CREDIT AUCTIONS AND BID CAPS

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Abstract

In this paper we offer two contributions to the field of credit auctions. First, we compare first- and second-price credit auctions and provide solvency-dependent conditions such that one mechanism dominates the other in terms of expected payoffs of all the parties involved. In addition, we present a new possibility of using bid caps in credit auctions. We study the equilibria in the capped mechanisms and show that bid caps can increase the expected payoffs of all sides, a win-win situation.

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1 Introduction

Credit auctions are auctions where the winning bidder may default after the price is set. Such defaults occur whenever the auctioned assets (or projects) are costly such that bidders rely on future income and financial markets to cover their expenses. Due to the potential grave damages, researchers are constantly searching for ways to minimize such occurrences. In this paper, we support the general effort by studying known and new methods to limit the expected losses.

The credit-auction problem begins similarly to any private-value standard auction. First, private values are distributed and bids are made, such that the winning bidder is set along with the agreed price. Next, with a slight deviation from the usual set-up, the winning bidder defaults with a certain probability, which depends on the agreed price. This probability function is referred to as a *solvency function.* A defaulting winner loses a fixed endowment, collected by the seller along with the auctioned asset.

Our solvency-function notion plays a key role in the first part of the paper, surrounding the comparison of first- and second-price credit auctions (FPCAs and SPCAs, respectively). We show that a 2-concave solvency function determines bidders' non-decreasing measure of absolute risk aversion. Such risk aversion leads to less aggressive bidding under the first-price mechanism, and higher expected solvency rates. Therefore, FPCAs prove superior than SPCAs in terms of expected payoffs, for *the seller and bidders combined.* On the other hand, we also prove that the above arguments and result completely reverse once the solvency function is 2-convex, implying a non-decreasing measure of absolute risk aversion, and second-price superiority.

The second part of the paper concerns new ways to deal with defaults. The three well-known methods are: renegotiation, reselling the asset, and penalties.[1] To differ, we propose a forth approach – using *bid caps.*

Our bid-cap approach has several unique features. First, bid caps are a preliminary formal part of the auction that restrict overbidding beforehand, while other solutions are applied ex-post. For example, most auctions do not (or, at least, should not) formally enable a renegotiation or a resale. When they do, the outcome tends to be poor since bidders can a-priori overbid only to later enforce a new deal at their own terms. Second, the implementation of bid caps is simple and cheap, whereas extracting damages or conducting renegotiations tend to be costly and involve many litigations. For example, in 1996 a block of radio frequencies was sold by the US Federal Communications Commission to companies that later defaulted since

[1] For more on these three methods see, e.g., Spulber (1990), Waehrer (1995), Rhodes-Kropf and Viswanathan (2000), Zheng (2001), Board (2007), and Decarolis (2009), among many others.

they could not sustain the high bids.[2] Another famous example is the case of ITV digital, a company that won the 2002 auction for the rights to broadcast the English Football League games with a £315 million winning-bid; a bid it later failed to pay.[3] In both cases, litigations proceeded for many years after the default had occurred. Both examples also illustrate the problem of timing. When dealing with broadcast rights, radio frequencies, or railways projects – time is money, literally. One cannot sell broadcast rights after the league ends, just as one cannot bring back the lost time for not commencing an infrastructure project. Thus, the ability to ex-ante implement bid caps is a significant advantage.

In the second part of the paper we prove that bid caps are profitable for the seller in case the latter's expected payoff is decreasing, from a certain point onwards, as a function of the agreed price. Moreover, we prove that bid caps can also increase bidders' expected payoff, with a slight exemption of high-valuations bidders, who are affected by the reduced probability of winning. Nevertheless, we exemplify how such bidders also profit from a cap, once the distribution of private values is convex such that the cap bares limited effect over their probability of winning.

1.1 Related literature

The literature on credit auctions has grown substantially in recent years,[4] so we will focus on papers most relevant to ours. Parlane (2003) and Board (2007), continued the work of Waehrer (1995), by studying first- and second-price credit auction. Both prove that the latter mechanism is superior for the seller in various scenarios, while Board (2007) also suggests that, under small-recovery rates, the seller may prefer a FPCA as it guarantees higher solvency rates. Though related, the two papers have several important differences with ours. We provide conditions such that the relevant mechanism is superior to *all* parties involved, and show that the first-price mechanism is superior for a broad set of solvency functions. This distinction is based on the different motivation for a default. Parlane (2003) and Board (2007) consider a strategic approach, where bidders evidently base their decision to default on the agreed price and private valuations. We, however, consider a price-dependent exogenous solvency function, motivated

[2]See Wall Street Journal, "NextWave's Tactics at Wireless Auction Are Under Fire" on May 6, 1996. Additional comments on the legal issues concerning this auction are given in Pardo (2001).

[3]See *The Financial Times*, "Turmoil as ITV Digital Owners Throw in Towel", March 28, 2002; and *BBC News*, "Football League loses damages bid", June 23, 2006.

[4]See, among many others, Spulber (1990), Rhodes-Kropf and Viswanathan (2000), Zheng (2001), Rhodes-Kropf and Viswanathan (2005), Engel et al. (2006), Spagnolo et al. (2006), Conley and Decarolis (2012), Decarolis (2013), Chillemi and Mezzetti (2014), and Chang et al. (2015).

by the information asymmetry in financial markets (which will be broadly explained later on).

Our distinction between mechanisms is based on bidders' risk preferences as in Matthews (1987) and Eso and White (2001). Matthews (1987) shows that bidders with decreasing absolute risk aversion (DARA) prefer the second-price mechanism, while reverse preferences point to the first-price mechanism as superior. In a similar spirit, Eso and White (2001) shows that DARA leads to less aggressive bidding, indicating that a seller facing such bidders would tend to reduce the imposed risk. We combine the two results through our credit-auction model. That is, we show that the solvency function determines whether bidders have a DARA or an increasing one. In return, bidders aggressiveness is set (as in Eso and White (2001)), directly affecting the expected solvency rates. Thus, our final conclusions are in-line with Matthews (1987), as the risk preferences determine the superior mechanism.

Moving on to the second part of the paper, there are a few recent papers that considered new ways to deal with defaulting. Calveras et al. (2004) suggest using surety bonds against abnormally low tenders in procurement auctions. This line of work is followed by Burguet et al. (2012) and Decarolis (2014) as they examine the way rents could be used to prevent low tenders from bidding too aggressively in limited-liability auctions. Nevertheless, to the best of our knowledge, bid caps were not formally considered in the literature in the context of FPAs or SPAs, but were considered in all-pay auctions (see, e.g., Che and Gale (1998), Gavious et al. (2002), and Sahuguet (2006)). In the all-pay framework, previous results indicate that caps can either improve bidders' expected payoff or the seller's. We, on the other hand, focus on increasing the expected payoff of all sides simultaneously. This goal is plausible due to the equilibrium sub-optimality in credit auctions. Moreover, a win-win theoretical result has practical implications, as well. For example, in 2003 the Tel Aviv city council auctioned the rights to operate the city's parking lots. The city council had an estimated total cost of operation (TCO) and every bid that exceeded the estimated TCO by more than 15% was eliminated.[5] A group of bidders appealed against the usage of bid caps claiming that the bound is biased. Our win-win outcome gives the required theoretical basis for the usage of bid caps.

Though the lack of theoretical studies, there are other known examples of bid-caps use. For example, in US auctions to explore and drill oil and gas on federal offshore lands. In these auctions, conducted by the U.S. Mineral Management Service (MMS) since 1954, rejections of the highest bids are not uncommon, particularly on drainage tracts (see Haile et al. (2010) for more information on these auctions). Though the policy of the MMS concerning risky bids differs from our binding bid cap, it produces the same effect. For example, in cases bidders have a rational expectations over the rejection threshold of the MMS, the two policies coincide.

[5] See Haaretz, "Whoever gives a too good offer, cannot win the auction", Feburary 2, 2003.

2 The model

Let $N = \{1, \ldots, n\}$ be a set of n risk-neutral buyers, who bid for a single indivisible good. Each bidder $i \in N$ has a private valuation $v_i \in V = [\underline{V}, \overline{V}] \subset \mathbb{R}_+$, drawn independently and randomly according to a non-atomic, cumulative distribution function (CDF) F with density f.

