SIGNALING IN CONTESTS

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Discussion Paper No. 17-08

November 2017

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May 15, 2017

Abstract

We analyze whether pre-contest communications would occur in contest models with asymmetric information. We find that in Tullock contests signals can be effectively used in equilibrium. We then study all-pay contests and show that such signals are not effective, and therefore pre-contest communications will not occur in equilibrium.

Keywords: Contests, signaling, asymmetric information, incomplete information.

JEL classification: C70, D72, D82, D44

1 Introduction

In signaling games some players are uninformed about the types of their opponents in which case their opponents may either send or not send signals to reveal their types. The power and the complexity of signaling games has already been demonstrated in the seminal works of Nobel prize-winners Akerlof (1970) and Spence (1973) as well as in many other research works. Our goal in this paper is to examine the power of signaling in the two main contest models: the Tullock contest[1] in which the contest success function is stochastic such that the probability of a player to win is equal to the ratio of his effort and the total effort

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[1] Numerous researchers have analyzed the Tullock contests with either complete and incomplete information. See, among others, Tullock (1980), Skaperdas (1996), Szidarovszky and Okuguchi (1997), Clark and Riis (1998), Fey (2008) and Wasser (2013).

exerted by all the players, and the all-pay contest[2] in which the contest success function is deterministic such that the player with the highest effort wins.[3] In this paper we ask whether or not pre-contest communication between contestants with asymmetric information is possible. In other words, is there a place for signaling in these two contest models under incomplete information. Alternatively, we ask whether the players who have private information about their types have any incentive to send costly signals in order to reveal their types and by doing so to change the contest with incomplete information into a contest with complete information. The incentive to send a costly signal could be whether the player is strong (has a high value of winning) or weak (has a low value of winning). In both cases, the role of the signal is to reduce the players' costs of effort.

We assume that one of the players (referred to as the informed player) has only two possible types (values of winning) and his type is private information. The type of the other player (referred to as the uninformed player) is common knowledge. In the first stage, the informed player decides whether or not to send a costly signal in order to reveal his type. In the second stage, both players compete in an asymmetric contest with either complete or incomplete information. We consider first the Tullock two-stage contest, analyze its perfect Bayesian equilibrium, and show that, depending on the parameters of the model, there may exist a separating perfect Bayesian equilibrium as well as a pooling perfect Bayesian equilibrium. We first find that the informed player (when he is relatively strong) might have an incentive to send a costly signal in order to reveal his type. In that case, the weak player who wishes to pretend to be strong may not find it profitable to send the same signal of the strong player. Specifically, we show that the informed player has an incentive to send such a signal when he is strong but the probability of his type is relatively low. Otherwise, when the probability of his type is relatively high, he does not have an incentive to send a costly signal since he does not need to convince his opponent that he is indeed strong. We also find that the weak player might have an incentive to send a costly signal, but the signal he needs to send in order to reveal his type is not

[2] Numerous researchers have analyzed the all-pay contest with either complete or incomplete information. See, among others, Hlilman and Riley (1989), Baye, Kovenock and de Vries (1996), Amann and Leininger (1996), Krishna and Morgan (1997), Che and Gale (1998), Moldovanu and Sela (2001, 2006) and Siegel (2009).

[3] The Tullock contest as well as the all-pay contest have several applications including rent-seeking and lobbying in organizations, R&D races, political contests, promotions in labor markets, trade wars, military purposes and biological wars of attrition.

profitable. Therefore there is only a separating equilibrium in which the strong player is the only one who might send a costly signal.

Then we consider the two-stage all-pay contest and analyze its perfect Bayesian equilibrium where the uninformed player's type is either higher or lower than the two possible types of the informed player. We demonstrate that in the all-pay contest, a separating perfect Bayesian equilibrium does not exist, which means that the informed player has no incentive to send a costly signal in order to reveal his type. The reason is that for both types of the informed player a separating equilibrium is not profitable compared to the pooling equilibrium and therefore each of the types does not have any incentive to send a costly signal. Furthermore, even when the uninformed player's type is higher than the informed player's high type and lower than his low type (for which we do not have an explicit characterization of the equilibrium strategies) it is clear that a separating equilibrium is not possible. These results emphasize the different effects of the deterministic and stochastic contest success functions on signalling in contests under incomplete information.

The issue of signaling in contests has already been studied (see, for example, Amegashie 2005 and Zhang and Wang 2009). However, there is a key difference between our model and the other ones. In the other papers, the signaling occurs through early period efforts that affect these players' probabilities of winning. On the other hand, in our paper the signaling occurs through a costly signal that has no effect on the players' probabilities of winning, but only on their expected payoffs.

The rest of the paper is organized as follows. In Section 2 we analyze the two-stage Tullock contest, in Section 3 we analyze the two-stage all-pay contest, and Section 4 concludes. The proofs appear in an Appendix.

