larger than $\frac{1}{n}$, the players exert the same effort of $\frac{v}{n\alpha}$ over all the first $n-1$ stages, and only in the final stage, they exert a lower effort of $\frac{v}{n}$.The explanation for this result is that a player's continuation value of winning increases in all the stages, but, his effort budget decreases. These two opposite forces, the continuation value of winning on the one hand, and the effort budget on the other, balance each other, such that the players equally allocate their efforts over all the stages except the last one.

## 2 The model

$2^n$ symmetric players (or teams) compete in an elimination tournament. In the first stage, they are allocated to $2^{n-1}$ pairs of players who simultaneously compete. Then, in each stage the winners of the previous stage are allocated to pairs of players and the winners advance to the next stage, until the last two players compete in the final, and the winner of the final wins the tournament. The players have the same value of winning which is normalized to 1. We model each match as a Tullock contest but without an effort-cost function: player $i$'s probability of winning in the match against player $j$ is $\frac{x_i}{x_i+x_j}$ where $x_i$ and $x_j$ are these players' effort allocations. In the first stage, each player has a budget of $v$ units of effort which he can allocate across all the stages. The effort budgets are reduced over the stages such that for each effort unit that a player allocates in any stage, he loses $\alpha$ units of effort from his budget, or formally, $v_{t+1}^i = v_t^i - \alpha x_t^i$, $0 \leq \alpha < 1$ where $v_t^i$ is the player $i$'s effort budgets in stage $t$. A player's effort allocation in each stage is smaller or equal to his effort budget in that stage. We refer to the parameter $\alpha$ as the fatigue parameter. It is assumed that each unit of effort up to the effort budget has a zero opportunity cost, so that the effort budget is "use it or lose it."

## 3 The equilibrium analysis

We begin with our main result about the effort allocation in elimination tournaments.

**Proposition 1** *In an elimination tournament with* $n$ *stages and* $2^n$ *symmetric players, if* $1 > \alpha > \frac{1}{n}$*, there is a subgame-perfect equilibrium in which every player exerts the same effort of* $\frac{v}{n\alpha}$ *in all the first* $n-1$ *stages, and only in the final stage, he exerts a lower effort of* $\frac{v}{n}$*.*

It is worth noting that if the fatigue parameter $\alpha$ is sufficiently small ($\alpha \leq \frac{1}{n}$), the players will allocate efforts that are equal to their effort budget in every stage. In order to prove the existence of the subgame-perfect equilibrium of the elimination tournament given by Proposition 1, we begin with the last (final) stage and go backwards to the first one.

### 3.1 The final stage

In the final stage (the $n$-th stage), each of the finalists exerts an effort that is equal to his effort budget in that stage,

$$x_n = v - \alpha \sum_{i=1}^{n-1} x_i. \tag{1}$$

By Proposition 1, the equilibrium efforts in the previous stages are $x_1 = x_2 =, ..., x_{n-1} = \frac{v}{n\alpha}$. Substituting these effort levels into (1) gives the equilibrium effort in the final stage, $x_n = \frac{v}{n}$. By (1), the expected payoff of each finalist is

$$u_n = \frac{v - \alpha \sum_{i=1}^{n-1} x_i}{v - \alpha \sum_{i=1}^{n-1} x_i + v - \alpha \sum_{i=1}^{n-1} y_i}, \tag{2}$$

where $x_i,\ i = 1, ..., n-1$denotes this finalist's effort in the $i$-th stage, and $y_i,\ i = 1, ..., n-1$ denotes his opponent's effort in that stage.