The auction begins when every bidder $i \in N$ places a non-negative fixed endowment $w \leqslant \underline{V}$ and a *bid* $b_i \in V$. Denote by $b = (b_1, \ldots, b_n)$ a vector of n bids. The winning bidder i_b is set to be the highest bidder, with a symmetric tie-breaking rule. In a FPCA the winning bidder pays his own bid, while in a SPCA the winner pays the second highest bid.[6] All other bidders neither pay nor receive, anything. Let p_b be the agreed price, which is the cost of the winning bidder.

After the winning bidder and cost are determined, the winner remains solvent with a certain probability, determined by a *solvency function* $S : \mathbb{R}_+ \to [0, 1]$. Formally, for every realized agreed price p_b, the quantity $S(p_b)$ is the probability that bidder i_b remains solvent, pays the agreed price p_b, and collects the object. If the winner defaults, he loses the endowment w, while the seller collects the endowment w and keeps the item. In both cases, the payoffs of all non-winning bidders are zero. We assume that S is a decreasing, twice continuously-differentiable function.

A strategy β_i of bidder $i \in N$ is a measurable function $\beta_i : V \to \mathbb{R}_+$ from the set of values to the set of non-negative real numbers. Fix a profile of strategies β. The expected payoff of the seller is

$$\begin{aligned} R_{\text{seller}}(\beta) &= \mathbf{E}[S(p_\beta)p_\beta + (1 - S(p_\beta))w] \\ &= \mathbf{E}[S(p_\beta)(p_\beta - w)] + w, \end{aligned}$$

where p_β is the random agreed price, given β. The payoff of the winning bidder, as a function of a realized agreed price p_b and a private value v, is

$$\begin{aligned} u(p_b, v) &= S(p_b)(v - p_b) - (1 - S(p_b))w \\ &= S(p_b)(v + w - p_b) - w. \end{aligned} \tag{1}$$

Therefore, the expected payoff of bidder i, with a private value v, is

$$R_i(\beta|v) = \mathbf{E}\left[u(p_\beta, v)\mathbb{1}_{\{\beta_i(v) > \max_{j \neq i} \beta_j\}}\right],$$

and a profile β is an *equilibrium* if $R_i(\beta|v) \geqslant R_i(b_i, \beta_{-i}|v)$, for every bidder i, for every private value v, and for every bid b_i.

[6] The provided results for SPCA also apply to the English auction.

2.0.1 The solvency function

In our model there are two related (and rather important) assumptions. The first concerns the i.i.d. private valuations, and the second concerns the stochastic solvency function that depends on the agreed price. Since private values are not uncommon in the context of credit auctions,[7] we focus on the less-obvious second assumption.

The basic idea behind a credit auction is that bidders cannot immediately cover their bids. Namely, bidders do not have enough funding to pay for their high valuations, and solvency depends on the ability to raise funds in financial markets. However, in financial markets bidders face the problem of credibly signalling their ability to repay the loan (i.e., signal their private value). In fact, an auction is organized specifically because one cannot credibly extract the private values of agents. Hence, bidders cannot rely on private values and differentiation occurs through the agreed price.

Another interpretation and motivation to our solvency function and model lay in the field of procurement auctions, where bidders bid to win some project (e.g., the drilling rights example mentioned in the introduction). In such cases, private values are the certainty equivalent of the project (or asset, in case of spectrum auctions), and there is an essential need for a third-party funding, since there is a basic time discrepancy between the project and the realization of value. Hence, the information asymmetry and uncertainty are evident, and conditioning on private values becomes impossible.

Our solvency-function model deals with this problem through two supporting efforts. On the one hand, the solvency function is general. It may depend on exogenous parameters, such as the distribution of private valuations, status of financial markets and financial institutions, number of bidders, and so on. On the other hand, the solvency function changes by the endogenously-derived agreed price. Therefore, it depends on the markets' available and credible information, and reflects the information asymmetry upon which the lenders' decisions are based. In addition, our use of a general probability function enables us to capture the essence of defaulting, without restricting ourselves neither to a specific information structure, nor to a specific financial structure.

3 FPCA vs. SPCA

The comparison between FPCAs and SPCAs starts with an equilibrium analysis of the two mechanisms. As most studies, we focus on a symmetric, increasing equilibrium. In the following

[7] See, e.g., Waehrer (1995), Rhodes-Kropf and Viswanathan (2005), and Board (2007).

proposition we prove that both equilibria are strictly increasing, continuous, and generate convex expected payoffs for bidders, where convexity is considered with respect to the private valuation. (All proofs are deferred to the Appendix.)

Proposition 1. *In* FPCAs *and* SPCAs *there exist symmetric, strictly increasing, continuous equilibria, denoted* β_{F} *and* β_{S} *respectively, such that bidders' expected payoffs are strictly increasing and convex in the private valuations. In* FPCAs*, the equilibrium* β_{F} *is also differentiable.*

The results of Proposition 1 are quite orthodox. First, the SPCA equilibrium β_{S} follows from the known phenomenon where, in SPAs, every bidder's bid generates a zero payoff had the bidder needed to pay his own bid. That is, the equilibrium sustains $u(\beta_{\mathrm{S}}(v), v) = 0$, which also shows that bidders bid more aggressively in the SCPA than in the FPCA. In addition, using an envelope argument, one can see that a bidder's expected payoff changes according to the effective probability of obtaining the asset. That is, the derivative of a bidder's expected payoff (w.r.t. the private valuation) equals the expected product of the probabilities to win and remain solvent. The latter effective probability must be an increasing function in the private valuation, thus explaining the convexity property.

Given the mentioned equilibria profiles, we can now compare the two mechanisms. We say that one mechanism *dominates* another if the former generates a higher expected payoff for the seller and bidders. The following theorem shows that dominance hinges on the convexity-concavity properties of S.

Theorem 1. *If the solvency function is* 2*-concave, then the* FPCA *dominates the* SPCA*. However, if the solvency function is* 2*-convex, then the* FPCA *is dominated by* SPCA.

The intuition behind Theorem 1 is traced to the influence of S over the risk preferences of bidders. If S is 2-concave, then bidders have a non-decreasing absolute risk aversion (ARA). Recalling the results of Matthews (1987) and Eso and White (2001), both building on Lemma 1 of Maskin and Riley (1984), we know that such bidders prefer the first-price mechanism over the second-price, by the reduced risk (i.e., in a standard setting, the price in the second-price auction is a mean-preserving spread of the price under the first-price mechanism). In our context, bidders' non-decreasing ARA leads to higher expected solvency rates under the first-price mechanism. Hence, the latter is preferable to all sides, seller and bidders alike. This intuition reverts completely once the solvency function is 2-convex, leading to a non-increasing ARA, and superiority of the second-price mechanism. Nevertheless, it is imperative to note that 2-concavity, defined by the concavity of $-1/S$, is weaker than log-concavity, whereas 2-convexity is stronger than convexity. The set of log-concave functions includes a long list of distributions such as: normal, exponential, uniform, logistic, beta, gamma, χ^2, and many others. Thus, the condition is met for a wide set of probability distributions, all lead to the first-price dominance.

4 Bid caps in credit auctions

In the following section, we track the second goal of this paper — the introduction of bid caps. For the sake of clarity, we describe the unfolding of a capped auction. As a preliminary stage, the seller posts a bid cap $\bar{b}$ that defines the set of acceptable bids. After bidders observe the cap and receive their private valuations, they submit bids. Since the bid cap $\bar{b}$ is binding, every bid strictly above the cap is eliminated, and the auction continues as in the uncapped mechanism.

The bid cap generates a pooling effect, as shown in Proposition 2. The simpler effect occurs in the SPCA as every bidder, who originally bid above the cap, now bids $\bar{b}$ while all other bidders remain unaffected. The intuition is clear. A deviation from a high bid to a lower one, strictly below the cap, is suboptimal in terms of winning the competition and remaining solvent.

