EFFORT MAXIMIZATION IN CONTESTS UNDER A BALANCE CONSTRAINT

Omri Haluta and Aner Sela

Discussion Paper No. 20-06

July 2020

Monaster Center for
Economic Research
Ben-Gurion University of the Negev
P.O. Box 653
Beer Sheva, Israel

Fax: 972-8-6472941
Tel: 972-8-6472286

Effort maximization in contests under a balance constraint

Omri Haluta*and Aner Sela†

May 26, 2020

Abstract

We study all-pay contests with complete information and two heterogeneous contestants who compete for a single prize. The contest is balanced if the difference between the contestants' efforts is not larger than a given threshold. We show that for every balanced all-pay contest, there is a maximum effort constraint that increases the contestants' expected total effort, while it is not necessarily increased by a minimum effort constraint.

1 Introduction

The goal of education systems is usually to maximize the achievements of the students, namely, to maximize their average grades. However, if half of the students have very high grades while the other half have low ones that are below the minimal required level such that they fail, even if the average grade of all the students is relatively high, the students' achievements would not be interpreted as successful. Thus, in this case, the goal would be to maximize the average grades subject to the constraint that their variance is not too large. Similarly, the goal in sport contests is to maximize the total output of the players or equivalently their average output, but if their variance is too large, then the contest is not competitively balanced and as such is not interesting. Therefore, in some real-life contests the goal of the designer could be to maximize

*Department of Economics, Ben-Gurion University of the Negev, 84105 Beer Sheva, Israel, haluta@post.bgu.ac.il.

†Corresponding author: Department of Economics, Ben-Gurion University of the Negev, Beer–Sheva 84105, Israel. Email: anersela@bgu.ac.il

the players' outputs under a balance constraint. In other words, the variance of the players' efforts should be bounded.

In the literature on contests, the designer usually wishes to maximize the players' expected total effort (output) or, alternatively, their average effort (see, for example, Glazer and Hassin 1988, Moldovanu and Sela 2001, Moldovanu et al. 2012, Ryvkin 2013, Franke et al. 2013, and Liu and Lu 2014), but there are contests in which the designer wishes to maximize the competitive balance, namely, he wishes to minimize the variance of the players' efforts (see, for example, Szymanski 2001, 2003, 2004, Runkel 2006, and Serena 2017). In this work, we combine these two common goals, effort maximization on the one hand, and, on the other, competitive balance. For this purpose, we study all-pay contests (auctions) with two players under complete information in which every player exerts an effort and the player with the highest effort wins, but all the players bear the cost of their effort (see, for example, Hillman and Samet 1987, Hillman and Riley 1989, Baye et al. 1996, Siegel 2009, and Sela 2012). In our model, the designer wishes to maximize the players expected total effort under the constraint that the difference between the players' efforts is smaller than a given threshold $\delta > 0$. We refer to this model as a balanced all-pay contest.

One way to increase the expected total effort under a balance constraint is to impose a maximum effort constraint. For example, in the US electoral campaign, there is a specific maximum campaign contribution that a single agent can make to a candidate. In addition, several sports leagues (e.g., the NBA) implement a salary cap that limits the total amount of money a team can spend on players' salaries. The usual reason for applying salary caps is that they facilitate even competition between pure and rich teams, since rich teams can pay more, buy the best talents, and ultimately remove any semblance of competition in the league.

A maximum effort constraint may enhance the total effort in the contest since, on the one hand, strong players (players with high values of winning) will exert less effort than they would in a contest without a minimum effort constraint, but, on the other hand, it could increase the effort of other players. The reason is that these players are aware that if they increase their efforts to be close to the level of the maximum effort constraint they will have a better chance to win. Hence, the effect of the maximum effort constraint on the players' total effort depends on the trade-off between the increase of the weaker players' efforts and the decrease of the stronger players' efforts. This trade-off has been extensively analyzed in all-pay contests.

For example, under incomplete information, Gavious et al. (2003) showed that, regardless of the number of bidders, if agents have linear or concave cost functions then setting a maximum effort constraint is not profitable for a designer who wishes to maximize the average effort. In all-pay contest under complete information, Che and Gale (1998) showed that if the maximum effort constraint is higher than half of the lower value of winning, the total effort will not change. On the other hand, Megidish and Sela (2013) found that for the sequential all-pay contest, the maximum effort constraint is profitable for a designer who wishes to maximize the players' expected total effort.[1]

We examine here if a maximum effort can increase total effort also under a competitive balance constraint, namely, when the designer wishes to maximize the aggregate effort but still wants to ensure a minimal level of competitive balance. Formally, we calculate the players' expected total effort for every value of the threshold δ, under the condition that if the difference between two possible efforts of the players is larger than δ, these efforts are not taken into account. We show that for every $\delta > 0$ there is a maximum effort constraint such that the expected total effort in the balanced all-pay contest is larger than in the same contest without any constraint. In other words, a designer who wishes to maximize the expected total effort under a balance constraint always has an incentive to impose a maximum effort constraint.

Another way to increase the expected total effort under a balance constraint is to impose a minimum effort constraint. For example, researchers at universities are required to achieve a minimal quality and quantity output in order to be promoted. Likewise, entry in professional sport competitions is often restricted, whereby only contestants who have achieved some minimal performance level are allowed to compete. A minimum effort constraint eliminates the weak players (players with low values of winning) and accordingly the competition becomes more balanced and intensive whereby the players increase their efforts. Indeed, Myerson (1981) found that the all-pay contest under incomplete information with the optimal participation constraint maximizes the contestants' expected total effort. Additionally, Laffont and Robert (1996) showed that an all-pay contest with a reserve price is a revenue-maximizing mechanism for selling an object to bidders who face linear costs and a common-knowledge fixed budget constraint.

Accordingly, we also examine the effect of the minimal effort constraint in our model and find that it is

[1] The effect of a maximum effort constraint in all-pay contests is analyzed, among others, by Dechenaux et al. (2006), Szech (2015), Hart (2016), and Cohen et al. (2019).

not as efficient as the maximum effort constraint for increasing the players' total effort. We first show that for relatively small or high values of the minimum effort constraint, the players' expected total effort decreases, while for middle values it increases. In particular, we show that there are balanced all-pay contests, in which for every value of the minimum effort constraint, the players' expected total effort is smaller than in the balanced all-pay contest without such a constraint. In other words, it is not effective in enhancing the players' expected total effort.

