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Theft in equilibrium*

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Abstract

We incorporate theft in a partial equilibrium model.

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1 Introduction

Early economic thinkers were well aware of the prevalence of crime, theft in particular, in society. For instance, Pareto [17] unequivocally states that "...the efforts of men are utilized in two different ways: they are directed to the production or transformation of economic goods, or else to the appropriation of goods produced by others." Likewise, J.S.Mill [16]

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writes "it is lamentable to think how a great proportion of all efforts and talents in the world are employed in merely neutralizing one another..." and claims that the role of government is "to reduce this wretched waste to the smallest amount". However, perhaps except for taxation, any appropriation activity is absent from the realm of the central model in economics. Only at the end of the 1960's, did economists begin to formally analyze the subject of crime, the seminal reference being Becker [1]. The first models focused on the decision-making process of a rational potential criminal, and although the aggregate behavior of economic agents is made mutually consistent through the adjustment of the relevant endogenous variables, they do not perfectly fit the standard Walrasian model, (see Ehrlich [7] for an overview). There is a related strand of literature that deals with conflict, whose seminal ideas can be traced back to Haavelmo [13] and Hirschleifer [14]. This literature adopts a game theoretic approach to conflict and appropriation and is related to the vast literature on contests.[1] An early general equilibrium model that incorporates appropriation is Grossman [12].[2] The paper that best fits the Walrasian model is Dal Bó and Dal Bó [6]. It introduces appropriation activities in the celebrated 2x2 production model and analyze, among other things, the effect of changes in the exogenous output prices and in the factor endowments on the level of crime. They also investigate the effect of tax, subsidies, and trade policies on crime.

Property theft is a pervasive phenomenon in all societies. In the US, the FBI estimates that Property crimes in 2018 resulted in losses at $16.4 billion. Although there is a vast theoretical literature that analyzes crime, there are very few papers that do this within a full Walrasian model. The purpose of this paper is to uncover the implications of this model once it is amended to admit the possibility of theft. In order to accomplish this, we introduce theft into a partial equilibrium model. We follow Dal Bó and Dal Bó [6] and allow agents to devote time to theft, the returns of which depend on the economy-wide crime level and the

[1] Notable examples include Skaperdas [18] and Garfinkel [9]. For a very thorough overview, see Garfinkel and Skaperdas [10].

[2] Other models that allow for appropriation activities are Burdett Lagos and Wright [3, 4], Imrohoroglu, Merlo and Rupert [15], Gonzalez [11] and Galiani, Cruz and Torrens [8].

wealth that is subject to theft. From the individual thief's point of view, he cannot affect the level of crime and the wealth subject to theft is a common resource. We also allow for the possibility of police protection, which can be either public or private. We study the existence and uniqueness of the competitive equilibrium, and analyze the nature of its inefficiency.

It turns out that the results depend on the wealth that is subject to theft. If the stealable wealth consists of the individuals' initial factor endowments, then a competitive equilibrium exists and is unique. As expected, the equilibium allocation is not efficient and a Pareto improvement can be achieved by means of an increase in output. Also, the supply of public police reduces the level of crime. If we allow for private police protection we obtain that although the competitive equilibrium is inefficient, conditional on the level of theft in the economy, the allocation of police protection is optimal.

However, when the wealth subject to theft is the gross domestic product, namely, when only produced goods can be stolen, most of the above results no longer hold. In particular, we show that a competitive equilibrium may not exist, and that when it does exist, it may not be unique. Equilibrium allocations are generally inefficient but incentives to output production are not necessarily Pareto improving. In fact, increases in police protection may lead to an increase in crime. Finally, unless the production technology is linear, equilibrium private police protection is no longer optimal, conditional on the equilibrium level of theft.

We also investigate, within the first model, the possibility of equilibria in which two or more regions coexist with different levels of crime and police protection and in which the tax rates are determined by majority voting. In particular, we show that when tax rates are constrained to be proportional, regions vote to supply the optimal level of police protection. When taxation is implemented by a head tax, the tax rates are not optimal.

The paper is organized as follows. Section 2 introduces the basic definitions of an economy with theft. Section 3 develops the model where all the initial endowment is subject to theft. Section 4 considers the case where the stealable wealth consists of produced goods. Finally, Section 5 concludes.

2 The model

The primitives of the model are the following. There is a firm that transforms labor into peanuts according to the technology $\mathcal{T} = \{(-Z, Q) : 0 \leq c(Q) \leq Z\}$,where $c : \mathbb{R}_+ \to \mathbb{R}_+$ is the cost function, which is assumed to be convex. There is a continuum of agents $I = [0, 1]$. Each agent $i \in I$ is characterized by a quasilinear utility function $u_i(x_i, m_i) = \phi_i(x_i) + m_i$, an initial endowment of labor ω_i and a share θ_i of the firm's profits. For simplicity, we assume that the consumption set is $\mathbb{R}_+ \times \mathbb{R}$, namely individuals can consume negative amounts of leisure. Further, we assume that ϕ_i is strictly increasing, concave, and that $\lim_{x\to\infty} \phi_i'(x) = 0$. We denote by ϕ the function defined on $[0, 1]$ that assigns to each agent i his utility function ϕ_i.

Individuals, apart from consuming peanuts and leisure, devote some time to theft and may obtain some police protection. A *bundle* for individual i is thus a four-tuple $(x_i, m_i, y_i, t_i) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$ whose components are the amount of peanuts, leisure, time devoted to theft, and time devoted to police protection.

For any real function f defined on $[0, 1]$, we will sometimes write $\int f$ for $\int_0^1 f_i \, di$. Aggregate (or per capita) values are denoted by capital letters. In particular, the per capita amount of resources in the economy is $\Omega = \int \omega$, the per capita consumption of leisure is $M = \int m$, the crime level is $Y = \int y$, and the average police protection is $T = \int t$. We assume that the agents in $[0, 1]$ are the sole owners of the firm: $\int \theta = 1$.

If individual i devotes y_i units of time to redistributive activities he gets a share y_i/Y of the booty. Police protection may be public or private, being public when t_i is decided by the government and private when it is decided by agent i. When police protection is public, it is usually allocated uniformly across individuals. However, public police protection may very well be discriminatory. When public, police protection is financed by means of compulsory taxation. When private, it is purchased voluntarily by the consumers themselves.

There is an appropriation technology described by $A : \mathbb{R}_+^2 \to [0, 1]$. The value $A(Y, t_i)$ is the proportion of individual i's stealable income that gets stolen when the crime level is Y and enjoys t_i units of police protection. We call $A(Y, t_i)$ the *excise rate* associated with Y and t_i. We assume that $A(0, t_i) = 0$, that A is increasing and strictly concave in its first argument, decreasing and strictly convex in its second argument, and that $A_{12} < 0$, namely the marginal excise rate of crime is decreasing in police protection.[3] These assumptions imply that

$$A_1(Y, t_i) < \frac{A(Y, t_i)}{Y}$$

and that $\lim_{Y \to 0} A(Y, t_i)/Y = A_1(0, t_i)$. We denote by $a(Y, t_i)$ the average appropriation, with the extension $a(0, t_i) = A_1(0, t_i)$. Namely, the marginal excise rate is lower than the average excise rate. Also, it follows from our assumptions that $a(Y, t_i)$ is decreasing in both its arguments, and convex in its second argument. We summarize the data of the economy by $\mathcal{E} = \langle (\phi, \omega, \theta), c, A \rangle$.

We denote the set of bundles by $\mathcal{X}$. A *feasible allocation* consists of a production plan $(-Z, Q) \in \mathcal{T}$ and a function $(x, m, y, t) : [0, 1] \to \mathcal{X}$ that assigns a bundle to each agent, such that

1. $\int x = Q$,

2. $\int m + Z + \int y + \int t = \Omega$.

A feasible allocation is *efficient* if there is no alternative feasible allocation that can make all agents better off. Given our assumptions on the individuals' consumption set and preferences, an allocation is efficient if it maximizes, among the feasible allocations, the social welfare, namely, the average utility, $W(x, m) = \int_0^1 u_i(x_i, m_i)\, di$, of the individuals.

In the next two sections we introduce and analyze the competitive equilibrium for an economy with theft. They differ in the way the wealth subject to theft is defined. In the next section, the stealable wealth consists of the whole factor endowment. In Section 4, in contrast, only earned income can be stolen. That is, time devoted to leisure cannot be alienated. Specifically, given a price of peanuts p, the firm's profits are $\Pi = pQ - Z$, and individual i's share in these profits is $\pi_i = \theta_i \Pi$. When all wealth is subject to theft, individual i's stealable wealth is $(\omega_i + \pi_i)$ and the aggregate stealable wealth is $(\Omega + \Pi)$.

[3] We denote by A_1 and A_2 the partial derivatives of A with respect to its first and second arguments. Also $A_{k\ell}$, for $k, \ell = 1, 2$, stand for the corresponding second derivatives.

When, alternatively, only earned income is subject to theft individual i's stealable wealth is $(\omega_i + \pi_i - m_i)$ and the aggregate stealable wealth is $(\Omega + \Pi - M)$.

3 All wealth can be stolen

In this section we assume that all wealth can be stolen. We first restrict attention to the case where there is no police protection. To simplify notation we will henceforth let $A(Y) = A(Y,0)$, $a(Y) = a(Y,0)$, etc.

3.1 Competitive equilibrium

When all wealth can be stolen, individual i's budget set is given by

$$\{(x_i, y_i, m_i) : px_i + m_i + y_i \leq (1 - A(Y))(\omega_i + \pi_i) + y_i a(Y)(\Omega + \Pi)\}$$

Note that the consumer takes not only the price p of peanuts as given, but the crime level Y and the returns to crime $a(Y)(\Omega + \Pi)$ as well.

The concept of competitive equilibrium is the usual one.

Definition 1 A *competitive equilibrium* consists of a feasible allocation $\langle(x^*, y^*, m^*), (-Z^*, Q^*)\rangle$ and a price p, such that

1. $(-Z^*, Q^*) \in \mathcal{T}$ maximize profits given p.

