THE THIRD PLACE GAME

Netanel Nissim and Aner Sela

Discussion Paper No. 17-09

November 2017

Monaster Center for
Economic Research
Ben-Gurion University of the Negev
P.O. Box 653
Beer Sheva, Israel

Fax: 972-8-6472941
Tel: 972-8-6472286

The Third Place Game

Netanel Nissim* and Aner Sela†

September 11, 2017

Abstract

We study an elimination tournament with four contestants, each of whom has either a high value of winning (a strong player) or a low value of winning (a weak player) and these values are common-knowledge. Each pair-wise match is modelled as an all-pay auction. The winners of the first stage (semifinal) compete in the second stage (final) for the first prize, while the losers of the first stage compete for the third prize. We examine whether or not the game for the third prize is profitable for the designer who wishes to maximize the total effort of the players. We demonstrate that if there are at least two strong players, there is always a seeding of the players such that the third place game is not profitable. On the other hand, if there are at least two weak players, then there is always a seeding of the players such that the third place game becomes profitable.

JEL Classification Numbers: D72, D82, D44.

Keywords: All-pay auctions, elimination tournaments, third place games.

1 Introduction

Prizes have a key role in contests as they provide the incentive for players to participate and exert efforts. Therefore, during the last decades, the contest literature has focused on the optimal prize structure. The main questions that have been raised include, when is a single prize optimal, and more generally, what is the

*Department of Economics, Ben-Gurion University of the Negev, Beer–Sheva 84105, Israel. Email: nissimn@post.bgu.ac.il

†Corresponding author: Department of Economics, Ben-Gurion University of the Negev, Beer–Sheva 84105, Israel. Email: anersela@bgu.ac.il

optimal allocation of prizes given that the prize sum is constrained? Clark and Riis (1996, 1998) found that in a symmetric Tullock contest with multiple prizes and linear cost functions, the contestants' total effort is maximized when only one prize is awarded. Schweinzer and Segev (2009) demonstrated that the latter result generalizes for non-linear cost functions; that is, in the symmetric Tullock with non-linear cost functions, the contestants' total effort is maximized when only one prize is allocated. In the all-pay contest under complete information, Barut and Kovenock (1998) found that if there are n players who are symmetric, then any allocation of the entire prize is optimal. However, in the all-pay contest under incomplete information, Moldovanu and Sela (2001) showed that when cost functions are linear or concave in effort it is optimal to allocate the entire prize sum to a single first prize (a prize for winning), but when cost functions are convex, several positive prizes may be optimal. In two-stage all-pay contests under incomplete information, Moldovanu and Sela (2006) showed that if the cost functions are linear in effort, it is optimal for a contest designer who maximizes the expected total effort to allocate a single first prize in the last (second) stage. Fu and Lu (2012) studied multi-stage sequential elimination Tullock contests and showed that the optimal contest eliminates one contestant at each stage until the final. Then, the winner of the final takes the entire prize sum.

In this paper we investigate the allocation of prizes in elimination tournaments in which teams or individual players play pair-wise matches, and the winner advances to the next round while the loser is eliminated from the competition. Many sportive events are organized in such a way. Examples include the ATP tennis tournaments; professional playoffs in US-basketball, football, baseball and hockey; NCAA college basketball; the FIFA (soccer) world-championship playoffs; the UEFA champions' league; Olympic disciplines such as fencing, boxing and wrestling; and top-level bridge and chess tournaments.[1] In elimination tournaments there is a third place game which is a single match to decide which competitor or team will be credited with finishing third. The teams that compete in the third place game are usually the two losing semi-finalists. Not all sports tournaments consider third place games to be of value, but others still use them. For example, FIFA world cup includes a third place game as well as do most elimination tournaments in the Olympic Games who use it for determining who wins the bronze medal. Our goal is to examine the contribution of

[1]Such elimination tournaments are, for example, also used within firms for promotions or budgeting decisions, and by committees who need to choose among several alternatives.

the third place game to the players' expected total effort in elimination tournaments and as such to decide whether the third place game is worthwhile or superfluous.

The elimination tournament was first studied in the statistical literature. The pioneering paper of David (1959) considered the winning probability of the top player in a four-player tournament with a random seeding (see also Glenn (1960) and Searles (1963) for early contributions). Most works in this literature suggest formulas for computing overall probabilities with which various players will win the tournament (see Horen and Reizman (1985) who consider general, fixed win probabilities and analyze tournaments with four and eight players) while others (see, for example, Hwang (1982), Horen and Reizman (1985) and Schwenk (2000)) consider various optimality criteria for choosing seedings.[2] These works assume that for each game among players i and j there is a fixed, exogenously given probability that i beats j. This probability does not depend on the stage of the tournament in which the particular game takes place nor on the identity of the expected opponent at the next stage. In contrast as opposed to the statistical literature, in the elimination tournaments studied in the economic literature the winning probabilities in each game become endogenous in that they result from mixed equilibrium strategies and are dependent on continuation values of winning. Moreover, the win probabilities depend on the stage of the tournament in which the game takes place as well as on the identity of the future expected opponents. For example, Rosen (1986) and Krakel (2014) studied an elimination tournament in which the probability of winning a match is a stochastic function of the players' efforts, Gradstein and Konrad (1999) and Stracke et al. (2014) studied an elimination tournament where players are matched in the Tullock contest, and Groh et al. (2012) studied an elimination tournament where players are matched in the all-pay auction in each of the stages.

We consider the elimination tournament model studied by Groh et al. (2012) in which four players are matched in the all-pay auction in each stage.[3] Each of the players is either strong (has a high value of winning) or weak (has a low value of winning) where the players' types are commonly known. In the first

[2] There are many possible seedings in an elimination tournament. In a tournament with 2^N players, there are $\frac{(2^N)!}{2^{(2^N-1)}}$ different seedings. This yields 3 seedings for 4 players, 315 seedings for 8 players, 638,512,875 seedings for 16 players and 1. 2253×10^{26} seedings for 32 players.

