THE RISE OF MANIPULATION IN STIFF COMPETITIONS

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The Rise of Manipulation in Stiff Competitions*

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Abstract:

This paper deals with the issue of manipulation. We formulate a general competition framework where private values are attributed to players, along with the ability to manipulate these values using costly noise. Although manipulation carries no clear advantages, our analysis shows that a stiff competition leads to a unique equilibrium where all players manipulate.

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1 Introduction

This paper studies the incentives behind manipulation. Whether it is “Fake News” in the media world, P-hacking in academic research, or various political races, the possibilities to come across some form of either high- or low-level manipulations are staggering. In the backdrop of this blooming phenomenon, one must wonder whether a manipulative strategy is, indeed, rational.

To this end we devise a general competition framework where private values are attributed to players, along with the ability to manipulate these values using costly noise. Only players with the highest realized valuations are considered winners and receive a positive payoff. Given this set-up, we ask whether players should manipulate in equilibrium

A priori, the answer is negative. Under the assumption that the cost is imminent and high, while the expected effect over players’ values is actually negative, then manipulation carries no clear advantages. Nevertheless, our analysis indicates that a stiff competition leads *all* players to manipulate. This result is based on a fundamental attribute of competitions, which is *the discontinuity of prizes.* The basic idea of competitions is to distinct between types such that a winner, in its individual meaning, cannot exist without a loser. The distinction is only meaningful by the discontinuity which, in turn, is tunnelled towards an all-manipulative equilibrium. However, discontinuity alone cannot account for the mentioned outcome, since manipulation is not a dominating strategy. Thus enters the role of stiff competition and limited liability.

Given that the competition is fierce such that prior chances of winning are slim, then low-value players get only the upside of manipulating. In particular, it increases their probability of winning, while the value loss and costs become irrelevant due to limited-liability constraints. This effect cascades upwards as the competition intensifies, gradually extending to capture all players and valuations. Interestingly, our result is not conditioned on the payoff function as long as the limited-liability condition is met. In other words, the rise of manipulation occurs although winning in this manner is extremely costly, essentially implying that players revert to a victory-at-all-costs strategy.

1.1 Motivating examples

Competitions, in their broadest form, exist in many fields, ranging from academic research to political races. Consider, for example, the problem of P-hacking and research tampering. In the academic world, one is often inclined to publish in prestige journals and present extravagant insights. In Economics, the strive to publish in Top-5 journals led Prof. Serrano from Brown University to write a rather entertaining article describing a disease called “Top5itis”.[1] Such goals are amplified by tenure-track criteria, inflicting additional pressure on researchers, which could lead to results tampering through

[1] See *ProMarket* - the blog of the Stigler Center at the University of Chicago, “Top5itis”, March 2, 2018.

P-hacking or other forms of manipulation. A recent well-known example is the theory of power posing by Amy Cuddy, Dana Carney, and Andy Yap. This theory was first highly acclaimed, only to be later considered problematic in follow-up research. Currently, some view it as an example of the replication crisis in psychology.[2] This incident shows how (an alleged) research manipulation is costly whether it is exposed during the refereeing process or afterwards.

Another recent example is accommodated by the popular notion of "Fake News". Media as a whole, and the news world specifically, are becoming more competitive as many find it hard to generate revenue through commercial ads and consumers.[3] Since news are time-dependant and there can only be a small number of news articles that open the evening news or appear at the front page, the intense competition could quite easily lead to manipulation such as plagiarism, "Fake News", and more. Note that this concern is not novel and was already depicted in a 2009 BBC article entitled "Economy 'threatens' news accuracy".

The last example immediately leads to the political world, where one is sometimes puzzled with the untruthfulness of politicians. The solution, in our context, is simple. A winner-take-all race to public office is, perhaps, the simplest example for a stiff competition leading to manipulation. As there can be only one president (or only one prime minister), politician must do everything in their power to gain and maintain their position. Thus, it is only reasonable that candidates do not always intend to follow through their statements and, in some cases, divert to extreme populism.[4]

1.2 Related literature

The theoretical study of manipulation and optimal enforcement dates back to the seminal work of Stigler (1970). In the spirit of our limited-liability constraint, Stigler argues that fines are rarely proportional to individuals' wealth. He states that uniform fines increase the expected severity of offences, and that extreme fines are suboptimal by the importance of marginal deterrence. Polinsky and Shavell (1979) follow this argument and prove that individuals' risk-aversion should lead to lower fines. Moreover, Malik (1990) claims that extreme penalties expedites offenders' will to engage in socially costly activities that lower the probability of being fined (e.g., radar detectors to avoid speeding tickets and manipulating politicians to change regulation). As we later point out (in Section 3), the last argument fits well with our modelling choices since it enables the players to regulate their actions according to the enforcement criteria, resulting in some form of plausible manipulation.