2. For each $i \in [0,1]$, (x_i^*, y_i^*, m_i^*) maximizes i's utility given his budget.

Characterization of the equilibrium. For simplicity, and since we want to focus on theft, we now characterize the competitive allocations that assign interior consumption bundles. Assume that $\langle(x^*, y^*, m^*), (-Z^*, Q^*)\rangle$ and p constitute such a competitive equilibrium. Then $(-Z^*, Q^*)$ must satisfy the necessary (and sufficient) conditions for profit maximization:

$$p = c'(Q) \text{ and } Z = c(Q).$$

Also, for all $i \in [0,1]$, (x_i^*, y_i^*, m_i^*) must satisfy the first order conditions for utility maximization:

$$\begin{aligned} \phi_i'(x) &= p \qquad i \in [0,1] \\ 1 &\geq a(Y)(\Omega + \Pi) \text{ with equality if } Y > 0 \\ px + y + m &= (1 - A(Y))(\omega + \pi) + ya(Y)(\Omega + \Pi) \end{aligned} \tag{1}$$

Finally, the allocation must be feasible:

$$\begin{aligned} \int x &= Q \\ M + Z + Y &= \Omega \end{aligned}$$

Condition (1) is a zero-profit condition for appropriation activities. It says that in equilibrium, if there is theft, individuals are indifferent between allocating an additional unit of time to leisure or to stealing. Note that condition (1) implies that

$$Y^* = A(Y^*)(\Omega + \Pi^*).$$

In other words, in equilibrium the time spent stealing equals the value of the stolen goods. For that reason it is justified to call Y^* the level of theft, or (property) crime.

It is routine to check that in order to find an equilibrium, it is enough to solve

$$p = c'(Q) \tag{2}$$

$$\phi_i'(x) = p \qquad i \in [0,1] \tag{3}$$

$$\int x = Q \tag{4}$$

$$1 \geq a(Y)(\Omega + \Pi) \text{ with equality if } Y > 0 \tag{5}$$

Note that the equilibrium production plan and price are determined by conditions (2–4) and therefore, given our assumptions about preferences and technology, are unique. Furthermore, they are independent of the appropriation technology and activity. Consequently, so is equilibrium level of profits Π^*. Given the equilibrium aggregate wealth, $(\Omega + \Pi^*)$, the equilibrium level of theft Y^* is characterized by equation (5). Therefore, given that $a(Y)$ is continuous, decreasing and converges to 0 as Y goes to infinity, an application of the intermediate value theorem leads to the following.

Observation 1 A competitive equilibrium exists and is unique. If $a(0) \leq 1/(\Omega + \Pi^*)$, the equilibrium level of crime is 0. If $a(0) > 1/(\Omega+\Pi^*)$ the equilibrium level of crime is positive.

Note that the equilibrium is *locally stable* in the sense that small perturbations of the level of theft unleash forces that return it to the equilibrium level. Indeed, if $Y < Y^*$, then since a is decreasing, the returns to theft are higher than 1, and induce an increase in theft. And, similarly, if $Y > Y^*$, the returns to theft are lower than 1, and induce a decrease in theft.

When positive, the amount, Y^*, of criminal activity is pure waste because it does not produce anything; it only transfers resources from victims to thieves. Moreover, even from the point of view of the thieves there is too much criminal activity. If they wanted to increase $A(Y)(\Omega + \Pi^*) - Y$, namely the booty in excess of the criminal effort, they would choose a crime level lower than Y^*. To see this, note that since $a(Y)$ is decreasing in Y, by (5) we have that $a(Y)(\Omega + \Pi^*) > 1$ for all $0 < Y < Y^*$, which implies that $A(Y)(\Omega + \Pi^*) - Y > 0 = A(Y^*)(\Omega + \Pi^*) - Y^*$ for all $0 < Y < Y^*$. Namely everybody could be made better off by simply reducing the level of crime. If it were possible, it would be beneficial for the thieves and for society as a whole to tax criminal activity. Unfortunately, as Professor G. H. Dorr [5] aptly puts it, "governmental regulations and civic safeguards cannot be applied to antisocial pursuits."

The above discussion shows that there are feasible allocations that can make all individuals better off. However, these allocations may not be enforceable by a social planner, because such a planner would not be able to implement arbitrary combinations of output and crime levels. Indeed, whereas a social planner would be able to control the output level, he would not be able to dictate the level of crime. The level of crime would be determined by the thieves themselves. We will now show, however, that a social planner can enforce a feasible allocation that makes all agents better off.

If we look closely at equation (5) we see that any policy that reduces the stealable wealth, will also reduce the level of crime. One such policy would be to command the firm to produce a quantity Q, larger than the equilibrium quantity Q^*.

Observation 2 Let $\langle (x(Q), m(Q), y(Q)), (-Z(Q), Q) \rangle$ be the equilibrium allocation when the firm is commanded to produce Q. Let $p(Q)$ be the equilibrium price, and let $W(Q) = \int (\phi(x(Q)) + m(Q))$ be the corresponding social welfare. Then $W'(Q^*) > 0$. Namely, starting from the equilibrium level of output, a slight increase in production increases social welfare.

Proof : See Appendix. $\square$

The idea of the proof is as follows. Starting from the competitive equilibrium, a small increase in output leads to a decrease in price and profits. The additional output leads to an increase in the sum of utilities that is completely offset by the increase in cost since at the equilibrium marginal utilities are equal to marginal cost. Therefore, whether or not social welfare increases depends on whether or not the level crime goes down. By the zero-profit condition for appropriation activities, crime will go down if and only if wealth goes down, which it does due to the decrease in profits. We conclude that welfare can be increased by increasing output.

3.2 Public police

Suppose now that the government levies a personalized tax $\hat{t}_i$ and that the total tax $T = \int \hat{t}$ is allocated to crime prevention. That is, $\hat{t}_i$ is the time that individual i contributes to the public police effort, and T is the per capita level of public police protection. Since public police is assumed to be enjoyed equally by all individuals, T is the actual time devoted to protecting individual i's wealth.

The definition of a competitive equilibrium is the same as before, except that now the budget of individual i consists of all the triples (x_i, m_i, y_i) that satisfy

$$px_i + m_i + y_i + \hat{t}_i \leq (1 - A(Y,T))(\omega_i + \pi_i) + y_i a(Y,T)(\Omega + \Pi)$$

The equilibrium allocation is still characterized by equations (2–5) with the proviso that $a(Y)$ is now replaced by $a(Y,T)$. The same argument as before shows that there is a unique equilibrium, which will exhibit positive levels of crime if $a(0,T) > 1/(\Omega + \Pi^*)$. In this case, the equilibrium level of redistributive activity is implicitly defined by

$$Y^*(T) = A(Y^*(T),T)(\Omega + \Pi^*)$$

By the implicit function theorem,

$$Y^{*\prime}(T) = \frac{(\Omega + \Pi^*)A_2(Y^*(T),T)}{1 - (\Omega + \Pi^*)A_1(Y^*(T),T)}.$$

Given that $A_2 < 0$ and that $(\Omega + \Pi^*)A_1(Y^*(T),T) < (\Omega + \Pi^*)A(Y^*(T),T)/Y^*(T) = 1$, we obtain the following.

Observation 3 When positive, the equilibrium level of theft $Y^*(T)$ is decreasing in T.

The *optimal tax level* is the one that minimizes $Y^*(T) + T$. Note that for all $T > Y^*(0)$, we have that $Y^*(T) + T > Y^*(0)$, and since $Y^*(T) + T$ is continuous, it attains its minimum in $[0, Y^*(0)]$. Therefore, the optimal tax is well defined and satisfies $Y^{*\prime}(T) \geq -1$, with equality if $T > 0$. More specifically, it satisfies

$$\frac{(\Omega + \Pi^*)\,A_2(Y^*(T),T)}{1 - (\Omega + \Pi^*)\,A_1(Y^*(T),T)} \geq -1 \qquad \text{with equality if } T > 0. \tag{6}$$

This condition is also sufficient if $Y^*(T)$ is convex. Needless to say, the optimal level of public police does not necessarily lead to zero crime and clearly depends on $(\Omega + \Pi^*)$.

The following example illustrates the concepts introduced so far.

Example 1 Assume that $\omega_i = 9(1+i)^2$ and that the appropriation technology is given by $A(Y,T) = \frac{Y}{(1+T)(1+Y)}$. Suppose that the production technology satisfies constant returns to scale and thus profits are 0 in equilibrium. As a result,

$$(\Omega + \Pi^*) = \int 9(1+i)^2 = 21$$

The zero-profit condition for appropriation activities is given by

$$1 = \frac{21}{(1+T)(1+Y)}$$

and the equilibrium level of crime is

$$Y^*(T) = \max\{\frac{20-T}{1+T}, 0\}$$

The optimal level of police per capita is given by $\hat{T} = \sqrt{21} - 1 = 3.58258$, and the associated optimal level of crime is $\hat{Y} = \sqrt{21} - 1 = 3.58258$.

3.3 Private police

Suppose now that there is no public police but one can hire private police. Alternatively, one can spend some time protecting his own wealth. Given a level Y of crime, if individual i spends t_i on private police, the proportion of his wealth that gets stolen is $A(Y, t_i)$.

The definition of a competitive equilibrium is the same as before, except that now the budget of individual i consists of all the bundles (x_i, m_i, y_i, t_i) that satisfy

$$px_i + m_i + y_i + t_i \leq (1 - A(Y, t_i))(\omega_i + \pi_i) + y_i \int a(Y, t)(\omega + \pi).$$

Note that the individual takes as given the price p, the level of crime Y and the return to theft $\int a(Y, t)(\omega + \pi)$. The equilibrium price and peanut output are still characterized by equations (2–4). The equilibrium allocation of private police, t^*, and the equilibrium level of theft, Y^*, are now characterized by the following conditions (note that since A is convex in t, the necessary conditions for utility maximization are also sufficient):

$$1 + (\omega_i + \pi_i) A_2(Y, t_i) \geq 0, \text{ with equality if } t_i > 0 \qquad i \in [0,1] \tag{7}$$

$$\int a(Y, t_i)(\omega_i + \pi_i) \leq 1, \text{ with equality if } Y > 0 \tag{8}$$

Condition (7) implicitly defines individual i's demand $t_i(Y)$ for private police as a function of crime level. It can be checked that given our assumptions on A, when $t_i(Y) > 0$, we have that $t_i'(Y) > 0$. Namely, the higher the crime rate, the higher the preferred level of private police protection. Further note that since $A(0, t_i) = 0$ for all $t_i \geq 0$, we have that $t_i(0) = 0$. Namely, when there is no crime, the individual does not demand police protection. All this, along with the fact that $a(Y, t_i(Y))$ is decreasing in Y and converges to 0 as Y goes to infinity, allows us to conclude the following.

Observation 4 An equilibrium with private police exists and is unique. If $a(0,0) \leq 1/(\Omega + \Pi^*)$, the equilibrium level of crime is 0. If $a(0,0) > 1/(\Omega + \Pi^*)$ the equilibrium level of crime is positive.