2 The two-stage Tullock contest

We first consider a two-stage Tullock contest with two players. The value of winning (type) for player j is v_j while the value of winning for player i is v_{iL} with a probability of p_L or v_{iH} with a probability of p_H. The type of player i is private information while the type of player j is commonly known. In the second stage, if players i and j exert efforts of x_i, x_j, then player i wins with a probability of $\frac{x_i}{x_i+x_j}$, player j wins with a probability of $\frac{x_j}{x_i+x_j}$, and the players' costs in that stage are x_i and x_j respectively. In the first stage, however, player i can send a signal s_i given his type in order to reveal it, and then his cost in that stage is equal to s_i. The players wish to maximize their utility functions which are given by

$$\begin{aligned} u_i(x_i, x_j, s_i) &= v_i \frac{x_i}{x_i + x_j} - x_i - s_i \\ u_j(x_i, x_j) &= v_j \frac{x_j}{x_i + x_j} - x_j \end{aligned}$$

We analyze the Perfect Bayesian Nash Equilibrium (PBNE) of the above two-stage Tullock contest that consists of strategy profiles of both players and the belief of player j (the uninformed player) after he observes the signal s_i of player i (the informed player). The players' strategies are sequentially rational given the beliefs and the strategies of their opponent, and the beliefs of the uninformed player that based on the signal received from the informed player satisfy the Bayes' rule whenever possible.

2.1 The second stage

In order to analyze the perfect Bayesian Nash Equilibrium of the two-stage Tullock contest we begin with the second stage and go backwards to the first one.

2.1.1 Pooling equilibrium

Consider first that both types of player i send the same signal in the first stage which can be either positive or zero. Then, the maximization problems of player i with types H and L in the second stage are

$$\begin{aligned} &\max_{x_{iH}} v_{iH} \frac{x_{iH}}{x_{iH} + x_j} - x_{iH} \\ &\max_{x_{iL}} v_{iL} \frac{x_{iL}}{x_{iL} + x_j} - x_{iL} \end{aligned} \tag{1}$$

In that case, player j does not know the type of player i and therefore his maximization problem is

$$\max_{x_j} v_j (p_H \frac{x_j}{x_j + x_{iH}} + P_L \frac{x_j}{x_j + x_{iL}}) - x_j \tag{2}$$

The F.O.C. of (1) and (2) are

$$\begin{aligned} v_{iH} \frac{x_j}{(x_{iH} + x_j)^2} &= 1 \\ v_{iL} \frac{x_j}{(x_{iL} + x_j)^2} &= 1 \end{aligned}$$

and

$$v_j(\frac{p_{iH}\cdot x_{iH}}{(x_{iH}+x_j)^2}+\frac{p_{iL}\cdot x_{iL}}{(x_{iL}+x_j)^2})=1$$

The solution of the F.O.C. yields that the players' efforts in the second stage are

$$\begin{aligned} x_{iL} &= \frac{v_{iL}(p_H\cdot v_j\sqrt{v_{iL}\cdot v_{iH}}+p_L\cdot v_j\cdot v_{iH})\cdot(v_{iL}\cdot v_{iH}+P_H\cdot v_j(v_{iL}-\sqrt{v_{iL}\cdot v_{iH}}))}{(v_{iL}\cdot v_{iH}+p_H\cdot v_{iL}\cdot v_j+p_L\cdot v_{iH}\cdot v_j)^2} \\ x_{iH} &= \frac{v_{iL}\cdot v_{iH}(p_H\cdot v_j\sqrt{v_{iL}\cdot v_{iH}}+p_L\cdot v_j\cdot v_{iH})\cdot(\sqrt{v_{iL}\cdot v_{iH}}-\frac{p_L\cdot v_j}{v_{iL}}(v_{iL}-\sqrt{v_{iL}\cdot v_{iH}}))}{(v_{iL}\cdot v_{iH}+p_H\cdot v_{iL}\cdot v_j+p_L\cdot v_{iH}\cdot v_j)^2} \end{aligned} \tag{3}$$

and

$$x_j=\frac{v_{iL}(p_H\cdot v_j\sqrt{v_{iL}\cdot v_{iH}}+p_L\cdot v_j\cdot v_{iH})^2}{(v_{iL}\cdot v_{iH}+p_H\cdot v_{iL}\cdot v_j+p_L\cdot v_{iH}\cdot v_j)^2} \tag{4}$$

We focus here on interior perfect Bayesian equilibrium and therefore we need the following sufficient condition that player i with type L exerts a positive effort (the efforts of player i with type H and player j are always positive):

$$v_{iL}\cdot v_{iH}+P_H\cdot v_j(v_{iL}-\sqrt{v_{iL}\cdot v_{iH}})>0$$

Then, player i's expected payoffs in the second stage are

$$\begin{aligned} u^p_{iH} &= v_{iH}\left(\frac{v_{iL}\cdot v_{iH}+p_L\cdot v_j(v_{iH}-\sqrt{v_{iL}\cdot v_{iH}})}{v_{iL}\cdot v_{iH}+p_H\cdot v_{iL}\cdot v_j+p_L\cdot v_{iH}\cdot v_j}\right)^2 \\ u^p_{iL} &= v_{iL}\left(\frac{v_{iL}\cdot v_{iH}+p_H\cdot v_j(v_{iL}-\sqrt{v_{iL}\cdot v_{iH}})}{v_{iL}\cdot v_{iH}+p_H\cdot v_{iL}\cdot v_j+p_L\cdot v_{iH}\cdot v_j}\right)^2 \end{aligned} \tag{5}$$

where u^p_{iK} is player i's expected payoff if his value is $v_{iK}, k\in\{H,L\}$.

