SEQUENTIAL CONTESTS WITH FIRST AND SECONDARY PRIZES

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Sequential contests with first and secondary prizes

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Abstract

We study a sequential two-stage Tullock contest with two asymmetric players. The players compete for two prizes; the player with the highest effort in the first stage wins the secondary prize while the player with the highest total effort in both stages wins the first prize. Both players have the same cost functions where the marginal cost in the first stage is higher than in the second one. We analyze the subgame perfect equilibrium of this contest and reveal a paradoxical behavior such that the players' utilities increase in their marginal effort cost.

Keywords: Multi-stage contests, multi-prize contests, variable costs.

JEL classification: C70, D82, D44

1 Introduction

A war might have several battles in which the winner is not necessarily the winner of the war. A war with several battles is an example of a multi-stage contest in which one of the contestants wins the first (main) prize at the end of the contest but each of the other contestants including the winner of the first prize may win secondary prizes during the contest. We can find several such real-life contests with secondary prizes. A well-known example of a contest with secondary prizes is the Tour de France which is an annual multi-stage bicycle race. In this contest, the rider with the lowest aggregate time over all the stages wins the first prize (the prize for the general classification). However, the rider who wins the race containing climbs wins a secondary prize (the prize for the mountain classification) and there are also other secondary prizes (the prizes for the minor classifications). Another example is a political race or an election in which a party

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member competes to be the party's candidate for head of government (first prize) and also competes to be elected together with his party as part of the government

In this paper, we study a two-stage Tullock contest with two contestants in which a contestant who exerts a higher total effort over the two stages has a higher probability to win the first prize, while a contestant who exerts a higher effort in the first stage only, has a higher probability to win the secondary prize.[1] In order that the contestants' decision about the effort allocation over the two stages be non-trivial we assume that the contestants' marginal effort cost in the first stage is higher than in the second one.[2] Note that if the relation between the contestants' effort cost in both stage is lower in the first stage than in the second one, then both contestants will allocate efforts in the first stage only since an effort in the first stage yields winning the first and secondary prizes while an effort in the second stage yields winning the first prize only.[3]

We assume that the contestants are asymmetric, namely, they have different values of winning for different secondary prizes, but they have the same value for the first prize. In that case, we have three forms of a subgame perfect equilibrium: 1. Both contestants are active in both stages, 2. both contestants are active in the first stage, 3. one contestant is active in both stages and the other is active in the first one only. We study only the two cases in which both contestants are active in both stages or both contestants are active in the first one, since the analysis of these cases is sufficient for arriving at conclusions. We obtain that the contestants' expected payoffs are not the same when they are both active in both stages as when they are both active in the first stage only. We compare these expected payoffs and find a paradoxical result according to which both contestants' expected payoffs are higher when they are active in both stages than when they are active in the first stage only although their marginal effort costs are higher when they both are active in both stages. In other words, both contestants' expected payoffs are higher when they have higher (marginal) effort costs.

[1] The present paper shows only one way of allocating two asymmetric prizes in a Tullock contest. In the literature, there are several ways to allocate k prizes in such a contest (see, for example, Berry 1993, and Clark and Riis 1996, 1998).

[2] There is much evidence that, as in our model, contestants strategically allocate their resources in multi-stage contests (see, for example, Harbaugh and Klump 2005, Amegashie et al. 2007, and Sela and Erez 2013).

[3] In our two-stage contest there is a synergy between the stages since the effort of the first stage affects winning the prize awarded in the first stage as well as the one awarded in the second stage. The literature suggests other reasons for the occurrence of synergy in multi-stage contests (see, for example, Kovenock and Roberson 2009, Ryvkin 2011, and Sela 2017).

2 The model

Consider a two-stage Tullock contest with two contestants, 1 and 2 (see Tullock 1980). In the first stage contestant i's value for the secondary prize is v_i. If both contestants exert efforts x_1, x_2 in the first stage, then the contestants' probabilities of winning are $p_1 = \frac{(x_1)^r}{(x_1)^r+(x_2)^r}, 0 < r < 2$ and $p_2 = 1 - p_1$. The cost of effort x_i for contestant i in the first stage is $c(x_i) = \beta x_i, \beta > 1$. Contestants 1 and 2 compete against each other also in the second stage where both contestants have the same value $w, w > v_i, i = 1,2$ for the first prize. If both contestants exert efforts of y_1, y_2 in the second stage then the contestants' probabilities of winning are $p_1 = \frac{(x_1+y_1)^r}{(x_1+y_1)^r+(x_2+y_2)^r}$ and $p_2 = 1 - p_1$ where x_1, x_2 are the contestants' efforts in the first stage. The cost of effort y_i for contestant i in the second stage is normalized to be $c(y_i) = y_i$.