Our work contributes to this line of research by depicting the intrinsic properties that lead to

[2]The academic debate could be tracked through Carney et al. (2010), Ranehill et al. (2015), Carney et al. (2015), Simmons and Simonsohn (2017), Cesario et al. (2017), and Gronau et al. (2017).

[3]See, e.g., *The Economist*, "Who killed the newspaper?", August 24, 2006.

[4]See, e.g., *The Washington Post*, "Why politicians lie", October 25, 2016; and *The Huffington Post*, "Why politicians lie", June 27, 2016.

manipulation, though such actions are not dominating ones. In addition, we propose a robust model for the analysis of manipulations, which carries clear resemblance to other forms of competitions such as auctions, but with a different set of actions. Evidently, the latter modification challenges well-known results, such as the impact of free entry (of competition) over outcomes.

Our wide interpretation of manipulation is well precedented in the economic literature. In the field of education, for example, Jacob and Levitt (2003) partially detect cheating and manipulation on the side of teachers concerning their students' test scores. Their results are supported by theoretical studies (as, e.g., Holmstrom and Milgrom (1991) and Baker (1992)) and empirical ones (see Cameron et al. (2009); Barr and Serra (2010); Gächter and Schulz (2016), among others). In sports, Duggan and Levitt (2002) studied corruption in Sumo wrestling and affiliate it to "sharp non-linearity in the payoff function for competitors". Similar non-linear effects are the focus of Elaad et al. (2017), who investigate potential corruption in soccer games, and find evidence of wide spread manipulations. This conjecture proves to be crucial in our conclusions, underlining the key rule of prizes' discontinuity.

1.3 Structure of the paper

The paper is organized as follows. In Section 2 we present the basic model, followed by the main result in Section 3. Extensions, further clarifications of our model, and a few concluding remarks are given in Section 4.

2 A manipulable competition

Before presenting our set-up, we review several famous sports scandals that motivate our modelling choices. The first occurrence is commonly known as the "Deflategate" incident, where the New England Patriots were accused of deliberately under-inflating footballs in their 2014–2015 championship-game victory over the Indiana Colts. This issue was resolved in a US Court on July 13, 2016, with a four-game no-pay suspension of Patriots quarterback, Tom Brady, a Patriots loss of two draft picks, and a one million dollar fine. Four years earlier, cyclist Lance Armstrong was stripped of his seven Tour-de-France titles and banned for life from all sports that follow the World Anti-Doping Code. This penalty was a response to his use of performance-enhancing drugs and his part in a widespread doping program. Similar fate awaited Olympic sprinters Ben Johnson and Marion Jones, both stripped of their Olympic titles for steroid use.

The first feature that follow from all mentioned examples is the use of a manipulative strategy to improve one's position. Though the offences concern different athletes, in different fields, status, and time, all tried to gain a foul advantage over their opponents. Next, all athletes were (and, in some cases, are still) considered among the best in their profession. So with a non-trivial probability, they need not use manipulation to win. In addition, all mentioned manipulations were caught ex post, sometimes a long time afterwards, while the penalties vary by the severity of the manipulative strategy.

Remark 1. *The formal presentation of the model is aimed to balance between simplicity and generality. Therefore, several straightforward extensions are given in* Section 4*, only after the basic model and main result are well established.*

2.1 The model

We capture the attributes of previously-mentioned examples through a strategic framework, referred to as *a manipulable competition*. It commences as an ordinary contest where individuals have personal independent capabilities. To win the competition, every player can either use his original value or apply a costly distortion of it, i.e., a manipulation. Winners get, in expectation, a positive payoff that generally depends on their ordinal ranking, cardinal result, and actions.

Formally, a *manipulable competition* is an N-player game which evolves as follows. First, all competitors (i.e., players) receive their private valuations, drawn independently according to a continuous random variable V, supported on $\left[\underline{V}, \overline{V}\right]$. Then, every player can either use his original value v, or use a costly manipulation of it, stochastically evaluated through $v + M$. That is, a *strategy* σ is a function that dictates an action, either v or $v + M$, for every possible value v. The strategic valuations are independently realized and the top k players are granted a payoff according to a payoff function $\mathcal{V}$, while others receive nothing. In what follows, we elaborate on the two key elements of our model - the manipulation M and the payoff function $\mathcal{V}$.

