THE INDIVIDUALLY ACCEPTABLE CHOICE CORRESPONDENCE

M. Mahajne and O. Volij

Discussion Paper No. 20-15

December 2020

Monaster Center for
Economic Research
Ben-Gurion University of the Negev
P.O. Box 653
Beer Sheva, Israel

Fax: 972-8-6472941
Tel: 972-8-6472286

The individually acceptable choice correspondence*

Muhammad Mahajne [†] Oscar Volij [‡]

November 11, 2020

Abstract: We study a setting in which there is an individual decision-maker who selects from every menu of feasible alternatives a non-empty subset of it. We characterize the choice procedure according to which the decision-maker has a preference relation and selects all the "individually acceptable" alternatives, namely the alternatives whose ranks are equal or less than the median rank of the menu.

Keywords Choice correspondence, axiomatization, responsive.
JEL Classification D01.

1 Introduction

A choice correspondence is a function that selects a nonepty subset of alternatives out of every menu. Eliaz, Richter, and Rubinstein (2011), and Chambers and Yenmez (2018) axiomatically characterize the family of q-responsive choice correspondences, where q is a fixed positive integer. These are choice correspondences for which there

*Mahajne acknowledges financial support from Université de Lyon (project INDEPTH Scientific Breakthrough Program of IDEX Lyon) within the program Investissement d'Avenir (ANR-16-IDEX-0005).

[†]Univ. Lyon, UJM Saint-Etienne, CNRS, GATE L-SE UMR 5824, F-42023 Saint-Etienne, France. mahajnah@gate.cnrs.fr. Declaration of Interest: None.

[‡]Ben Gurion University, Beer-Sheba, Israel. ovolij@bgu.ac.il. Declaration of Interest: None.

is a linear order on the universe of alternatives such that only the q highest ranked alternatives are selected from any menu, with the proviso that if the size of the menu is less than q all its alternatives are chosen. Eliaz, Richter, and Rubinstein (2011) mention several circumstances in which choice correspondences arise naturaly, and Chambers and Yenmez (2018) point out that q-responsive choice correspondences in particular arise in matching theory. In the context of social choice, Sertel and Yılmaz (1999) focus on social choice correspondences that select alternatives which are considered by a majority of voters to be on the better half of the alternatives. Mahajne and Volij (2018) call these alternatives socially acceptable and characterize the social choice function that selects them. In this paper we characterize the class of individually acceptable choice correspondences. These are the ones for which there is a linear order on the universe of alternatives such that only those that are placed on the better half are selected from any menu. In other words, those elements of the menu such that the number of alternatives that are ranked below it, is greater or equal than the number of alternatives ranked above it. The difference between the q-responsive choice correspondences and the individually acceptable ones is that whereas the former select a fixed number of alternatives out of any menu, the latter select a fixed percentile, the median, of the menu's alternatives. As in Chambers and Yenmez (2018), we use two axioms in our characterization, the main one being the Weaker Axiom of Revelad Preference, introduced by Jamison and Lau (1973). The other axiom requires that the size of the choice set be the median number of the menus's alternatives. Despite the similarity of the results, the proofs are different. In particular, we exploit the fact that the size of the choice set is variable to make an inductive argument.

In what follows, and after some basic definitions, we present an axiomatization of the individually acceptable choice correspondences.

2 Definitions

Let $X = \{x_1, ..., x_K\}$ be a finite set of alternatives and let $\mathcal{X}$ be the set of non-empty subsets of X. Elements of $\mathcal{X}$ are called *menus.* For any menu $A \in \mathcal{X}$, n_A denotes the cardinality of A, and $M_A = \lfloor (n_A + 1)/2 \rfloor$ is the *median rank* of the elements of A. For an individual preference relation (linear order) $\succ$ on X, and a menu $A \in \mathcal{X}$, alternatives whose ranks are equal or less than M_A, are said to be *individually acceptable*.[1]

A *choice correspondence* is a function that assigns to every menu $A \in \mathcal{X}$, a non-empty subset of A. A choice correspondence D is an *Individually Acceptable Choice Correspondence*, or IACC, if there exists a linear order $\succ$ such that D assigns to each $A \in \mathcal{X}$ the subset of its M_A highest ranked alternatives according to $\succ$. In this case, we say that D is the Individually Acceptable Choice Correspondence associated with $\succ$.

A well-known consistency requirement for choice correspondences is the Weak Axiom of Revealed Preference (WARP). A choice correspondence D satisfies the WARP if $a, b \in A \cap B$, $a \in D(A)$ and $b \in D(B)$ imply that $a \in D(B)$.

The following example shows that Individually Acceptable Choice Correspondences do not satisfy WARP.