The rest of the paper is organized as follows: In Section 2, we introduce our balanced all-pay contest, and in Sections 3 and 4, we analyze the effect of the maximum and minimum effort constraints on the players' expected total effort. Section 4 concludes. The proofs appear in the Appendix.

2 The balanced all-pay contest

We begin with an analysis of the standard all-pay contest with two players, called 1 and 2, where the players' values for winning are $v_1 \geq v_2 > 0$. Valuations are common knowledge. Each player exerts an effort $x_i \in [0, \infty)$ and if the players exert efforts of x_1, x_2, player 1's utility function is

$$u_1(x_1, x_2) = \begin{cases} v_1 - x_1 \text{ if } x_1 > x_2 \\ \frac{1}{2}v_1 - x_1 \text{ if } x_1 = x_2 \\ -x_1 \text{ if } x_1 < x_2 \end{cases}.$$

The utility function of player 2 is similar. According to Hillman and Riley (1989) and Baye, Kovenock and de Vries (1996), there is always a unique mixed-strategy equilibrium in which the players randomize on the interval $[0, v_2]$ according to their effort cumulative distribution functions which are given by

$$\begin{aligned} v_1 F_2(x) - x &= v_1 - v_2 \\ v_2 F_1(x) - x &= 0. \end{aligned}$$

Thus, player 1's effort is uniformly distributed such that

$$F_1(x) = \frac{x}{v_2},$$

while player 2's effort is distributed according to the cumulative distribution function

$$F_2(x) = \frac{v_1 - v_2 + x}{v_1}.$$

The total expected effort is

$$TE^* = \frac{v_2}{2}(1 + \frac{v_2}{v_1}). \tag{1}$$

We say that the contest is balanced if and only if the players' equilibrium strategies satisfy

$$|x_1 - x_2| \leq \delta,$$

where δ is a constant larger than zero. Accordingly, the all-pay contest is referred to as a balanced all-pay contest with a threshold of δ. The players' expected total effort is then given by

Proposition 1 *The players' expected total effort in a balanced all-pay contest with a threshold of δ is*

$$TE_A = \frac{4(v_2)^2\delta - 3v_2\delta^2 + v_1\delta^2}{2v_1v_2}. \tag{2}$$

Proof. *See Appendix.* ■

By (2), if $\delta = v_2$, namely, the contest is completely unbalanced, the players' expected total effort is $TE = \frac{v_2}{2}(1 + \frac{v_2}{v_1})$ which is exactly the expected total effort in the standard all-pay contest. It is also worth mentioning that the total effort given by (2) is not monotonically increasing in the value of the threshold δ.

3 The balanced all-pay contest with a maximum effort constraint

We now assume that there is a maximum effort constraint $d \in [0, v_2]$. Note that if $d > v_2$ the maximum effort constraint is not effective since it is not binding. According to Che and Gale (1998), if $d \in \left[0, \frac{v_2}{2}\right]$ there is an equilibrium with pure strategies in which each player exerts an effort that is equal to the effort cap $\frac{v_2}{2}$. Then, each of the players wins with a probability of one-half and the expected payoff of player i is $\frac{v_i}{2} - d$. On the other hand, if $d \in (\frac{v_2}{2}, v_2]$, there is a mixed-strategy equilibrium in which players 1 and 2 randomize on the interval $[0, 2d - v_2] \cup \{d\}$ according to their effort cumulative distribution functions which are given by

$$\begin{aligned} v_1 F_2(x) - x &= v_1 \left[F_2(2d - v_2) + \frac{1 - F_2(2d - v_2)}{2} \right] - d \\ v_2 F_1(x) - x &= v_2 \left[F_1(2d - v_2) + \frac{1 - F_1(2d - v_2)}{2} \right] - d. \end{aligned}$$

Here, the LHS of the equations are the expected payoffs of the players if they exert an effort of $x \in [0, 2d - v_2]$ and the RHS are the expected payoffs if they exert an effort that is equal to d. Thus, player 1's equilibrium effort is distributed according to the cumulative distribution function

$$F_1(x) = \begin{cases} \frac{x}{v_2} \text{ if } x \in [0, 2d - v_2] \\ \frac{2d - v_2}{v_2} \text{ if } x \in (2d - v_2, d) \\ 1 \quad \text{if} \quad x = d \end{cases},$$

while player 2's equilibrium effort is distributed according to the cumulative distribution function

$$F_2(x) = \begin{cases} 1 - \frac{v_2 - x}{v_1} \text{ if } x \in [0, 2d - v_2] \\ 1 - \frac{2v_2 - 2d}{v_1} \text{ if } x \in (2d - v_2, d) \\ 1 \quad \text{if} \quad x = d \end{cases}.$$

Then, player $1'$s expected payoff is $v_1 - v_2$, while player 2's expected payoff is zero. In the case where $d \in (\frac{v_2}{2}, v_2]$ the players' probabilities of winning and their expected payoffs are the same as in the standard all-pay contest without a maximum effort constraint.

Now, if the all-pay contest is balanced we split our analysis into two cases as follows: 1) There is a relatively low threshold, $\delta \leq d - (2d - v_2) = v_2 - d$ and 2) There is a relatively high threshold, $\delta > d - (2d - v_2) = v_2 - d$. In the first case when the threshold δ is relatively low we have

Proposition 2 *The players' expected total effort in the balanced all-pay contest with a threshold $\delta \leq v_2 - d$ and a maximum effort constraint d is*