[3] The all-pay auction under complete information has been studied, among others, by Hilman and Samet (1987), Hilman and Riley (1989), Baye et al. (1993, 1996) and Sela (2012).

stage, two pairs of players simultaneously compete in two semifinals. The two winners (one in each semifinal) compete in the final, and the winner of the final obtains the first prize while the loser of the final obtains the second prize. The losers of the semifinals compete in the third place game for the third prize.

We show that the third prize has two opposite effects on the players' expected total effort. On the one hand, the third place game is an additional game in which the players exert effort and as such the expected total effort in the tournament increases. On the other hand, the players' expected values of winning in the semifinal are the differences of their expected payoffs in the final and in the third place. Therefore, the third place game decreases the players' expected values in the semifinals and as such decreases their expected efforts in that stage. When the players are symmetric, namely, they have the same type, either strong or weak, their efforts in the semifinals are relatively small since their expected payoffs in the final are small too and then the positive effect of the third place game on the expected total effort is higher than its negative effect. As such, it is obvious that the designer who wishes to maximize the expected total effort in the elimination tournament should consider a third place game. Consequently, we will assume that the players are asymmetric and that the ratio of their types (strong\weak) is significant. In that case, whether or not the third place game has a positive contribution to the players' expected total effort is not at all clear.

We tackle this issue by first assuming that there is one strong player (a dominant player) and three weak players, and show that if the first prize is sufficiently larger than the other prizes, then the existence of the third place game increases the players' expected total effort in the tournament. When we assume, however, that there is one weak player (an inferior player) and three strong players, we show that if the inferior player's value of winning is sufficiently small, then the existence of the third place game decreases the players' expected total effort in the tournament.

We consider next the case of two strong players and two weak players for which the seeding of the players in the semifinals plays a key role. When the two strong players as well as the two weak players compete against each other in the semifinals, we find that if the weak players' value of winning is sufficiently small, then the third place game decreases the players' expected total effort. On the other hand, when each strong player competes against a weak player in a semifinal, then if the weak players' values is sufficiently small, then the third place game increases the players' expected total effort in the tournament.

Based on our findings we can conclude that in an elimination tournament with four players, if there are at least two strong players the third place game does not necessarily increase the expected total effort. On the other hand, if there are at least two weak players, by choosing the right seeding of players, the third place game does increase the players' expected total effort. In sum, even if the third prize is an extra prize that does not decrease the values of the higher prizes, it still may not increase the players' total effort in elimination tournaments.

The paper is organized as follows. Section 2 presents the elimination tournament model. In Sections 3 and 4 we analyze the model with one dominant player and with one inferior player respectively. In Section 5 we analyze the model with the same number of weak and strong players. Section 6 concludes.

2 The model

The model consists of four players (or teams) $i = 1, ..., 4$, competing for three different prizes in an elimination tournament. In the first stage, two pairs of players simultaneously compete in two semifinals. In the second stage, the two winners (one in each semifinal) compete in the final, and the winner obtains the first prize while the loser obtains the second prize. The losers of the semifinals then compete in the third place game for the third prize where the prize for the loser is normalized to zero. We model each match among two players as an all-pay auction: both players exert effort, and the one exerting the higher effort wins.

Player $i's$ value for the first prize is v_i; for the second prize it is αv_i, $0 < \alpha \leq 1$; and for the third prize βv_i, $0 < \beta \leq \alpha$. The players' values are common knowledge. We assume that each player's value v_i has two possible types, either strong (v_H) or weak (v_L) where $v_H > v_L$. In the following, we assume that $v_H >> v_L$, namely, the strong player's value is much higher than the weak player's value. The reason is that if the difference between these values is sufficiently small, then, because of the (almost) symmetry, the players do not expect a meaningful payoff in the final and therefore they will exert a negligible effort in the semifinal. As such, the efforts of the players in the third place game will increase their expected total effort. However, if the difference between the players' value is sufficiently large, the players' effort in the semifinal is not negligible and as a result of the third place game it decreases. Thus, it is not clear whether or not the third place game increases or decreases the players' expected total effort.

If in a final, players i and j exert efforts of e_i^F , e_j^F, the payoff for player i is given by

$$u_i^F(e_i^F, e_j^F) = \begin{cases} -e_i^F & \text{if } e_i^F < e_j^F \\ \frac{v_i}{2} - e_i^F & \text{if } e_i^F = e_j^F \\ v_i - e_i^F & \text{if } e_i^F > e_j^F \end{cases} \tag{1}$$

and analogously for player j. In the third place game between players i and j, if they exert efforts of e_i^T, $e_{j,}^T$ the payoff for player i is given by

$$u_i^T(e_i^T, e_j^T) = \begin{cases} -e_i^T & \text{if } e_i^T < e_j^T \\ \frac{\beta v_i}{2} - e_i^T & \text{if } e_i^T = e_j^T \\ \beta v_i - e_i^T & \text{if } e_i^T > e_j^T \end{cases} \tag{2}$$

and, analogously for player j. Player $i's$ payoff in a semifinal between players i and j is given by

$$u_i^S(e_i^S, e_j^S) = \begin{cases} Eu_i^T - e_i^S & \text{if } e_i^S < e_j^S \\ \frac{Eu_i^F + Eu_i^T}{2} - e_i^S & \text{if } e_i^S = e_j^S \\ Eu_i^F - e_i^S & \text{if } e_i^S > e_j^S. \end{cases} \tag{3}$$

and analogously for player j. Note that each player's payoff in a semifinal depends on the expected utility associated with participation in the final (Eu_i^F) as well as in the the third place game (Eu_i^T).