Example 1. *let $X = \{a, b, c\}$ and A be the preference relation given by $a \succ b \succ c$. Consider $A = \{a, b, c\}$ and $B = \{b, c\}$, then according to the IACC associated with $\succ$, $D(A) = \{a, b\}$ and $D(B) = \{b\}$. This shows also that the IACC associated with $\succ$ violates WARP.*

The reason why individually acceptable choice correspondences do not satisfy WARP is that the number of alternatives they choose varies with the size of the menu. In the next section we consider an axiom weaker than WARP and use it to characterize the class of individually acceptable choice correspondences.

3 Axiomatization

It can be easily checked that all the individually acceptable choice correspondences satisfy the following axioms.

[1] In other words, for a preference relation $\succ$ and a menu A, an alternative is individually acceptable if it is placed among the most preferred "half" of the alternatives in A, and more precisely, if the number alternatives that are placed below it by $\succ$, is at least as large as the number of alternatives which are placed above it.

Median For all $A \in \mathcal{X}$, $|D(A)| = M_A$.

WrARP For all $A, B \in \mathcal{X}$, and $x, y \in A, B$, $[x \in D(A)$ and $y \notin D(A)] \Longrightarrow$ not $[y \in D(B)$ and $x \notin D(B)]$.

Dominant Element There is an alternative $x \in X$ (called *the dominant element*) such that for all $A \in \mathcal{X}, x \in A \Longrightarrow x \in D(A)$.

WrARP, appears in Jamison and Lau (1973) and in Ehlers and Sprumont (2008), is a weakening of WARP. It requires that if two elements belong to each one of two menus and only the first element is selected from the first menu then it can't be that only the second element is selected from the second menu. The Dominant Element axiom requires the existence of an alternative that is chosen whenever available. It turns out that this axiom is implied by the previous two.

Claim 1. *If a choice correspondence satisfies Median and WrARP, it satisfies Dominant Element as well.*

Proof. The proof is by induction on the cardinality of X. The claim is trivially satisfied by all choice correpondences defined on domains of cardinality 1. Assume that Median and WrARP imply Dominant Element for all choice functions defined on domains of cardinality less than K, and let D be a choice function that satisfies Median and WrARP on $X = \{x_1, \ldots, x_K\}$. Assume w.l.o.g. that

$$D(\{x_1, ..., x_K\}) = \{x_{\lfloor \frac{K+1}{2} \rfloor}, ..., x_K\} \tag{1}$$

Since D satisfies Median and WrARP on X, so does the restriction of D on the set $X' = \{x_{\lfloor \frac{K+1}{2} \rfloor}, \ldots, x_K\}$. By the induction hypothesis, this restriction satisfies Dominant Element on X'. Assume w.l.o.g. that the dominant element is x_K. Therefore, for any $x_i \in X'$, we have that $D(\{x_i, x_K\}) = \{x_K\}$. By WrARP, for any A and $x_i \in X'$ such that $x_i, x_K \in A$, if $x_i \in D(A)$ we have that $x_K \in D(A)$ as well. On the other hand, since D satisfies WrARP, it follows from (1) that for any A and $x_i \notin X'$ such that $x_i, x_K \in A$, if $x_i \in D(A)$ we have $x_K \in D(A)$ as well. We conclude that for any A such that $x_K \in A$, $x_K \in D(A)$. $\square$

The following proposition provides an axiomatization of the class of individually acceptable choice correspondences.

Theorem 1. *A choice correspondence D satisfies Median and WrARP if and only if it is a Individually acceptable choice correspondence.*

Proof. It is easy to see that any IACC satisfies the forgoing axioms. Let D be a choice correspondence that satisfies the axioms. We need to find a linear order $\succ$ on X such that for all $A \in \mathcal{X}$, $D(A)$ is the subset of its M_A highest ranked alternatives according to $\succ$. The proof is by induction on the cardinality of X. If $|X| = 1$ then, there is only one choice correspondence and it can be seen that is a IACC. It is easy to see that the statement of the proposition also holds for any X such that $|X| = 2$.

Let $K > 2$ and assume that for all sets of alternatives X' with cardinality less than K the statement of the proposition holds: Any choice correspondence D' defined on X' that satisfies the Median, and WrARP is a IACC. Let $X = \{x_1, \ldots, x_K\}$ be a set of K alternatives, and assume that D satisfies the axioms on X. By Claim 1, there is an element, say $x_K \in X$ such that for all $A \in \mathcal{X}$, $x_K \in A \Longrightarrow x_K \in D(A)$. Consider the set of alternatives

$$X' = X \setminus \{x_K\}.$$

and let D' be the restriction of D to X'. Since D satisfies Median and WrARP on X, so does D' on X'. By the induction hypothesis, there is $\succ'$ on X' such that for all $A' \in \mathcal{X}'$, $D'(A')$ is the subset of its $M_{A'}$ highest ranked alternatives according to $\succ'$. Let $\succ$ be the order on X that is obtained by extending $\succ'$ so that $x_K \succ x$ for all $x \in X'$ and let $A \in \mathcal{X}$. We need to show that $D(A)$ is the subset of its M_A highest ranked alternatives according to $\succ$. If $x_K \notin A$, then $D(A) = D'(A)$ and we are done. If, on the other hand, $x_K \in A$. There are two cases to consider:

Case 1: $A = \{a_1, \ldots, a_T, b_1, ..., b_T, x_K\}$ for some $0 \leq T < K/2$. If $T = 0$, then $A = \{x_K\}$ and the result is immediate. Therefore, assume that $T > 0$. Assume w.l.o.g. that $D(A \backslash \{x_K\}) = \{b_1, \ldots, b_T\}$. By the induction hypothesis $\{b_1, ..., b_T\}$ is the set of the $M_{A \backslash \{x_K\}}$ highest ranked members of $A \backslash \{x_K\}$ according to $\succ$. Then, by Median and WrARP $\{b_1, ..., b_T\} \subset D(A)$. Since $x_K \in D(A)$, we have that $D(A) = \{b_1, \ldots, b_T, x_K\}$. Since $x_K \succ x$ for all $x \in X'$, $D(A)$ is the set of the M_A highest ranked members of A according to $\succ$.

Case 2: $A = \{a_1, \ldots, a_T, b_1, \ldots, b_T, b_{T+1}, x_K\}$ for some $0 \leq T < (K-1)/2$. Assume w.l.o.g. that $D(A \backslash \{x_k\}) = \{b_1, \ldots, b_{T+1}\}$. By the induction hypothesis, $\{b_1, \ldots, b_{T+1}\}$ is the set of the $M_{A \backslash \{x_K\}}$ highest ranked members of $A \backslash \{x_K\}$ according to $\succ$. By Median and WrARP, $a_t \notin D(A)$ for $t = 1, \ldots, T$. Since $x_K \in D(A)$, we have that $b_t \notin D(A)$ for some $t \in \{1, \ldots, T+1\}$. Assume w.l.o.g. that $b_{T+1} \notin D(A)$. That is,

$$D(A) = \{b_1, \ldots, b_T, x_K\} \ .$$

We need to show that $\{b_1, \ldots, b_T, x_K\}$ the set of the M_A highest ranked members of A according to $\succ$. Given that $b_t \succ a_{t'}$ for all $t, t' = 1, \ldots, T$ and that $x_k \succ x$ for all $x \in X'$, it is enough to show that $b_t \succ b_{T+1}$ for all $t = 1, \ldots, T$.

Consider $A^* = A \backslash \{a_1, x_K\}$. Note that $b_t \in D(A)$ for all $t = 1, \ldots, T$ and that $b_{T+1} \in A \backslash D(A)$. Then, by WrARP, if $b_{T+1} \in D(A^*)$ we would also have $b_t \in D(A^*)$ for $t = 1, \ldots, T$. But in this case we would have $|D(A^*)| = T + 1 > M_{A^*} = T$. We conclude that $b_{T+1} \notin D(A^*)$. An analogous reasoning shows that $a_t \notin D(A^*)$ for $t = 2, \ldots, T$. Therefore $D(A^*) = \{b_1, \ldots, b_T\}$. By the induction hypothesis, $b_t \succ b_{T+1}$ for all $t = 1, \ldots, T$.

$\square$

Theorem 1 can be extended to any percentile p other than the median (for instance, the first quartile) as follows. Given a percentile p, let $P_A = \lceil pn_A \rceil$. The integer P_A is the percentile-rank associated with p and A. For example, if p is the first quartile and $n_A = 5$, then $P_A = \lceil n_A/4 \rceil = 2$. A choice correspondence D is a p-IACC if there exists a linear order $\succ$ such that D assings to each $A \in \mathcal{X}$, the subset of its P_A highest ranked alternatives according to $\succ$. The percentile-version of axiom Median is:

***p*-responsiveness** For all $A \in \mathcal{X}$, $|D(A)| = P_A$.

An analogous proof to that of Theorem 1 shows that a choice correspondence satisfies p-responsiveness and WrARP if and only if it is a p-IACC.

References

CHAMBERS, C. P., AND M. B. YENMEZ (2018): "A simple characterization of responsive choice," *Games and Economic Behavior*, 111, 217–221.

EHLERS, L., AND Y. SPRUMONT (2008): "Weakened WARP and top-cycle choice rules," *Journal of Mathematical Economics*, 44(1), 87–94.

ELIAZ, K., M. RICHTER, AND A. RUBINSTEIN (2011): "Choosing the two finalists," *Economic Theory*, 46(2), 211–219.

JAMISON, D. T., AND L. J. LAU (1973): "Semiorders and the theory of choice," *Econometrica: Journal of the Econometric Society*, pp. 901–912.

MAHAJNE, M., AND O. VOLIJ (2018): "The socially acceptable scoring rule," *Social Choice and Welfare*, 51(2), 223–233.

SERTEL, M. R., AND B. YILMAZ (1999): "The majoritarian compromise is majoritarian-optimal and subgame-perfect implementable," *Social Choice and Welfare*, 16(4), 615–627.