2.1.2 Separating equilibrium

Consider now that both types of player i send different signals in the first stage. Then, the maximization problems of player i with types H and L are

$$\max_{x_{iH}} v_{iH}\frac{x_{iH}}{x_{iH}+x_j}-x_{iH}$$
$$\max_{x_{iL}} v_{iL}\frac{x_{iL}}{x_{iL}+x_j}-x_{iL}$$

In that case, player j is able to distinguish between the types of player i and therefore his maximization problem is

$$\max_{x_j} v_j\frac{x_j}{x_j+x_i}-x_j \tag{6}$$

where $x_i = x_{iH}$ or $x_i = x_{iL}$ according to the type of player i. Then, the equilibrium efforts are given by the solution of the standard two-player Tullock contest as follows:

$$\begin{aligned} x_i &= \frac{v_i^2 v_j}{(v_i + v_j)^2} \\ x_j &= \frac{v_j^2 v_i}{(v_i + v_j)^2} \end{aligned} \tag{7}$$

where $x_i = x_{iH}$ and $v_i = v_{iH}$ or $x_i = x_{iL}$ and $v_i = v_{iL}$ according to the type of player i. Then, player i's expected payoffs in the second stage are

$$\begin{aligned} u_{iH}^s &= \frac{v_{iH}^3}{(v_{iH} + v_j)^2} \\ u_{iL}^s &= \frac{v_{iL}^3}{(v_{iL} + v_j)^2} \end{aligned} \tag{8}$$

where u_{iK}^s is player i's expected payoff if his value is $v_{iK}, k \in \{H, L\}$.

Denote now by $\widetilde{u}_{iL}^s$ the expected payoff of player i with type L when player j believes that he has type H. By (7), the strategy of player j will be $x_j = \frac{v_j^2 v_{iH}}{(v_{iH}+v_j)^2}$. Then, the maximization problem of player i with type L is

$$\max_{x_{iL}} v_{iL} \frac{x_{iL}}{x_{iL} + \frac{v_j^2 v_{iH}}{(v_{iH}+v_j)^2}} - x_{iL} \tag{9}$$

Similarly, denote by $\widetilde{u}_{iH}^s$ the expected payoff of player i with type H when player j believes that he has type L. By (7), the strategy of player j will be $x_j = \frac{v_j^2 v_{iL}}{(vL+v_j)^2}$. Then, the maximization problem of player i with type H is

$$\max_{x_{iH}} v_{iH} \frac{x_{iH}}{x_{iH} + \frac{v_j^2 v_{iL}}{(v_{iL}+v_j)^2}} - x_{iH} \tag{10}$$

The solution of the maximization problems (9) and (10) yields

Proposition 1 *In the two-stage Tullock contest*

1. *If player j believes that player i with type L has type H, then player i has an expected payoff of*

$$\widetilde{u}_{iL}^s = \left(\frac{\sqrt{v_{iL}}(v_j + v_{iH}) - v_j \sqrt{v_{iH}}}{v_j + v_{iH}} \right)^2 \tag{11}$$

2. *If player j believes that player i with type H has type L, then player i has an expected payoff of*

$$\widetilde{u}_{iH}^s = \left(\frac{\sqrt{v_{iH}}(v_j + v_{iL}) - v_j \sqrt{v_{iL}}}{v_j + v_{iL}} \right)^2 \tag{12}$$

Proof. See Appendix. ■

The following result provides the conditions under which player i has an incentive to pretend that he has a different type.

Proposition 2 *In the two-stage Tullock contest*

1. $\widetilde{u}^s_{iL} > u^s_{iL}$ *iff* $v_j < \sqrt{v_{iH} \cdot v_{iL}}$, *i.e., for such players' values of winning, player i with type L has a higher expected payoff if player j believes that he has type H.*

2. $\widetilde{u}^s_{iH} > u^s_{iH}$ *iff* $v_j > \sqrt{v_{iH} \cdot v_{iL}}$, *i.e., for such players' values of winning, player i with type H has a higher expected payoff if player j believes that he has type L.*

Proof. See Appendix. ■

2.1.3 Separating equilibrium vs. pooling equilibrium

In the following we compare player i's expected payoffs under the pooling and the separating equilibrium. We first denote the differences of both types of player i's expected payoffs when player j has the true beliefs about the types of player i, namely, he believes that type k is indeed type k, $k \in \{H, L\}$. Formally,

$$\begin{aligned} \Delta^H_{s-p} &= u^s_{iH} - u^p_{iH} \\ \Delta^L_{s-p} &= u^s_{iL} - u^p_{iL} \end{aligned} \tag{13}$$

We then show that the types of player i necessarily have different preferences about these two types of equilibrium (pooling or separating).

Proposition 3 *In the two-stage Tullock contest it is not possible that both types of player i prefer the separating equilibrium over the pooling equilibrium and vice versa. In particular,*

1. *If* $v_j < \sqrt{v_{iH} \cdot v_{iL}}$ *then* $\Delta^H_{s-p} > 0$ *and* $\Delta^L_{s-p} < 0$, *i.e., player i with type H (L) has a higher (lower) expected payoff under the separating equilibrium than under the pooling equilibrium.*

2. *If* $v_j > \sqrt{v_{iH} \cdot v_{iL}}$ *then* $\Delta^H_{s-p} < 0$ *and* $\Delta^L_{s-p} > 0$, *i.e., player i with type L (H) has a higher (lower) expected payoff under the separating equilibrium than under the pooling equilibrium.*

3. *If* $v_j = \sqrt{v_{iH} \cdot v_{iL}}$ *then* $\Delta^H_{s-p} = \Delta^L_{s-p} = 0$, *i.e., both types of player i, H and L, have the same expected payoff under the separating and the pooling equilibrium.*

Proof. See Appendix. ■

We now denote the differences of both types of player i's expected payoffs when player j has wrong beliefs about player i's type (namely, he believes that type L is type H and vice versa) as follows:

$$\begin{aligned}\widetilde{\Delta}^H_{s-p} &= \widetilde{u}^s_{iH} - u^p_{iH} \\ \widetilde{\Delta}^L_{s-p} &= \widetilde{u}^s_{iL} - u^p_{iL}\end{aligned}$$

In the following we show that both types of player i may prefer the separating equilibrium over the pooling equilibrium if player j has the wrong beliefs about their types.