The more complicated effect occurs in FPCAs since some bidders, who originally bid below the cap, deviate to $\bar{b}$ in equilibrium. That is, the pooling effect occurs downwards and upwards. The upwards effect follows from the increase in the expected payoff of bidders who originally bid $\bar{b}$. The bid cap eliminates higher bids, thus increases their probability of winning, while the agreed price is fixed at $\bar{b}$. Consequently, even a bidder with a smaller, yet sufficiently close, valuation gains from bidding $\bar{b}$. In equilibrium, the upwards-pooled bidders balance the discontinuous increase in the probability of winning with the excess agreed price they are willing to pay.

Proposition 2. *For every bid cap $\bar{b}$ and for every bidder with valuation v:*

(i) *an equilibrium in the capped* SPCA *is* $\beta^*(v) = \min\{\beta_{\mathrm{S}}(v), \bar{b}\}$*;*

(ii) *an equilibrium in the capped* FPCA *is*

$$\beta^*(v) = \begin{cases} \beta_{\mathrm{F}}(v), & \text{if } v < \bar{c}, \\ \bar{b}, & \text{if } v \geqslant \bar{c}, \end{cases} \tag{2}$$

for some $\bar{c} \leqslant \beta_{\mathrm{F}}^{-1}(\bar{b})$*, where* $\bar{c}$ *increases with* $\bar{b}$*.*

The value $\bar{c}$, which we refer to as the *upwards-pooling bound* of $\bar{b}$, is the value of the bidder who is indifferent between bidding $\bar{b}$ and giving his original bid $\beta_{\mathrm{F}}(\bar{c})$. This indifference is derived from the fact that every pooled bidder has a negative marginal effect on the probability of winning the auction with a bid of $\bar{b}$. That is, the more bidders bidding $\bar{b}$, the smaller the probability of winning with such a bid. Thus, the upwards pooling effect weakens with every additional pooled bidder. It should be noted that for a sufficiently low cap, such an indifferent bidder may not exist, since $\bar{b}$ might be the optimal bid for all bidders.

4.1 Increasing payoffs using bid caps

We can now address the issue of bid caps in credit auctions. For the purpose of our analysis, we consider the seller's payoff $u_{\text{seller}}(p_b) = S(p_b)(p_b - w) + w$. The monotonicity of u_{seller} is crucial for the implementation of a cap, in the sense that no cap should be used if $u_{\text{seller}}(p_b)$ is a strictly increasing function. So, if $u_{\text{seller}}(p_b)$ is decreasing from a certain price onwards, a bid cap can increase the seller's expected payoff. Formally, define p_0 to be the *threshold price*, such that $u_{\text{seller}}(p_b)$ is increasing if and only if $p_b \leqslant p_0$. Theorem 2 states that, in a SPCA, every cap $\bar{b} \geqslant p_0$ increases (potentially weakly) the seller's expected payoff.

Theorem 2. *In a* SPCA *and if a threshold price p_0 exists, then every bid cap $\bar{b} \geqslant p_0$ increases the seller's expected payoff.*

An additional conclusion, based on the SPCA's pooling effect, concerns the optimal cap $\bar{b}_{\text{opt}}$. Since bidders are only pooled downwards towards p_0, the seller's optimal cap is the threshold price $\bar{b}_{\text{opt}} = p_0$. Nevertheless, assuming that S is unknown to the seller (otherwise he could simply rank bids using S), his ability to fix the optimal cap is limited. Yet, the seller can still fix a sufficiently high cap and guarantee a weak increase in expected revenue, in case a threshold price p_0 exists. In addition, the continuity of the seller's payoff suggests that even a cap below (but sufficiently close to) p_0 increases the seller's expected payoff in equilibrium. The following theorem shows that similar results hold in FPCAs, as well.

Theorem 3. *In a* FPCA*, there exists a neighbourhood of p_0 in which every bid cap $\bar{b}$ strictly increases the seller's expected payoff. In addition, the optimal cap $\bar{b}_{\text{opt}}$ is lower than p_0.*

The result that the optimal cap is bounded away from p_0 is somewhat puzzling. It suggests that the seller actually prefers pooling bidders at a price below p_0, although p_0 maximizes his payoff function. The solution is traced to the influence of the threshold price over the payoff function. If indeed a threshold price exists and the seller's payoff function is differentiable, it implies that u_{seller} is concave in the neighbourhood of p_0. In other words, a small decrease in the pooling price carries a limited effect over the payoff. On the other hand, by fixing a cap below p_0, the seller pools additional low-evaluation bidders, thus increasing the probability that the agreed price equals the cap. The additional pooling compensates for the marginal loss in the agreed price.

4.1.1 The bidders' payoffs

The effect of bid caps on bidders' expected payoffs depends on their private valuations. The pooling effect along with the convexity of expected payoffs divide the set of bidders into three categories: *low-valuation bidders*, *average-valuation pooled bidders*, and *high-valuation pooled bidders*. The low-valuation bidders are bidders who bid below the cap in the capped auction. These bidders' payoffs are unaffected by the cap in any way. Their equilibrium strategies and payoffs are identical in the capped and uncapped auctions. The set of pooled bidders consists of high- and average-valuation bidders, who bid the cap in the capped mechanisms. Their payoffs are significantly affected by the cap. On the one hand, high-valuation bidders may lose from the introduction of a cap, since they cannot give high bids and win the auction with a high probability. On the other hand, average-valuation pooled bidders gain from the introduction of a cap by the increased probability of winning. The following lemma summarizes these results.

Lemma 1. *Fix a cap $\bar{b}$, and denote $v_{\bar{b}}$ as the valuation that induce a bid of $\bar{b}$ in the uncapped mechanism. There exists a private value $v_0 > v_{\bar{b}}$ such that $\bar{b}$ increases (potentially, weakly) the expected payoff of a bidder with valuation v if and only if $v \leqslant v_0$.*

The intuition and proof of Lemma 1 follow from two previous results: (i) bidders' expected payoffs are convex (Proposition 1); and (ii) the set of pooled bidders is convex (Proposition 2). In addition, note that pooled bidders pay the same price when winning, so their expected payoffs are linearly increasing in private valuations. Therefore, by the single-crossing property, the set of bidders who strictly gain from a cap is convex. Note that the increase in expected payoffs, due to the implementation of a cap $\bar{b}$, is weak for the non-pooled bidders and strict for a convex set of average-valuation pooled bidders.

Although Lemma 1 states that a set of high-valuation bidders may lose from the introduction of a cap, there is another possibility. The following example shows that a cap can increase the expected payoff of all bidders.

Example 1. *High-bids* SPCA*.*

To increase the payoffs of high-valuation bidders using a cap, we need to ensure that the cap generates a relatively low agreed price to compensate for the loss in the probability of winning the auction. We achieve this effect using two assumptions. First, we assume that no endowment is needed ($w = 0$), which increases equilibrium bids, as can be seen in Proposition 1. Second, we assume that the CDF F on the private valuation is convex. This convexity ensures that a winning high-valuation bidder pays a relatively high agreed price (with high probability). Alternatively, the convexity property secures a significant amount of high-valuation bidders, such that the cap's influence over the probability of winning is mild.

Claim 1. *In a* SPCA*, assuming that no endowment is needed* ($w = 0$) *and F is convex, every bid cap increases all bidders' expected payoffs.*

This result carries some resemblance to the work of Che et al. (2017) which studies an auction-collusion context, and shows how the advantages of pooling hinges on the convexity-concavity of private valuations. Note that a unification of Claim 1 and Theorem 2 implies that a cap can simultaneously increase the seller's and the bidders' expected payoffs (a win-win situation).

5 Summary and discussion

In this paper we offered two main contributions for designers of credit auctions. First, we motivated the usefulness of first-price mechanisms relative to the second-price mechanisms. Second, we studied the possibility of using bid caps in first- and second-price auctions. We studied the equilibria in both auctions and showed that caps can increase the expected payoffs of all sides.

Though we made a significant progress in the study of bid caps in FPCAs and SPCAs, there is more to be done. For example, one can try extending Theorem 3 to cases where the cap is bounded away from the threshold price p_0. In such cases, one needs to show that the upwards-pooling effect is weaker than the downwards-pooling effect, in terms of the seller's revenue, so that every bid cap above the threshold price increases the seller's revenue. Another possible question is whether the upwards-pooling effect can be used to increase seller's revenue in cases where the seller's expected payoff is not necessarily decreasing.