2.1 Case A: The contestants are active in both stages

2.1.1 The second stage

The maximization problem of contestant 1 in the second stage is

$$\max_{y_1} w \frac{(x_1+y_1)^r}{(x_1+y_1)^r + (x_2+y_2)^r} - y_1$$

Similarly, the maximization problem of contestant 2 is

$$\max_{y_1} w \frac{(x_2+y_2)^r}{(x_1+y_1)^r + (x_2+y_2)^r} - y_2$$

The F.O.C. are[4]

$$w \frac{r(x_1+y_1)^{r-1}(x_2+y_2)^r}{((x_1+y_1)^r + (x_2+y_2)^r)^2} = 1 \tag{1}$$
$$w \frac{r(x_1+y_1)^{r}(x_2+y_2)^{r-1}}{((x_1+y_1)^r + (x_2+y_2)^r)^2} = 1$$

By symmetry of the contestants in the second stage, we have, $x_1 + y_1 = x_2 + y_2$, and then from (1) we obtain

$$\frac{wr}{4(x_2+y_2)} = 1$$

Thus, the contestants' equilibrium strategies in the second stage are

$$y_i = \frac{wr}{4} - x_i \ , \ i = 1,2$$

The necessary conditions that both contestants exert efforts in the second stage are $x_i < \frac{wr}{4}, i = 1,2$. Then, we obtain that

$$x_i + y_i = \frac{wr}{4}$$

[4] For all the maximization problems, the S.O.C. is satisfied for the same values of r as in the standard Tullock contest.

The contestants' expected payoffs in the second stage are

$$u_i(x_i) = \frac{w(2-r)}{4} + x_i \ , \ i = 1, 2$$

2.1.2 The first stage

The maximization problem of contestant 1 in the first stage is

$$\begin{aligned} & \max_{x_1} v_1 \frac{(x_1)^r}{(x_1)^r + (x_2)^r} + u_1(x_1) - \beta x_1 \\ = \; & \max_{x_1} v_1 \frac{(x_1)^r}{(x_1)^r + (x_2)^r} + \frac{w(2-r)}{4} + x_1 - \beta x_1 \end{aligned}$$

and the maximization problem of contestant 2 is

$$\max_{x_2} v_2 \frac{(x_2)^r}{(x_1)^r + (x_2)^r} + \frac{w(2-r)}{4} + x_2 - \beta x_2$$

The F.O.C. are

$$\begin{aligned} v_1 \frac{r(x_1)^{r-1}(x_2)^r}{((x_1)^r + (x_2)^r)^2} &= \beta - 1 \\ v_2 \frac{r(x_1)^r(x_2)^{r-1}}{((x_1)^r + (x_2)^r)^2} &= \beta - 1 \end{aligned}$$

Thus, the contestants' equilibrium strategies in the first stage are

$$\begin{aligned} x_1 &= \frac{(v_1)^{r+1}(v_2)^r}{(\beta - 1)((v_1)^r + (v_2)^r)^2} \\ x_2 &= \frac{(v_1)^r(v_2)^{r+1}}{(\beta - 1)((v_1)^r + (v_2)^r)^2} \end{aligned}$$

The necessary and sufficient conditions that $y_i > 0, i = 1, 2$ are

$$\begin{aligned} x_1 &= \frac{(v_1)^{r+1}(v_2)^r}{(\beta - 1)((v_1)^r + (v_2)^r)^2} < \frac{wr}{4} \\ x_2 &= \frac{(v_1)^r(v_2)^{r+1}}{(\beta - 1)((v_1)^r + (v_2)^r)^2} < \frac{wr}{4} \end{aligned}$$

Thus, the subgame perfect equilibrium in which both contestants exert efforts in both stages exists if

$$\beta > \max\{1 + \frac{4(v_1)^{r+1}(v_2)^r}{wr((v_1)^r + (v_2)^r)^2}, 1 + \frac{4(v_1)^r(v_2)^{r+1}}{wr((v_1)^r + (v_2)^r)^2}\} \tag{2}$$