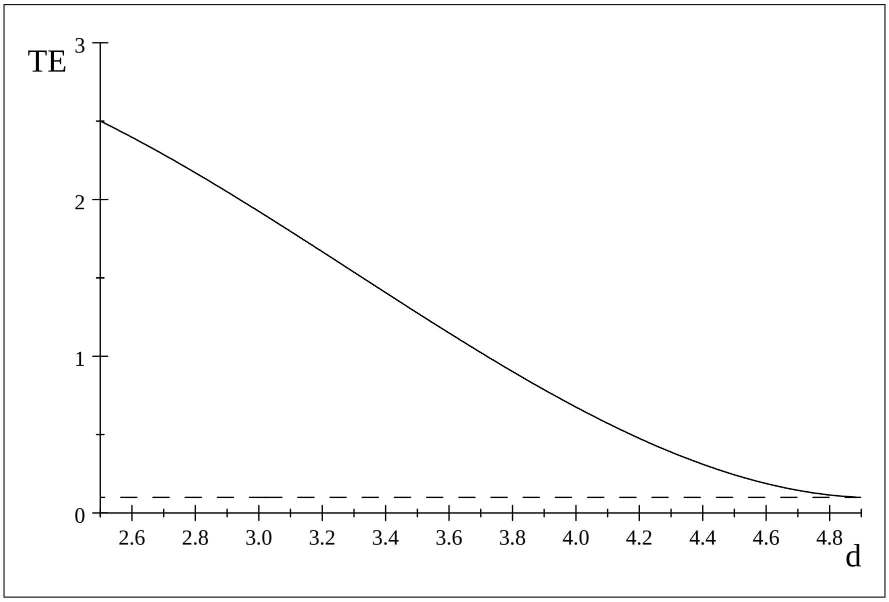
$$TE_B = \frac{4(2d - v_2)^2 \delta - 2(2d - v_2)\delta^2 + (v_1 - v_2)\delta^2 + 4d(v_2 - (2d - v_2))^2}{2v_1 v_2}. \tag{3}$$

Proof. See Appendix. ■

Note that if $d = 0$, namely, there is no maximum effort constraint, we obtain by (3) that the players' expected total effort is $TE_B = \frac{4(v_2)^2\delta - 3v_2\delta^2 + v_1\delta^2}{2v_1 v_2} = TE_A$.

The following example illustrates how in this case the expected total effort varies in the value of the maximum effort constraint.

Example 1 *Consider a balanced all-pay contest with two players 1 and 2 who have values of winning* $v_1 = 10$ *and* $v_2 = 5$*, respectively. Let* $\delta = 0.1$*. The following figure depicts the players' expected total effort as a function of the maximum effort constraint* d*.*



Solid line - with a maximal effort constraint. Dashed line - without any constraint.

We can see that the players' expected total effort decreases in the value of the maximum effort constraint, but the players' expected total effort is always larger than in the same contest without any constraint.

In the second case, when the threshold is relatively high we have

Proposition 3 *The players' expected total effort in the balanced all-pay contest with a threshold* $\delta > v_2 - d$ *and a maximum effort constraint* d *is*

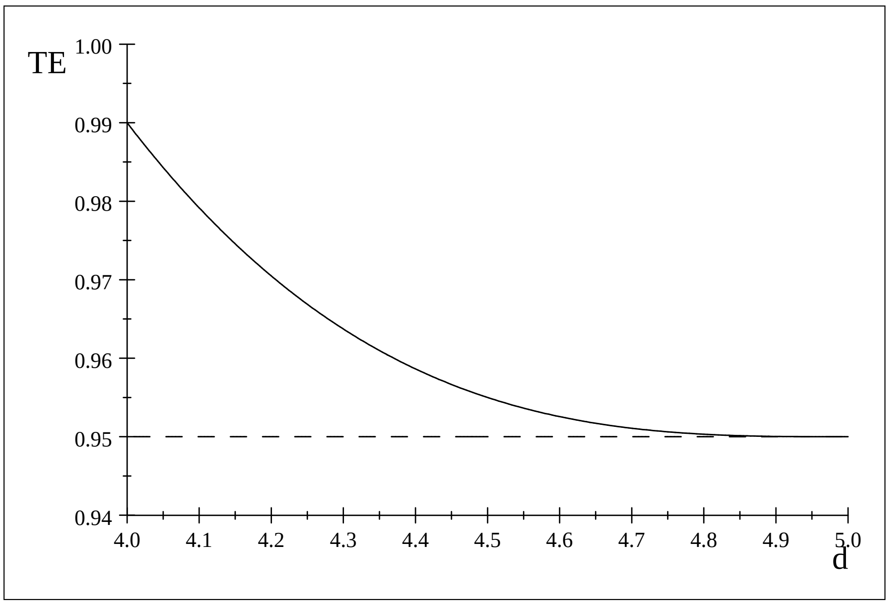
$$\begin{aligned} TE_C &= \frac{4(2d-v_2)^2\delta - 2(2d-v_2)\delta^2 + (v_1-v_2)\delta^2 + 4d(v_2-(2d-v_2))^2}{2v_1v_2} \\ &\quad +4(v_2-d)\frac{5d^2+(v_2)^2-6dv_2-\delta^2+4d\delta}{2v_1v_2} \end{aligned} \tag{4}$$

Proof. See Appendix. ■

Note that if $\delta = v_2$, and $d = v_2$, namely, the contest is completely unbalanced and there is no maximum effort constraint, we obtain by (4) that the players' expected total effort is $TE = \frac{v_2}{2}(1+\frac{v_2}{v_1})$ which is exactly the expected total effort in the standard all-pay contest. Moreover, if $d = v_2$, for every δ, we obtain by (4) that the players' expected total effort is exactly the expected total effort obtained by (2).

The following example illustrates how in this case the expected total effort varies in the value of the maximum effort constraint.

Example 2 *Consider a balanced all-pay contest with two players 1 and 2 who have values of winning $v_1 = 10$ and $v_2 = 5$, respectively. Let $\delta = 1$. The following figure depicts the players' expected total effort as a function of the maximum effort constraint d.*



Solid line - with a maximal effort constraint. Dashed line - without any constraint.

We can see that similarly to the previous example, the players' expected total effort decreases in the value of the maximum effort constraint, but the players' expected total effort is always larger than in the same contest without any constraint.

The following result demonstrates the efficiency of imposing a maximum effort constraint on a balanced all-pay contest in order to increase the players' expected total effort.