The use of a public cap, compared to a private one, is another significant aspect for future research. In our model, the cap is common knowledge. This generates the pooling equilibrium and assures that bidders do not make the mistake of bidding above the cap. One can consider a different set-up where the cap remains a private assessment of the seller. A private cap carries a non-trivial influence over bidders' strategies since they depend on bidders' ex ante assessments of the seller's cap. For example, the uncertainty regarding the cap may lead to cases where bidders mistakenly bid above it.

In general, the main question of an optimal credit auction remains unanswered. Note that the optimal solution may not be based on bid caps, but on other methods, such as higher endowments or sequential auctions. Nevertheless, we hope that the current analysis sheds some light on applicative methods for credit auctions.

References

Board, Simon, "Bidding into the red: a model of post-auction bankruptcy," *The Journal of Finance*, 2007, *62* (6), 2695–2723.

Burguet, Roberto, Juan-José Ganuza, and Esther Hauk, "Limited liability and mechanism design in procurement," *Games and Economic Behavior*, 2012, *76* (1), 15–25.

Calveras, Aleix, Juan-Jose Ganuza, and Esther Hauk, "Wild bids. Gambling for resurrection in procurement contracts," *Journal of Regulatory Economics*, 2004, *26* (1), 41–68.

Chang, Wei-Shiun, Bo Chen, and Timothy C. Salmon, "An Investigation of the Average Bid Mechanism for Procurement Auctions," *Management Science*, 2015, *61* (6), 1237–1254.

Che, Yeon-Koo and Ian L. Gale, "Caps on political lobbying," *American Economic Review*, 1998, *88* (3), 643—-651.

_ , Daniele Condorelli, and Jinwoo Kim, "Weak Cartels and Collusion-Proof Auctions," 2017.

Chillemi, Ottorino and Claudio Mezzetti, "Optimal procurement mechanisms: bidding on price and damages for breach," *Economic Theory*, 2014, *55* (2), 335–355.

Conley, Timothy G. and Francesco Decarolis, "Detecting bidders groups in collusive auctions," 2012.

Decarolis, Francesco, "When the Highest Bidder Loses the Auction: Theory and Evidence from Public Procurement," *Working paper*, 2009.

_ , "Comparing Public Procurement Auctions," 2013.

— , "Awarding Price, Contract Performance, and Bids Screening: Evidence from Procurement Auctions," *American Economic Journal: Applied Economics*, 2014, *6* (1), 108–32.

Engel, Andreas R., Juan-José Ganuza, Esther Hauk, and Achim Wambach, *Managing risky bids (Handbook of Procurement)*, Cambridge: Cambridge University Press, 2006.

Eso, Péter and Lucy White, "Precautionary Bidding in Auctions," 2001.

Gavious, Arieh, Benny Moldovanu, and Aner Sela, "Bid costs and endogenous bid caps," *RAND Journal of Economics*, 2002, *33* (4), 709–722.

Haile, Philip, Kenneth Hendricks, and Robert Porter, "Recent U.S. offshore oil and gas lease bidding: A progress report," *International Journal of Industrial Organization*, 2010, *28* (4), 390–396.

Maskin, Eric S and John G Riley, "Optimal auctions with risk averse buyers," *Econometrica*, 1984, *52* (6), 1473–1518.

Matthews, Steven, "Comparing Auctions for Risk Averse Buyers: A Buyer's Point of View," *Econometrica*, may 1987, *55* (3), 633.

Pardo, Rafael I., "Bankruptcy court jurisdiction and agency action: resolving the Next Wave of conflict," *New York University Law Review*, 2001, *76* (3), 945–972.

Parlane, Sarah, "Procurement contracts under limited liability," *Economic and Social Review*, 2003, *34* (1), 1–22.

Rhodes-Kropf, Matthew and S. Viswanathan, "Corporate reorganizations and non-cash auctions," *The Journal of Finance*, 2000, *55* (4), 1807–1849.

— **and** — , "Financing auction bids," *The RAND Journal of Economics*, 2005, *36* (4), 789–815.

Sahuguet, Nicolas, "Caps in asymmetric all-pay auctions with incomplete information," *Economics Bulletin*, 2006, *3* (9), 1–8.

Spagnolo, Giancarlo, Gian Luigi Albano, and Milo Bianchi, "Bid avarage methods in Procurement," *Rivista di Politica Economica*, 2006, *96* (1), 41–62.

Spulber, Daniel F., "Auctions and contract enforcement," *Journal of Law, Economics, & Organization*, 1990, *6* (2), 325–344.

Waehrer, Keith, "A model of auction contracts with liquidated damages," *Journal of Economic Theory*, 1995, *67* (2), 531–555.

Zheng, Charles Z., "High bids and broke winners," *Journal of Economic Theory*, 2001, *100* (1), 129–171.

6 Appendix

Proposition 1. *In* FPCAs *and* SPCAs *there exist symmetric, strictly increasing, continuous equilibria, denoted* β_{F} *and* β_{S} *respectively, such that bidders' expected payoffs are strictly increasing and convex in the private valuations. In* FPCAs*, the equilibrium* β_{F} *is also differentiable.*

Proof. We begin our analysis with the second-price mechanism, where the equilibrium β_{S} is given by

$$\beta_{\mathrm{S}}(v) = v + w - \frac{w}{S(\beta_{\mathrm{S}}(v))}. \tag{3}$$

Fix a private value (PV) $v \in V$. We first prove that a solution for $y = v + w - \frac{w}{S(y)}$ exists. If $v = \underline{V}$, then $\beta_{\mathrm{S}}(v) = v$ solves the equation. Moreover, by the monotonicity and continuity of S and the intermediate value theorem, a solution exists for every $v > \underline{V}$. Denote this solution by $\beta_{\mathrm{S}}(v)$. Note that it solves the zero-profit condition $u(\beta_{\mathrm{S}}(v), v) = 0$, where $u(\beta_{\mathrm{S}}(v), v)$ is the payoff of a winning bidder as a function of an agreed price $\beta_{\mathrm{S}}(v)$. The function u is strictly decreasing in its first argument, so a deviation upwards to $y > \beta_{\mathrm{S}}(v)$ is suboptimal. In addition, the dependence of the agreed price on the second highest bid assures that deviating downwards, to $y < \beta_{\mathrm{S}}(v)$, is also suboptimal. Hence, β_{S} is an equilibrium. The monotonicity and continuity of β_{S} are straightforward from the symmetric equilibria and properties of u: continuous; decreasing in first coordinate ; increasing in second coordinate.

Now consider the expected payoff $u_{\mathrm{S}}(v)$ of a bidder with PV v. If u_{S} is not strictly increasing in v, there exist $v' < v$ such that $u_{\mathrm{S}}(v') \geqslant u_{\mathrm{S}}(v)$. In this case, the bidder has a profitable deviation to $\beta_{\mathrm{S}}(v')$, contradicting the equilibrium analysis. Thus, we are left with the convexity of bidders' expected payoff. Let $Y = \max_{j \neq i} v_j$ be the random variable defined by the maximal valuation of all bidders excluding the given bidder with PV v. Hence,

$$\begin{aligned} u_{\mathrm{S}}(v) &= \mathbf{E}\left[[S(\beta_{\mathrm{S}}(Y))(v + w - \beta_{\mathrm{S}}(Y)) - w]\mathbb{1}_{\{Y<v\}}\right] \\ &= \mathbf{E}\left[u(\beta_{\mathrm{S}}(Y), Y)\mathbb{1}_{\{Y<v\}}\right] + \mathbf{E}\left[S(\beta_{\mathrm{S}}(Y))(v - Y)\mathbb{1}_{\{Y<v\}}\right] \\ &= \mathbf{E}\left[S(\beta_{\mathrm{S}}(Y))(v - Y)\mathbb{1}_{\{Y<v\}}\right], \end{aligned}$$

where the last equality follows from the relation $u(\beta_{\mathrm{S}}(v), v) = 0$. Differentiating w.r.t. v under the integral sign, we get $\frac{du_{\mathrm{S}}(v)}{dv} = \mathbf{E}\left[S(\beta_{\mathrm{S}}(Y))\mathbb{1}_{\{Y<v\}}\right]$. Since S is non-negative, the derivative is increasing in v, establishing the convexity, as needed.