Suppose that players i and j compete in the semifinal and that if player i wins this game, his conditional expected payoff is w_i given the possible opponents in the final. Similarly, if player i loses this game, his conditional expected payoff is l_i. Without loss of generality, we assume that $w_i - l_i \geq w_j - l_j$. Then, according to Baye, Kovenock and de Vries (1996), there is always a unique mixed-strategy equilibrium, $F_i(x), F_j(x)$ in which players i and j randomize on the interval $[0, w_j - l_j]$ according to their effort cumulative distribution functions, which are given by

$$\begin{aligned} w_i F_j(x) + l_i(1 - F_j(x)) - x &= l_j + w_i - w_j \\ w_j F_i(x) + l_j(1 - F_i(x)) - x &= l_j \end{aligned}$$

Thus, player i's equilibrium effort in this game is uniformly distributed; that is

$$F_i(x) = \frac{x}{w_j - l_j} \tag{4}$$

while player j's equilibrium effort is distributed according to the cumulative distribution function

$$F_j(x) = \frac{l_j - l_i + w_i - w_j + x}{w_i - l_i} \tag{5}$$

Player j's probability of winning against player i is then

$$p_{ji} = \frac{w_j - l_j}{2(w_i - l_i)} \tag{6}$$

The players' expected payoffs are

$$\begin{aligned} u_i &= l_j + w_i - w_j \\ u_j &= l_j \end{aligned} \tag{7}$$

and the players' expected total effort is

$$TE = \frac{w_j - l_j}{2}(1 + \frac{w_j - l_j}{w_i - l_i}) \tag{8}$$

3 An elimination tournament with a dominant player

Assume that there is a dominant player such that the players' values for the first prize are $v_H = v_1 >> v_2 = v_3 = v_4 = v_L$. In that case, the seeding of the players in the first stage is irrelevant, and, without loss of generality, we consider the seeding 1-2,3-4; namely, players 1 and 2 compete against each other in one of the semifinals, and players 3 and 4 compete against each other in the other semifinal.

3.1 The players' expected payoffs

The final: If player 1 (the dominant player or the only strong one) wins in the semifinal, he competes against a weak player in the final, and therefore by (7), his expected payoff is

$$EP_1^F = (1 - \alpha)(v_H - v_L) + \alpha v_H = v_H - (1 - \alpha)v_L \tag{9}$$

If player 2 (a weak player) wins in the semifinal, he competes against a weak player in the final, and therefore by (7), his expected payoff is

$$EP_2^F = \alpha v_L \tag{10}$$

If either player 3 or player 4 (both of them are weak players) wins in the semifinal, he competes either against a strong player (player 1) or against a weak one (player 2) in the final, and, in both cases, by (10), his expected payoff is

$$EP_{3,4}^F = \alpha v_L \tag{11}$$

The third place game: If player 1 loses in the semifinal, he competes against a weak player in the third place game, and therefore by (10) his expected payoff is

$$EP_1^T = \beta(v_H - v_L) \tag{12}$$

If player 2 loses in the semifinal, he competes against a weak player in the third place game, and therefore by (10), his expected payoff is

$$EP_2^T = 0 \tag{13}$$

If player 3 (or player 4) loses in the semifinal, he competes either against a strong player (player 1) or against a weak player (player 2) in the third place game, and, in both cases, by (10), his expected payoff is

$$EP_{3,4}^T = 0 \tag{14}$$

The semifinals: Since player 1 will compete against a weak player either in the final or in the third place game, his expected payoff from winning in the semifinal is the difference between his expected payoffs in these events, and by (9) and (12), his expected payoff is

$$EP_1^S = EP_1^F - EP_1^T = (v_H - v_L + \alpha v_L) - \beta(v_H - v_L) \tag{15}$$

Since player 2 will compete against a weak player either in the final or in the third place game, by (10) and (13), his expected payoff is

$$EP_2^S = EP_2^F - EP_2^T = \alpha v_L \tag{16}$$

Player 3 (or player 4) competes in the final against player 1 with probability $q_{1,2}^S$ and then his expected payoff from winning in the semifinal is the difference between his expected payoff in the final when he competes against the dominant player and his expected payoff in the third place game when he competes against player 2. On the other hand, player 3 competes in the final against player 2 with probability $1-q^S_{1,2}$, and then his expected payoff from winning in the semifinal is the difference between his expected payoff in the final when he competes against player 2 and his expected payoff in the third place game when he competes against player 1. Thus, by (11) and (14), the expected payoff of players 3 and 4 in the semifinal is

$$EP^S_{3,4} = q^S_{1,2}(\alpha v_L - 0) + (1-q^S_{1,2})(\alpha v_L - 0) = \alpha v_L$$

where by (6), the probability that player 1 wins against player 2 in the semifinal is given by

$$q^S_{1,2} = 1 - \frac{EP^S_2}{2EP^S_1} = 1 - \frac{\alpha v_L}{2((v_H - v_L + \alpha v_L) - \beta(v_H - v_L))} \tag{17}$$

and by symmetry, $q^S_{3,4}$, the probability that player 3 wins against player 4 in the semifinal, is $q^S_{3,4} = \frac{1}{2}$.

3.2 The players' total efforts

The final: Player 1 competes with probability $q^S_{1,2}$ in the final, and then, by (8), the expected total effort is $\frac{v_L(1-\alpha)}{2}(1+\frac{v_L}{v_H})$. Likewise, player 2 competes with probability $1-q^S_{1,2}$ in the final, and then, by (8), the expected total effort is $v_L - \alpha v_L$. Thus, the expected total effort is

$$TE^F = q^S_{1,2}\frac{v_L(1-\alpha)}{2}(1+\frac{v_L}{v_H}) + (1-q^S_{1,2})(v_L - \alpha v_L)$$

The third place game: Player 1 competes with probability $1-q^S_{1,2}$ in the third place game, and then, by (8), the expected total effort is $\frac{\beta v_L}{2}(1+\frac{v_L}{v_H})$. Likewise, player 2 competes with probability $q^S_{1,2}$ in the third place game, and then, by (8), the expected total effort is βv_L. Thus, the expected total effort is

$$TE^T = q^S_{1,2}\beta v_L + (1-q^S_{1,2})\frac{\beta v_L}{2}(1+\frac{v_L}{v_H})$$