Proposition 4 *In the two-stage Tullock contest*

1. If $v_j < \sqrt{v_{iH} \cdot v_{iL}}$, then $\widetilde{\Delta}^L_{s-p} > 0$, i.e., player i with type L has a lower expected payoff under the pooling equilibrium than under the separating equilibrium when player j believes that he has type H.

2. If $v_j > \sqrt{v_{iH} \cdot v_{iL}}$, then $\widetilde{\Delta}^H_{s-p} > 0$, i.e., player i with type H has a lower expected payoff under the pooling equilibrium than under the separating equilibrium when player j believes that he has type L.

Proof. See Appendix. ■

We also denote the differences of both types of player i's expected payoffs when player j has the wrong and the right beliefs by

$$\begin{aligned}\overline{\Delta}^H_{s-s} &= \widetilde{u}^s_{iH} - u^s_{iH} \\ \overline{\Delta}^L_{s-s} &= \widetilde{u}^s_{iL} - u^s_{iL}\end{aligned} \tag{14}$$

Then, we can conclude from Propositions (3) and (4) that

Conclusion 1 *In the two stage Tullock contest*

1. If $v_j < \sqrt{v_{iH} \cdot v_{iL}}$, player i with type H prefers the separating equilibrium over the pooling equilibrium, i.e., $\Delta^H_{s-p} > 0$. In that case, player i with type L prefers a separating equilibrium to the pooling equilibrium iff player j believes that he has type H, i.e., $\overline{\Delta}^L_{s-s} > 0$.

2. If $v_j > \sqrt{v_{iH} \cdot v_{iL}}$, player i with type L prefers the separating equilibrium over the pooling equilibrium, i.e., $\Delta^L_{s-p} > 0$. In that case, player i with type H prefers a separating equilibrium to the pooling equilibrium iff player j believes that he has type L, i.e., $\overline{\Delta}^H_{s-s} > 0$.

2.2 The first stage

In order to characterize the perfect Bayesian equilibrium of the two-stage Tullock contest we define the following beliefs of player j about the type of player i according to the signal sent by player i in the first stage.

Definition 2 *In the two-stage Tullock contest, if* $v_j < \sqrt{v_{iH} \cdot v_{iL}}$*, player* j*'s beliefs are as follows: If* $\Delta^H_{s-p} > -\Delta^L_{s-p}$*, then player* j *believes that each signal* $s_i \geq \overline{\Delta}^L_{s-s}$ *is sent by player* i *with type* H*, and each signal* $s_i < \overline{\Delta}^L_{s-s}$ *is sent by type* L*. And, if* $\Delta^H_{s-p} \leq -\Delta^L_{s-p}$*, then player* j *believes that each signal* s_i *is sent by both types of player* i *according to their priors.*

The rationale behind Definition 2 is that when $v_j < \sqrt{v_{iH} \cdot v_{iL}}$, by Proposition 3, type H of player i prefers the separating equilibrium to the pooling equilibrium, while type L of player i has the opposite preference. If $\Delta^H_{s-p} > -\Delta^L_{s-p}$, the difference of type H's payoff from the separating equilibrium compared to the pooling equilibrium is higher than the difference of type L's payoff from the pooling equilibrium compared to the separating equilibrium. Then, if type H sends a signal higher than $\overline{\Delta}^L_{s-s}$ it is clear that this signal was sent by him since type L will have a negative payoff if he would send the same signal. Any signal lower than $\overline{\Delta}^L_{s-s}$ can be sent by both types of player i and therefore player j does not distinguish between player i's types for such signals. If, on the other hand, $\Delta^H_{s-p} \leq -\Delta^L_{s-p}$, the difference between type H's payoff from the separating equilibrium compared to the pooling equilibrium is lower than the difference of type L's payoff from the pooling equilibrium compared to the separating equilibrium. Then, every signal that type H will send can be sent by type L as well. Thus, player j believes that any signal could be sent by each of the types of player i according to their priors.

Definition 3 *In the two-stage Tullock contest, if* $v_j > \sqrt{v_{iH} \cdot v_{iL}}$*, player* j*'s beliefs are as follows: If* $\Delta^L_{s-p} > -\Delta^H_{s-p}$ *then player* j *believes that each signal* $s_i \geq \overline{\Delta}^H_{s-s}$ *is sent by player* i *with type* L*, and that each signal* $s_i < \overline{\Delta}^H_{s-s}$ *is sent by type* H*. And, if* $\Delta^L_{s-p} \leq -\Delta^H_{s-p}$*, then player* j *believes that each signal* s_i *is sent by both types of player* i *according to their priors.*

The rationale behind Definition 3 is that when $v_j > \sqrt{v_{iH} \cdot v_{iL}}$, by Proposition 3, type L of player i prefers the separating equilibrium to the pooling equilibrium while type H of player i has the opposite preference. If $\Delta^L_{s-p} > -\Delta^H_{s-p}$, the difference of type L's payoff from the separating equilibrium compared to the pooling equilibrium is higher than the difference of type L's payoff from the pooling equilibrium and the separating equilibrium. If type L sends a signal higher than $\overline{\Delta}^H_{s-s}$ it is clear that this signal was sent by him since type H will have a negative payoff if he would send the same signal. Any signal lower than $\overline{\Delta}^H_{s-s}$ can be sent by both types of player i and therefore player j does not distinguish between player i's types for such signals. If, on the other, $\Delta^L_{s-p} \leq -\Delta^H_{s-p}$, the difference of type H's payoff from the separating equilibrium compared to the pooling equilibrium is lower than the difference of type H's payoff from the pooling equilibrium compared to the separating equilibrium and then every signal that type L will send can be sent by type H as well. Thus, player j believes that any signal could be sent by each of the types of player i according to their priors.