The contestants' expected payoffs are then

$$\begin{aligned} \pi_1^A &= \frac{(v_1)^{2r+1} + (1-r)(v_1)^{r+1}(v_2)^r}{((v_1)^r + (v_2)^r)^2} + \frac{w(2-r)}{4} \\ \pi_2^A &= \frac{(v_2)^{2r+1} + (1-r)(v_2)^{r+1}(v_1)^r}{((v_1)^r + (v_2)^r)^2} + \frac{w(2-r)}{4} \end{aligned} \tag{3}$$

Proposition 1 *In the asymmetric two-stage Tullock contest if the ratio of the contestants' marginal effort costs in the first and second stages is larger than or equal to* $\max\{1+\frac{4(v_1)^{r+1}(v_2)^r}{wr((v_1)^r+(v_2)^r)^2}, 1+\frac{4(v_1)^r(v_2)^{r+1}}{wr((v_1)^r+(v_2)^r)^2}\}$, *there is a subgame perfect equilibrium in which both contestants are active in both stages.*

In the subgame perfect equilibrium of the asymmetric two-stage Tullock contest when both contestants are active in both stages their expected payoffs do not depend on the ratio of the contestants' marginal effort costs, and is equal to their total expected payoffs in the two independent one-stage Tullock contests: one with the symmetric first prize and the second with the asymmetric secondary ones.

2.2 Case B: The contestants are active only in the first stage

2.2.1 The second stage

In this case, both contestants are not active in the second stage, namely, $y_i = 0, i = 1, 2$. In the following we find the necessary condition that contestant 1 does not want to exert a positive effort in the second stage given that contestant 2 exerts an effort only in the first stage.

The maximization problem of contestant 1 in the second stage when contestant 2 does not exert any effort in that stage is

$$\max_{y_1} w\frac{(x_1+y_1)^r}{(x_2)^r+(x_1+y_1)^r} - y_1$$

The F.O.C. is

$$w\frac{r(x_1+y_1)^{r-1}(x_2)^r}{((x_2)^r+(x_1+y_1)^r)^2} - 1 \leq 0$$

Similarly, the maximization problem of contestant 2 is

$$\max_{y_2} w\frac{(x_2+y_2)^r}{(x_1)^r+(x_2+y_2)^r} - y_1$$

and the F.O.C. is

$$w\frac{r(x_2+y_2)^{r-1}(x_1)^r}{((x_1)^r+(x_2+y_2)^r)^2} - 1 \leq 0$$

Thus, by symmetry of the contestants in the second stage, we have $x_1 + y_1 = x_2 + y_2$. Note that

$$\begin{aligned}
& \frac{d}{dy_1}(w\frac{r(x_1+y_1)^{r-1}(x_2)^r}{((x_2)^r+(x_1+y_1)^r)^2} - 1) \\
= \; & \frac{r(r-1)(x_1+y_1)^{r-2}(x_2)^r((x_2)^r+(x_1+y_1)^r)^2 - 2((x_2)^r+(x_1+y_1)^r)r^2(x_1+y_1)^{2r-2}(x_2)^r}{((x_2)^r+(x_1+y_1)^r)^4} \\
= \; & \frac{r(x_1+y_1)^{r-2}(x_2)^r((x_2)^r+(x_1+y_1)^r)((r-1)(x_2)^r+(r-1)(x_1+y_1)^r - 2r(x_1+y_1)^r}{((x_2)^r+(x_1+y_1)^r)^4} \\
= \; & \frac{r(x_1+y_1)^{r-2}(x_2)^r((x_2)^r+(x_1+y_1)^r)((r-1)((x_2)^r-(r+1)(x_1+y_1)^r)}{((x_2)^r+(x_1+y_1)^r)^4} < 0
\end{aligned}$$

The last inequality holds since $(x_2) < (x_1 + y_1)$. Therefore, we obtain that

Lemma 1 *In the asymmetric two-stage Tullock contest, sufficient conditions that both contestants do not exert efforts in the second stage are*

$$\begin{aligned} w\frac{r(x_1)^{r-1}(x_2)^r}{((x_2)^r + (x_1)^r)^2} - 1 &\leq 0 \\ w\frac{r(x_2)^{r-1}(x_1)^r}{((x_2)^r + (x_1)^r)^2} - 1 &\leq 0 \end{aligned}$$

where x_1 and x_2 are the contestants' equilibrium efforts in the first stage.