Proposition 4 *For every $\delta > 0$ there is a maximum effort constraint $d > 0$ such that the expected total effort in the balanced all-pay contest is larger than in the same contest without any constraint.*

Proof. See Appendix. ■

4 The all-pay contest with a minimum effort constraint

We now assume that there is a minimal effort of $b \in [0, v_2]$. Then, player 1's equilibrium effort is distributed according to the cumulative distribution function

$$F_1(x) = \begin{cases} \frac{b}{v_2} \text{ if } x = b \\ \frac{x}{v_2} \text{ if } x \in [b, v_2) \\ 1 \quad \text{if} \quad x \geq v_2 \end{cases},$$

while player 2's equilibrium bid is distributed according to the cumulative distribution function

$$F_2(x) = \begin{cases} \frac{v_1 - v_2 + b}{v_1} \text{ if } x = 0 \\ \frac{v_1 - v_2 + x}{v_1} \text{ if } x \in [b, v_2) \\ 1 \quad \text{if} \quad x \geq v_2 \end{cases}.$$

Then, the players' expected total effort in the balanced all-pay contest is

$$\begin{aligned} TE_D &= \int_b^{v_2-\delta} \int_x^{x+\delta} (x+y) f_2(y) f_1(x) dy dx + \int_{b+\delta}^{v_2} \int_{x-\delta}^{x} (x+y) f_2(y) f_1(x) dy dx \\ &+ \frac{b}{v_2} \int_b^{b+\delta} (b+x) f_1(x) dx \\ &= \frac{1}{v_1 v_2} \left(-2b^2\delta - 2b\delta^2 - 2\delta^2 v_2 + 2\delta v_2^2 + 2b^2\delta + \frac{\delta^2 b}{2} \right). \end{aligned}$$

Now, we split our analysis into two cases as follows: 1) There is a relatively low threshold $\delta \leq b$ and 2) There is a relatively high threshold $\delta > b$. In the first case when the threshold δ is relatively low we have

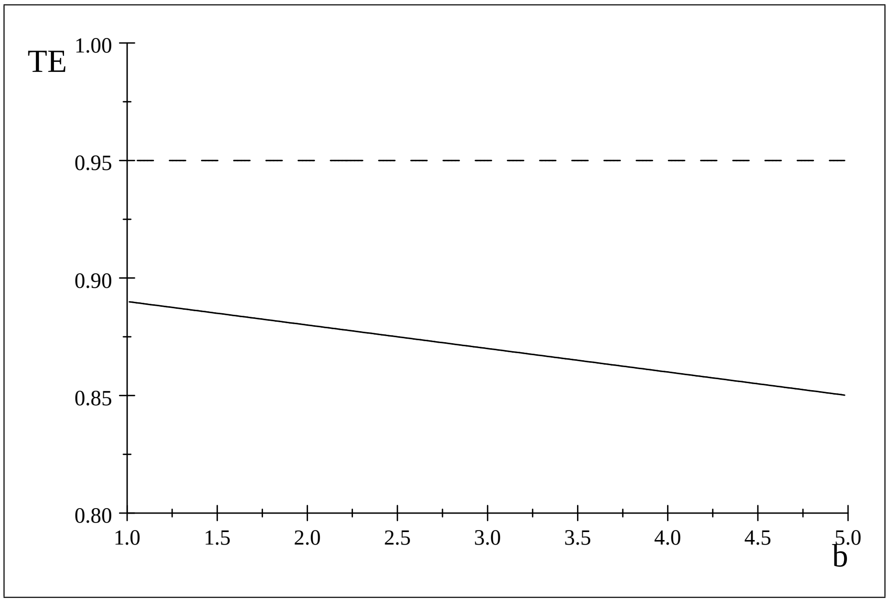
Proposition 5 *The players' expected total effort in the balanced all-pay contest with a threshold $\delta \leq b$ and a minimum effort constraint b is*

$$TE_D = \frac{4{v_2}^2\delta - 2v_2\delta^2 - b\delta^2}{2v_1 v_2}. \tag{5}$$

Proof. See Appendix. ■

The following example illustrates how in this case the expected total effort varies in the value of the minimum effort constraint.

Example 3 *Consider a balanced all-pay contest with two players, 1 and 2, who have values of winning $v_1 = 10$ and $v_2 = 5$, respectively. Let $\delta = 1$ and the minimum effort constraint $b \in (1, 5)$. The following figure depicts the players' expected total effort as a function of the minimum effort constraint b.*



Solid line - with a minimal effort constraint. Dashed line - without any constraint.

We can see that the players' expected total effort decreases in the value of the minimum effort constraint, but the players' expected total effort is always smaller than in the same contest without any constraint.

In the second, case when the threshold is relatively high we have

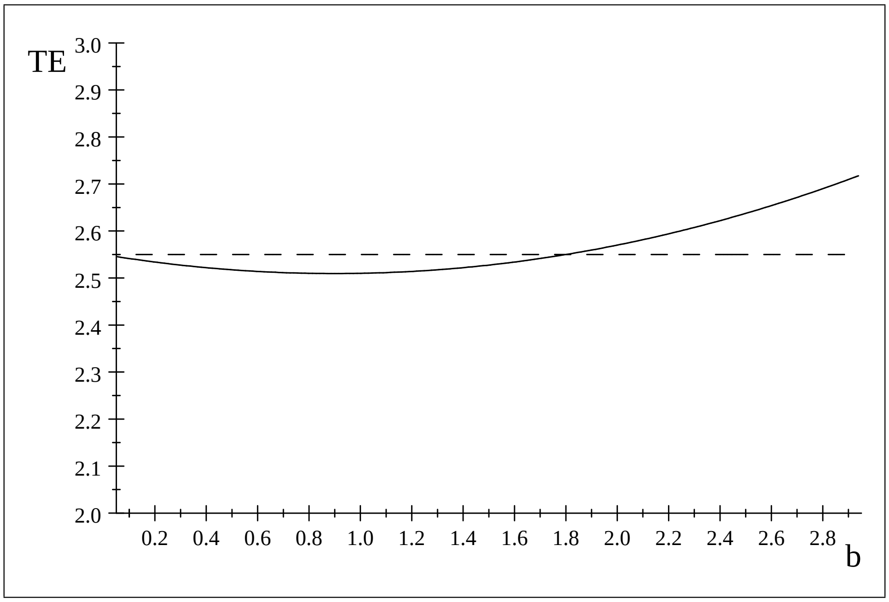
Proposition 6 *The players' expected total effort in the balanced all-pay contest with a threshold $\delta > b$ and a minimum effort constraint b is*

$$TE_E = \frac{4(v_2)^2\delta - 3v_2\delta^2 - b\delta^2 + v_1\delta^2 + b^2(v_1 - v_2)}{2v_1v_2}. \tag{6}$$

Proof. See Appendix. ■

We can see that if $b = 0$, for every δ,we obtain by (6) that the players' expected total effort is exactly the expected total effort obtained by (2). The following example illustrates how in this case the expected total effort varies in the value of the minimum effort constraint.