### 3.2 The semifinal stage

In the semifinal stage (the $n-1$-th stage), if a player wins, his expected payoff in the final stage is given by (2). Thus, a player's maximization problem in the semifinal stage is

$$\max_{x_{n-1}} \frac{x_{n-1}}{x_{n-1} + \widehat{y}_{n-1}} \frac{v - \alpha \sum_{i=1}^{n-1} x_i}{v - \alpha \sum_{i=1}^{n-1} x_i + v - \alpha \sum_{i=1}^{n-1} y_i}, \tag{3}$$

where $y_i$ denotes the effort in the $i$-th stage of his opponent's effort in the final stage, while $\widehat{y}_{n-1}$ denotes his opponent's effort in the semifinal stage. By symmetry, $y_{n-1} = \widehat{y}_{n-1}$. Then, the first-order condition (FOC) is

$$\begin{aligned} & \frac{y_{n-1}}{(x_{n-1} + y_{n-1})^2} \frac{v - \alpha \sum_{i=1}^{n-1} x_i}{(v - \alpha \sum_{i=1}^{n-1} x_i + v - \alpha \sum_{i=1}^{n-1} y_i)} - \frac{x_{n-1}}{x_{n-1} + y_{n-1}} \frac{\alpha(v - \alpha \sum_{1=i}^{n-1} y_i)}{(v - \alpha \sum_{1=i}^{n-1} x_i + v - \alpha \sum_{i=1}^{n-1} y_i)^2} \\ = & \frac{1}{(x_{n-1} + y_{n-1})(v - \alpha \sum_{i=1}^{n-1} x_i + v - \alpha \sum_{i=1}^{n-1} y_i)} \left( \frac{y_{n-1}(v - \alpha \sum_{i=1}^{n-1} x_i)}{x_{n-1} + y_{n-1}} - \frac{-\alpha x_{n-1}(v - \alpha \sum_{i=1}^{n-1} y_i)}{(v - \alpha \sum_{i=1}^{n-1} x_i + v - \alpha \sum_{i=1}^{n-1} y_i)} \right) \\ = & 0. \end{aligned} \tag{4}$$

By symmetry, $x_i = y_i,\ i = 1, ..., n-1$. Then, by (4), we obtain that

$$\frac{1}{2} - \frac{-\alpha x_{n-1}}{2(v - \alpha \sum_{i=1}^{n-1} x_i)} = 0.$$

Thus, the equilibrium effort in the semifinal stage is

$$x_{n-1} = \frac{v}{2\alpha} - \frac{\sum_{i=1}^{n-2} x_i}{2}. \tag{5}$$

By Proposition 1, the equilibrium efforts in the previous stages are $x_1 = x_2 =, ..., x_{n-2} = \frac{v}{n\alpha}$. Substituting these effort levels in (5) gives the equilibrium effort in the semifinal, $x_{n-1} = \frac{v}{n\alpha}$. By (3) and (5), a player's utility in the semifinal stage is

$$u_{n-1} = (\frac{(v - \alpha \sum_{i=1}^{n-2} x_i)}{(v - \alpha \sum_{i=1}^{n-2} x_i) + (v - \alpha \sum_{i=1}^{n-2} y_i)})^2 \tag{6}$$

### 3.3 Stage $t$ $(t = 1, ..., n-2\ )$

Given the previous results (2) and (6), by induction, we assume that a player's utility in stage $t+1, 1 \le t < n-1$ is

$$u_{t+1} = (\frac{(v - \alpha \sum_{i=1}^{t} x_i)}{(v - \alpha \sum_{i=1}^{t} x_i) + (v - \alpha \sum_{i=1}^{t} y_i)})^{n-t},$$

where $x_i$, $i = 1, 2, ..., t$ denotes this player's effort in the $i$-th stage, and $y_i$, $i = 1, 2, ..., t$ denotes the effort in the $i$-th stage of his opponent in stage $t+1$. Then, the maximization problem of a player in stage $t$ is

$$\max_{x_t} \frac{x_t}{x_t + \widehat{y}_t}(\frac{(v - \alpha \sum_{i=1}^{t} x_i)}{(v - \alpha \sum_{i=1}^{t} x_i) + (v - \alpha \sum_{i=1}^{t} y_i)})^{n-t}, \tag{7}$$

where $y_i$ denotes the effort in the $i$-th stage, $i = 1, ...., t$, of his opponent's effort in stage $t+1$ , while $\widehat{y}_t$ denotes his opponent's effort in stage $t$. By symmetry, $y_t = \widehat{y}_t$. Then, the FOC is