Definition 4 *In the two-stage Tullock contest, if $v_j = \sqrt{v_{iH} \cdot v_{iL}}$, player j believes that each signal s_i is sent by both types of player i according to their priors.*

The rationale behind Definition 4 is that when $v_j = \sqrt{v_{iH} \cdot v_{iL}}$, by Proposition 3, types H and L of player i are indifferent between the separating equilibrium and the pooling equilibrium. Thus they both do not have an incentive to send a costly signal, and therefore player j believes that each signal s_i is sent by both types of player i according to their priors.

Given player j's beliefs, we can characterize the perfect Bayesian equilibrium in the two-stage Tullock model. The equilibrium characterization is divided into three parts (Theorems 5, 6 and 7) according to the relation between the players' values of winning.

1. If $v_j < \sqrt{v_{iH} \cdot v_{iL}}$ we have a separating equilibrium as well as a pooling equilibrium.

Theorem 5 *In the two-stage Tullock contest, if $v_j < \sqrt{v_{iH} \cdot v_{iL}}$ and if the players' beliefs are given by Definition 2, then*

(i). If $\Delta^H_{s-p} \leq -\Delta^L_{s-p}$, there is a pooling perfect Bayesian equilibrium in which both types of player i, L and H, do not send any signal in the first stage. Then, in the second stage, the players' strategies are given by (3) and (4).

(ii). If $\Delta^H_{s-p} > -\Delta^L_{s-p}$, there is a separating perfect Bayesian equilibrium in which player i with type H sends a signal in the first stage $s_{iH} = \overline{\Delta}^L_{s-s}$ and player i with type L does not send any signal. Then, in

the second stage, the players strategies are given by (7). In that case, a sufficient condition for a separating perfect Bayesian equilibrium is a sufficiently small value of p_H.

Proof. See Appendix. ■

According to Theorem 5, we can see that there is a separating perfect Bayesian equilibrium in the two-stage Tullock contest if the probability of type H of player i is relatively small and his value is significantly larger than the value of his opponent. Note that only if the probability of type H is small, player i with type H has an incentive to send a signal. Otherwise, if the probability of type H is high, there is no need to send a signal since player j already believes that his opponent probably has type H.

2. If $v_j > \sqrt{v_{iH} \cdot v_{iL}}$ we have only the pooling equilibrium.

Theorem 6 *In the two-stage Tullock contest, if $v_j > \sqrt{v_{iH} \cdot v_{iL}}$ and if the players' beliefs are given by Definition 3, there is only a pooling perfect Bayesian equilibrium in which both types of player i, L and H, do not send any signal in the first stage. Then, in the second stage, the players' strategies are given by (3) and (4).*

Proof. See Appendix. ■

3. If $v_j = \sqrt{v_{iH} \cdot v_{iL}}$, as in the previous case, we have only the pooling equilibrium.

Theorem 7 *In the two-stage Tullock contest, if $v_j = \sqrt{v_{iH} \cdot v_{iL}}$ and if the players' beliefs are given by Definition 4, there is a pooling perfect Bayesian equilibrium in which both types of player i, L and H, do not send any signal in the first stage. Then, in the second stage, the players' strategies are given by (3) and (4).*

Proof. See Appendix. ■

3 The two-stage all-pay contest

We now consider a two-stage all-pay contest with two players. The value of winning (type) for player j is v_j while the value of winning for player i is v_L with probability p_L, or v_H with probability p_H. The type of player i is private information and the type of player j is commonly known. In the second stage, each player i submits a bid (effort) $x \in [0, \infty)$ and the player with the highest bid wins the first prize and all the players pay their bids. In the first stage, however, player i can send a signal s_i in order to reveal his type and then his cost in that stage is equal to s_i. The players wish to maximize their utility functions which are given by

$$u_i(x_i, x_j, s_i) = \begin{cases} v_i - x_i - s_i & \text{if} \quad x_i > x_j \\ \frac{1}{2}v_i - x_i - s_i & \text{if} \quad x_i = x_j \\ -x_i - s_i & \text{if} \quad x_i < x_j \end{cases}$$

$$u_j(x_i, x_j) = \begin{cases} v_j - x_i & \text{if} \quad x_j > x_i \\ \frac{1}{2}v_j - x_i & \text{if} \quad x_j = x_i \\ -x_i & \text{if} \quad x_j < x_i \end{cases}$$

We analyze the Perfect Bayesian Nash Equilibrium (PBNE) of the above two-stage all-pay contest that consists of strategy profiles of both players and the belief of player j (the uninformed player) after he observes the signal s_i of player i (the informed player). The players' strategies are sequentially rational given the beliefs and the strategies of their opponent, and the beliefs of the uninformed player that based on the signal received from the informed player satisfy the Bayes' rule whenever possible.