Here again there is too much criminal activity even from the thieves' point of view. Since $\int a(Y, t_i(Y))(\omega_i + \pi_i^*)$ is decreasing in Y, we have that $\int a(Y, t_i(Y))(\omega_i + \pi_i^*) > \int a(Y^*, t_i(Y^*))(\omega_i + \pi_i^*) = 1$ for all $Y < Y^*$. Therefore, $\int A(Y, t_i(Y))(\omega_i + \pi_i^*) - Y > 0 = \int A(Y^*, t_i(Y^*))(\omega_i + \pi_i^*) - Y^*$ for all $Y < Y^*$. This means that $\int A(Y, t_i)(\omega_i + \pi_i^*) - Y$, namely the thieves' net benefit from appropriation activities could be increased by decreasing the level of criminal activity. That is, the decrease in the rewards to crime due to the reduction in criminal activity will be more than compensated by the increase in these rewards resulting from the induced lower police protection plus the time saved by the thieves.

3.3.1 Optimal allocation of private police

We now investigate whether the competitive equilibrium allocates private police efficiently. The optimal allocation of private police minimizes total waste. Namely it solves

$$\min_{Y,t} Y + \int t$$
$$\text{s.t. } 1 \geq \int a(Y, t_i)(\omega_i + \pi_i^*) \quad \text{with equality if } Y > 0$$

It can be seen that, since a is decreasing in its second argument, any solution to this problem satisfies the constraint with equality. We can solve this problem in two steps. First we solve for the optimal allocation of police resources given an arbitrary level of crime, and then we solve for the optimal level of crime. Specifically, the *optimal allocation of police protection given a level of crime* Y is the solution to

$$V(Y) = \min_t \int t \tag{9}$$
$$\text{s.t. } \; 1 = \int a(Y, t_i)(\omega_i + \pi_i^*)$$

and the *optimal level of crime* solves

$$\min_Y Y + V(Y).$$

The problem (9) is equivalent to

$$\min_{t} \int t$$

$$\text{s.t. } k_i' = a(Y, t_i)(\omega_i + \pi_i^*)$$

$$k_0 = 0$$

$$k_1 = 1$$

The associated Hamiltonian is $H(t_i, k_i, \lambda_i, Y) = t_i + \lambda_i a(Y, t_i)(\omega_i + \pi_i^*)$, and the necessary conditions for a solution are

$$1 + \lambda_i(\omega_i + \pi_i^*)a_2(Y, t_i) \geq 0 \quad \text{with equality if } t_i > 0 \tag{10}$$

$$\lambda_i' = 0 \tag{11}$$

$$k_i' = a(Y, t_i)(\omega_i + \pi_i^*) \tag{12}$$

$$k_0 = 0, \;\; k_1 = 1. \tag{13}$$

It follows from (10–11) that λ_i does not depend on i and that $\lambda_i \geq 0$. As a result, given that $a(Y, t_i)$ is convex in the second argument, the above conditions are also sufficient for the optimality of the distribution of police protection.

It follows from (10) that for any level of crime Y, the optimal police function satisfies

$$(\omega_i + \pi_i^*)A_2(Y, t_i) = (\omega_j + \pi_j^*)A_2(Y, t_j) \qquad \text{for all } i, j \in [0, 1] \text{ with } t_i, t_j > 0$$

namely, the property loss prevented by an additional unit of police protection is independent of whom this additional unit is allocated to. This means that it is never optimal to allocate police effort equally unless incomes are equal. In fact, richer people should be allocated higher police effort.

Inspecting equations (7–8) which characterize the competitive equilibrium, we see that the competitive private police allocation t_i^*, along with $\lambda_i = Y^*$ also satisfies conditions (10–13) when the level of crime is fixed to be at the competitive level Y^*. Furthermore, t_i^* is the only distribution of police protection that solves (9) for $Y = Y^*$. To see this, note that for each $\lambda > 0$, there is a unique distribution t that satisfies (10). This t is increasing in λ, which implies that, $\int a(Y^*, t_i(\lambda))(\omega_i + \pi_i^*)$ is decreasing in λ. This means that $\int a(Y^*, t_i(\lambda))(\omega_i + \pi_i^*) = 1$ has at most one solution. Summing up:

Observation 5 Given the competitive crime level Y^*, the only optimal allocation of police protection is the competitive one.

Private police exerts a positive externality; it reduces the returns to theft, which induces people to spend less time stealing from the whole population. This externality is not taken into account by the individual and as a result, the competitive level of crime is not globally optimal. In fact, we can show the following.

Observation 6 At the competitive equilibrium, the level of crime is too high. Namely, the total waste could be reduced by increasing spending on police protection, thereby reducing crime.

Proof : It is enough to show that $-V'(Y^*) < 1$. Indeed, the value of $-V'(Y^*)$ is the additional spending on police required to reduce crime by one small unit. If at the equilibrium we had $-V'(Y^*) < 1$, there would be too much crime; reducing the time spent on crime by one unit costs less than one unit. The Lagrangian associated with problem (9) is

$$L = \int \big(t_i - \lambda_i(1 - a(Y, t_i)()).$$

By the envelope theorem

$$\begin{aligned}
V'(Y^*) &= \int \lambda^* A_1(Y^*, t_i^*)(\omega_i + \pi_i^*) \\
&= \int Y^* A_1(Y^*, t_i^*)(\omega_i + \pi_i^*) \\
&= \int A_1(Y^*, t_i^*)(\omega_i + \pi_i^*) - \int a(Y^*, t_i^*)(\omega_i + \pi_i^*) \\
&\geq \int A_1(Y^*, t_i^*)(\omega_i + \pi_i^*) - 1 \\
&> -1
\end{aligned}$$

where we have used the fact that when the level of crime is the competitive one, $\lambda^* = Y^*$. $\square$

If the excise rate $a(Y, t_i)$ is convex, then the function V is convex as well. In that case, the optimal level of crime $\hat{Y}$ is lower than the competitive level. In order to induce individuals to spend the optimal level of police protection, the government could subsidize the cost of police by means of a quantity subsidy of $s = (\hat{\lambda} - Y^*)/\hat{\lambda}$, where $\hat{\lambda}$ is the value of the costate variable when the level of crime is the optimal one.[4] We should note that an income subsidy of $\sigma = \hat{\lambda} - 1$ (so that wealth becomes $\hat{\lambda}(\omega_i + \pi_i)$) does not achieve the optimal level of crime and private police protection because it also increases the stealable wealth with the resulting enhanced incentives to theft.

The following example illustrates the concepts developed so far.

Example 2 Consider the economy described in Example 1, where the the appropriation technology is given by $A(Y, t_i) = \frac{Y}{(1+t_i)(1+Y)}$. We now calculate its competitive equilibrium. The utility maximizing level of private police, if positive, satisfies the first-order condition

$$1 - \frac{9(i+1)^2 Y}{(t_i+1)^2(Y+1)} = 0$$

which yields a demand of police protection given by

$$t_i(Y) = \max\{\frac{3(1+i)\sqrt{Y(Y+1)}}{Y+1} - 1, 0\}.$$

The condition of zero-profitability of crime is

$$1 = \frac{9\sqrt{Y(Y+1)}}{2Y(Y+1)}$$

which yields $Y^* = 4.02769$ and $t_i^* = 1.68513 + 2.68513\, i$.

We now calculate the optimal police allocation conditional on a positive level Y of crime. To avoid corner solutions, we also assume that $Y \leq 25/2$. The Hamiltonian is

$$H(t_i, k_i, \lambda) = t_i + \lambda \frac{1}{(1+t_i)(1+Y)} 9(1+i)^2$$

Assuming that there is an interior minimum, it satisfies

$$\frac{\partial H}{\partial t_i} = 1 - \frac{9(1+i)^2 \lambda}{(t_i+1)^2(Y+1)} = 0$$

[4]In this case, since the price of a unit of police protection is 1, a quantity subsidy and an ad-valorem subsidy amount to the same thing.

which gives

$$t_i = \frac{3(1+i)\sqrt{\lambda(Y+1)}}{Y+1} - 1$$

We must also have that $\int a(Y, t_i)(\omega_i + \pi_i) = 1$, which gives

$$\lambda(Y) = \frac{81}{4(Y+1)}$$

Substituting back into the formula of t_i we obtain that

$$\hat{t}_i(Y) = \frac{25 + 27i - 2Y}{2(Y+1)}$$

Note that when the level of theft is the equilibrium one, namely when $Y = Y^*$, the optimal allocation of police protection is also the equilibrium one:

$$\hat{t}_i(Y^*) = 1.68513 + 2.68513\, i = t_i^*.$$

The total waste is

$$Y + \int \hat{t}_i(Y) = \frac{77 + 4Y^2}{4(Y+1)}.$$

Consequently, the optimal level of crime and that the associated private police function are

$$\widehat{Y} = 3.5, \qquad \hat{t}_i = 2 + 3i.$$

Since $\lambda(\hat{Y}) = \hat{\lambda} = 9/2$, we obtain that a subsidy on private police protection of $s = (\hat{\lambda} - \hat{Y})/\hat{\lambda} = 2/9$ will yield an equilibrium with the optimal level of crime.

3.4 Voting equilibrium

In this section, we introduce the notion of a voting equilibrium. A voting equilibrium consists of a competitive equilibrium in which individuals are partitioned into groups, each one residing in a different community with its own crime level and its own level of public police protection. In such an equilibrium, nobody wants to leave his community and furthermore, the tax rate used to finance police protection is preferred to any other tax rate by a majority of the residents. Different tax regimes may lead to different voting equilibria.

3.4.1 Voting equilibrium with a head tax

Given a partition of the individuals $\mathcal{P} = \{R_1, \ldots, R_K\}$ into K nonempty groups with associated head tax rates T_k for each group $k = 1, \ldots, K$, a feasible allocation consists of an assignment of bundles (x, y, m) and a production plan $(-Z, Q)$ such that

1. $\int x = Q$,

2. $\int m + Z + \int y + \sum_{k=1}^{K} \int_{R_k} T_k = \Omega$

The interpretation is as follows. The individuals are partitioned into different groups and reside in different regions. Each region k sets a head tax T_k which the residents must pay, and which is used to finance a local public police. The size of group k is $\mu(R_k) = \int_{R_k} 1$. The wealth per capita in region k is $(\Omega_k + \Pi_k)$ where $\Omega_k = \int_{R_k} \omega/\mu(R_k)$ and $\Pi_k = \int_{R_k} \pi/\mu(R_k)$. The crime rate in region R_k is $Y_k = \int_{R_k} y/\mu(R_k)$.