Example 4 *Consider a balanced all-pay contest with two players, 1 and 2, who have values of winning $v_1 = 10$ and $v_2 = 5$, respectively. Let $\delta = 3$ and the minimum effort constraint be $b \in [0.05, 2.95]$. The following figure depicts the players' expected total effort as a function of the minimum effort constraint b.*



Solid line - with a minimal effort constraint. Dashed line - without any constraint.

We can see that for small (large) values of the minimum effort constraint, the players' expected total effort is smaller (larger) than in the same contest without any constraint.

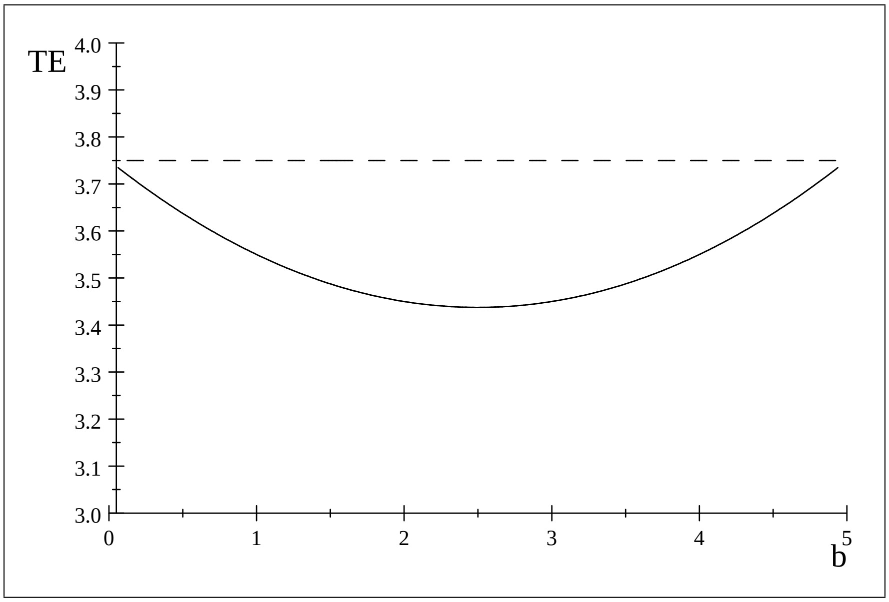
The following result demonstrates the inefficiency of imposing a minimum effort constraint if the goal is to increase the players' expected total effort.

Proposition 7 *If $b > \delta$ and $b \leq \frac{\delta}{2(v_1 - v_2)}$, the expected total effort in the balanced all-pay contest decreases in the value of the minimum effort constraint and if $\delta \geq b > \frac{\delta}{2(v_1 - v_2)}$, the expected total effort increases in the value of the minimum effort constraint*

Proof. See Appendix. ■

By Proposition 7, there are all-pay contests in which for every value of the minimum effort constraint, the players' expected total effort is smaller than in the all-pay contest without any constraint. The following example illustrates that the minimum effort constraint is not efficient for increasing the players' expected total effort.

Example 5 *Consider a balanced all-pay contest with two players, 1 and 2, who have values of winning $v_1 = 10$ and $v_2 = 5$, respectively. Let $\delta = 5$, and the minimum effort constraint be $b \in [0.05, 4.95]$. The following figure depicts the players' expected total effort as a function of the minimum effort constraint b.*



Solid line - with a minimal effort constraint. Dash line - without any constraint.

We can see that for all $b \leq \delta$ the players' expected payoff in the balanced all-pay contest is smaller than in the all-pay contest without a minimal constraint. Since for every $b > \delta$, the players 'expected payoff deceases in the value of b, we obtain that the players' expected payoff is always smaller than in the balanced all-pay contest without any minimum effort constraint.

5 Conclusion

In numerous contests the designer wants to maximize the players' total effort, or, alternatively, the players' average effort. We show that the well-known methods to increase the players' total effort in the standard all-pay contest are not necessarily useful when the designer maximizes the players' total effort under a balance constraint. In particular, we show that a maximum effort constraint is always efficient in enhancing the players' total effort and, on the other hand, a minimum effort constraint might not be efficient at all. Hence, our findings raise the question about the robustness of the well known results in contest theory, namely, it is not clear if the established methods to maximize the total effort in contests are still efficient if the maximization effort is done under a balance constraint or under different constraints.

6 Appendix

6.1 Proof of Proposition 1

We separate the analysis of the expected total effort into five cases as follows:

- Case $A1$: Player 1 exerts an effort of $x \in [0, v_2 - \delta]$ and player 2 an effort of $y \in [x, x + \delta]$, such that player 2 exerts a higher effort than player 1.
- Case $A2$: Player 1 exerts an effort of $x \in [v_2 - \delta, v_2]$ and player 2 an effort of $y \in [x, v_2]$, such that player 2 exerts a higher effort than player 1.
- Case $A3$: Player 1 exerts an effort of $x \in [\delta, v_2]$ and player 2 an effort of $y \in [x - \delta, x]$, such that player 1 exerts a higher effort than player 2.
- Case $A4$: Player 1 exerts an effort of $x \in [0, \delta]$ and player 2 an effort of $y \in [0, x]$, such that player 1 exerts a higher effort than player 2.
- Case $A5$: Player 1 exerts an effort of $x \in [0, \delta]$ and player 2 an effort of $y = 0$, such that player 1 exerts a higher effort than player 2.