3.1 The second stage

3.1.1 Separating equilibrium

Consider first that both types of player i send different signals in the first stage. Then, assume that player i's value in the second stage is v_i where $v_i = v_{iL}$ or $v_i = v_{iH}$, and, without loss of generality, assume also that the players' values satisfy $v_i > v_j$. According to Baye, Kovenock and de Vries (1996), there is always a unique mixed-strategy equilibrium in which players i and j randomize on the interval $[0, v_j]$ according to their effort cumulative distribution functions F_i, F_j, which are given by

$$\begin{aligned} v_i F_j(x) - x &= v_i - v_j \\ v_j F_i(x) - x &= 0 \end{aligned}$$

Thus, player i's equilibrium effort in the second stage is uniformly distributed as follows:

$$F_i(x) = \frac{x}{v_j}$$

while player j's equilibrium effort in the second stage is distributed according to the following cumulative distribution function

$$F_j(x) = \frac{v_i - v_j + x}{v_i}$$

The respective expected payoffs in the second stage are

$$\begin{aligned} u_i &= v_i - v_j \\ u_j &= 0 \end{aligned} \tag{15}$$

3.1.2 Pooling equilibrium

Consider now that both types of player i send the same signal in the first stage where this signal could be either positive or zero. Then, we consider two possible scenarios as follows:

1. Assume first that $v_j > v_{iH} > v_{iL}$. In that case, there is a mixed-strategy equilibrium in which player i with types H and player j randomize on the interval $[0, v_{iH}]$ according to their effort cumulative distribution functions F_{iH}, F_j, which are given by

$$\begin{aligned} v_j(p_L + p_H \cdot F_{iH}(x)) - x &= v_j - v_{iH} \\ v_{iH} \cdot F_j(x) - x &= 0 \end{aligned}$$

However, player i with type L chooses to stay out of the contest. Thus,

Proposition 5 *In the two-stage all-pay contest, if $v_j > v_{iH} > v_{iL}$, player i with type L chooses $x_{iL} = 0$ with probability one in the second stage, while the equilibrium effort of player i with type H in the second stage is distributed according to the cumulative distribution function*

$$F_{iH}(x) = \begin{cases} 0 \quad if \quad x \le \max\{0, v_{iH} - p_H \cdot v_j\} \\ \frac{p_H \cdot v_j - v_{iH} + x}{p_H \cdot v_j} \quad if \quad \max\{0, v_{iH} - p_H \cdot v_j\} < x \le v_{iH} \\ 1 \quad if \quad x > v_{iH} \end{cases} \tag{16}$$

Player j's equilibrium effort in the second stage is uniformly distributed as follows:

$$F_j(x) = \begin{cases} 0 \quad \text{if} \quad x \leq \max\{0, v_{iH} - p_H \cdot v_j\} \\ \frac{x}{v_{iH}} \quad \text{if} \quad \max\{0, v_{iH} - p_H \cdot v_j\} < x \leq v_{iH} \\ 1 \quad \text{if} \quad x > v_{iH} \end{cases} \tag{17}$$

The respective expected payoffs in the second stage are then

$$\begin{aligned} u_{iL} &= u_{iH} = 0 \\ u_j &= v_j - v_{iH} \end{aligned} \tag{18}$$

Proof. See Appendix. ■

2. Assume now that $v_j < v_{iL} < v_{iH}$. Then, there is a mixed-strategy equilibrium in which player i with type L and player j randomize on the interval $[0, p_L v_j]$ according to their effort cumulative distribution functions F_{iL}, F_j, which are given by

$$\begin{aligned} v_j \cdot p_L \cdot F_{iL}(x) - x &= 0 \\ v_{iL} \cdot F_j(x) - x &= v_{iL} - v_j\left(\frac{p_H \cdot v_{iL}}{v_{iH}} + p_L\right) \end{aligned}$$

And player i with types H and player j randomize on the interval $[p_L v_j, v_j]$ according to their effort cumulative distribution functions F_{iH}, F_j, which are given by

$$\begin{aligned} v_j(p_L + p_H F_{iH}(x)) - x &= 0 \\ v_{iH} \cdot F_j(x) - x &= v_{iH} - v_j \end{aligned}$$

Thus, we obtain

Proposition 6 *In the two-stage all-pay contest, if $v_j < v_{iL} < v_{iH}$, the equilibrium effort of player i with type L in the second stage is distributed according to the cumulative distribution function*

$$F_{iL}(x) = \begin{cases} \frac{x}{p_L \cdot v_j} \quad \text{if} \quad 0 \leq x \leq p_L \cdot v_j \\ 1 \quad \text{if} \quad x > p_L \cdot v_j \end{cases} \tag{19}$$

and the equilibrium effort of player i with type H in the second stage is distributed according to the cumulative *distribution function*

$$F_{iH}(x) = \begin{cases} 0 \quad if \;\; 0 \leq x \leq p_L \cdot v_j \\ \frac{x - p_L \cdot v_j}{p_H \cdot v_j} \quad if \;\; p_L \cdot v_j < x \leq v_j \\ 1 \quad if \quad x > v_j \end{cases} \tag{20}$$

Player j's equilibrium effort in the second stage is distributed according to the cumulative distribution function

$$F_j(x) = \begin{cases} \frac{v_{iL} - v_j(\frac{p_H \cdot v_{iL}}{v_{iH}} + p_L) + x}{v_{iL}} \quad if \;\; 0 \leq x \leq p_L \cdot v_j \\ \frac{v_{iH} - v_j + x}{v_{iH}} \quad if \;\; p_L \cdot v_j < x \leq v_j \\ 1 \quad if \quad x > v_j \end{cases} \tag{21}$$