The semifinals: In the semifinal in which player 1 competes against player 2, by (8), (15) and (16), the expected total effort is

$$TE^{S1} = \frac{EP^S_2}{2}(1+\frac{EP^S_2}{EP^S_1}) = \frac{\alpha v_L}{2}(1+\frac{\alpha v_L}{(v_H - v_L + \alpha v_L) - \beta(v_H - v_L)})$$

and in the semifinal in which players 3 and 4 compete against each other, the expected total effort is

$$TE^{S2} = EP^S_{3,4} = \alpha v_L$$

Therefore, if we combine the expected total efforts in all the above stages, we obtain that the expected total effort in the tournament is

$$\begin{aligned} TE &= TE^F + TE^T + TE^{S1} + TE^{S2} = \\ & q^S_{1,2}\frac{v_L(1-\alpha)}{2}(1+\frac{v_L}{v_H}) + (1-q^S_{1,2})(v_L - \alpha v_L) \\ & + q^S_{1,2}\beta v_L + (1-q^S_{1,2})\frac{\beta v_L}{2}(1+\frac{v_L}{v_H}) \\ & + \frac{\alpha v_L}{2}(1+\frac{\alpha v_L}{(v_H - v_L + \alpha v_L) - \beta(v_H - v_L)}) + \alpha v_L \end{aligned} \tag{18}$$

3.3 Results

By (18) we have

$$\frac{dTE(\beta, q^S_{1,2})}{d\beta} = q^S_{1,2}v_L + (1-q^S_{1,2})\frac{v_L}{2}(1+\frac{v_L}{v_H}) + \frac{\alpha v_L}{2}\frac{\alpha v_L(v_H - v_L)}{((v_H - v_L + \alpha v_L) - \beta(v_H - v_L))^2} > 0$$

and

$$\frac{dTE(\beta, q^S_{1,2})}{dq^S_{1,2}} = v_L(1-\alpha-\beta)(\frac{v_L}{2v_H} - \frac{1}{2})$$

Note that if $1-\alpha-\beta > 0$ since $\frac{v_L}{2v_H} - \frac{1}{2} < 0$, then $\frac{dTE(\beta, q^S_{1,2})}{dq^S_{1,2}} < 0$. By (17) we also have

$$\frac{dq^S_{1,2}}{d\beta} = \frac{-\alpha v_L(v_H - v_L)}{4((v_H - v_L + \alpha v_L) - \beta(v_H - v_L))^2} < 0$$

Thus, we obtain that $\frac{dTE(\beta, q^S_{1,2})}{d\beta} + \frac{dTE(\beta, q^S_{1,2})}{dq^S_{1,2}}\frac{dq^S_{1,2}}{d\beta} > 0$. This yields the following result:

Proposition 1 *In an elimination tournament with three weak players and a dominant player who has a higher value of winning, if the first prize is larger than the sum of the other prizes ($\alpha + \beta < 1$), then the third place game increases the players' expected total effort.*

Proposition 1 shows that since the dominant player has a high chance to win the tournament, the players do not exert high efforts in the semifinal, and therefore the increase of the effort there is higher than the decrease of the effort in the semifinals. Thus, the third place game increases the players' total effort in the elimination tournament.

4 An elimination tournament with an inferior player

Assume now that there is an inferior player such that the players' values for the first prize are $v_H = v_1 = v_2 = v_3 >> v_4 = v_L$. In that case, the seeding of the players in the first stage is irrelevant, and without loss of generality, we consider the seeding 1-2,3-4; namely, players 1 and 2 compete against each other in one of the semifinals, and players 3 and 4 compete against each other in the other one.

4.1 The players' expected payoffs

The final: If either player 1 or player 2 (both of them are strong players) wins in the semifinal, he competes with probability $q^S_{3,4}$ against player 3 (a strong player) and with probability $1 - q^S_{3,4}$ against player 4 (the inferior player or the only weak one) in the final. Therefore, by (7), his expected payoff is

$$EP^F_{1,2} = q^S_{3,4}\alpha v_H + (1 - q^S_{3,4})((v_H - \alpha v_H) - (v_L - \alpha v_L) + \alpha v_H) \tag{19}$$

If player 3 wins in the semifinal, he will compete against a strong player in the final, and therefore, by (7), his expected payoff is

$$EP^F_3 = \alpha v_H \tag{20}$$

If player 4 wins in the semifinal, he will compete against a strong player in the final, and therefore, by (7), his expected payoff is

$$EP^F_4 = \alpha v_L \tag{21}$$

The third place game: If player 1 (or player 2) loses in the semifinal, he competes with probability $q^S_{3,4}$ against player 4 in the third place game, and with probability of $q^S_{3,4}$ against player 3 in the third place game. Therefore, by (7), his expected payoff is

$$EP^T_{1,2} = q^S_{3,4}\beta(v_H - v_L) \tag{22}$$

If player 3 loses in the semifinal, he will compete against a strong player in the third place game, and therefore, by (7), his expected payoff is

$$EP^T_3 = 0 \tag{23}$$

If player 4 loses in the semifinal, he will compete against a strong player in the third place game, and therefore, by (7), his expected payoff is

$$EP_4^T = 0 \tag{24}$$

The semifinals: Player 3 wins with probability $q_{3,4}^S$ in the semifinal, and then the expected payoff of player 1 or player 2 is the difference between their expected payoff in the final when they compete against player 3 and their expected payoff when they compete against player 4 in the third place game. On the other hand, player 3 loses with probability $1 - q_{3,4}^S$ in the semifinal, and then the expected payoff of player 1 or player 2 is the difference between their expected payoff when they compete against player 4 in the final and their expected payoff when they compete against player 3 in the third place game. Thus, by (19) and (22), the expected payoff of player 1 and player 2 in the semifinal is

$$EP_{1,2}^S = q_{3,4}^S(\alpha v_H - \beta(v_H - v_L)) + (1 - q_{3,4}^S)((v_H - \alpha v_H) - (v_L - \alpha v_L) + \alpha v_H)$$