We move on to the first-price mechanism, where the equilibrium β_{F} is defined by the differential equation and boundary condition

$$\begin{aligned} \beta_F^{'}(v) &= -\frac{(n-1)f(v)}{F(v)} \cdot \frac{u(\beta_{\mathrm{F}}(v), v)}{u_1(\beta_{\mathrm{F}}(v), v)}, \\ \beta_{\mathrm{F}}(\underline{V}) &= \underline{V} + w - \frac{w}{S(\beta_{\mathrm{F}}(\underline{V}))}. \end{aligned} \tag{4}$$

Fix a PV v. Since u_1 is negative, β_{F} is strictly increasing and differentiable, by definition. In case $v = \underline{V}$, then the boundary condition implies that $\beta_{\mathrm{F}}(v)$ sustains $u(\beta_{\mathrm{F}}(\underline{V}), \underline{V})) = 0$. By the previous SPCA analysis, this is an optimal action given the equilibrium profile. So we focus on a bidder with a PV $v > \underline{V}$ who bids $\beta_{\mathrm{F}}(t)$, while all other bidders bid truthfully according to β_{F}. Taking the first-order derivative, we get

$$\begin{aligned} \frac{\partial R(t,v)}{\partial t} &= F^{n-1}(t)u_1(\beta_{\mathrm{F}}(t), v)\left[\beta_{\mathrm{F}}{}'(t) + \frac{(n-1)f(t)}{F(t)} \cdot \frac{u(\beta_{\mathrm{F}}(t), v)}{u_1(\beta_{\mathrm{F}}(t), v)}\right] \\ &= -u_1(\beta_{\mathrm{F}}(t), v)(n-1)F^{n-2}(t)f(t)\left[\frac{u(\beta_{\mathrm{F}}(t), t)}{u_1(\beta_{\mathrm{F}}(t), t)} - \frac{u(\beta_{\mathrm{F}}(t), v)}{u_1(\beta_{\mathrm{F}}(t), v)}\right], \end{aligned}$$

where the second equality follows from plugging in $\beta_{\mathrm{F}}{}'(t)$. A straightforward examination shows that $\frac{u(\beta_{\mathrm{F}}(t),v)}{u_1(\beta_{\mathrm{F}}(t),v)}$ is decreasing in v. Hence,

$$\frac{\partial R(t,v)}{\partial t} \gtreqless 0, \quad \text{as} \quad t \lesseqgtr v, \tag{5}$$

and $t = v$ maximizes the bidder's expected payoff, proving that β_{F} is a symmetric, strictly-increasing, differentiable equilibrium.

The monotonicity of a bidder's expected payoff is, again, straightforward from the symmetric equilibrium, so it remains to verify convexity. Differentiating the expected payoff w.r.t. v and using the definition of β_{F} yields

$$\frac{d}{dv}\{u(\beta_{\mathrm{F}}(v), v)F^{n-1}(v)\} = F^{n-1}(v)S(\beta_{\mathrm{F}}(v)).$$

Note that the term on the RHS is the bidder's effective probability of obtaining the object. If $F^{n-1}(v)S(\beta_{\mathrm{F}}(v))$ is non-increasing in v, there exists a profitable deviation to a lower value $t < v$ (a lower agreed price with a higher effective probability), contradicting the equilibrium analysis. ∎

Theorem 1. *If the solvency function is* 2*-concave, then the* FPCA *dominates the* SPCA*. However, if the solvency function is* 2*-convex, then the* FPCA *is dominated by* SPCA.

Proof. Fix a bidder with a PV v. Let $(\beta_{\mathrm{F}}, u_{\mathrm{F}})$ and $(\beta_{\mathrm{S}}, u_{\mathrm{S}})$ be the bidder's strategy and expected payoff in equilibrium, in a FPCA and in a SPCA, respectively. Note that both equilibria dictate a zero expected payoff for a bidder with PV $v = \underline{V}$. We will prove that $u'_{\mathrm{F}}(v) \gtreqless u'_{\mathrm{S}}(v)$ whenever $u_{\mathrm{F}}(v) = u_{\mathrm{S}}(v)$, according to the condition over S, thus establishing the part concerning the bidders' expected payoffs.

Recall that bidders' payoff equals $u(p_b, v) = S(p_b)(v + w - p_b) - w$, and Proposition 1 shows that $u'_{\mathrm{F}}(v) = F^{n-1}(v)u_2(\beta_{\mathrm{F}}(v), v)$. To differ, the expected payoff of a SPCA winning bidder is $u_{\mathrm{S}}(v) = \mathbf{E}[u(\beta_{\mathrm{S}}(Y_v), v)]F^{n-1}(v)$, where Y_v is the second-highest PV, given that v is the highest. By a standard envelope argument, we get $u'_{\mathrm{S}}(v) = F^{n-1}(v)\mathbf{E}[u_2(\beta_{\mathrm{S}}(Y_v), v)]$, where we used the fact that $u(\beta_{\mathrm{S}}(v), v) = 0$ by the definition of β_{S}. Hence, we need to prove that $u_2(\beta_{\mathrm{F}}(v), v) \gtreqless \mathbf{E}[u_2(\beta_{\mathrm{S}}(Y_v), v)]$ if $u(\beta_{\mathrm{F}}(v), v) = \mathbf{E}[u(\beta_{\mathrm{S}}(Y_v), v)]$. This follows directly from Lemma 1 of Maskin and Riley (1984) where u is decreasing in the agreed price p_b, and $-\frac{u_{11}}{u_1}$ is either a non-increasing or a non-decreasing function of v. The last condition implies that bidders have either a non-decreasing or a non-increasing absolute risk aversion. Taking the relevant derivatives yields

$$\begin{aligned}
-\frac{u_{11}(t,v)}{u_1(t,v)} &= -\frac{S''(t)(v+w-t) - 2S'(t)}{S'(t)(v+w-t) - S(t)}, \\
\frac{\partial}{\partial v}\left[-\frac{u_{11}(t,v)}{u_1(t,v)}\right] &= \frac{S(t)S''(t) - 2(S'(t))^2}{(S'(t)(v+w-t) - S(t))^2},
\end{aligned}$$

and the statements follow from the condition over S, since $S(t)S''(t) - 2(S'(t))^2 \gtreqless 0$ is determined by the convexity-concavity properties of $-1/S$.

For the second part of the proof, recall that the expected payoff of the seller, conditional on the agreed price p_b, is $u_{\mathrm{seller}}(p_b) = S(p_b)(p_b - w) + w$. Let $R_{\mathrm{F}}(v) = u_{\mathrm{seller}}(\beta_{\mathrm{F}}(v)) - w$ and $R_{\mathrm{S}}(v) = \mathbf{E}[u_{\mathrm{seller}}(\beta_{\mathrm{S}}(Y_v))] - w$ denote the seller's expected payoff, deducted the initial endowment w and conditional on a highest valuation v, in a FPCA and in a SPCA respectively. Since $R_{\mathrm{F}}(v) = R_{\mathrm{S}}(v)$ in case $v = \underline{V}$, we only need to prove that $R'_{\mathrm{F}}(v) \gtreqless R'_{\mathrm{S}}(v)$, according to S, whenever $R_{\mathrm{F}}(v) = R_{\mathrm{S}}(v)$. To simplify the notation, denote $y_1 = \beta_{\mathrm{F}}(v)$, $y_2 = \beta_{\mathrm{S}}(Y_v)$, and $\bar{y_2} = \beta_{\mathrm{S}}(v)$. The first-order derivatives of R_{F} and R_{S} are

$$\begin{aligned}
R'_{\mathrm{F}}(v) &= -\frac{(n-1)f(v)}{F(v)} \cdot \frac{u(y_1,v)}{u_1(y_1,v)}[S'(y_1)(y_1 - w) + S(y_1)], \\
R'_{\mathrm{S}}(v) &= -\frac{(n-1)f(v)}{F(v)}[R_{\mathrm{S}}(v) - S(\bar{y_2})(\bar{y_2} - w)].
\end{aligned}$$