Given a price of peanuts, a level of crime Y_k, and police protection T_k, the budget of a resident i of community k is

$$\{(x_i, m_i, y_i) : px_i + m_i + y_i + T_k \leq (1 - A(Y_{R_k}, T_k))(\omega_i + \pi_i) + y_i a(Y_k, T_k)(\Omega_k + \Pi_k)\}.$$

A *voting equilibrium with head income tax* consists of a partition of the individuals $\mathcal{P} = \{R_1, \ldots, R_K\}$ with associated head tax rates $T_1, \ldots T_K$, a feasible allocation (x^*, y^*, m^*), and a production plan $(-Z^*, Q^*)$ and a price p such that

1. (Z^*, Q^*) maximize profits given p.

2. For each $i \in R_k$, (x_i^*, y_i^*, m_i^*) maximizes his utility given his budget.

3. Residents of region R_k do not prefer to move to other regions.

4. The tax rate T_k is preferred by a majority of residents in R_k to any other tax rate.

Conditions (1–2) are the usual profit and utility maximizing conditions, which along with the peanut market clearing condition determine the equilibrium price, p, production plan, $(-Z^*, Q^*)$, and corresponding profits, Π^*, which, as noted earlier, are independent of the appropriation technology and activity. Condition 3 says that, given the tax rates and crime levels of the various communities, individuals do not want to emigrate from their own community.[5] Condition 4 says that the tax rate in each community is a Condorcet winner given the preferences over tax rates of its residents.

Let R_k be a nonempty subset of individuals. The equilibrium level of theft in community k is implicitly defined by

$$1 \geq a(Y_k^*(T), T)(\Omega_k + \Pi_k^*), \qquad \text{with equality if } Y_k^*(T) > 0.$$

For $i \in R_k$, his preferred level of public police is the one that minimizes the sum of wealth that is stolen from him plus the tax that he pays. That is, it is the level T that minimizes $A(Y_k^*(T), T)(\omega_i + \pi_i^*) + T$. Since $A(Y_k^*(T), T)(\Omega_k + \Pi_k^*) = Y_k^*(T)$, the preferred level is the one that minimizes $Y_k^*(T)(\omega_i + \pi_i^*)/(\Omega_k + \Pi_k^*) + T$. Therefore, the preferred level of public police satisfies $Y_k^{*\prime}(T)(\omega_i + \pi_i^*)/(\Omega_k + \Pi_k^*) \geq -1$. In other words, it satisfies

$$\frac{(\omega_i + \pi_i^*)A_2(Y_k^*(T), T)}{1 - (\Omega_k + \Pi_k^*)A_1(Y_k^*(T), T)} \geq -1 \quad \text{with equality if } T > 0. \tag{14}$$

Note that $\frac{(\Omega_k+\Pi_k^*)A_2(Y_k^*(T),T)}{1-(\Omega_k+\Pi_k^*)A_1(Y_k^*(T),T)}$ is the slope of $Y_k^*(T)$, which by Observation 3 is negative. Note also that it is increasing in T. Therefore, $\frac{A_2(Y_k^*(T),T)}{1-(\Omega_k+\Pi_k^*)A_1(Y_k^*(T),T)}$ is also negative and increasing in T. Consequently, the higher $(\omega_i + \pi_i^*)$, the higher the preferred tax rate. Namely, wealthier people prefer higher taxes. As a result, by the median voter theorem (Black [2]), the tax preferred by a median voter, m_k, of R_k is a Condorcet winner.

Based on the above, the equilibrium allocations are characterized by the peanut market equilibrium conditions (2-4) along with the following conditions:

$$\begin{aligned}
a(Y_k, T_k)(\Omega_k + \Pi_k^*) &\leq 1 \qquad k = 1, \ldots, K \\
A(Y_k, T_k)(\omega_i + \pi_i^*) + T_k &\leq A(Y_{k'}, T_{k'})(\omega_i + \pi_i^*) + T_{k'} \quad \text{for all } i \in R_k, k, k' = 1, \ldots, K \\
\frac{(\omega_{m_k} + \pi_{m_k}^*)A_2(Y_k^*(T_k), T_k)}{1 - (\Omega_k + \Pi_k^*)A_1(Y_k^*(T_k), T_k)} &\geq -1 \quad \text{with equality if } T_k > 0 \quad \text{for all } k = 1, \ldots K.
\end{aligned}$$

[5] If the partition consists of a single group, this condition is superfluous.

The next observation shows that ordering regions in increasing order of tax rates or in decreasing order of excise rates amounts to the same thing. It also shows that equilibrium regions are classified by income, and establishes the corresponding equilibrium income brackets.

Observation 7 Let $\{R_1, \ldots, R_K\}, \{T_1, \ldots, T_K\}$ be an equilibrium partition such that $T_1 < \cdots < T_K$. Then $A(Y^*_{R_1}, T_1) > \cdots > A(Y^*_{R_K}, T_K)$. Furthermore,

$$(\omega_i + \pi^*_i) \leq \frac{T_{k+1} - T_k}{A(Y^*_{R_k}, T_k) - A(Y^*_{R_{k+1}}, T_{k+1})} \leq (\omega_j + \pi^*_j) \qquad \text{for all } i \in R_k, j \in R_{k+1}$$

Proof : Let $i \in R_k$ and $j \in R_{k+1}$. Since no individual wishes to move to another region, we must have that

$$\Big(A(Y^*_{R_k}, T_k) - A(Y^*_{R_{k+1}}, T_{k+1})\Big)(\omega_i+\pi^*_i) \leq T_{k+1}-T_k \leq \Big(A(Y^*_{R_k}, T_k) - A(Y^*_{R_{k+1}}, T_{k+1})\Big)(\omega_j+\pi^*_j).$$

And since $T_{k+1} > T_k$ and $(\omega_j + \pi^*_j) > 0$, we have that $A(Y^*_{R_k}, T_k) - A(Y^*_{R_{k+1}}, T_{k+1}) > 0$. Therefore, dividing the above inequalities by this expression we obtain the desired result. $\square$

The following example illustrates an equilibrium partition with two regions.

Example 3 Assume that $\omega_i = 4$ for $i \in [0, 1/3)$, $\omega_i = 25$ for $i \in [1/3, 5/9)$ and $\omega_i = 100$ for $i \in [5/9, 1]$, and that the appropriation technology is given by $A(Y, T) = \frac{Y}{(1+T)(1+Y)}$. Suppose that the production technology satisfies constant returns to scale and thus profits are 0 in equilibrium. Consider now a partition $R_1 = [0, 1/2)$, $R_2 = [1/2, 1]$. The average wealths of each region are $\Omega_1 = 11$ and $\Omega_2 = 91.667$. The regions' median voters are $m_1 = 1/4$ and $m_2 = 3/4$. Their respective incomes are, $\omega_{m_1} = 4$ and $\omega_{m_2} = 100$. The zero-profit conditions for appropriation activities are given by

$$1 = \frac{\Omega_k}{(1+T)(1+Y)} \qquad k = 1, 2$$

and the equilibrium levels of crime are

$$Y_k(T) = \frac{\Omega_k - 1 - T}{1+T} \qquad k = 1, 2$$

Using (14), we obtain that the levels of police per capita preferred by the median voter is $T_k = \sqrt{\omega_{m_k}} - 1$, and the associated optimal level of crime is $Y_k = \Omega_k / \sqrt{\omega_{m_k}} - 1$. Consequently, the corresponding police protection and crime rates are given by

$$Y_1 = 4.5 \qquad T_1 = 1 \qquad Y_2 = 8.167 \qquad T_2 = 9.$$

which results in the following proportions of wealth being redistributed:

$$A(Y_1, T_1) = 9/22 \qquad A(Y_2, T_2) = 49/550.$$

Notice that $\big(T_1 - T_2\big)/\big(A(Y_2, T_2) - A(Y_1, T_1)\big) = 25$ and therefore

$$\omega_i \leq \frac{T_1 - T_2}{A(Y_2, T_2) - A(Y_1, T_1)} \leq \omega_j \qquad \text{for all } i \in R_1, j \in R_2$$

As a result, $\{R_1, R_2\}$ along with tax rates $\{T_1, T_2\}$ constitutes an equilibrium partition.

3.4.2 Voting equilibrium with proportional income tax

We now introduce a voting equilibrium where taxation is restricted to be proportional to income. Given a partition of the individuals $\mathcal{P} = \{R_1, \ldots, R_K\}$ into K nonempty groups with associated proportional tax rates τ_k for each group $k = 1, \ldots, K$, a feasible allocation consists of an assignment of triples (x, y, m), and a production plan $(-Z, Q)$ such that

1. $\int x = Q$,
2. $\int m + Z + \int y + \sum_{k=1}^{K} \int_{R_k} \tau_k(\omega_i + \pi_i) = \Omega$

The interpretation is similar to that of a voting equilibrium with a head tax. The only difference is that now, the level of police protection per capita in region k is $T_k = \tau_k(\Omega_k + \Pi_k)$.

Given a price of peanuts, a level of crime Y_{R_k}, and police protection T_k, a resident i of community R_k's budget is

$$\{(x_i, m_i, y_i) : px_i + m_i + y_i + \tau_k(\omega_i + \pi_i) \leq (1 - A(Y_{R_k}, T_k))(\omega_i + \pi_i) + y_i a(y_{R_k}, T_k)(\Omega_k + \Pi_k)\}.$$

A *voting equilibrium with proportional income tax* consists of a partition of the individuals $\mathcal{P} = \{R_1, \ldots, R_K\}$ with associated proportional tax rates $\tau_1, \ldots \tau_K$, a feasible allocation (x^*, y^*, m^*), and a production plan $(-Z^*, Q^*)$ and a price p such that

1. (Z^*, Q^*) maximize profits given p.

2. For each $i \in R_k$, (x_i^*, y_i^*, m_i^*) maximizes his utility given his budget.

3. Residents of region R_k do not prefer to move to other regions.

4. The tax rate τ_k is preferred by a majority of residents in R_k to any other tax rate.

Conditions (1–2) are the usual profit and utility maximizing conditions, which along with the peanut market clearing condition determine the equilibrium price, p, production plan, $(-Z^*, Q^*)$, and corresponding profits, Π^*, which, as noted earlier, are independent of the appropriation technology and activity. Condition 3 is the usual free mobility condition, and condition 4 requires that the region's tax rate be a Condorcet winner. In order to characterize it we need to figure out the agents' preferred level of public police, which we do next.