Since player 3 will compete against a strong player either in the final or in the third place game, his expected payoff from winning in the semifinal is the difference between his expected payoff in these events, and by (20) and (23), his expected payoff is

$$EP_3^S = EP_3^F - EP_3^T = \alpha v_H \tag{25}$$

Since player 4 will compete against a strong player either in the final or in the third place game, his expected payoff from winning in the semifinal is the difference between his expected payoff in these events, and by (21) and (24), his expected payoff is

$$EP_4^S = EP_4^F - EP_4^T = \alpha v_L \tag{26}$$

where, by (6), the probability that player 3 wins against player 4 in the semifinal is given by

$$q_{3,4}^S = 1 - \frac{v_L}{2v_H}$$

4.2 The players' expected total effort

The final: Player 3 competes with probability $q_{3,4}^S$ in the final, and then, by (8), the expected total effort is $(v_H - \alpha v_H)$. Player 4 competes with probability $1 - q_{3,4}^S$ in the final, and then, by (8), the expected total effort is $\frac{v_L-\alpha v_L}{2}(1+\frac{v_L}{v_H})$. Thus, the expected total effort is

$$TE^F = q^S_{3,4}(v_H - \alpha v_H) + (1-q^S_{3,4})\frac{v_L(1-\alpha)}{2}(1+\frac{v_L}{v_H})$$

The third place game: Player 3 competes with probability $1-q^S_{3,4}$ in the third place game, and then, by (8), the expected total effort is βv_H. Player 4 competes with probability $q^S_{3,4}$ in the third place game, and then, by (8), the expected total effort is$\frac{\beta v_L}{2}(1+\frac{v_L}{v_H})$. Thus, the expected total effort is

$$TE^T = q^S_{3,4}\frac{\beta v_L}{2}(1+\frac{v_L}{v_H}) + (1-q^S_{3,4})\beta v_H$$

The semifinals: In the semifinal in which player 1 competes against player 2, by (8), the expected total effort is

$$TE^{S1} = EP^S_{1,2} = q^S_{3,4}(\alpha v_H - \beta(v_H - v_L)) + (1-q^S_{3,4})((v_H - v_L + \alpha v_L)$$

and in the semifinal in which players 3 and 4 compete against each other, by (8), (25) and (26), the expected total effort is

$$TE^{S2} = \frac{EP^S_4}{2}(1+\frac{EP^S_4}{EP^S_3}) = \frac{\alpha v_L}{2}(1+\frac{v_L}{v_H})$$

Therefore, if we combine the expected total efforts in all the above stages we obtain that the expected total effort in the tournament is

$$\begin{aligned} TE &= TE^F + TE^T + TE^{S1} + TE^{S2} = \\ & (1-\frac{v_L}{2v_H})(v_H - \alpha v_H) + \frac{v_L}{2v_H}\frac{v_L(1-\alpha)}{2}(1+\frac{v_L}{v_H}) \\ & +(1-\frac{v_L}{2v_H})\frac{\beta v_L}{2}(1+\frac{v_L}{v_H}) + \frac{v_L}{2v_H}\beta v_H \\ & +(1-\frac{v_L}{2v_H})(\alpha v_H - \beta(v_H - v_L)) + \frac{v_L}{2v_H}((v_H - v_L + \alpha v_L) \\ & +\frac{\alpha v_L}{2}(1+\frac{v_L}{v_H}) \end{aligned} \tag{27}$$

4.3 Results

By (27) we have

$$\frac{dTE}{d\beta} = (1-\frac{v_L}{2v_H})\frac{v_L}{2}(1+\frac{v_L}{v_H}) + \frac{v_L}{2} - (1-\frac{v_L}{2v_H})(v_H - v_L)$$

Note that

$$\lim_{v_L \to 0} \frac{dTE}{d\beta} = -v_H$$
$$\lim_{v_L \to v_H} \frac{dTE}{d\beta} = v_H$$

This yields the following result:

Proposition 2 *In an elimination tournament with three strong players and one inferior player who has a lower value of winning, if this value is sufficiently small, then the third place game decreases the players' expected total effort.*

Proposition 2 shows that as a result of the third place game the increase of the effort there is lower then the decrease of effort in the semifinals. Therefore, the third place game decreases the players' total effort.

5 A balanced elimination tournament

We assume now that the tournament is balanced, namely, there are two strong and two weak players such that the players' values for the first prize are $v_H = v_1 = v_2 >> v_3 = v_4 = v_L$. In that case, we have two possible seedings of the players in the first stage: 1-2, 3-4 and 1-3,2-4. We begin the analysis with the first seeding, namely, the strong players (players 1 and 2) compete in one semifinal and the weak players (players 3 and 4) compete in the other one.

5.1 The players' expected payoffs (1-2,3-4)

The final: If either player 1 or player 2 (the strong players) wins in the first stage, he will compete against a weak player in the final, and then, by (7), his expected payoff is

$$EP_{1,2}^F = v_H(1-\alpha) - v_L(1-\alpha) + \alpha v_H = v_H - v_L(1-\alpha) \tag{28}$$

If either player 3 or player 4 (the weak players) wins in the semifinal, he will compete against a strong player in the final, and then, by (7), his expected payoff is

$$EP_{3,4}^F = \alpha v_L \tag{29}$$

The third place game: If player 1 (or player 2) loses in the semifinal, he will compete against a weak player in the third place game, and then, by (7), his expected payoff is

$$EP^T_{1,2} = \beta(v_H - v_L) \tag{30}$$

If player 3 (or player 4) loses in the semifinal, he will compete against a strong player in the third place game, and then, by (7), his expected payoff is

$$EP^T_{3,4} = 0 \tag{31}$$

The semifinals: Player 1(or player 2) will compete against a weak player either in the final or in the third place game, and therefore, by (7), his expected payoff is

$$EP^S_{1,2} = (v_H - v_L + \alpha v_L) - \beta(v_H - v_L)$$

Player 3 (or player 4) will compete against a strong player either in the final or in the third place game, and therefore, by (7), his expected payoff is

$$EP^S_{3,4} = \beta v_L$$

5.2 The players' expected total effort (1-2,3-4)

The final: One of the strong players (player 1 or player 2) competes against a weak player (player 3 or player 4) in the final, and therefore, by (8), the expected total effort is

$$TE^F = \frac{v_L(1-\alpha)}{2}(1 + \frac{v_L}{v_H})$$

The third place game: One of the strong players (player 1 or player 2) competes against a weak player (player 3 or player 4) in the third place game, and therefore, by (8), the expected total effort is

$$TE^T = \frac{\beta v_L}{2}(1 + \frac{v_L}{v_H})$$

The semifinals: In the semifinal between players 1 and 2, the expected effort is equal to the difference between these players' expected payoff in the final and in the third place game, and therefore, by (8), (28) and (30), the expected total effort is

$$TE^S_{1,2} = EP^F_{1,2} - EP^T_{1,2} = (v_H - v_L + \alpha v_H) - \beta(v_H - v_L)$$