Using the equality $u_1(y_1, v) = S'(y_1)(v + w - y_1) - S(y_1)$, the definition of $\beta_{\mathrm{S}}(v)$ such that $u(\bar{y_2}, v) = 0$, and the assumption that $R_{\mathrm{S}}(v) = R_{\mathrm{F}}(v)$, where $R_{\mathrm{F}}(v) = vS(y_1) - w - u(y_1, v)$, yields the following equalities:

$$
\begin{aligned}
R'_{\mathrm{F}}(v) &= \frac{(n-1)f(v)}{F(v)} \cdot \frac{u(y_1, v)}{u_1(y_1, v)}[u_1(y_1, v) - S'(y_1)v], \\
R'_{\mathrm{S}}(v) &= -\frac{(n-1)f(v)}{F(v)}[vS(y_1) - w - u(y_1, v) + w - vS(\bar{y_2})].
\end{aligned}
$$

Thus, we need to show that $-S'(y_1)\frac{u(y_1,v)}{u_1(y_1,v)} \gtreqless S(\bar{y_2}) - S(y_1)$. For that purpose, define the function $H(t) = -S'(t)\frac{u(t,v)}{u_1(t,v)} - S(\bar{y_2}) + S(t)$. Note that $H(t)$ is either decreasing or increasing by the condition over S, since $H'(t) = \frac{u(t,v)}{(u_1(t,v))^2}\left[S(t)S''(t) - 2(S'(t))^2\right]$. In addition, $u(\bar{y_2}, v) = 0$ suggests that $H(\bar{y_2}) = 0$. Thus, $y_1 \leqslant \bar{y_2}$ implies that $H(y_1) \gtreqless 0$, and the result follows. ■

Lemma 2. (Monotonicity lemma). *For every bidder with valuation v in a* FPCA *and given the equilibrium β_{F}, bidding $\beta_{\mathrm{F}}(y)$ dominates bidding $\beta_{\mathrm{F}}(x)$ when $x < y < v$ and when $v < y < x$.*

Proof. Consider a bidder with valuation v in a FPCA when all other bidders bid according to β_{F}. Assume $x < y < v$ and define the function $\Phi(t, x, y)$ by

$$
\begin{aligned}
\Phi(t, x, y) &= F^{n-1}(y)u(\beta_{\mathrm{F}}(y), t) - F^{n-1}(x)u(\beta_{\mathrm{F}}(x), t) \\
&= F^{n-1}(y)[S(\beta_{\mathrm{F}}(y))(t + w - \beta_{\mathrm{F}}(y)) - w] \\
&- F^{n-1}(x)[S(\beta_{\mathrm{F}}(x))(t + w - \beta_{\mathrm{F}}(x)) - w].
\end{aligned}
$$

We know that $F^{n-1}(x) < F^{n-1}(y)$ and β_{F} is strictly increasing, implying that $t + w - \beta_{\mathrm{F}}(x) > t + w - \beta_{\mathrm{F}}(y)$. Thus, the probability $F^{n-1}(y)S(\beta_{\mathrm{F}}(y))$ of obtaining the object by bidding $\beta_{\mathrm{F}}(y)$ is strictly greater than the probability $F^{n-1}(x)S(\beta_{\mathrm{F}}(x))$ of obtaining the object while bidding $\beta_{\mathrm{F}}(x)$. Otherwise, a bidder with valuation y could increase his expected payoff by bidding $\beta_{\mathrm{F}}(x)$. Since β_{F} is an equilibrium, it follows that $\Phi(y, x, y) \geqslant 0$. The fact that $\Phi(t, x, y)$ is a linear function in t along with the inequality $F^{n-1}(y)S(\beta_{\mathrm{F}}(y)) - F^{n-1}(x)S(\beta_{\mathrm{F}}(x)) > 0$ suggests that $\Phi(v, x, y) > 0$ for every $v > y > x$, as needed.

Now assume $v < y < x$. For similar reasons, we know that $F^{n-1}(x)S(\beta_{\mathrm{F}}(x)) > F^{n-1}(y)S(\beta_{\mathrm{F}}(y))$ when $x > y$. Therefore, the function $\Phi(t, x, y)$ is linearly decreasing in t, and the fact that $\Phi(y, x, y) \geqslant 0$ suggests that $\Phi(v, x, y) > 0$ for every $v < y < x$. ■

Proposition 2. *For every bid cap $\bar{b}$ and for every bidder with valuation v:*

(i) *an equilibrium in the capped* SPCA *is $\beta^*(v) = \min\{\beta_{\mathrm{S}}(v), \bar{b}\}$;*

(ii) *an equilibrium in the capped* FPCA *is*

$$\beta^*(v) = \begin{cases} \beta_{\mathrm{F}}(v), & \text{if } v < \bar{c}, \\ \bar{b}, & \text{if } v \geqslant \bar{c}, \end{cases} \tag{2}$$

for some $\bar{c} \leqslant \beta_{\mathrm{F}}^{-1}(\bar{b})$*, where* $\bar{c}$ *increases with* $\bar{b}$*.*

Proof. We consider only the case where the bid cap $\bar{b}$ lies in the closed interval of possible equilibrium bids. Whenever the cap is lower (or higher) than the minimal (resp. maximal) possible bid, the result is trivial.

The proof for the capped SPCA is immediate. Once a cap $\bar{b}$ is implemented in a SPCA, every player with PV $v < \beta_{\mathrm{S}}^{-1}(\bar{b})$ cannot gain by bidding $\bar{b}$. If such a player bids $\bar{b}$, then the probability of winning increases whenever the conditional expected payoff is negative. Furthermore, bidding $b(v) \notin \{\beta_{\mathrm{S}}(v), \bar{b}\}$ is sub-optimal, as β_{S} is an equilibrium in the original SPCA and bidders with valuation close to v bid according to β_{S}. If the PV of the bidder is $v \geqslant \beta_{\mathrm{S}}^{-1}(\bar{b})$, he can either bid $\bar{b}$ or less. Bidding less than $\bar{b}$ is suboptimal by the same reasoning that β_{S} is an equilibrium in the uncapped SPCA (i.e., it does not affect the agreed price, conditional on winning, while decreasing the probability of winning the auction). Thus concluding the proof for the SPCA.

Moving on to the capped FPCA, we prove that a pooling equilibrium exists such that every bidder with $v > \bar{c}$ bids $\bar{b}$ for some $\bar{c} < \beta_{\mathrm{F}}^{-1}(\bar{b})$, while all other bid according to β_{F}. We start by defining $\bar{c}$. Denote by $v_{\bar{b}} = \beta_{\mathrm{F}}^{-1}(\bar{b})$ the valuation of a bidder who bids $\bar{b}$ in the β_{F}-equilibrium of the uncapped FPCA, and consider a bidder with PV $c \leqslant v_{\bar{b}}$. Let β_c be a strategy where all bidders with valuation below c bid according to β_{F} and all the other bidders bid $\bar{b}$. Given all bidders bid according to β_c, the probability that the c-valuation bidder wins the auction is

$$G(c) = \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{F^{n-1-k}(c)(1-F(c))^k}{k+1} = \frac{1-F^n(c)}{n(1-F(c))} > F^{n-1}(c),$$

where the second equality and the strict inequality hold for every $F(c) \in (0,1)$. Recall the expected payoff $u_{\mathrm{F}}(c) = F^{n-1}(c)u(\beta_{\mathrm{F}}(c), c)$ of a bidder with valuation c in the uncapped FPCA, and define $u_{\bar{\mathrm{F}}}(c) = G(c)u(\bar{b}, c)$ to be the c-valuation bidder's expected payoff assuming all bidders bid according to β_c.

By the increased winning probability, it follows that $u_{\bar{\mathrm{F}}}(v_{\bar{b}}) - u_{\mathrm{F}}(v_{\bar{b}}) > 0$, while an agreed price of $\bar{b} > \underline{V}$ assures that $u_{\bar{\mathrm{F}}}(\underline{V}) - u_{\mathrm{F}}(\underline{V}) < 0$. Since $u_{\bar{\mathrm{F}}}$ and u_{F} are continuous, there exists $c \in (\underline{V}, v_{\bar{b}})$ such that $u_{\bar{\mathrm{F}}}(c) = u_{\mathrm{F}}(c)$. Continuity also suggests that one can fix a maximal value $\bar{c} < v_{\bar{b}}$ such that $u_{\bar{\mathrm{F}}}(\bar{c}) = u_{\mathrm{F}}(\bar{c})$. We refer to $\bar{c}$ as the *upwards-pooling bound* of $\bar{b}$. In simple terms, a bidder with private valuation $\bar{c}$ is unaffected by the introduction of the bid cap $\bar{b}$.