Let R_k be a nonempty subset of individuals. The equilibrium level of theft in community k is implicitly defined by

$$1 \geq a(Y_k^*(T), T)(\Omega_k + \Pi_k^*), \qquad \text{with equality if } Y_k^*(T) > 0.$$

For $i \in R_k$, his preferred level of public police is the one that minimizes the wealth that is stolen from him plus the tax that he pays. Therefore, individual i's preferred level of police protection is the level T that minimizes $A(Y_k^*(T), T)(\omega_i + \pi_i^*) + (T/(\Omega_k + \Pi_k^*))(\omega_i + \pi_i^*)$. Multiplying by the constant $(\Omega_k + \Pi_k^*)/(\omega_i + \pi_i^*)$ and taking into account that $A(Y_k^*(T), T)(\Omega_k + \Pi_k^*) = Y_k^*(T)$, we obtain that this level is the one that minimizes $Y_k^*(T) + T$. That is, all agents prefer the region's socially optimal level of public police. As a result, taking into account that the optimal level of public police is given by (6), the equilibrium allocation is characterized now by the peanut market equilibrium conditions as well as by the following conditions:

$$\begin{aligned}
a(Y_k^*(T), T)(\Omega_k + \Pi_k^*) &\leq 1, \qquad \text{with equality if } Y_k^*(T) > 0 \\
A(Y_k, T_k) + \tau_k &= A(Y_{k'}, T_{k'}) + \tau_{k'} \qquad k, k' = 1, \ldots, K \\
\frac{(\Omega_k + \Pi_k^*) A_2(Y_k^*(T_k), T_k)}{1 - (\Omega_k + \Pi_k^*) A_1(Y_k^*(T_k), T_k)} &\geq -1 \quad \text{with equality if } T_k > 0 \quad \text{for all } k = 1, \ldots K.
\end{aligned}$$

Since under proportional income taxation, the optimal tax per capita is unanimously preferred to any other level of tax, there exist trivial equilibrium partitions in which the economy-wide optimal tax rate (see Section 3.2) is applied to every region. Formally,

Observation 8 Any partition $\{R_1, \ldots, R_K\}$ such that all the mean wealths are equal, namely, $(\Omega_1 + \Pi_1^*) = \cdots = (\Omega_K + \Pi_K^*) = (\Omega + \Pi^*)$, and such that the tax per capita is the economy-wide optimal one, is an equilibrium partition.

There may however be, non-trivial equilibrium partitions, as the following example illustrates.

Example 4 Assume that $\omega_i = 16\,i$ and that the appropriation technology is given by $A(Y,T) = \frac{Y}{(1+4T)(1+Y)}$. Suppose that the production technology satisfies constant returns to scale and thus profits are 0 in equilibrium. As a result, for any region R , the per capita wealth is given by

$$\Omega_R = \frac{1}{\mu(R)} \int_R 16\,i$$

The zero-profit condition for appropriation activities is given by

$$1 = \frac{\Omega_R}{(1+4T)(1+Y)}$$

and the equilibrium level of crime is

$$Y_R(T) = \frac{\Omega_R - 1 - 4T}{1+4T}$$

Using (6), the optimal level of police per capita is given by $T_R = \sqrt{\Omega_R}/2 - 1/4$, and the associated optimal level of crime is $Y_R = \sqrt{\Omega_R}/2 - 1$. Consider now a partition $R_1 = [1/32, 15/32)$, $R_2 = [0, 1/32) \cup [15/32, 1]$. The average wealths of each region are $\Omega_1 = 4$ and $\Omega_2 = 100/9$. Consequently, the corresponding optimal police and crime rates are given by

$$Y_1 = 0 \qquad T_1 = 3/4 \qquad Y_2 = 2/3 \qquad T_2 = 17/2$$

which results in the following excise rates:

$$A(Y_1, T_1) = 0 \qquad A(Y_2, T_2) = .06$$

Since income is taxed proportionally, we have that

$$\tau_1 = \frac{T_1}{\Omega_1} = .1875 \qquad \tau_2 = \frac{T_2}{\Omega_2} = .1275$$

Since $A(Y_1, T_1) + \tau_1 = A(Y_2, T_2) + \tau_2$, we conclude that $(\{R_1, R_2\}, \{\tau_1, \tau_2\})$ is an equilibrium partition.

4 Only earned income can be stolen

In this section, we assume that leisure cannot be stolen. That is, individual i's stealable wealth is $(\omega_i + \pi_i - m_i)$ and the aggregate stealable wealth is $(\Omega + \Pi - M)$. We now reintroduce the equilibrium concepts, first when there is no police protection, and later we allow for public and private police protection.

4.1 Competitive equilibrium

According to our assumptions, given a price of peanuts p individual i's budget set is given by

$$\{(x_i, m_i, y_i) : px_i + y_i \leq (1 - A(Y))(\omega_i + \pi_i - m_i) + y_i a(Y)(\Omega + \Pi - M)\}$$

That is, the wealth available to individual i for spending on peanuts and theft is the sum of the part of his stealable wealth that has not been stolen, and the other people's wealth that he has appropriated for himself. Note that the parameters that the individual takes as given are the price p, the crime rate Y and the returns to theft $a(Y)(\Omega+\Pi-M)$. Note also that the relative price of peanuts (in terms of leisure) faced by the consumers is $p/(1 - A(Y))$. This is so because for every unit of time that they devote to work, $A(Y)$ is stolen and therefore only $(1 - A(Y))$ can be used to purchase peanuts. Equivalently, a consumer who wants to bring home one unit of peanuts needs to buy $1/(1 - A(Y))$ units because a proportion $A(Y)$ of them will be stolen.

Since for each individual i his share of the booty is proportional to y_i, we can equivalently assume that neither leisure nor the time spent stealing can be stolen. Indeed, his budget can be equivalently written as

$$\{(x_i, m_i, y_i) : px_i \leq (1 - A(Y))(\omega_i + \pi_i - m_i - y_i) + y_i a(Y)(\Omega + \Pi - M - Y)\}$$

That is, the wealth available to individual i for peanut consumption is the sum of two terms. One is the part of his legitimate income that has not been stolen and the other is the share of his neighbors' legitimate income that he stole. We will henceforth call $\omega_i - \pi_i - m_i - y_i$ agent *i's net stealable wealth* and we will call $\Omega - \Pi - M - Y$ the *gross net stealable wealth*.

The definition of a competitive equilibrium is, *mutatis mutandis*, the one introduced in Section 3.1.

Characterization of the equilibrium. We now characterize the competitive allocations that assign interior consumption bundles. Assume that $\langle (x^*, m^*, y^*), (-Z^*, Q^*) \rangle$ and p constitute such a competitive equilibrium. Then $(-Z^*, Q^*)$ satisfies the following necessary (and sufficient) conditions for profit maximization:

$$p = c'(Q) \text{ and } Z = c(Q).$$

Also, (x_i, m_i, y_i) must satisfy the first-order conditions for utility maximization: there is $\lambda \geq 0$ such that

$$\begin{aligned} \phi_i'(x_i) &= \lambda p \qquad i \in [0,1] \\ 1 &= \lambda(1 - A(Y)) && (15) \\ 1 &\geq a(Y)(\Omega + \Pi - M) \quad \text{with equality if } Y > 0 && (16) \\ px + y &= (1 - A(Y))(\omega + \pi - m) + ya(Y)(\Omega + \Pi - M) && (17) \end{aligned}$$

Finally, the allocation must be feasible:[6]

$$\begin{aligned} \int x &= Q && (18) \\ M + Z + Y &= \Omega && (19) \end{aligned}$$

Note that in equilibrium $A(Y^*) < 1$. This follows from condition (15).

[6] In fact, by Walras's law, condition 19 is reduntant.

Condition (16) is a zero-profit condition for appropriation activities. It says that in equilibrium either theft doesn't pay and nobody engages in crime, or individuals are indifferent between allocating an additional unit of time to leisure or to stealing. Specifically, if an individual spends one additional unit of time on theft, he gives up one unit of leisure. Namely, the opportunity cost of theft is 1. On the other hand, the benefit of that same unit of time devoted to theft is its share in the stolen wealth, $a(Y)(\Omega + \Pi - M)$. If this share is less than one, nobody wants to engage in theft. Only when this share is one, will an individual devote part of his time to theft. Note that the aggregate wealth subject to theft considered in condition (16) is the gross stealable wealth, namely the one that includes the time spent stealing. That is, in their calculation of the marginal benefit of theft, the thieves include not only the income legitimately earned but also the one acquired by stealing. We will discuss this point later.

Integrating (17) and using (18) we see that in equilibrium

$$pQ^* + Y^* = (\Omega + \Pi^* - M^*) \tag{20}$$

namely, the gross stealable wealth equals the sum of the GDP and the value of the stolen goods. We also have that

$$Y^* = A(Y^*)(\Omega + \Pi^* - M^*). \tag{21}$$

This equality is trivially satisfied if $Y^* = 0$, and if $Y^* > 0$ it follows from (16). But using equations (20–21) we obtain that the value of the stolen goods is

$$Y^* = \frac{A(Y^*)}{1 - A(Y^*)} pQ^*. \tag{22}$$

Using (20) and (22) we can see that at the equilibrium level of crime, the gross stealable wealth can be written as the sum of two components:

$$\Omega + \Pi^* - M^* \quad = \quad pQ^* + \frac{A(Y^*)}{1 - A(Y^*)} pQ^*$$

One component is the economy's GDP, namely the value of the peanuts actually produced. The other is the value of the stolen peanuts, where stolen peanuts are counted as many times as they are stolen. Indeed,

$$\begin{aligned}\Omega + \Pi^* - M^* &= pQ^* + \frac{A(Y^*)}{1 - A(Y^*)} pQ^* \\ &= pQ^* + A(Y^*)pQ^* + A(Y^*)^2 pQ^* + \cdots\end{aligned}$$

Namely, the gross stealable wealth considered in condition (16) consists of the income legitimately earned and that illegitimately earned. Referring to equation (22), we see that as in the previous section, the equilibrium time devoted to theft equals the value of the stolen goods. For this reason, as in Section (3), Y^* is aptly referred to as the level of crime.

4.2 Existence

It is routine to check that in order to find an interior equilibrium, it is enough to solve

$$p = c'(Q) \tag{23}$$

$$\phi_i'(x_i) = \frac{p}{1 - A(Y)} \qquad i \in [0, 1] \tag{24}$$

$$\frac{a(Y)}{1 - A(Y)} p \int x \leq 1 \quad \text{with equality if } Y > 0 \tag{25}$$

$$\int x = Q \tag{26}$$

Once this system is solved, the remaining variables are obtained by mere substitution.