Similarly, in the semifinal between players 3 and 4, by (8), (29) and (31), the expected total effort is

$$TE_{3,4}^{S} = EP_{3,4}^{F} - EP_{3,4}^{T} = \alpha v_L$$

Therefore, if we combine the expected total efforts in all the above stages, we obtain that the expected total effort in the tournament is

$$\begin{aligned} TE &= TE^F + TE^T + TE_{1,2}^S + TE_{3,4}^S \\ &= \frac{v_L(1-\alpha)}{2}(1+\frac{v_L}{v_H}) + \frac{\beta v_L}{2}(1+\frac{v_L}{v_H}) \\ &\quad +(v_H - v_L + \alpha v_H) - \beta(v_H - v_L) + \alpha v_L \end{aligned} \tag{32}$$

5.3 Results

By (32), we have

$$\frac{dTE}{d\beta} = \frac{3}{2}v_L + \frac{v_L^2}{2v_H} - v_H$$

Note that

$$\begin{aligned} \lim_{v_L \to 0} \frac{dTE}{d\beta} &= -v_H \\ \lim_{v_L \to v_H} \frac{dTE}{d\beta} &= v_H \end{aligned}$$

This yields the following result:

Proposition 3 *In an elimination tournament with two strong players who compete against each other in one of the semifinals and two weak players who compete against each other in the other one, if the weak players' value of winning is sufficiently small, the third place game decreases the players' expected total effort.*

Proposition 3 shows that since the strong players who compete against each other exert high efforts in the semifinal, as a result of the third place game, the increase of the effort there is lower than the decrease of the effort in the semifinals. Therefore, the third place game decreases the players' total effort in the elimination tournament.

5.4 The players' expected payoffs (1-3,2-4)

We assume that there are two strong players and two weak players such that the players' values for the first prize are $v_H = v_1 = v_2 >> v_3 = v_4 = v_L$ and the players' seeding in the first stage is now 1-3,2-4; namely, in each of the semifinals, a strong player competes against a weak one.

The final: If player 2 (player 1) wins in the first stage, he competes with probability $q^S_{1,3}$ ($q^S_{2,4}$) against a strong player in the final, and then, by (7), his expected payoff is αv_H. On the other hand, player 2 (player 1) competes with probability $1 - q^S_{1,3}$ ($1 - q^S_{2,4}$) against a weak player in the final, and then by (7), his expected payoff is $(v_H - \alpha v_H) - (v_L - \alpha v_L) + \alpha v_H$. Thus, the expected payoff of player 2 (player 1) is

$$EP^F_{1,2} = (1 - q^S_{1,3})((v_H - \alpha v_H) - (v_L - \alpha v_L) + \alpha v_H) + q^S_{1,3}\alpha v_H \tag{33}$$

If player 4 (player 3) wins in the semifinal, he competes with probability $1 - q^S_{1,3}$ ($1 - q^S_{2,4}$) against a weak player and with probability of $q^S_{1,3}$ ($q^S_{2,4}$) against a strong player in the final. In both cases, by (7), his expected payoff is αv_L. Thus, the expected payoff of player 4 (player 3) is

$$EP^F_{3,4} = \alpha v_L \tag{34}$$

The third place game: If player 2 (player 1) loses in the semifinal, he competes with probability $q^S_{1,3}$ ($q^S_{2,4}$) against a weak player in the third place game ,and then, by (7), his expected payoff is $\beta(v_H - v_L)$. On the other hand, player 2 (player 1) competes with probability $1 - q^S_{1,3}$ ($1 - q^S_{2,4}$) against a strong player in the third place game, and then, by (7), his expected payoff is zero. Thus, the expected payoff of player 2 (or player 1) is

$$EP^T_{1,2} = q^S_{1,3}\beta(v_H - v_L) \tag{35}$$

If player 4 (player 3) loses in the semifinal, he competes with probability $q^S_{1,3}$ ($q^S_{2,4}$) against a weak player and with probability $1 - q^S_{1,3}$ ($1 - q^S_{2,4}$) against a strong player in the third place game, and in both cases, by (7), his expected payoff is zero. Thus, the expected payoff of player 4 (or player 3) is

$$EP^T_{3,4} = 0 \tag{36}$$

The semifinals: Player 2 (player 1) competes with probability $q^S_{1,3}$ ($q^S_{2,4}$) against a strong player in the final, and then his expected payoff from winning the semifinal is the difference between his expected payoff in the final when he competes against a strong player and his expected payoff in the third place game when he competes against a weak player. Similarly, player 2 (player 1) competes with probability $1-q^S_{1,3}$ $(1-q^S_{2,4})$ against a weak player in the final, and then his expected payoff from winning the semifinal is the difference between his expected payoff in the final when he competes against a weak player and his expected payoff in the third place game when he competes against a strong player. Therefore, by (7), (33) and (35), the expected payoff of players 1 and 2 is

$$EP^S_{1,2} = q^S_{1,3}(\alpha v_H - \beta(v_H - v_L) + (1-q^S_{1,3})\beta(v_H - v_L + \alpha v_L) \tag{37}$$