The respective expected payoffs in the second stage are then

$$\begin{aligned} u_{iL} &= v_{iL} - v_j(\frac{p_H \cdot v_{iL}}{v_{iH}} + p_L) \\ u_{iH} &= v_{iH} - v_j \\ u_j &= 0 \end{aligned} \tag{22}$$

Proof. *See Appendix.* ■

3.2 The first stage

Based on the analysis of the second stage we show that in contrast to the two-stage Tullock contest we obtain:

Proposition 7 *In the two-stage all-pay contest there is no separating perfect Bayesian equilibrium.*

Proof. See Appendix. ■

The result of Proposition 7, according to which there is no separating equilibrium in the two-stage all-pay contest, is proved only for the case when $v_j > v_{iH} > v_{iL}$ and $v_j < v_{iL} < v_{iH}$ since there we explicitly calculate the equilibrium strategies. For the other case when $v_{iL} < v_j < v_{iH}$ it is quite complex to explicitly calculate the equilibrium strategies. However, by similar arguments used in the proof of Proposition 7, even without such a calculation, it can be shown that a separating perfect Bayesian equilibrium does not exist.

4 Concluding remarks

We analyzed the Tullock and the all-pay contest when the uninformed player has a commonly known type while the informed player has two possible types which are private information. We demonstrated that while in the Tullock contest the informed player may have an incentive to send a costly signal to reveal his type, in the all-pay contest he never has such an incentive. One of the reasons that there is a separating perfect Bayesian equilibrium in the Tullock contest and not in the all-pay contest is that the distributions of the players' revenues are completely different. While in the all-pay contest only one of the players has a positive expected payoff, in the Tullock contest both players have positive expected payoffs and as such pre-contest communication might be useful only in the latter form of contest. Because of the complexity in analyzing multi-stage contests with signaling, we assumed the simplest case of two possible types of players. It would be of interest to examine whether our results hold when the set of types is larger or even continuous.

5 Appendix

5.1 Proof of Proposition 1

If player j believes that he plays against player i with type H, by (7) his effort will be

$$x_j = \frac{v_{iH} v_j^2}{(v_{iH} + v_j)^2} \tag{23}$$

We want to find the optimal effort for player i with type L who has the maximization problem

$$\max_{x_{iL}} \widetilde{u}_{iL}^s = v_{iL} \frac{x_j}{(x_j + x_{iL})} - x_{iL} \tag{24}$$

The F.O.C. is

$$\begin{aligned} v_{iL} \frac{x_j}{(x_{iL} + x_j)^2} - 1 &= 0 \\ &\Rightarrow x_{iL}^2 + 2x_{iL}x_j + x_j^2 - x_j v_{iL} = 0 \end{aligned}$$

The solution of this quadratic equation is

$$x_{iL} = \frac{-2x_j + \sqrt{(2x_j)^2 - 4(x_j^2 - x_j v_{iL})}}{2} \tag{25}$$

By substituting (23) in (25), we obtain that

$$x_{iL} = \frac{v_j v_{iH}\sqrt{v_{iH}v_{iL}} + v_j^2\sqrt{v_{iH}v_{iL}} - v_{iH}v_j^2}{(v_{iH}+v_j)^2} \tag{26}$$

Substituting (26) in (24) yields that the expected payoff of player i with type L when player j believes that he has type H is

$$\widetilde{u}_{iL}^s = \left(\frac{\sqrt{v_{iL}}(v_j+v_{iH}) - v_j\sqrt{v_{iH}}}{v_j+v_{iH}}\right)^2$$

Similarly, we obtain that the expected payoff of player i with type H when player j believes that he has type L is

$$\widetilde{u}_{iH}^s = \left(\frac{\sqrt{v_{iH}}(v_j+v_{iL}) - v_j\sqrt{v_{iL}}}{v_j+v_{iL}}\right)^2$$

Q.E.D.

5.2 Proof of Proposition 2

By (7), when player j believes that he plays against player i with type H, he will exert an effort of $x_{j-H} = \frac{v_{iH}v_j^2}{(v_{iH}+v_j)^2}$, and when he believes that he plays against player i with type L he will exert an effort of $x_{j-L} = \frac{v_{iL}v_j^2}{(v_{iL}+v_j)^2}$. It can be easily verified that

$$\begin{aligned} x_{j-H} &= \frac{v_{iH}v_j^2}{(v_{iH}+v_j)^2} \geq \frac{v_{iL}v_j^2}{(v_{iL}+v_j)^2} = x_{j-L} \\ \text{iff}\ \ v_j &\geq \sqrt{v_{iH}v_{iL}} \end{aligned} \tag{27}$$

Now, by the Envelope Theorem we obtain that if

$$V = \max_{x_i} v_i \frac{x_i}{x_i+x_j} - x_i$$

Then,

$$\frac{dV}{dx_j} = -\frac{v_i x_i}{(x_i+x_j)^2} < 0$$

In other words, player i's expected payoff decreases in player j's effort. Thus, by (27), if $v_j < \sqrt{v_{iH}v_{iL}}$, player i with type L prefers that player j will believe that he has type H since then player j will exert a lower effort. Similarly, if $v_j > \sqrt{v_{iH}v_{iL}}$, player i with type H prefers that player j will believe that he has type L since then player j will exert a lower effort. *Q.E.D.*