Let $x_i^d : \mathbb{R}_+ \to \mathbb{R}_+$ be the demand function of individual i as a function of the effective relative price of peanuts $p^d = p/(1 - A(Y))$. Namely, $x_i^d(p^d)$ solves $\phi_i'(x_i) = p^d$. Also let $X^d = \int x_i^d$ be the aggregate demand as a function of the effective relative price of peanuts. Similarly, let $Q^s : \mathbb{R}_+ \to \mathbb{R}_+$ be the supply function of peanuts. Namely, $Q^s(p)$ solves $c'(Q) = p$. Then, the conditions of equilibrium can be written as

$$X^d\left(\frac{p}{1 - A(Y)}\right) = Q^s(p) \tag{27}$$

$$\frac{a(Y)}{1 - A(Y)} pQ^s(p) \leq 1 \text{ with equality if } Y > 0 \tag{28}$$

Figure 1 depicts the equilibrium in the peanut market. Crime has a similar effect to that of an ad valorem tax of $A(Y^*)/(1 - A(Y^*))$. It introduces a wedge between the effective price paid by the consumers and the one received by the firm. The difference is the value of the peanuts being stolen when one ends up acquiring one unit of peanuts. However, since the value of the stolen peanuts equals the value of the time spent on appropriation activities, this value ultimately dissipates.

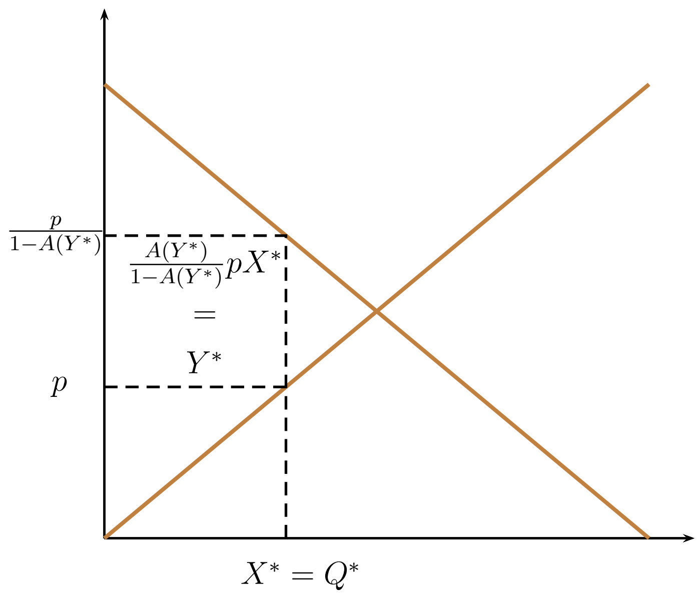

Figure 1: The peanut market.

Equation (27) implicitly defines p as a function of Y. Let $p(Y)$ denote this function. Also let $X(Y) = X^d(\frac{p(Y)}{1-A(Y)})$ denote the associated aggregate demand. Note that $p(Y)$ is non increasing and that $X(Y)$ is strictly decreasing. As a result, $p(Y)X(Y)$ is strictly decreasing. Making use of $p(Y)$ and $X(Y)$, the conditions for equilibrium are reduced to:

$$\frac{a(Y)}{1-A(Y)}p(Y)X(Y) \leq 1 \quad \text{with equality if } Y > 0 \tag{29}$$

Namely, the returns to theft as a function of the crime level when the peanut market is in equilibrium should be 1, unless the crime level is 0, in which case it should not exceed 1.

As opposed to the case of Section 3, an equilibrium is not guaranteed to exist as the following example illustrates.

Example 5 Consider an economy where consumers have a common utility function given by $\phi_i(x) = -1/x^2$, the cost function is $c(Q) = 2Q$, and the appropriation technology is given by $A(Y) = 1 - e^{-Y}$. Given the linear technology, the equilibrium price must be $p(Y) = 2$, and therefore we have that $X(Y) = \big(1 - A(Y)\big)^{1/3}$. Hence, in equilibrium we must have that $p(Y)X(Y) = 2(1 - A(Y))^{1/3}$. It can be checked that

$$\frac{a(Y)}{1 - A(Y)} = \begin{cases} 1 & Y = 0 \\ \frac{e^Y - 1}{Y} & Y > 0 \end{cases}$$

As a result, the returns to crime as a function of Y is given by

$$\frac{a(Y)}{1 - A(Y)} p(Y)X(Y) = \begin{cases} 2 & Y = 0 \\ 2\frac{e^Y - 1}{Y} e^{-Y/3} & Y > 0 \end{cases}$$

which is greater than 1 for all $Y \geq 0$. We conclude that this economy has no equilibrium.

In this model, an equilibrium is *locally stable* if the returns to theft $\frac{a(Y)}{1-A(Y)} p(Y)X(Y)$ are decreasing at the equilibrium level of theft. When an equilibrium does exist, it may be neither unique nor stable. The following examples illustrate this point.

Example 6 Consider an economy where consumers have a common utility function defined on $[0, 10]$ given by $\phi_i(x) = x(10 - x/2)$, the cost function is $c(Q) = Q/20$, and the appropriation technology is given by $A(Y) = 1 - e^{-Y}$. Given the linear technology, the equilibrium price must be $p(Y) = 1/20$. Therefore,

$$\begin{aligned} X(Y) &= \max\{10 - \frac{p(Y)}{1 - A(Y)}, 0\} \\ &= \max\{10 - \frac{1}{20e^{-Y}}, 0\}. \end{aligned}$$

and hence in equilibrium we must have that

$$p(Y)X(Y) = \max\{\frac{1}{2} - \frac{1}{400e^{-Y}}, 0\}.$$

It can be checked that

$$\frac{a(Y)}{1 - A(Y)} = \begin{cases} 1 & Y = 0 \\ \frac{e^Y - 1}{Y} & Y > 0 \end{cases}$$

As a result, the returns to crime as a function of Y is given by

$$\frac{a(Y)}{1-A(Y)}p(Y)X(Y) = \begin{cases} 199/400 & Y=0 \\ -\frac{\left(e^Y-200\right)\left(e^Y-1\right)}{400Y} & 0<Y\leq \ln(200) \\ 0 & Y>\ln(200) \end{cases}$$

which intersects 1 at $Y = 1.28$ and $Y = 5.24$. We conclude that this economy has three equilibria. One, with no theft, one with $Y = 1.28$ and one with $Y = 5.24$. Only the first and third equilibrium are locally stable.

Observation 9 A sufficient condition for the existence of an equilibrium is that the returns to theft, $\frac{a(Y)}{1-A(Y)}p(Y)X(Y)$ be less than 1 for some Y. In particular, this condition holds if $\lim_{Y\to\infty}\frac{a(Y)}{1-A(Y)} = 0$. For example, if A is bounded away from 1 an equilibrium exists. If, furthermore, $\frac{a(Y)}{1-A(Y)}$ is non-increasing, the equilibrium is unique.

Proof : If $\frac{a(Y)}{1-A(Y)}p(Y)X(Y) < 1$ for all Y, then $Y^* = 0$ satisfies the equilibrium condition (29). Otherwise, if there is some Y, for which this inequality does not hold, then by an application of the intermediate value theorem there must be a Y^* for which condition (29) holds with equality. The rest of the proof follows from the fact that $p(Y)X(Y)$ is decreasing in Y.

□

4.3 Inefficiency of the equilibrium

If the equilibrium level of theft is 0, then it coincides with the equilibrium of a standard economy in which theft is not allowed and therefore it is efficient. Since we are interested in equilibria with positive theft, in this section we assume that the equilibrium level of theft is positive. Furthermore, we will assume that there is a unique equilibrium and therefore

$$\frac{a(Y)}{1-A(Y)}p(Y)X(Y) > 1 \qquad \text{for all } Y < Y^*. \tag{30}$$

In this case, it is clear that the market equilibrium is inefficient. There are two reasons for this inefficiency. First, the equilibrium conditions (23–24) imply that the marginal utility of peanuts is higher than its marginal cost, and hence in equilibrium there is underproduction and underconsumption of peanuts. Second, as in the previous section, the amount Y^* of criminal activity is pure waste; it only transfers resources from victims to thieves. But worse than that, even from the point of view of the thieves there is too much criminal activity. Note that when the crime level is Y, and taking into account (21) and (22), the resulting booty is

$$A(Y)\,(\Omega + \Pi - M) \quad = \quad \frac{A(Y)}{1 - A(Y)} p(Y) X(Y)$$

namely, the value of the stolen goods. If the thieves, as a union, wanted to maximize $\frac{A(Y)}{1-A(Y)} p(Y)X(Y) - Y$, conditional on consumers choosing their consumption bundles (x_i, m_i) optimally, namely the booty in excess of their criminal effort, they would choose a criminal level that is lower than the equilibrium one. This follows from (30), which implies that for all $0 < Y < Y^*$, we have that $\frac{A(Y)}{1-A(Y)} p(Y)X(Y) - Y > 0 = \frac{A(Y^*)}{1-A(Y^*)} p(Y^*)X(Y^*) - Y^*$, where the equality follows from the definition of equilibrium. Namely, any level $0 < Y < Y^*$ attains a better result for the thieves, meaning that everybody could be made better off by simply reducing the level of crime.

The above discussion shows that there are feasible allocations that can make all individuals better off. However, these allocations may not be enforceable by a social planner, because he would not be able to implement arbitrary combinations of output and crime levels. The above discussion, however, suggests that any tool, such as a quantity subsidy, that induces a slight increase in the production and consumption of peanuts should be welfare improving. We will now see that this is not always the case. While sometimes, a subsidy on peanuts induces an increase in social welfare, there are instances when a tax on peanuts is the appropriate policy. Furthermore, it is possible that neither a tax nor a subsidy can lead to a welfare improvement.

When a government imposes a quantity subsidy σ on peanuts, the relevant equilibrium

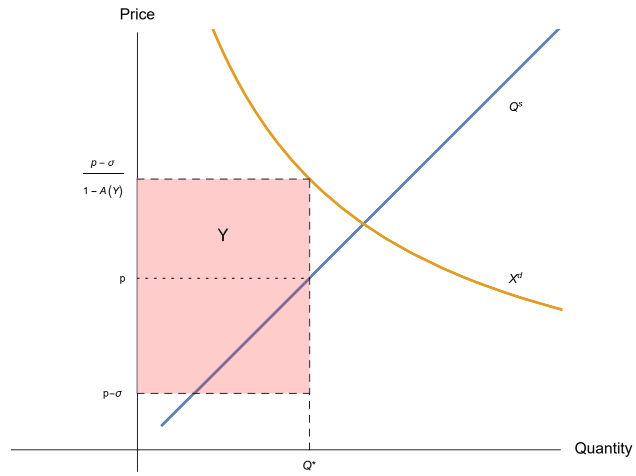

Figure 2: The equilibrium level of crime with a quantity subsdidy.

conditions become

$$X^d\left(\frac{p-\sigma}{1-A(Y)}\right) = Q^s(p) \tag{31}$$

$$\frac{a(Y)}{1-A(Y)}(p-\sigma)Q^s(p) = 1 \tag{32}$$

Figure 2 depicts such an equilibrium. The resulting level of crime is depicted by the shaded area.