The manipulation M is a binary random variable that equals $+m$ or $-m$ with positive probabilities p and $1 - p$, respectively. Note that the positive distortion $m < \overline{V} - \underline{V}$ and its probability p are potentially small. Hence, a manipulation could be mild, and actually lead to a value loss during the evaluation process.

The payoff function $\mathcal{V} : \left[\underline{V}, \overline{V}\right]^k \times \{0, M\}^k \to \mathbb{R}_+^k$ transforms a profile of the top k-players' valuations and actions into a positive reward for every winning player. The essential properties of $\mathcal{V}$ follow from a limited-liability assumption such that $\mathcal{V}$ is positive, bounded away from zero, and bounded. Note that we make no assumptions regarding continuity or monotonicity, but (motivation-wise) one can consider functions where the exercise of a manipulation significantly lowers a winner's expected payoff. In other words, the cost of manipulation could be extremely high, conditional on winning. Moreover, the payoff function depends on the actual actions rather than the realizations, thus capturing the idea that the potential loss (due to manipulation) is imminent. Doing so, we relate to cases where manipulations are not caught in time, but eventually revealed later on.

3 Main result

As customary, we consider the symmetric case where all players use the same strategy,[5] and ask whether players manipulate in equilibrium. A priori, it would appear suboptimal to do so. First, it is costly, potentially reducing the expected prize close to zero, independently of the realized value. Second, it grants no assurances and could even be harmful during the contest. Yet, by fixing the number of winners and employing free entry (i.e., increasing the number of competitors), the stiff competition leads to a unique outcome where all players to manipulate. (The proof is deferred to the Appendix.)

Theorem 1. *For a sufficiently high number of players, all players manipulate in equilibrium.*

Since we made a great deal of effort to make manipulation suboptimal, we wish to clarify the intuition behind Theorem 1. The immediate motivation is the players' desire to win, or in the words of Sir Winston Churchill: "Victory at all costs".[6] Though manipulations are costly, the players' primary concern is to win the competition since winning outweighs losing.

But the desire to win is merely the short explanation, whereas the extensive one is more complex and involves two vital elements: (i) limited liability; and (ii) manipulation cascade. Manipulation is not a dominating strategy for high-value players (since their probability of winning is already high). To differ, low-value players get only the upside of manipulating. It increases their probability of winning, while the expected value loss and costs are irrelevant from their perspective, by the limited-liability assumption. Now, this effect cascades upwards as the competition intensifies, gradually extending to capture all valuations.

In turn, one can rightfully argue that weakening the limited-liability constraint to inflict severe penalties, on either winning or losing manipulations, could prove useful in this context. Though we broadly relate to this issue in the extensions provided in Section 4, there is one crucial point we wish to emphasize. In practice, manipulations are devised only after the rules of the competition are formed, and the former are more flexible in terms of applicability and severity. Therefore, even under a stricter set of rules and penalties, competitors would still find a way to manipulate the competition to the extent that the expected costs are mild.[7] The fact that there is a basic time-discrepancy between the formulation of a competition and the application of a manipulation, along with the immense manipulation possibilities, ensure that the given model accommodates for a wide set of scenarios. To put in terms of the previous examples: politicians can limit themselves to small lies; researchers can cut corners to a limited extent; and athletes can mildly challenge the rules to gain a foul advantage.

[5] Henceforth, we assume that all measurability requirements, considering $(V, \mathcal{V}, M, \sigma)$, are met.

[6] Taken from Churchill's speech, from May 13, 1940, which is best-known for the phrase "blood, toil, tears and sweat".

[7] Note that we did not account for false positives and marginal deterrence, as suggested by Stigler (1970).