Given that all other bidders bid according to $\beta_{\bar{c}}$, which is the strategy given in Eq. (2), the $\bar{c}$-bidder can either bid $\beta_{\mathrm{F}}(\bar{c})$ or bid $\bar{b}$, and get the same expected payoff as in the uncapped auction.

Now we need to prove that every bidder with valuation $v \neq \bar{c}$ bids according to the proposed strategy. Note that any bid $b(v) \in (\beta_{\mathrm{F}}(\bar{c}), \bar{b})$ is suboptimal since a deviation to $b(v) - \epsilon > \beta_{\mathrm{F}}(\bar{c})$ for some small enough $\epsilon > 0$, induces the same probability of winning the auction, but with an ϵ-reduced price. Thus, given all other bidders use the strategy $\beta_{\bar{c}}$, a bidder with valuation v will bid $b \in [\beta_{\mathrm{F}}(\underline{V}), \beta_{\mathrm{F}}(\bar{c})] \cup \{\bar{b}\}$. From this point on, and with a slight abuse of notation, we assume that $u_{\bar{\mathrm{F}}}(v)$ is the expected payoff of a bidder with valuation v where all bidders bid according to strategy $\beta^* = \beta_{\bar{c}}$. That is,

$$u_{\bar{\mathrm{F}}}(v) = \begin{cases} F^{n-1}(v)u(\beta_{\mathrm{F}}(v), v), & \text{if } v < \bar{c}, \\ G(\bar{c})u(\bar{b}, v), & \text{if } v \geqslant \bar{c}. \end{cases}$$

Define the function $\Psi(v, \bar{b}) = G(\bar{c})u(\bar{b}, v) - F^{n-1}(\bar{c})u(\beta_{\mathrm{F}}(\bar{c}), v)$. By the indifference of a bidder with valuation $\bar{c}$, it follows that $\Psi(\bar{c}, \bar{b}) = 0$, and explicitly

$$\left[G(\bar{c})[S(\bar{b})(v + w - \bar{b}) - w] - F^{n-1}(\bar{c})[S(\beta_{\mathrm{F}}(\bar{c}))(v + w - \beta_{\mathrm{F}}(\bar{c})) - w]\right]_{v=\bar{c}} = 0.$$

Since $\bar{b} > \beta_{\mathrm{F}}(\bar{c})$ and $G(\bar{c}) > F^{n-1}(\bar{c})$, it follows that $G(\bar{c})S(\bar{b}) > F^{n-1}(\bar{c})S(\beta_{\mathrm{F}}(\bar{c}))$. Therefore $\Psi(v, \bar{b})$ is linearly increasing in v. Thus, a bidder with valuation $v \neq \bar{c}$ prefers bidding $\beta_{\mathrm{F}}(\bar{c})$ instead of bidding $\bar{b}$ if and only if $v < \bar{c}$. By the mentioned monotonicity lemma (given in the Appendix) and the fact that β_{F} is an equilibrium in the uncapped FPCA, the result follows.

A few remarks on the value of $\bar{c}$ are in order. Since the equality $u_{\bar{\mathrm{F}}}(\bar{c}) = u_{\mathrm{F}}(\bar{c})$ implies $G(\bar{c})S(\bar{b}) > F^{n-1}(\bar{c})S(\beta_{\mathrm{F}}(\bar{c}))$, one can use an envelope argument to show that $u'_{\bar{\mathrm{F}}}(\bar{c}) > u'_{\mathrm{F}}(\bar{c})$. Thus, a similar comparison of $u_{\bar{\mathrm{F}}}(\bar{c} + \epsilon)$ and $u_{\mathrm{F}}(\bar{c} + \epsilon)$ for a small enough $\epsilon > 0$ will show that $u_{\bar{\mathrm{F}}}(\bar{c} + \epsilon) > u_{\mathrm{F}}(\bar{c} + \epsilon)$ and $u'_{\bar{\mathrm{F}}}(\bar{c} + \epsilon) > u'_{\mathrm{F}}(\bar{c} + \epsilon)$. For this reason, it is easy to verify that every cap $\bar{b}$ has a unique upwards-pooling bound $\bar{c}$. In addition, $\bar{c}$ is an increasing, implicit function of $\bar{b}$. That is, under a small increase in $\bar{b}$ to $\bar{b} + \epsilon$, the function $u_{\bar{\mathrm{F}}}(\bar{c}) = G(\bar{c})u(\bar{b} + \epsilon, \bar{c})$ decreases, and the equality $u_{\bar{\mathrm{F}}}(\bar{c}) = u_{\mathrm{F}}(\bar{c})$ does not hold. Thus, the upwards-pooling bound of $\bar{b} + \epsilon$ is greater than the upwards-pooling bound of $\bar{b}$, as suggested. ■

Theorem 2. *In a* SPCA *and if a threshold price p_0 exists, then every bid cap $\bar{b} \geqslant p_0$ increases the seller's expected payoff.*

Proof. In case a threshold price p_0 exists, but the maximal possible bid in equilibrium $\beta(\overline{V}) \leqslant p_0$ is lower than p_0, then any cap $\bar{b} \geqslant p_0$ has no affect on the auction. Thus, the case where p_0 exists is only considered when some bidders can bid above p_0 in equilibrium. Since the pooling effect occurs only downwards, bounding bids above $\bar{b} \geqslant p_0$ does not affect bids below $\bar{b}$, and assures that any agreed price $p_b > \bar{b}$ shifts to $\bar{b}$. Under the assumption that $u_{\text{seller}}(p_b)$ is decreasing if and only if $p_b \geqslant p_0$, the cap increases the seller's expected payoff R_{seller}. ■

Theorem 3. *In a* FPCA*, there exists a neighbourhood of p_0 in which every bid cap $\bar{b}$ strictly increases the seller's expected payoff. In addition, the optimal cap $\bar{b}_{\text{opt}}$ is lower than p_0.*

Proof. Assume that a threshold price p_0 exists. If $\bar{b} = p_0$, then the pooling effect shifts bids below and above the cap to $\bar{b}$. Since p_0 is an extreme point of $u_{\text{seller}}(p_b)$, the pooling upwards and downwards strictly increases the seller's expected payoff. By the continuity of u_{seller} the same holds for every bid cap in the neighbourhood of p_0.

To examine the optimal cap, define $H(\bar{c})$ to be the difference between the seller's expected payoff in the capped and the uncapped FPCA, where $\bar{c}$ is defined in Eq. (2), and get

$$H(\bar{c}) = (1 - F^n(\bar{c}))[u_{\text{seller}}(\bar{b}) - \mathbf{E}[u_{\text{seller}}(\beta_{\text{F}}(\tilde{v}_1))|\tilde{v}_1 \geqslant \bar{c}]].$$

Note that $H(p_0) > 0$ and $H(\overline{V}) = 0$. In addition, each of the terms $\bar{c}$ and $\bar{b}$ is an increasing and implicit function of the other, based on the equality $\Psi(\bar{c}, \bar{b}) = 0$, given in the proof of Proposition 2. Using the implicit function theorem, the first-order derivative of $H(\bar{c})$ is

$$\begin{aligned} H'(\bar{c}) &= -nF^{n-1}(\bar{c})f(\bar{c})u_{\text{seller}}(\bar{b}) + (1 - F^n(\bar{c}))\frac{d}{d\bar{c}}[u_{\text{seller}}(\bar{b})] \\ &+ nF^{n-1}(\bar{c})f(\bar{c})u_{\text{seller}}(\beta_{\text{F}}(\bar{c})). \end{aligned}$$