A quantity subsidy will improve social welfare if its social marginal benefit is higher than its social marginal cost. The social marginal benefit is the marginal utility of the increase in peanuts induced by the subsidy. The social marginal cost of the subsidy has two components. One is the the marginal production cost and the other is the additional crime level induced by the subsidy. Since, at the equilibrium, the marginal utility of peanuts is $\frac{p}{1-A(Y)}$, and the marginal cost of peanuts is p, a quantity subsidy is welfare-improving if and only if

$$\frac{A(Y)}{1-A(Y)}pQ' > Y'.$$

At the competitive equilibrium, this conditions does not necessarily hold. Specifically, while a small increase in output leads to an increase in individuals' utility which is higher than the additional production cost, it also leads to a change in the level of crime that may offset the mentioned increase in social surplus. The following example illustrates this point.

Example 7 Consider an economy in which $\phi_i(x) = x(6 - x/d)$, $c(Q) = Q^2/2$, and $A(Y) = Y/(1+Y)$, where $d \in [1,4]$. The peanut supply function is, therefore, given by $Q(p) = p$. If there is a quantity subsidy σ on peanuts, aggregate demand is

$$X^d(\frac{p-\sigma}{1-A(Y)}) \quad = \quad 3d - \frac{p-\sigma}{2(1-A(Y))}$$

It can be checked that the price, quantity, and crime level that satisfy equilibrium conditions (31–32) are given by

$$\begin{aligned} p(\sigma) = Q(\sigma) &= \frac{1}{2}\left(\sigma + \sqrt{\sigma^2+4}\right) \\ Y(\sigma) &= \frac{d\left(-\sqrt{\sigma^2+4}+\sigma+12\right) - 2\left(\sqrt{\sigma^2+4}+\sigma\right)}{d\left(\sqrt{\sigma^2+4}-\sigma\right)}. \end{aligned}$$

In particular, when there is no subsidy, the equilibrium price, quantity and crime levels are

$$p^* = 1,\ X^* = Q^* = 1,\ Y^* = 5 - \frac{2}{d}.$$

Social welfare is given by

$$\frac{(d+2)\sqrt{\sigma^2+4} + (2-3d)\sigma}{2d\left(\sqrt{\sigma^2+4}-\sigma\right)}.$$

It can be checked that when $d = 2$, social welfare is constant as a function of σ. As a result, no subsidy can improve social welfare. When $d < 2$ a small postitive subsidy improves welfare, and when $d > 2$, a small quantity tax improves welfare.

4.4 Public police

Under the regime of public police, a level of protection $t_i = T$ is allocated uniformly across individuals and is financed by a personalized compulsory contribution $\hat{t}_i$ such that $\int \hat{t} = T$. We assume that, as leisure, individuals' tax payments are not subject to theft. Therefore, individual i's budget set is now

$$\left\{(x_i, m_i, y_i) : px_i + y_i \leq (1 - A(Y,T))(\omega_i + \pi_i - \hat{t}_i - m_i) + y_i a(Y,T)(\Omega + \Pi - T - M)\right\}$$

The parameters that the individual takes as given are the price p, his contribution to police protection $\hat{t}_i$, the crime rate Y and the returns to theft $a(Y,T)(\Omega + \Pi - M - T)$.

A competitive allocation consists of a feasible allocation $\langle (x^*, m^*, y^*, T), (-Z^*, Q^*) \rangle$ and a price p, such that

1. $(-Z^*, Q^*)$ maximize profits given p.

2. For each $i \in [0,1]$, (x_i^*, m_i^*, y_i^*) maximize utility given his budget set.

We can see that a competitive equilibrium with public police T and tax schedule $\hat{t}$ is equivalent to a competitive equilibrium with no police of the economy $\left\langle \left(\phi, \omega - \hat{t}, \theta\right), c, A(\cdot, T) \right\rangle$. Furthermore, given T, and except for equilibrium leisure m^*, the tax schedule $\hat{t}$ does not affect the equilibrium outcome. As a result, the equilibrium is still characterized, *mutatis mutandis*, by conditions (23–26).

As opposed to the model in Section 3, in which all wealth is subject to theft, the equilibrium level of crime is not necessarily decreasing in the police level. The reason is that, other things being equal, more police reduces crime, which induces individuals to work and produce more, which itself increases crime. Since, in equilibrium, the value of the stolen goods equals the level of crime, it may well be that an increase in public police protection leads to an increase in the value of the appropriated goods. The following example illustrates this point.

Example 8 Assume that $\phi_i(x) = x\,(10(1+i) - x/2)$ and that the appropriation technology is given by $A(Y,T) = \frac{Y}{(1+T)(1+Y)}$. Suppose that the cost of peanuts in terms of work hours is given by $c(q) = q^2/2$. Then, individual i's demand, as a function of the excise rate and peanut price is

$$x_i^d(\frac{p}{1-A}) = 10(1-i) - \frac{p}{(1-A)}$$

Consequently, the aggregate demand is given by

$$X^d(\frac{p}{1-A}) = \int x = 15 - \frac{p}{1-A}$$

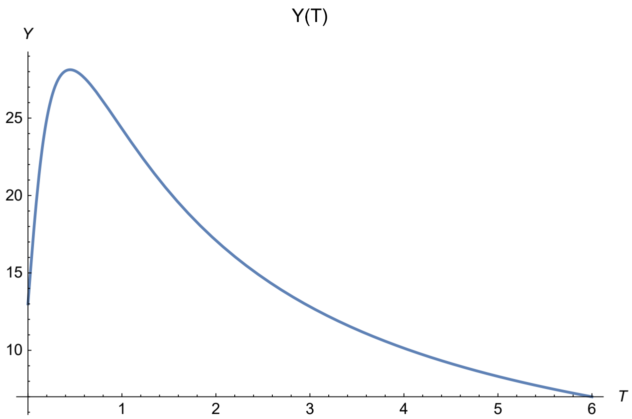

Figure 3: The equilibrium level of crime as a function of police protection.

The peanut supply function is given by $Q(p) = p$. Therefore, the equilibrium conditions (27–28) are

$$\begin{aligned} 15 - \frac{p(1+T)(1+Y)}{1+T(1+Y)} &= p \\ \frac{p^2}{1+T(1+Y)} &= 1 \end{aligned}$$

This system of equations implicitly define the equilibrium level of crime and price as functions of police protection. It can be checked that the equilibrium level of crime is not decreasing in the police level. Specifically, for low levels of police protection, the crime rate is increasing in T. (See Figure 3). The optimal level of police per capita is given by $T^* = 6.70109$, and the associated level of crime is $Y^* = 6.27845$.

Even crime is not necessarily decreasing in the level of police protection, the following proposition shows that if the equilibrium is stable, the equilibrium excise rate is decreasing and the equilibrium level of output is increasing in the level of public police protection.

Proposition 1 Let $\langle (x^*(T), m^*(T), y^*(T)), (-Z^*(T), Q^*(T)) \rangle$ be a competitive equilibrium allocation of the economy with public police protection T. Further assume that this equilibrium is locally stable. Then, $Q^*(T)$ is increasing in T and $A(Y^*(T), T)$ is decreasing in T.

Proof : See Appendix. $\square$

4.5 Private police

Suppose now that there is no public police but one can hire private police protection. The definition of a competitive equilibrium is the same as before, except that now the budget of individual i consists of all the bundles (x_i, m_i, y_i, t_i) that satisfy

$$px_i + y_i \leq (1 - A(Y, t_i))(\omega_i + \pi_i - t_i - m_i) + y_i \int a(Y, t)(\omega + \pi - t - m),$$

where the data that the agent considers as given are the price p, his share of the profits π_i, and the returns to crime $\int a(Y, t)(\omega + \pi - t - m)$. As a result, the equilibrium allocation is now characterized by the following conditions (assuming an interior solution):

$$\begin{aligned}
c'(Q) &= p && (33)\\
(1 - A(Y, t_i)) + (\omega_i + \pi_i - t_i - m_i)A_2(Y, t_i) &= 0 \qquad i \in [0, 1] && (34)\\
(1 - A(Y, t_i))\phi_i'(x_i) &= p \qquad i \in [0, 1] && (35)\\
(1 - A(Y, t))(\omega + \pi - t - m) &= px && (36)\\
\int a(Y, t)(\omega + \pi - t - m) &= 1 && (37)\\
\int x &= Q && (38)
\end{aligned}$$

Condition (34) is the condition that the choice of private police must satisfy if it is to be utility-maximizing.

4.5.1 Optimal allocation of private police

Let $\mathcal{E} = \langle(\phi, \omega, \theta), c, A\rangle$ be an economy and let $\langle(x^*, m^*, y^*, t^*), (-Z^*, Q^*)\rangle$ be a competitive allocation. The corresponding crime level and tax collection are $Y^* = \int y^*$ and $T^* = \int t^*$. For the reasons discussed in the previous sections, this equilibrium is not efficient. However, one may ask whether, as was the case in the model of Section 3, the allocation of police protection is efficient, *given the equilibrium level of crime.* It turns out that this is not the case. As the following example illustrates, the government can impose a different allocation of police protection whose resulting equilibrium attains a higher level of social welfare.

Example 9 Consider an economy with two types of consumers. For $i \in [0, 1/2]$, the utility of peanuts is given by $\phi_i(x) = x(10 - x/2)$, and for $i \in (1/2, 1]$, it is given by $\phi_i(x) = x(50 - x/2)$. The cost function is $c(Q) = Q^2/27$ and the appropriation technology is given by $A(Y, t_i) = \frac{Y}{(1+Y)(1+t_i)}$. It can be checked that the competitive equilibrium consists of $p^* = 2.04554$, $Q^* = 27.6148$, $Y^* = 6.41443$ along with an allocation of peanuts given by $x_i^* = 7.46766$ for $i \leq 1/2$ and $x_i^* = 47.762$ for $i > 1/2$, and an allocation of private police protection given by $t_i^* = 3.5004$ for $i \leq 1/2$ and $t_i^* = 9.05872$ for $i > 1/2$. However, a social planner can impose public but discriminatory police protection given by $\hat{t}_i = 3.47$ if $i \leq 1/2$ and $\hat{t}_i = 9.08691$ if $i > 1/2$ and the resulting competitive equilibrium (the one that solves the system of equation (35–38) would yield $\hat{p} = 2.04542$, $\hat{Q} = 27.6138$, $\hat{Y} = Y^*$, along with the peanut allocation $\hat{x}_i = 7.46737$ if $i \leq 1/2$ and $\hat{x}_i = 47.7627$ if $i > 1/2$. As we can see, in both equilibria the level of crime is the same, but it can be checked that the equilibrium with discriminatory public police protection attains a higher level of social welfare.