Player 4 (player 3) competes with probability $q^S_{1,3}$ $(q^S_{2,4})$ against a strong player in the final, and then his expected payoff from winning the semifinal is the difference between his expected payoff in the final when he competes against a strong player and his expected payoff in the third place game when he competes against a weak player. Similarly, player 4 (player 3) competes with probability $1-q^S_{1,3}$ $(1-q^S_{2,4})$ against a weak player in the final, and then his expected payoff from winning the semifinal is the difference between his expected payoff in the final when he competes against a weak player and his expected payoff in the third place game when he competes against a strong player. In both cases, by (7), (34) and (36), player 4's expected payoff is αv_L. Therefore, the expected payoff of players 3 and 4 is

$$EP^S_{3,4} = \alpha v_L \tag{38}$$

By (6), (37) and (38), the probability that player 1 (player 2) wins against player 3 (player 4) in the semifinal is

$$q^S_{1,3} = q^S_{2,4} = 1 - \frac{EP^S_{3,4}}{2EP^S_{1,2}} = 1 - \frac{\alpha v_L}{2(q^S_{1,3}(\alpha v_H - \beta(v_H - v_L) + (1-q^S_{1,3})\beta(v_H - v + \alpha v_L))}$$

The solution of the last equation is

$$q^S_{1,3} = \frac{1}{2((1-\alpha+\beta)(v_L - v_H)}((2+\beta)(v_L - v_H) + \alpha(v_H - 2v_L) + K) \tag{39}$$

where

$$K = \sqrt{v_L^2(-2\alpha + 2\alpha^2 + \beta^2 - 2\alpha\beta) + v_H^2(\alpha-\beta)^2 + v_L v_H(2\alpha + 4\alpha\beta - 2\alpha^2 - 2\beta^2}$$

5.5 The players' expected total effort (1-3,2-4)

The final: The expected total effort in the final depends on the identity of the finalists which is unknown. If the two strong players (players 1 and 2) compete against each other in the final, then, by (8), the expected total effort is $v_H(1-\alpha)$, and if the two weak players (players 3 and 4) compete against each other in the final, then, by (8), the expected total effort is $v_L(1-\alpha)$. On the other hand, if a strong player and a weak player compete against each other in the final, the expected total effort is $\frac{v_L(1-\alpha)}{2}(1+\frac{v_L}{v_H})$. Thus, the expected total effort is

$$\begin{aligned} TE^F &= (q_{1,3}^S)^2 v_H(1-\alpha) + (1-q_{1,3}^S)^2 v_L(1-\alpha) + \\ & 2q_{1,3}^S(1-q_{1,3}^S)\frac{v_L(1-\alpha)}{2}(1+\frac{v_L}{v_H}) \end{aligned}$$

The third place game: The expected total effort in the third place game as well as in the final depends on the identity of the finalists, which is unknown. If the two strong players (players 1 and 2) compete against each other in the third place game, then, by (8), the expected total effort is βv_H, and if the two weak players (players 3 and 4) compete against each other in the third place game, then, by (8), the expected total effort is βv_L. On the other hand, if a strong and a weak player compete against each other in the third place game, the expected total effort is $\frac{\beta v_L}{2}(1+\frac{v_L}{v_H})$. Thus, the expected total effort is

$$TE^T = (q_{1,3}^S)^2 \beta v_L + (1-q_{1,3}^S)^2 \beta v_H + 2q_{1,3}^S(1-q_{1,3}^S)\frac{\beta v_L}{2}(1+\frac{v_L}{v_H})$$

The semifinals: In both semifinals, a strong player competes against a weak player, and therefore, by (8), the expected total efforts are

$$TE_{1,3}^S = TE_{2,3}^S = \frac{EP_{3,4}^S}{2}(1+\frac{EP_{3,4}^S}{EP_{1,2}^S})$$

By (37) and (38), we obtain that

$$TE_{1,3}^S = TE_{2,4}^S = \frac{\alpha v_L}{2}(1+\frac{\alpha v_L}{q_{1,3}^S(\alpha v_H - \beta(v_H - v_L) + (1-q_{1,3}^S)\beta(v_H - v + \alpha v_L)})$$

Therefore, if we combine the expected total efforts in all the above stages, we obtain that the expected total effort in the tournament is

$$
\begin{aligned}
TE &= TE^F + TE^T + TE^S_{1,3} + TE^S_{2,4} \qquad (40)\\
&= (q^S_{1,3})^2 v_H(1-\alpha) + (1-q^S_{1,3})^2 v_L(1-\alpha) + \\
&\quad 2q^S_{1,3}(1-q^S_{1,3})\frac{v_L(1-\alpha)}{2}(1+\frac{v_L}{v_H}) + \\
&\quad (q^S_{1,3})^2\beta v_L + (1-q^S_{1,3})^2\beta v_H + 2q^S_{1,3}(1-q^S_{1,3})\frac{\beta v_L}{2}(1+\frac{v_L}{v_H}) \\
&\quad \alpha v_L(1+\frac{\alpha v_L}{q^S_{1,3}(\alpha v_H - \beta(v_H-v_L) + (1-q^S_{1,3})(v_H - v + \alpha v_L)})
\end{aligned}
$$

5.6 Results (1-3,2-4)

By (39), we have

$$
\begin{aligned}
\frac{dq^S_{1,3}(\beta)}{d\beta} &= \frac{1}{2(v_L-v_H)(1+\alpha-\beta)^2}(v_H - v_L(1-\alpha) + v_L v_H(2\alpha - 4\beta + 4\alpha\beta - 2\alpha^2 - 2\beta^2) \\
&\quad + \frac{1}{2(v_L-v_H)(1+\alpha-\beta)^2 Q}(v_H^2(\beta-\alpha) + v_L^2(\alpha+\beta-\alpha^2) + 2\alpha^2 v_L v_H
\end{aligned}
$$

where

$$Q = \sqrt{v_L^2(2\alpha^2 - 2\alpha\beta - 2\alpha + \beta^2) + v_H^2(\alpha^2 - 2\alpha\beta} + \beta^2) - 2\alpha^2 v_L v_H$$

When v_L approaches zero we have

$$\lim_{v_L\to 0}\frac{dq^S_{1,3}(\beta)}{d\beta} = -\frac{1}{2(1+\alpha-\beta)^2} - \frac{v_H^2(\beta-\alpha)}{2v_H(1+\alpha-\beta)^2 v_H(\alpha-\beta)} = 0$$