$$\begin{aligned} & \frac{y_t}{(x_t + y_t)^2}(\frac{(v - \alpha \sum_{i=1}^{t} x_i)}{(v - \alpha \sum_{i=1}^{t} x_i) + (v - \alpha \sum_{i=1}^{t} y_i)})^{n-t} \\ & - \frac{x_t}{x_t + y_t}(n-t)(\frac{(v - \alpha \sum_{i=1}^{t} x_i)}{(v - \alpha \sum_{i=1}^{t} x_i) + (v - \alpha \sum_{i=1}^{t} y_i)})^{n-t-1} \frac{\alpha(v - \alpha \sum_{i=1}^{t} x_i)}{((v - \alpha \sum_{i=1}^{t} x_i) + (v - \alpha \sum_{i=1}^{t} y_i))^2} \\ = \;& \frac{(v - \alpha \sum_{i=1}^{t} x_i)^{n-t-1}}{(x_t + y_t)((v - \alpha \sum_{i=1}^{t} x_i) + (v - \alpha \sum_{i=1}^{t} y_i))^{n-t}}(\frac{y_t(v - \alpha \sum_{i=1}^{t} x_i)}{x_t + y_t} - \frac{(n-t)\alpha x_t(v - \alpha \sum_{i=1}^{t} y_i)}{((v - \alpha \sum_{i=1}^{t} x_i) + (v - \alpha \sum_{i=1}^{t} y_i))}) \\ = \;& 0. \end{aligned} \tag{8}$$

By symmetry, $x_i = y_i$, $i = 1, ..., t$. Then, by (8), we obtain that

$$\frac{1}{2} - (\frac{n-t}{2})\frac{\alpha x_t}{(v - \alpha \sum_{i=1}^{t} x_i)} = 0.$$

Thus, for all $n-1 > t \ge 1$

$$x_t = \frac{v}{(n-t+1)\alpha} - \frac{\sum_{i=1}^{t-1} x_i}{(n-t+1)}. \tag{9}$$

By Proposition 1, the equilibrium efforts in the previous stages are $x_1 = x_2 =, ..., x_{t-1} = \frac{v}{n\alpha}$. Substituting these effort levels in (9) gives the equilibrium effort in stage $t$, $t = 1, 2, ...n-2$,

$$\begin{aligned} x_t &= \frac{v}{(n-t+1)\alpha} - \frac{(t-1)}{(n-t+1)} \frac{v}{n\alpha} \\ &= \frac{v}{(n-t+1)\alpha}(1 - \frac{t-1}{n}) = \frac{v}{n\alpha}. \end{aligned}$$

Then, substituting (9) in (7) confirms our induction assumption that

$$\begin{aligned} u_t &= \frac{x_t}{x_t + \widehat{y}_t} u_{t+1} = \frac{x_t}{x_t + \widehat{y}_t} (\frac{(v - \alpha \sum_{i=1}^{t} x_i)}{(v - \alpha \sum_{1=1}^{t} x_i) + (v - \alpha \sum_{i=1}^{t} y_i)})^{n-t} \\ &= (\frac{(v - \alpha \sum_{1=1}^{t} x_i)}{(v - \alpha \sum_{1=1}^{t} x_i) + (v - \alpha \sum_{i=1}^{t} y_i)})^{n-t+1}. \end{aligned}$$

□

# 4 Conclusion

We studied elimination tournaments with $n$ stages and $2^n$ symmetric players, and showed that players equally allocate their efforts over all the first $n-1$ stages, but only in the final stage, they exert a lower effort. In a related work, Sela and Erez (2013) studied a dynamic model in which there are $n$ matches over $n$ stages, where there is a prize for winning in each stage that is equal over all the stages. They found that while a player allocates a resource that is weakly decreasing over the stages, if the value of the fatigue parameter ($\alpha$) is sufficiently high, he allocates the same level of resource in the first stages and then decreases the resource allocation over the stages. Ryvkin (2011), on the other hand, studied a best-of-$k$ contest under the presence of fatigue as a reduction in a player's probability of winning resulting from previous resources. He found that agents are more likely to allocate higher resources in the later stages of the competition. These findings indicate that our results according to which players equally allocate their efforts over all the stages (except the last one) holds for elimination tournaments, but do not necessarily hold for other forms of contest.

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