The above example shows that Observation 5 cannot be extended to the case in which only produced output can be stolen. However, if the production technology is linear, under certain conditions the competitive allocation of police protection will be efficient conditional on crime being at the equilibrium level.

Assume now that the production technology is linear and let $p = c'(Q)$ be the corresponding constant marginal cost. Conditional on the level of crime being the competitive one, Y^*, the optimal allocation of private police maximizes social welfare, given that the other variables are determined in equilibrium. Formally, it solves

$$\begin{aligned} & \max_{t,x,m} \int (m_i + \phi_i(x_i)) \\ \text{s.t.} \quad & (35) - (38) \end{aligned}$$

We will show that the competitive allocation, $\langle (x^*, m^*, y^*, t^*), (-Z^*, Q^*) \rangle$, which satisfies condition (35), solves the simpler problem

$$\begin{aligned}
&\max_{t,x,m} \int (m_i + \phi_i(x_i)) \\
\text{s.t.} \quad & 1 = \int a(Y^*, t)(\omega - t - m) \\
& px = (1 - A(Y^*, t))(\omega - t - m) \\
& \int x = Q
\end{aligned}$$

The above problem's constraints readily imply that $\Omega = M + pQ + T + Y^*$. As a result, this problem can be rewritten as

$$\begin{aligned}
&\max_{t,x} \int \phi(x) + \omega - px - t \\
\text{s.t.} \quad & k' = \frac{a(Y^*, t)}{1 - A(Y^*, t)} px \\
& k_0 = 0, \ k_1 = 1
\end{aligned} \tag{39}$$

The associated Hamiltonian is

$$H = \phi_i(x_i) + \omega_i - t_i - px_i - \lambda\big(a(Y^*, t)\frac{px}{1 - A(Y^*, t)}\big)$$

and the necessary conditions for a solution are

$$\phi_i'(x_i) - p\left(1 + \lambda \frac{a(Y^*, t)}{(1 - A(Y^*, t))}\right) = 0 \tag{40}$$

$$-1 - \lambda \frac{A_2(Y^*, t)px}{Y^*(1 - A(Y^*, t))^2} = 0 \tag{41}$$

$$\lambda' = 0 \tag{42}$$

$$\int a(Y^*, t)\frac{px}{(1 - A(Y^*, t))} = 1 \tag{43}$$

Let now $\langle (x^*, m^*, y^*, t^*), (-Z^*, Q^*) \rangle$ be a competitive equilibrium allocation along with $\lambda = Y^*$. Since $p = (1 - A(Y^*, t_i^*))\phi_i'(x_i^*)$ we have that condition (40) is satisfied. Also, since $(1 - A(Y^*, t_i^*)) + \frac{px_i^*}{1 - A(Y^*, t_i^*)} A_2(Y^*, t_i^*) = 0$ we have that

$$\frac{px_i^*}{(1 - A(Y^*, t_i^*))^2} A_2(Y^*, t_i^*) = -1$$

and condition (41) is satisfied as well. Since Y^* is constant, $\lambda' = 0$ and condition (42) is also satisfied. Finally, by definition, condition (43) is also satisfied by the competitive allocation.

We conclude that the competitive equilibrium satisfies the necessary condition for an optimal allocation of police, conditional on the level of crime. We cannot conclude, however, that the allocation is optimal, because the necessary conditions may not be sufficient. But even if the allocation of police protection is constrained efficient, the competitive level of crime is not globally optimal. Indeed, as Observation 10 below shows, social welfare can be increased by reducing the level of crime. The reason is the same as that of the model in Section 3. Namely, private police exerts a positive externality; it reduces the returns to theft, which induces people to spend less time stealing from the whole population. This externality is not taken into account by the individual. By an argument analogous to the one used in Section 3 we can show the following, whose proof appears in the appendix.

Observation 10 At the competitive equilibrium, the level of crime is too high. Namely, the total waste could be reduced by increasing spending in police protection, thereby reducing crime.

5 Concluding remarks

We have introduced theft into the standard partial equilibrium model of an economy. We considered two models that differ in the kind of goods that are subject to theft. In the first model, we allow thieves to steal from the initial endowments of factors of production. In the second one, only produced goods can be stolen. In particular, time devoted to leisure, theft and property protection is not subject to appropriation. The two models generate different conclusions. While in the first an equilibrium exists and is unique, in the second there may be non-existence and multiplicity of equilibria. In both models, theft generates the obvious inefficiency associated with the fact that time spent stealing is itself a waste of resources. In the second model, there is an additional source of inefficiency due to the fact that theft acts as a quantity tax that introduces a wedge between the consumers' marginal utility and the firms' marginal cost of production. The two models also differ in their policy recommendations. While in the first model, a policy that increases output is beneficial in the sense that it reduces crime, this is not necessarily so in the second model. Also, whereas in the first model, public police protection reduces crime, in the second one it may very well increase it. The allocation of protection granted by public police is typically inefficient since equal protection is awarded to all individuals, independent of their stealable wealth. Although allowing for private police does not induce an optimal level of crime, it has the potential to distribute police efficiently (conditional on the level of crime). It turns out that in the first model, the competitive equilibrium does allocate police protection efficiently (conditional on the level of crime). In the second model, however, private police is typically inefficient, unless the technology of peanut production is linear. Finally, within the first model, we have investigated the notion of a voting equilibrium under different tax regimes. Under a regime of a head tax, the equilibrium communities are classified by income and typically the tax rates are not the optimal ones. Under a regime of proportional taxation, communities are not necessarily classified by income brackets, but in all of them, the sum of the excise and tax rates are the same. Furthermore, the equilibrium crime tax rates in each community are the optimal ones.

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A Appendix

Proof of Observation 2. By definition, $(p(Q), x(Q))$ solves

$$\begin{aligned}\phi'(x(Q)) &= p(Q)\\ \int x(Q) &= Q.\end{aligned}$$

It is routine to show that these conditions imply that $p' < 0$. Letting $\Pi = p(Q)Q - c(Q)$, the level of crime $Y(Q) = \int y(Q)$ is the solution to the equation

$$1 = a(Y)(\Omega + \Pi) \tag{44}$$

Since the equilibrium allocation satisfies $\int m = \Omega - Z - Y$, and $Z = c(Q)$, we have that

$$W(x(Q), m(Q)) = \int \phi(x(Q)) + \Omega - c(Q) - Y(Q).$$

As a result,

$$\frac{\partial W}{\partial Q} = \int \phi'(x)x' - c' - Y'$$

Since in equilibrium, when $Q = Q^*$, we have that $\phi'(x) = c'$, and $\int x' = 1$ we conclude that

$$\frac{\partial W}{\partial Q}(x(Q^*), m(Q^*)) = -Y'(Q^*),$$

That is, the increase in welfare is exactly the decrease in time devoted to theft. In order to calculate this value, note that $Y(Q)$ solves equation (44). Therefore, since $a(Y)$ is a decreasing function, Y is decreasing in Q if and only if $\Pi' < 0$. By Hotelling's lemma, $\Pi' = Qp'$. Therefore, since $p' < 0$, we have that $\Pi' < 0$ and we can conclude that

$$\frac{\partial W}{\partial Q}(x(Q^*), m(Q^*)) = -Y'(Q^*) > 0.$$

□

Proof of Proposition 1. For any level of public police T, the equilibrium is stable if at the equilibrium level of crime $Y^*(T)$, the partial derivative of

$$\frac{a(Y,T)}{1 - A(Y,T)} p(Y,T)Q(Y,T)$$

with respect to Y is negative, where $p(Y,T)$ and $Q(Y,T)$ are implicitly defined, *mutatis mutandis*, by equations 15 and 16. Letting $A^* = A(Y^*(T),T)$, $p^* = p(Y^*,T)$, $X' = X'(\frac{p^*}{1-A^*})$, and $Q^* = Q(Y^*(T),T)$, routine calculations show that this happens if and only

$$(1-A^*)Q^*\left[(Y^*A_1^*-A^*)\left(X'c''(Q^*)-1\right)-(A^*)^2\right]-X'p^*\,Y^*A^*A_1^*<0 \tag{45}$$

Also, tedious calculations show that the derivative of $Q^*(Y^*(T),T)$, which is defined, *mutatis mutandis*, by conditions (23–26) is positive if and only if inequality 45 holds. As a result, we obtain that $Q^*(T)$ is increasing on T if an only if the equilibrium is stable.

Similarly, equally tedious calculations show that the equilibrium excise rate $A(Y^*(T),T)$ is decreasing in T if and only if inequality (45) holds. As a result, we obtain that $A(Y^*(T),T)$ is decreasing in T if an only if the equilibrium is stable. □

Proof of Observation 10. The Lagrangian associated with problem (39) is

$$L=\int\Big(\phi(x)+\omega-px-t+\lambda(1-\frac{a(Y,t)}{1-A(Y,t)}px)\Big).$$

By the envelope theorem

$$\begin{aligned}
V'(Y^*) &= -\int\lambda^*\frac{\partial(\frac{a}{1-A})}{\partial Y}px^*\\
&= -\int\frac{Y^*}{(1-A)^2}\left[\frac{A_1Y^*-A}{Y^{*2}}(1-A)+A_1a\right]px^*\\
&= -\int\left[\frac{1}{(1-A)}A_1-\frac{a}{1-A}+A_1\frac{A}{(1-A)^2}\right]px^*\\
&= 1-\int\left[\frac{1}{(1-A)}A_1+A_1\frac{A}{(1-A)^2}\right]px^*\\
&= 1-\int\left[A_1\frac{1}{(1-A)^2}\right]px^*\\
&< 1
\end{aligned}$$

where we have used the facts that $\int\frac{a(Y^*,t^*)}{1-A(Y^*,t^*)}px^*=1$ and that when the level of crime is the competitive one, $\lambda^*=Y^*$. □