Note that $\frac{d}{d\bar{c}}[u_{\text{seller}}(\bar{b})] = 0$ and $u_{\text{seller}}(\beta_{\text{F}}(\bar{c})) < u_{\text{seller}}(\bar{b})$ in case $\bar{b} = p_0$, since p_0 is a local extreme point of u_{seller}. Thus, $H'(p_0) < 0$ and the optimal cap $\bar{b}_{\text{opt}} < p_0$, as stated. ■

Lemma 1. *Fix a cap $\bar{b}$, and denote $v_{\bar{b}}$ as the valuation that induce a bid of $\bar{b}$ in the uncapped mechanism. There exists a private value $v_0 > v_{\bar{b}}$ such that $\bar{b}$ increases (potentially, weakly) the expected payoff of a bidder with valuation v if and only if $v \leqslant v_0$.*

Proof. A bidder who bids below the cap, in equilibrium of the capped mechanism, is unaffected by the cap. Thus, we can solely relate to bidders who bid the cap itself. The lowest bidder who bids $\bar{b}$ gets the same payoff in the capped and the uncapped mechanism. Moreover, in each capped mechanism the probabilities and prices coincide to all $\bar{b}$-bidders, so their expected payoffs are linearly increasing w.r.t. PVs. On the other hand, Proposition 1 shows that in the uncapped mechanism the expected payoffs are convex w.r.t. PVs. Hence, there could be only one crossing, denoted v_0, between the expected payoff functions of the capped and uncapped mechanism. For every $v \leqslant v_0$, every bidder gains weakly from the introduction of the cap, whereas bidders with PVs above v_0 lose in expectation, as required. We now need to prove that $v_0 > v_{\bar{b}}$.

Starting with the SPCA, denote $v_{\bar{b}} = \beta_{\mathrm{S}}^{-1}(\bar{b})$ and let $\Phi_{\mathrm{SP}}(v)$ be the difference between the expected payoffs of a v-valuation bidder in the capped and the uncapped mechanism, given $v \geqslant v_{\bar{b}}$. Formally,

$$\begin{aligned}\Phi_{\mathrm{SP}}(v) &= (G(v_{\bar{b}}) - F^{n-1}(v_{\bar{b}}))u(\bar{b}, v) \\ &- (F^{n-1}(v) - F^{n-1}(v_{\bar{b}}))E[u(\beta_{\mathrm{S}}(\tilde{v}_2), v)|v_{\bar{b}} \leqslant \tilde{v}_2 \leqslant v],\end{aligned}$$

where G is defined in the proof of Proposition 2 and $\tilde{v}_2$ is the second highest bid. Note that $G(v_{\bar{b}}) > F^{n-1}(v_{\bar{b}})$, and u is decreasing in its first coordinate, so $u(\bar{b}, v) \geqslant E[u(\beta_{\mathrm{S}}(\tilde{v}_2), v)|v_{\bar{b}} \leqslant \tilde{v}_2 \leqslant v]$ for v sufficiently close to $v_{\bar{b}}$. Thus, if v is sufficiently close to $v_{\bar{b}}$, the v-valuation bidder strictly gains from the cap, implying $v_0 > v_{\bar{b}}$.

We turn to the FPCA. Let $\Phi_{\mathrm{FP}}(v)$ be the difference between the expected payoffs of a v-valuation bidder in the capped and the uncapped FPCA where $v \geqslant \bar{c}$ and $\bar{c}$ is the upwards-pooling bound of $\bar{b}$. Specifically,

$$\begin{aligned}\Phi_{\mathrm{FP}}(v) &= G(\bar{c})u(\bar{b}, v) - F^{n-1}(v)u(\beta_{\mathrm{F}}(v), v) \\ &= G(\bar{c})[S(\bar{b})(v + w - \bar{b}) - w] - F^{n-1}(v)[S(\beta_{\mathrm{F}}(v))(v + w - \beta_{\mathrm{F}}(v)) - w],\end{aligned}$$

and $\Phi'_{\mathrm{FP}}(v) = G(\bar{c})S(\bar{b}) - F^{n-1}(v)S(\beta_{\mathrm{F}}(v))$. In the proof of Proposition 2, we showed that $\Phi_{\mathrm{FP}}(\bar{c}) = 0$ and concluded that $\Phi'_{\mathrm{FP}}(\bar{c}) > 0$. Similarly, for every $v \in (\bar{c}, v_{\bar{b}})$ one can show that $\Phi_{\mathrm{FP}}(v) \geqslant 0$ implies that $\Phi'_{\mathrm{FP}}(v) \geqslant 0$, and every bidder with valuation $v \in (\bar{c}, v_{\bar{b}})$ strictly gains from the introduction of the cap $\bar{b}$. ∎

Claim 1. *In a* SPCA*, assuming that no endowment is needed* $(w = 0)$ *and* F *is convex, every bid cap increases all bidders' expected payoffs.*

Proof. By Lemma 1, it is sufficient to prove that every bid cap $\bar{b} < \overline{V}$ increases the expected payoff of a $\overline{V}$-valuation bidder. The difference between the expected payoff of a $\overline{V}$-valuation bidder in the capped and the uncapped auctions is $\Phi_{\mathrm{SP}}(\overline{V})$, defined in the proof of Lemma 1. We have

$$\begin{aligned}\Phi_{\mathrm{SP}}(\overline{V}) &= (G(\bar{b}) - F^{n-1}(\bar{b}))u(\bar{b}, \overline{V}) - (1 - F^{n-1}(\bar{b}))E[u(\beta_{\mathrm{S}}(\tilde{v}_2), v)|\bar{b} \leqslant \tilde{v}_2 \leqslant \overline{V}] \\ &\geqslant (G(\bar{b}) - F^{n-1}(\bar{b}))S(\bar{b})(\overline{V} - \bar{b}) - S(\bar{b}) \int_{\bar{b}}^{\overline{V}} (n-1)f(t)F^{n-2}(t)(\overline{V} - t)dt,\end{aligned}$$

where the inequality follows from the assumption that S is decreasing. Dividing by $S(\bar{b})$, where $S(\bar{b}) > S(\overline{V}) \geqslant 0$, and integrating by parts yields

$$\begin{aligned} \frac{\Phi_{\mathrm{SP}}(\overline{V})}{S(\bar{b})} &\geqslant (G(\bar{b}) - F^{n-1}(\bar{b}))(\overline{V} - \bar{b}) + F^{n-1}(\bar{b})(\overline{V} - \bar{b}) - \int_{\bar{b}}^{\overline{V}} F^{n-1}(t)dt \\ &= \frac{(\overline{V} - \bar{b})}{n} \sum_{k=0}^{n-1} F^k(\bar{b}) - \int_{\bar{b}}^{\overline{V}} F^{n-1}(t)dt. \end{aligned}$$

Now define $\phi(\bar{b}) = \frac{(\overline{V}-\bar{b})}{n} \sum_{k=0}^{n-1} F^k(\bar{b}) - \int_{\bar{b}}^{\overline{V}} F^{n-1}(t)dt$. Taking the first-order derivative of ϕ yields

$$\begin{aligned} \phi'(\bar{b}) &= \frac{(\overline{V} - \bar{b})}{n} \sum_{k=0}^{n-1} kF^{k-1}(\bar{b}) f(\bar{b}) - \frac{1}{n} \sum_{k=0}^{n-1} F^k(\bar{b}) + F^{n-1}(\bar{b}) \\ &\leqslant \frac{1}{n} \sum_{k=0}^{n-1} kF^{k-1}(\bar{b})(1 - F(\bar{b})) - \frac{1}{n} \sum_{k=0}^{n-1} F^k(\bar{b}) + F^{n-1}(\bar{b}) \\ &= \frac{1}{n} \left[\sum_{k=0}^{n-1} \Big(kF^{k-1}(\bar{b}) - (k+1)F^k(\bar{b}) \Big) \right] + F^{n-1}(\bar{b}), \end{aligned}$$

where the inequality follows from the convexity of F, which implies that $\frac{1-F(\bar{b})}{\overline{V}-\bar{b}} \geqslant f(\bar{b})$. Note that the sum is telescoping and yields $\phi'(\bar{b}) \leqslant 0$. Therefore, $\phi(\bar{b})$ is a decreasing function and $\phi(\overline{V}) = 0$ implies that $\phi(\bar{b}) \geqslant 0$ for $\bar{b} < \overline{V}$, as needed. ∎