Remark 2. *In the proof of* Theorem 1 *we provide a lower bound on N to reach the all-manipulative equilibrium. In particular, we state that one should choose $N > k$ such that for every $n > N$ it follows that*

$$\left(1+\frac{p_0}{2}\right)^{n-1} > \max\left\{\frac{k\Theta_+}{p\Theta_-}\left[\frac{(1+p_1)en}{(1-p_1)(k-1)}\right]^{k-1}\mathbf{1}_{\{k\geqslant 2\}}, \frac{\Theta_+}{p\Theta_-}\left[\frac{2+p_0}{(1-p_1)p_2}\right]^{k-1}\right\},$$

where the p_is are positive probabilities that depend on the distributions of V and M, and $\Theta_\pm$ are upper and lower bounds on the payoff function. Note that N depends logarithmically on the payoff function, and increases linearly with the number of winners. In addition, the intensified competitiveness, attributed to the number of players, has an exponential effect over the chances of winning. Thus, even a low number of competitors could prove sufficient for Theorem 1 *to hold.*

4 Extensions & discussion

There are several payoff-related generalizations of our model that impose no technical problems on the statement Theorem 1. First, the proof of Theorem 1 is based on the upper and lower bounds of the reward function. Thus, the proof holds uniformly for all functions that meet the same bounds. Second, we can apply a stochastic reward function that also induces negative rewards. As long as the positive bounds are met in expectation, the result remains valid. Moreover, one can assume that a manipulative player is taken out of the competition with probability of $1-p$, instead of competing with a value of $v-m$. The last extension requires mild adjustments for the result to hold (e.g., an assumption that the manipulation has no effect with a small positive probability is sufficient).

Another subtle change concerns the payoff of non-winning players. The current model incorporates no costs for a manipulative lose. Nevertheless, if one assumes that manipulation is costly with small probability (decreasing in the number of players) given that a player loses, then the previous result remains valid. Such an extension holds, e.g., in case a losing manipulation is observed with a probability proportional to the probability of a fair win. The latter assumption is consistent with the fact that inspections are generally considered to be costly observations and, therefore, limited.

The next level of (non-payoff-related) extensions deals with the distribution values and noise. In practice, valuations need not be continuous and manipulations are possibly smooth. Such modifications impose technical and mathematical difficulties that require additional research. We conjecture that a discrete version of V is plausible, as long as the manipulation is not redundant relative to V.

Another interesting line of research is the inclusion of various manipulation opportunities, or an evolutionary process where manipulations constantly change along the dynamics. In light of current results, it appears reasonable that players would revert to a manipulation that maximizes their probability of winning. Yet, this cannot be determined without further analysis.

To conclude, we emphasize that the current work should not be viewed as criticism against competition and free entry. The main goal of this paper, which we hope to have accomplished, is to present the underline incentives that yield manipulations, and possibly instigate future solutions.

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5 Appendix

Theorem 1. *For a sufficiently high number of players, all players manipulate in equilibrium.*

Proof. Let Θ_+ and Θ_- be maximal and minimal positive bounds on the reward function. Consider a strategy σ and let G_σ be its induced CDF. The probability of being a winning player (i.e., among the top k realized valuations), subject to σ and private value v, is

$$\phi_N(v) = \sum_{l=0}^{k-1} \binom{N}{l} (1 - G_\sigma(v))^l \, G_\sigma^{N-1-l}(v),$$

without manipulation, and $p\phi_N(v+m) + (1-p)\phi_N(v-m)$ with manipulation. We will prove there exists an $N > k > 0$, such that for every σ, every v, and every $n \geqslant N$, it follows that $\frac{\phi_n(v+m)}{\phi_n(v)} > \frac{\Theta_+}{p\Theta_-}$, which is sufficient for this proof.

For every $v \in \left[\underline{V}, \overline{V}\right]$, denote $I_v = [v, v+m]$, $I_v^- = [v-m, v]$, and $I_v^+ = [v+m, v+2m]$. Define

$$\begin{aligned} p_0 &= \min_v \left[p\Pr\left(V \in I_v^-\right) + (1-p)\Pr\left(V \in I_v^+\right)\right], \\ p_1 &= \max_v \left[p\Pr\left(V \in I_v^-\right) + (1-p)\Pr\left(V \in I_v^+\right) + \Pr\left(V \in I_v\right)\right], \\ p_2 &= (1-p)\Pr(V \leqslant \underline{V} + m), \end{aligned}$$

and choose $N > k$ such that for every $n \geqslant N$,

$$\left(1+\tfrac{p_0}{2}\right)^{n-1} > \max\left\{\tfrac{k\Theta_+}{p\Theta_-}\left[\tfrac{(1+p_1)en}{(1-p_1)(k-1)}\right]^{k-1}\mathbf{1}_{\{k\geqslant 2\}}, \tfrac{\Theta_+}{p\Theta_-}\left[\tfrac{2+p_0}{(1-p_1)p_2}\right]^{k-1}\right\}.$$