2.2.2 The first stage

Given that the contestants are active in the first stage only, the maximization problem of contestant 1 in the first stage is

$$\max_{x_1}(v_1 + w)\frac{(x_1)^r}{(x_1)^r + (x_2)^r} - \beta x_1$$

Similarly, the maximization problem of contestant 2 is

$$\max_{x_1}(v_2 + w)\frac{(x_2)^r}{(x_1)^r + (x_2)^r} - \beta x_2$$

The F.O.C. are

$$\begin{aligned} (v_1 + w)\frac{r(x_1)^{r-1}(x_2)^r}{((x_1)^r + (x_2)^r)^2} &= \beta \\ (v_2 + w)\frac{r(x_2)^{r-1}(x_1)^r}{((x_1)^r + (x_2)^r)^2} &= \beta \end{aligned}$$

Thus, the contestants' equilibrium strategies in the first stage are

$$\begin{aligned} x_1 &= \frac{r(v_1 + w)^{r+1}(v_2 + w)^r}{\beta((v_1 + w)^r + (v_2 + w)^r)^2} \\ x_2 &= \frac{r(v_2 + w)^{r+1}(v_1 + w)^r}{\beta((v_1 + w)^r + (v_2 + w)^r)^2} \end{aligned}$$

The sufficient conditions that $y_i = 0, i = 1, 2$ in the second stage are

$$\begin{aligned} w\frac{r(x_1)^{r-1}(x_2)^r}{((x_2)^r + (x_1)^r)^2} - 1 &= wr\frac{(\frac{r(v_1+w)^{r+1}(v_2+w)^r}{\beta((v_1+w)^r+(v_2+w)^r)^2})^{r-1}(\frac{r(v_2+w)^{r+1}(v_1+w)^r}{\beta((v_1+w)^r+(v_2+w)^r)^2})^r}{((\frac{r(v_1+w)^{r+1}(v_2+w)^r}{\beta((v_1+w)^r+(v_2+w)^r)^2})^r + (\frac{r(v_2+w)^{r+1}(v_1+w)^r}{\beta((v_1+w)^r+(v_2+w)^r)^2})^r)^2} - 1 \\ &= \frac{w\beta}{(v_1 + w)} - 1 \leq 0 \end{aligned}$$

and

$$\begin{aligned} w\frac{r(x_1)^r(x_2)^{r-1}}{((x_2)^r + (x_1)^r)^2} - 1 &= wr\frac{(\frac{r(v_1+w)^{r+1}(v_2+w)^r}{\beta((v_1+w)^r+(v_2+w)^r)^2})^r(\frac{r(v_2+w)^{r+1}(v_1+w)^r}{\beta((v_1+w)^r+(v_2+w)^r)^2})^{r-1}}{((\frac{r(v_1+w)^{r+1}(v_2+w)^r}{\beta((v_1+w)^r+(v_2+w)^r)^2})^r + (\frac{r(v_2+w)^{r+1}(v_1+w)^r}{\beta((v_1+w)^r+(v_2+w)^r)^2})^r)^2} - 1 \\ &= \frac{w\beta}{(v_2 + w)} - 1 \leq 0 \end{aligned}$$

Thus, a subgame perfect equilibrium in which both contestants exert effort in the first stage only satisfies iff

$$\beta \leq \min\{\frac{v_2 + w}{w}, \frac{v_1 + w}{w}\} \tag{4}$$

The contestants' expected payoffs are

$$\begin{aligned} \pi_1^B &= \frac{(v_1 + w)^{r+1}((v_1 + w)^r + (v_2 + w)^r(1 - r))}{((v_1 + w)^r + (v_2 + w)^r)^2} \\ \pi_2^B &= \frac{(v_2 + w)^{r+1}((v_2 + w)^r + (v_1 + w)^r(1 - r))}{((v_1 + w)^r + (v_2 + w)^r)^2} \end{aligned} \tag{5}$$

Proposition 2 *In the asymmetric two-stage Tullock contest if the ratio of the contestants' marginal effort costs in the first and second stages is smaller than or equal to* $\min\{\frac{v_2+w}{w}, \frac{v_1+w}{w}\}$*, there is a subgame perfect equilibrium in which both contestants are active in the first stage only.*

In the subgame perfect equilibrium of the asymmetric two-stage Tullock contest when both contestants are active in the first stage only, their expected payoffs do not depend on the ratio of the marginal costs in both stages, and is the same as their expected payoff in the one-stage Tullock contest with a prize that is equal to the sum of the first and the secondary prizes.