The sum of the expected total effort in the above cases is

$$
\begin{aligned}
TE_A &= TE_{A1} + TE_{A2} + TE_{A3} + TE_{A4} + TE_{A5} = \\
& \int_0^{v_2-\delta} \int_x^{x+\delta} (x+y) f_1(x) f_2(y) dx dy + \int_{v_2-\delta}^{v_2} \int_x^{v_2} (x+y) f_1(x) f_2(y) dx dy \\
& + \int_\delta^{v_2} \int_{x-\delta}^{x} (x+y) f_1(x) f_2(y) dx dy + \int_0^{\delta} \int_0^{x} (x+y) f_1(x) f_2(y) dx dy \\
& + \frac{v_1 - v_2}{v_1} \int_0^x x f_1(x) dx \\
&= \int_0^{v_2-\delta} \int_x^{x+\delta} \frac{x+y}{v_1 v_2} dx dy + \int_{v_2-\delta}^{v_2} \int_x^{v_2} \frac{x+y}{v_1 v_2} dx dy \\
& + \int_\delta^{v_2} \int_{x-\delta}^{x} \frac{x+y}{v_1 v_2} dx dy + \int_0^{\delta} \int_0^{x} \frac{x+y}{v_1 v_2} dx dy \\
& + \frac{v_1 - v_2}{v_1} \int_0^x \frac{x}{v_2} dx \\
&= \left(\frac{\delta^3 + 2(v_2)^2\delta - 3v_2\delta}{2v_1v_2}\right) + \left(\frac{2v_2\delta - \delta^3}{2v_1v_2}\right) \\
& + \left(\frac{2(v_2)^2\delta - v_2\delta^2 - \delta^3}{2v_1v_2}\right) + \left(\frac{\delta^3}{2v_1v_2}\right) + \left(\frac{\delta^2(v_1 - v_2)}{2v_1v_2}\right) \\
&= \frac{4(v_2)^2\delta - 3v_2\delta^2 + v_1\delta^2}{2v_1v_2}.
\end{aligned}
$$

Q.E.D.

6.2 Proof of Proposition 2

We separate the analysis of the expected total effort into six cases as follows:

- Case $B1$: Player 1 exerts an effort of $x \in [0, 2d - v_2 - \delta]$ and player 2 an effort of $y \in [x, x + \delta]$, such that player 2 exerts a higher effort than player 1.

- Case $B2$: Player 1 exerts an effort of $x \in [2d - v_2 - \delta, 2d - v_2]$ and player 2 an effort of $y \in [x, 2d - v_2]$, such that player 2 exerts a higher effort than player 1.

- Case $B3$: Player 1 exerts an effort of $x \in [\delta, 2d - v_2]$ and player 2 an effort of $y \in [x - \delta, x]$, such that player 1 exerts a higher effort than player 2.

- Case $B4$: Player 1 exerts an effort of $x \in [0, \delta]$ and player 2 an effort of $y \in [0, x]$, such that player 1 exerts a higher effort than player 2.

- Case $B5$: Player 1 exerts an effort of $x \in [0, \delta]$ and player 2 an effort of $y = 0$, such that player 1 exerts a higher effort than player 2.

- Case $B6$: Player 1 exerts an effort of $x = d$ and player 2 an effort $y = d$.

The sum of the expected total effort in the above cases is

$$\begin{aligned}
TE_B &= TE_{B1} + TE_{B2} + TE_{B3} + TE_{B4} + TE_{B5} + TE_{B6} \\
&= \int_0^{2d-v_2-\delta} \int_x^{x+\delta} (x+y) f_1(x) f_2(y) dx dy + \int_{2d-v_2-\delta}^{2d-v_2} \int_x^{2d-v_2} (x+y) f_1(x) f_2(y) dx dy \\
&\quad + \int_\delta^{2d-v_2} \int_{x-\delta}^{x} (x+y) f_1(x) f_2(y) dx dy + \int_0^\delta \int_0^x (x+y) f_1(x) f_2(y) dx dy \\
&\quad + \frac{v_1 - v_2}{v_1} \int_0^\delta x f_1(x) dx + 2d(1 - \frac{2d - v_2}{v_2})(1 - \frac{v_1 - v_2 + 2d - v_2}{v_1}) \\
&= \int_0^{2d-v_2-\delta} \int_x^{x+\delta} \frac{(x+y)}{v_1 v_2} dx dy + \int_{2d-v_2-\delta}^{2d-v_2} \int_x^{2d-v_2} \frac{(x+y)}{v_1 v_2} dx dy \\
&\quad + \int_\delta^{2d-v_2} \int_{x-\delta}^{x} \frac{(x+y)}{v_1 v_2} + \int_0^\delta \int_0^x \frac{(x+y)}{v_1 v_2} dx dy \\
&\quad + \frac{v_1 - v_2}{v_1} \int_0^\delta \frac{x}{v_2} dx + 2d(1 - \frac{2d - v_2}{v_2})(1 - \frac{v_1 - v_2 + 2d - v_2}{v_1}) \\
&= \frac{\delta^3 + 2(2d - v_2)\delta - 3(2d - v_2)\delta^2}{2 v_1 v_2} + \frac{2(2d - v_2)\delta^2 - \delta^3}{2 v_1 v_2} \\
&\quad + \frac{2(2d - v_2)^2 \delta - (2d - v_2)\delta^2 - \delta^3}{2 v_1 v_2} + \frac{\delta^3}{2 v_1 v_2} \\
&\quad + \frac{\delta^2 (v_1 - v_2)}{2 v_1 v_2} + 2d \frac{(v_2 - (2d - v_2))^2}{v_1 v_2} \\
&= \frac{4(2d - v_2)^2 \delta - 2(2d - v_2)\delta^2 + (v_1 - v_2)\delta^2 + 4d(v_2 - (2d - v_2))^2}{2 v_1 v_2}.
\end{aligned}$$

Q.E.D.

6.3 Proof of Proposition 3

We separate the analysis of the expected total effort into eight cases, the first six cases being $B1 - B6$ from the proof of Proposition 2. The last two are:

- Case $C1$: Player 1 exerts an effort of $x = d$ and player 2 an effort of $y \in [d - \delta, 2d - v_2]$, such that player 1 exerts a higher effort than player 2.

- Case $C2$: Player 1 exerts an effort of $x \in [d - \delta, 2d - v_2]$ and player 2 an effort of $y = d$, such that player 2 exerts a higher effort than player 1.