Assume, by contradiction, there exists a set of valuations (potentially of 0-measure w.r.t. V) that do not manipulate according to an equilibrium strategy σ. Choose an interval I_{v_0} with a maximal measure of non-manipulating authors, and assume w.l.o.g. that $\sigma(v_0) = v_0$. Otherwise, the optimality requirement for I_{v_0} could be weakened to ϵ-optimality, for some small $\epsilon \in \left(0, \frac{p_0}{2}\right)$. Denote the measure of non-manipulating authors in I_{v_0} and $I^{\pm}_{v_0}$ by μ_0 and $\mu_1^{\pm}$, respectively. The optimality of I_{v_0} suggests that $\mu_0 > p\mu_1^- + (1-p)\mu_1^+ - \frac{p_0}{2}$. Hence,

$$\begin{aligned}
\frac{G_\sigma(v_0+m)}{G_\sigma(v_0)} &= \frac{G_\sigma(v_0) + G_\sigma(v_0+m) - G_\sigma(v_0)}{G_\sigma(v_0)} \\
&\geqslant 1 + G_\sigma(v_0+m) - G_\sigma(v_0) \\
&= 1 + \mu_0 + p\left[\Pr(V \in I^-_{v_0}) - \mu_1^-\right] + (1-p)\left[\Pr(V \in I^+_{v_0}) - \mu_1^+\right] \\
&> 1 - \frac{p_0}{2} + p\Pr(V \in I^-_{v_0}) + (1-p)\Pr(V \in I^+_{v_0}) \\
&\geqslant 1 + \frac{p_0}{2},
\end{aligned}$$

where the second and third inequalities follow from the optimality of μ_0 and p_0, respectively. Note that

$$\begin{aligned}
\frac{\phi_N(v_0+m)}{\phi_N(v_0)} &= \frac{\sum_{l=0}^{k-1}\binom{N}{l}\left(1-G_\sigma(v_0+m)\right)^l G_\sigma^{N-1-l}(v_0+m)}{\sum_{l=0}^{k-1}\binom{N}{l}\left(1-G_\sigma(v_0)\right)^l G_\sigma^{N-1-l}(v_0)} \\
&= \frac{\sum_{l=0}^{k-1}\binom{N}{l}\left[\frac{1-G_\sigma(v_0+m)}{G_\sigma(v_0+m)}\right]^l}{\sum_{l=0}^{k-1}\binom{N}{l}\left[\frac{1-G_\sigma(v_0)}{G_\sigma(v_0)}\right]^l} \cdot \left[\frac{G_\sigma(v_0+m)}{G_\sigma(v_0)}\right]^{N-1} \\
&> \frac{\sum_{l=0}^{k-1}\binom{N}{l}\left[\frac{1-G_\sigma(v_0+m)}{G_\sigma(v_0+m)}\right]^l}{\sum_{l=0}^{k-1}\binom{N}{l}\left[\frac{1-G_\sigma(v_0)}{G_\sigma(v_0)}\right]^l} \cdot \left[1 + \frac{p_0}{2}\right]^{N-1}.
\end{aligned}$$

If $k = 1$, then the choice of N guarantees that the statement $\frac{\phi_n(v+m)}{\phi_n(v)} > \frac{\Theta_+}{p\Theta_-}$ holds, and the result follows.

Otherwise, fix $k \geqslant 2$. Consider the two cases of either $G_\sigma(v_0+m) > p_1 + \frac{1-p_1}{2}$, or $G_\sigma(v_0+m) \leqslant p_1 + \frac{1-p_1}{2}$. The value p_1 is the maximal possible measure (w.r.t. V) that could be induced by any σ in an interval of length m. Thus, if $G_\sigma(v_0+m) > p_1 + \frac{1-p_1}{2}$, it follows that $G_\sigma(v_0) > \frac{1-p_1}{2}$. Indeed, the last two inequalities suggest