3 A comparison of the contestants' expected payoffs

If the ratio of the contestants' marginal effort costs in the first and second stages satisfies $\beta > \max\{\beta_1^A, \beta_2^A\} = \max\{1 + \frac{4v_1^2 v_2}{w(v_1+v_2)^2}, 1 + \frac{4v_2^2 v_1}{w(v_1+v_2)^2}\}$, both contestants are active in both stages, and if the ratio of their marginal effort costs in both stages satisfies $\beta \leq \min\{\beta_1^B, \beta_2^B\} = \min\{\frac{v_1+w}{w}, \frac{v_2+w}{w}\}$ both contestants are active in the first stage only. Assume without loss of generality that $v_1 > v_2$. Then,

$$\begin{aligned} \beta_1^A - \beta_1^B &= (1 + \frac{v_1}{w}) - (1 + \frac{4v_1^2 v_2}{w(v_1 + v_2)^2}) = \frac{v_1}{w}(1 - \frac{4v_1 v_2}{(v_1 + v_2)^2}) \\ &= \frac{v_1}{w}(\frac{(v_1 + v_2)^2 - 4v_1 v_2}{(v_1 + v_2)^2}) = \frac{v_1}{w}(\frac{(v_1 - v_2)^2}{(v_1 + v_2)^2}) > 0 \end{aligned}$$

and

$$\begin{aligned} \beta_2^A - \beta_2^B &= (1 + \frac{v_2}{w}) - (1 + \frac{4v_1^2 v_2}{w(v_1 + v_2)^2}) = \frac{v_2}{w}(1 - \frac{4v_1^2}{(v_1 + v_2)^2}) = \\ &= \frac{v_2}{w}(\frac{(v_1 + v_2)^2 - 4v_1^2}{(v_1 + v_2)^2}) < 0 \end{aligned}$$

Thus, we obtain that the order of the critical values of the marginal effort costs is $\beta_1^A > \beta_1^B > \beta_2^A > \beta_2^B$. Hence, when contestants are asymmetric such that $v_1 > v_2$, if the ratio of the contestants' marginal effort costs in the first and second stages satisfies $\beta > \beta_1^A$ then both contestants are active in both stages and their expected payoffs are given by (3). If, on the other hand, the ratio of the contestants' marginal effort costs satisfies $\beta \leq \beta_2^B$ then both contestants are active in the first stage only and their expected payoffs are given by (5). By comparing the contestants' expected payoffs when both contestants are active in both stages, π_1^A, π_2^A, and when they are active in the first stage only, π_1^B, π_2^B, we obtain that

$$\begin{aligned}
\pi_1^A - \pi_1^B &= (\frac{v_1^3}{(v_1+v_2)^2} + \frac{w}{4}) - \frac{(v_1+w)^3}{(v_1+v_2+2w)^2} \\
&= \frac{1}{4}w\frac{(v_1-v_2)^2}{(v_1+v_2)^2(2w+v_1+v_2)^2}\left(5v_1^2+6v_1v_2+8wv_1+v_2^2+4wv_2\right) > 0
\end{aligned}$$

Similarly,

$$\begin{aligned}
\pi_2^A - \pi_2^B &= (\frac{v_2^3}{(v_1+v_2)^2} + \frac{w}{4}) - \frac{(v_2+w)^3}{(v_1+v_2+2w)^2} \\
&= \frac{1}{4}w\frac{(v_1-v_2)^2}{(v_1+v_2)^2(2w+v_1+v_2)^2}\left(v_1^2+6v_1v_2+4wv_1+5v_2^2+8wv_2\right) > 0
\end{aligned}$$

Since $\pi_i^A > \pi_i^B, i = 1, 2$, we have

Proposition 3 *In the asymmetric two-stage Tullock contest the contestants' expected payoffs depend on the value of the marginal effort cost. Moreover, the contestants have higher expected payoffs with a higher marginal effort cost ($\beta > \beta_1^A$) than with a lower marginal effort cost ($\beta < \beta_2^B$).*

We can now explain the above paradox that occurs in our two-stage Tullock contest according to which higher effort costs yield higher expected payoffs for the contestants as follows. By Propositions 1 and 2, when the ratio of the contestants' marginal effort costs in both stages is relatively small, the contestants actually compete in a one-stage contest in which the winner wins a prize that is equal to the sum of the first and the secondary prizes. On the other hand, when the ratio of the contestants' marginal effort costs is relatively high, the contestants actually compete in two separate contests, one with the first prize and the second with the secondary one. In that case, it could be that one contestant wins both prizes but also that each of the contestants wins one of the prizes. Then, according to Proposition 3, asymmetric contestants might prefer to compete for several prizes separately and not compete for a single prize in a winner-take-all contest although the value of the single prize in the winner-take-all contest is higher than the sum of the prizes in the separate two contests.

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