The sum of the expected total effort in the above cases is

$$
\begin{aligned}
TE_C &= TE_B + TE_{C1} + TE_{C2} \\
&= TE_B + \int_{d-\delta}^{2d-v_2}\int_{2d-v_2}^{v_2}(d+y)f_1(x)f_2(y)dxdy + \int_{d-\delta}^{2d-v_2}\int_{2d-v_2}^{v_2}(x+d)f_1(x)f_2(y)dxdy \\
&= TE_B + \int_{d-\delta}^{2d-v_2}\int_{2d-v_2}^{v_2}\frac{(d+y)}{v_1v_2}dxdy + \int_{d-\delta}^{2d-v_2}\int_{2d-v_2}^{v_2}\frac{(x+d)}{v_1v_2}dxdy \\
&= \frac{4(2d-v_2)^2\delta - 2(2d-v_2)\delta^2 + (v_1-v_2)\delta^2 + 4d(v_2-(2d-v_2))^2}{2v_1v_2} \\
&\quad + \frac{d(v_2-(2d-v_2))((2d-v_2)-(d-\delta))}{v_1v_2} + \frac{((2d-v_2)^2-(d-\delta)^2)((v_2-(2d-v_2))}{2v_1v_2} \\
&\quad + \frac{(v_2-(2d-v_2))((2d-v_2)^2-(d-\delta)^2)}{2v_1v_2} + \frac{((2d-v_2)-(d-\delta))d((v_2-(2d-v_2))}{2v_1v_2} \\
&= \frac{4(2d-v_2)^2\delta - 2(2d-v_2)\delta^2 + (v_1-v_2)\delta^2 + 4d(v_2-(2d-v_2))^2}{2v_1v_2} \\
&\quad +4(v_2-d)\frac{5d^2+(v_2)^2-6dv_2-\delta^2+4d\delta}{2v_1v_2}.
\end{aligned}
$$

Q.E.D.

6.4 Proof of Proposition 4

By Proposition 3, the players' expected total effort in the balanced all-pay contest with a threshold of $\delta > v_2 - d$ is

$$
\begin{aligned}
TE_C &= \frac{4(2d-v_2)^2\delta - 2(2d-v_2)\delta^2 + (v_1-v_2)\delta^2 + 4d(v_2-(2d-v_2))^2}{2v_1v_2} \\
&\quad +4(v_2-d)\frac{5d^2+(v_2)^2-6dv_2-\delta^2+4d\delta}{2v_1v_2}
\end{aligned}
$$

The derivative of the expected total effort with respect to the maximum effort constraint is

$$\frac{dTE_C}{dd} = -\frac{6}{v_1v_2}(d-v_2)^2$$

Thus, for every $d \leq v_2$ the expected total effort decreases in the value of the maximum effort constraint. In other words, if $\delta > v_2 - d$ every value of the maximum effort constraint d increases the players' expected total value with respect to the same contest without any effort constraint ($d = v_2$). Therefore, for every $\delta > 0$, we can choose a value of the maximum effort constraint d such that there exists $\delta > v_2 - d$, and then we obtain an expected payoff larger than in the balanced all-pay contest without any constraint.

6.5 Proof of Proposition 5

We separate the analysis of the expected total effort into five cases as follows:

- Case $D1$: Player 1 exerts an effort of $x \in [b, v_2 - \delta]$ and player 2 exerts an effort of $y \in [x, x + \delta]$, such that player 2 exerts a higher effort than player 1.

- Case $D2$: Player 1 exerts an effort of $x \in [v_2 - \delta, v_2]$ and player 2 an effort of $y \in [x, v_2]$, such that player 2 exerts a higher effort than player 1.

- Case $D3$: Player 1 exerts an effort of $x \in [b + \delta, v_2]$ and player 2 an effort of $y \in [x - \delta, x]$, such that player 1 exerts a higher effort than player 2.

- Case $D4$: Player 1 exerts an effort of $x \in [b, b + \delta]$ and player 2 an effort of $y \in [b, x]$, such that player 1 exerts a higher effort than player 2.

- Case $D5$: Player 1 exerts an effort of $x = b$ and player 2 an effort of $y \in [b, b + \delta]$, such that player 2 exerts a higher effort than player 1.

The sum of the expected total effort in the above cases is

$$
\begin{aligned}
TE_D &= TE_{D1} + TE_{D2} + TE_{D3} + TE_{D4} + TE_{D5} = \\
& \int_b^{v_2-\delta} \int_x^{x+\delta} (x+y) f_1(x) f_2(y) dx dy + \int_{v_2-\delta}^{v_2} \int_x^{v_2} (x+y) f_1(x) f_2(y) dx dy \\
& + \int_{b+\delta}^{v_2} \int_{x-\delta}^{x} (x+y) f_1(x) f_2(y) dx dy + \int_b^{b+\delta} \int_b^{x} (x+y) f_1(x) f_2(y) dx dy \\
& + \frac{b}{v_2} \int_0^{b+\delta} (b+y) f_2(y) dy \\
&= \int_b^{v_2-\delta} \int_x^{x+\delta} \frac{x+y}{v_1 v_2} dx dy + \int_{v_2-\delta}^{v_2} \int_x^{v_2} \frac{x+y}{v_1 v_2} dx dy \\
& + \int_{b+\delta}^{v_2} \int_{x-\delta}^{x} \frac{x+y}{v_1 v_2} dx dy + \int_b^{b+\delta} \int_b^{x} \frac{x+y}{v_1 v_2} dx dy \\
& + \frac{b}{v_2} \int_0^{b+\delta} \frac{b+y}{v_1} dx \\
&= \left(\frac{\delta^3 + 2(v_2)^2\delta - 3v_2\delta^2 - 2b^2\delta - b\delta^2}{2v_1v_2}\right) + \left(\frac{2v_2\delta^2 - \delta^3}{2v_1v_2}\right) \\
& + \left(\frac{2(v_2)^2\delta - v_2\delta^2 - 2b^2\delta - 3b\delta^2 - \delta^3}{2v_1v_2}\right) + \left(\frac{2b\delta^2 + \delta^3}{2v_1v_2}\right) + \left(\frac{4b^2\delta + b\delta^2}{2v_1v_2}\right) \\
&= \frac{4(v_2)^2\delta - 2v_2\delta^2 - b\delta^2}{2v_1v_2}.
\end{aligned}
$$

Q.E.D.