By (40), we have

$$
\begin{aligned}
\frac{dTE(\beta)}{d\beta} &= (q^S_{1,3})^2 v_L + (1-q^S_{1,3})^2 v_H + 2q^S_{1,3}(1-q^S_{1,3})\frac{v_L}{2}(1+\frac{v_L}{v_H}) \\
&\quad - \frac{q^S_{1,3}(v_H - v_L)\alpha^2 v_L^2}{(q^S_{1,3}(\alpha v_H - \beta(v_H-v_L) + (1-q^S_{1,3})\beta(v_H - v + \alpha v_L))^2}
\end{aligned}
$$

When v_L approaches zero, we have

$$\lim_{v_L\to 0}\frac{dTE(\beta)}{d\beta} = (1-q^S_{1,3})^2 v_H \geq 0$$

Thus, we obtain that

$$\lim_{v_L\to 0}\frac{dTE(\beta, q^S_{1,3})}{d\beta} = \lim_{v_L\to 0}(\frac{dTE(q^S_{1,3})}{dq^S_{1,3}}\frac{dq^S_{1,3}(\beta)}{d\beta} + \frac{dTE(\beta)}{d\beta}) \geq 0$$

This yields the following result:

Proposition 4 *In an elimination tournament with two strong players and two weak players , if in each semifinal a strong player competes against a weak player, and if the weak player's value of winning is sufficiently small, the third place game increases the players' expected total effort in the tournament.*

Proposition 4 shows that the strong players do not exert high efforts against the weak players in the semifinal, and therefore as a result of the third place game, the increase of the effort there is higher than the decrease of the effort in the semifinals. Therefore, the third place game increases the players' total effort.

6 Conclusion

We showed (Proposition 1) that in an elimination tournament with a dominant player, namely, one strong player and three weak players, independent of the relation between the players' values (strong/weak), the third place game increases the players' expected total effort in the tournament, but with three strong players and one inferior player who has a lower value of winning (Proposition 2), if the inferior player 's value is sufficiently small, then the third place game decreases the players' expected total effort in the tournament. When the players are balanced, namely, there are two strong and two weak players, we found that the players' seeding in the semifinal plays a key role on the effect of the third place game on the players' expected total effort. In addition, in an elimination tournament with two strong players who compete against each other in one of the semifinals and two weak players who compete against each other in the other one, we showed (Proposition 3) that if the weak players' value of winning is sufficiently small, then the third place game decreases the players' expected total effort. However, if in each semifinal players with different types compete against each other, and in addition, if the weak player's value of winning is sufficiently small, we found (see Proposition 4) that the third place game increases the players' expected total effort.

The implication of these results is that in elimination tournaments with two types of players (strong and weak) if there are at least two weak players, by choosing the correct players' seeding in the semifinal, the third place game has a positive effect on the players' expected total effort; however, if there are at least two strong players, the third place game might have a negative effect on the players' expected total effort. In other words, if there is a dominant player such that the identity of the winner is quite clear, the players exert relatively low efforts in the semifinals such that the third place game increases the players' efforts in the second stage (final), but also significantly decreases the effort in the first stage (semifinals). In that case, we may find that the third place game is not efficient for a designer who wishes to maximize the expected total effort in elimination tournaments.

References

[1] Baye, M., Kovenock, D., de Vries, C.: Rigging the lobbying process. *American Economic Review* 83, 289-294 (1993)

[2] Baye, M., Kovenock, D., de Vries, C.: The all-pay auction with complete information. *Economic Theory* 8, 291-305 (1996)

[3] Clark, D., Riis, C.: A multi-winner nested rent-seeking contest. *Public Choice* 77, 437-443 (1996)

[4] Clark, D., Riis, C.: Influence and the discretionary allocation of several prizes. *European Journal of Political Economy* 14 (4), 605-625 (1998)

[5] David, H.: Tournaments and paired comparisons. *Biometrika* 46, 139-149 (1959)

[6] Glenn, W.: A comparison of the effectiveness of tournaments. *Biometrika* 47, 253-262 (1960)

[7] Gradstein, M., Konrad, K.: Orchestrating rent seeking contests. *The Economic Journal* 109, 536-545 (1999)

[8] Groh, C., Moldovanu, B., Sela, A., Sunde, U.: Optimal seedings in elimination tournaments. *Economic Theory*, 49, 59-80 (2012)

[9] Hillman, A., Riley, J.: Politically contestable rents and transfers. *Economics and Politics* 1, 17-40 (1989)

[10] Hillman, A., Samet, D.: Dissipation of contestable rents by small numbers of contenders. *Public Choice* 54(1), 63-82 (1987)

[11] Horen, J., Riezman, R.: Comparing draws for single elimination tournaments. *Operations Research* 3(2), 249-262 (1985)

[12] Hwang, F.: New concepts in seeding knockout tournaments. *American Mathematical Monthly* 89, 235-239 (1982)

[13] Klaassen, F., Magnus, J.: Forecasting the winner of a tennis match. *European Journal of Operational Research* 148, 257-267 (2003)

[14] Krakel, M.: Optimal seeding in elimination tournaments revisited. *Economic Theory Bulletin* 2, 77-91 (2014)

[15] Rosen, S.: Prizes and incentives in elimination tournaments. *American Economic Review* 74, 701-715 (1986)

[16] Searles, D.: On the probability of winning with different tournament procedures. *Journal of the American Statistical Association* 58, 1064-1081 (1963)

[17] Schweinzer, P., Segev, E.: The optimal prize structure of symmetric Tullock contests. *Public Choice* 153(1), 69-82 (2012)

[18] Schwenk, A.: What is the correct way to seed a knockout tournament. *American Mathematical Monthly* 107, 140-150 (2000)

[19] Sela, A.: Sequential two-prize contests. *Economic Theory* 51, 383-395 (2012)

[20] Stracke, R., Hochtl, W., Kerschbamer, R., Sunde, U.: Optimal prizes in dynamic elimination contests: theory and experimental evidence. *Journal of Economic Behavior & Organization* 102, 43-58 (2014)