$$\begin{aligned}
\frac{\phi_N(v_0+m)}{\phi_N(v_0)} &> \frac{1}{\sum_{l=0}^{k-1}\binom{N}{l}\left[\frac{1-G_\sigma(v_0)}{G_\sigma(v_0)}\right]^l}\left[1+\tfrac{p_0}{2}\right]^{N-1} \\
&> \left[1+\sum_{l=1}^{k-1}\left(\tfrac{eN}{l}\right)^l\left[\frac{1-\frac{1-p_1}{2}}{\frac{1-p_1}{2}}\right]^l\right]^{-1}\left[1+\tfrac{p_0}{2}\right]^{N-1} \\
&> \left[1+\left[\tfrac{1+p_1}{1-p_1}\right]^{k-1}\sum_{l=1}^{k-1}\left(\tfrac{eN}{l}\right)^l\right]^{-1}\left[1+\tfrac{p_0}{2}\right]^{N-1} \\
&> \left[1+(k-1)\left[\tfrac{1+p_1}{1-p_1}\cdot\tfrac{eN}{k-1}\right]^{k-1}\right]^{-1}\cdot\tfrac{k\Theta_+}{p\Theta_-}\cdot\left[\tfrac{(1+p_1)eN}{(1-p_1)(k-1)}\right]^{k-1} \\
&> k^{-1}\left[\tfrac{(1+p_1)eN}{(1-p_1)(k-1)}\right]^{1-k}\cdot\tfrac{k\Theta_+}{p\Theta_-}\cdot\left[\tfrac{(1+p_1)eN}{(1-p_1)(k-1)}\right]^{k-1} = \frac{\Theta_+}{p\Theta_-},
\end{aligned}$$

where the first inequality follows from the positivity of all terms; the second follows from the bound on $G_\sigma(v_0)$ and Sterling's bounds for the Binomial coefficient; the third is due to simple monotonicity requirements; the forth follows from the monotonicity of $\left(\frac{eN}{l}\right)^l$ and the choice of N.

Otherwise, $G_\sigma(v_0+m) \leqslant \frac{1+p_1}{2}$, or equivalently $1-G_\sigma(v_0+m) \geqslant \frac{1-p_1}{2}$ and we get

$$\begin{aligned}
\frac{\phi_N(v_0+m)}{\phi_N(v_0)} &= \frac{\sum_{l=0}^{k-1}\binom{N}{l}\left[\frac{1-G_\sigma(v_0+m)}{G_\sigma(v_0+m)}\right]^l}{\sum_{l=0}^{k-1}\binom{N}{l}\left[\frac{1-G_\sigma(v_0)}{G_\sigma(v_0)}\right]^l}\left[\frac{G_\sigma(v_0+m)}{G_\sigma(v_0)}\right]^{N-1} \\
&> \frac{\left[1-G_\sigma(v_0+m)\right]^{k-1}\sum_{l=0}^{k-1}\binom{N}{l}}{\left[\frac{1}{G_\sigma(v_0)}\right]^{k-1}\sum_{l=0}^{k-1}\binom{N}{l}}\left[\frac{G_\sigma(v_0+m)}{G_\sigma(v_0)}\right]^{N-1} \\
&= \left[(1-G_\sigma(v_0+m))G_\sigma(v_0)\right]^{k-1}\left[\frac{G_\sigma(v_0+m)}{G_\sigma(v_0)}\right]^{N-1} \\
&= \left[(1-G_\sigma(v_0+m))G_\sigma(v_0+m)\right]^{k-1}\left[\frac{G_\sigma(v_0+m)}{G_\sigma(v_0)}\right]^{N-k} \\
&\geqslant \left[\tfrac{(1-p_1)p_2}{2}\right]^{k-1}\left[1+\tfrac{p_0}{2}\right]^{N-k} \\
&= \left[\tfrac{(1-p_1)p_2}{2}\right]^{k-1}\left[1+\tfrac{p_0}{2}\right]^{N-1}\left[1+\tfrac{p_0}{2}\right]^{1-k} \\
&\geqslant \left[\tfrac{(1-p_1)p_2}{2}\right]^{k-1}\cdot\tfrac{\Theta_+}{p\Theta_-}\cdot\left[\tfrac{2+p_0}{(1-p_1)p_2}\right]^{k-1}\left[\tfrac{2+p_0}{2}\right]^{1-k} = \frac{\Theta_+}{p\Theta_-},
\end{aligned}$$

where the first inequality follows from monotonicity of the numerator and the denominator; the second inequality follows from bounds on the probability (i.e., $G_\sigma(v_0+m) \geqslant p_2$ for every strategy and every value v_0); and the third inequality follows from the choice of N.

The proof, up to this point, accommodated for a unique SGP equilibrium, as Nash equilibria are set up to a zero-measure deviation. However, I_{v_0} is fixed as ϵ-optimal where $\epsilon \in \left(0, \frac{p_0}{2}\right)$. Thus, we can take a (small) positive-measure set of non-manipulating valuations close to v_0, that sustain the same ϵ-optimality condition. The analysis according to v_0 holds for every valuation in that set, and the result follows. ■