7 Proof of Proposition 6

We separate the analysis of the expected total effort into seven cases. The first five are $D1 - D5$ which appear in the proof of Proposition 5. The last two cases are:

- Case $E1$: Player 1 exerts an effort of $x = b$ and player 2 an effort of $y = 0$, such that, player 1 exerts a higher effort than player 2.

- Case $E2$: Player 1 exerts an effort of $x \in [b, \delta]$, and player 2 an effort of $y = 0$, such that, player 1 exerts a higher effort than player 2.

The sum of the expected total effort in the above cases is

$$\begin{aligned}
TE_E &= TE_D + TE_{E1} + TE_{E2} \\
&= TE_D + b\frac{b}{v_2}\frac{(v_1 - v_2)}{v_1} + \frac{(v_1 - v_2)}{v_1}\int_b^\delta x f_1(x)dx \\
&= \frac{4(v_2)^2\delta - 2v_2\delta^2 - b\delta^2}{2v_1v_2} + \frac{b^2}{v_2}\frac{(v_1 - v_2)}{v_1} + \frac{(v_1 - v_2)}{v_1}\frac{(\delta^2 - b^2)}{2v_2} \\
&= \frac{4(v_2)^2\delta - 3v_2\delta^2 - b\delta^2 + v_1\delta^2 + b^2(v_1 - v_2)}{2v_1v_2}.
\end{aligned}$$

Q.E.D.

7.1 Proof of Proposition 7

By Proposition 5, the players' expected total effort in the balanced all-pay contest with a threshold $\delta > b$ is

$$TE_D = \frac{4(v_2)^2\delta - 2v_2\delta^2 - b\delta^2}{2v_1v_2}.$$

The derivative of the expected total effort with respect to the minimum effort constraint is

$$\frac{dTE_D}{db} = -\frac{\delta^2}{2v_1v_2}.$$

Thus, for every $\delta > b$ we obtain that $\frac{dTE_D}{db} < 0$. That is, if $\delta > b$, the expected payoff decreases in the value of the minimum effort constraint b.

Now, if $\delta \leq b$, by Proposition 6, the players' expected total effort in the balanced all-pay contest is

$$TE_E = \frac{4(v_2)^2\delta - 3v_2\delta^2 - b\delta^2 + v_1\delta^2 + b^2(v_1 - v_2)}{2v_1v_2}.$$

The derivative of the expected total effort with respect to the minimum effort constraint is

$$\frac{dTE_E}{db} = \frac{2b(v_1 - v_2) - \delta^2}{2v_1v_2}.$$

Thus, if $\frac{\delta^2}{2(v_1-v_2)} \geq b$ we obtain that $\frac{dTE_E}{db} < 0$, namely, the players' expected total effort decreases in the value of the minimum effort constraint b, and, otherwise it increases. $Q.E.D.$

References

[1] Baye, M., Kovenock, D., de Vries, C.: The all-pay auction with complete information. *Economic Theory* 8, 291-305 (1996)

[2] Che, Y-K., Gale, I.: Caps on political lobbying. *American Economic Review* 88, 643-651 (1998)

[3] Gavious, A., Moldovanu, B., Sela, A.: Bid costs and endogenous bid caps. *Rand Journal of Economics* 33(4), 709-722 (2003)

[4] Cohen, C., Levi, O., Sela, A.: All-pay auctions with asymmetric effort constraints. *Mathematical Social Sciences* 97, 18-23 (2019)

[5] Dechenaux, E., Kovenock, D., Lugovskyy, V.: Caps on bidding in all-pay auctions: Comments on the experiments of A. Rapoport and W. Amaldoss, *Journal of Economic Behavior & Organization* 61, 276–283 (2006)

[6] Franke, J., Kanzow, C., Leininger, W., Schwartz A.: Effort maximization in asymmetric contest games with heterogeneous contestants. *Economic Theory* 52, 589-630 (2013)

[7] Glazer, A., Hassin, R.: Optimal contests. *Economic Inquiry* 26(1), 133-143 (1988)

[8] Hart, S.: Allocation games with caps: from Captain Lotto to all-pay auctions. *Games and Economic Behavior* 45, 37-61 (2016)

[9] Hillman, A., Riley, J.: Politically contestable rents and transfers. *Economics and Politics* 1, 17-39 (1989)

[10] Hillman, A., Samet, D.: Dissipation of contestable rents by small numbers of contenders. *Public Choice* 54(1), 63-82 (1987)

[11] Liu, X., Lu, J.: The effort maximizing contests with heterogeneous prizes. *Economics Letters* 125(3), 422-425 (2014)

[12] Megidish, R., Sela, A.: Caps in sequential contests. *Economic Inquiry* 52(2), 608-617 (2014)

[13] Moldovanu, B., Sela, A.: The optimal allocation of prizes in contests. *American Economic Review* 91, 542-558 (2001)

[14] Moldovanu, B., Sela, A., Shi, X.: Carrots and sticks: prizes and punishments in contests. *Economic Inquiry* 50(2), 453-462 (2012)

[15] Myerson, R.: Optimal auction design. *Mathematics of Operations Research* 6(1), 58-73 (1981)

[16] Runkel, M,: Total effort, competitive balance and the optimal contest success function. *European Journal of Political Economy* 22(4), 1009-1013 (2006)

[17] Ryvkin, D.: Heterogeneity of players and aggregate effort in contests. *Journal of Economics & Management Strategy* 22(4), 728-743 (2013)

[18] Sela, A.: Sequential two-prize contests. *Economic Theory* 51, 383-395 (2012)

[19] Serena, M.: Quality contests. *European Journal of Political Economy* 46, 15-25 (2017)

[20] Siegel, R.: All-pay contests. *Econometrica* 77(1), 71-92 (2009)

[21] Szech, N.: Tie-breaking and bid-caps in all-pay auctions. *Games and Economic Behavior* 92, 138-149 (2015)

[22] Szymanski, S.: Income inequality, competitive balance and the attractiveness of team sports: some evidence and a natural experiment from English soccer. *Economic Journal* 111(469), 69-84 (2001)

[23] Szymanski, S.: The economic design of sporting contests. *Journal of Economic Literature* 41(4), 1137-1187 (2003)

[24] Szymanski, S.: Competitive balance and gate revenue sharing in team sports. *Journal of Industrial Economics* 52(1), 165-177 (2004)