5.3 Proof of Proposition 3

By (5) and (8), we have

$$\Delta^H_{s-p} = v_{iH}\left(\frac{v_{iH}^2}{(v_{iH}+v_j)^2} - \left(\frac{v_{iL}\cdot v_{iH} + p_L\cdot v_j(v_{iH} - \sqrt{v_{iL}\cdot v_{iH}})}{v_{iL}\cdot v_{iH} + p_H\cdot v_{iL}\cdot v_j + p_L\cdot v_{iH}\cdot v_j}\right)^2\right)$$

Since $v_{iH} > v_{iL}$, we obtain that $\Delta^H_{s-p} \geq 0$ iff

$$v_{iH}(v_{iL}\cdot v_{iH} + p_H\cdot v_{iL}\cdot v_j + p_L\cdot v_{iH}\cdot v_j) \geq (v_{iH}+v_j)(v_{iL}\cdot v_{iH} + p_L\cdot v_j(v_{iH} - \sqrt{v_{iL}\cdot v_{iH}}))$$

The last inequality holds iff

$$v_j\left(1-\sqrt{\frac{v_{iH}}{v_{iL}}}\right) \geq \sqrt{v_{iH}}(\sqrt{v_{iL}} - \sqrt{v_{iH}}$$

Since $\left(1-\sqrt{\frac{v_{iH}}{v_{iL}}}\right) < 0$, if we divide both sides by this term we obtain that

$$v_j \leq \sqrt{v_{iH}\cdot v_{iL}}$$

Thus, we obtain that player i with type H prefers a separating equilibrium over a pooling equilibrium iff $v_j \leq \sqrt{v_{iH}\cdot v_{iL}}$.

Similarly, by (5) and (8), we have

$$\Delta^L_{s-p} = v_{iL}\left(\frac{v_{iL}^2}{(v_{iL}+v_j)^2} - \left(\frac{v_{iL}\cdot v_{iH} + p_H\cdot v_j(v_{iL} - \sqrt{v_{iL}\cdot v_{iH}})}{v_{iL}\cdot v_{iH} + p_H\cdot v_{iL}\cdot v_j + p_L\cdot v_{iH}\cdot v_j}\right)^2\right)$$

Since $v_{iH} > v_{iL}$, we obtain that $\Delta^L_{s-p} \geq 0$ iff

$$v_{iL}(v_{iL}\cdot v_{iH} + p_H\cdot v_{iL}\cdot v_j + p_L\cdot v_{iH}\cdot v_j) \geq (v_{iL}+v_j)(v_{iL}\cdot v_{iH} + p_H\cdot v_j(v_{iL} - \sqrt{v_{iL}\cdot v_{iH}}))$$

The last inequality holds iff

$$v_j \geq \frac{v_{iL}(\sqrt{v_{iH}\cdot v_{iL}} - v_{iH})}{-\sqrt{v_{iH}\cdot v_{iL}} + v_{iL}} = \sqrt{v_{iH}\cdot v_{iL}}$$

Thus, we obtain that player i with type L prefers the separating equilibrium over the pooling equilibrium iff $v_j \geq \sqrt{v_{iH}\cdot v_{iL}}$. *Q.E.D.*

6 Proof of Proposition 4

1. We first need to show that if $v_j < \sqrt{v_{iH}\cdot v_{iL}}$ then

$$\widetilde{\Delta}^L_{s-p} = \widetilde{u}^s_{iL} - u^p_{iL} > 0$$

By (5) and (11)

$$\widetilde{\Delta}^L_{s-p} = \left(\frac{\sqrt{v_{iL}}(v_j + v_{iH}) - v_j\sqrt{v_{iH}}}{v_j + v_{iH}}\right)^2 - v_{iL}\left(\frac{v_{iL}\cdot v_{iH} + p_H \cdot v_j(v_{iL} - \sqrt{v_{iL}\cdot v_{iH}})}{v_{iL}\cdot v_{iH} + p_H\cdot v_{iL}\cdot v_j + p_L\cdot v_{iH}\cdot v_j}\right)^2$$

Thus, we need to show that

$$\begin{aligned} &(\sqrt{v_{iL}}(v_j + v_{iH}) - v_j\sqrt{v_{iH}})(v_{iL}\cdot v_{iH} + p_H\cdot v_{iL}\cdot v_j + p_L\cdot v_{iH}\cdot v_j) \\ &-\sqrt{v_{iL}}(v_{iL}\cdot v_{iH} + p_H\cdot v_j(v_{iL} - \sqrt{v_{iL}\cdot v_{iH}}))(v_j + v_{iH}) \\ \geq\ & 0 \end{aligned}$$

The last inequality holds iff

$$\sqrt{v_{iL}\cdot v_{iH}}(\sqrt{v_{iH}} - \sqrt{v_{iL)}}) - v_j(\sqrt{v_{iH}} - \sqrt{v_{iL)}}) \geq 0$$

Thus, $\widetilde{\Delta}^L_{s-p} \geq 0$ iff $v_j \leq \sqrt{v_{iH}\cdot v_{iL}}$.

2. Now we need to show that if $v_j > \sqrt{v_{iH}\cdot v_{iL}}$ then

$$\widetilde{\Delta}^H_{s-p} = \widetilde{u}^s_{iH} - u^p_{iH} > 0$$

By (5) and (12)

$$\widetilde{\Delta}^H_{s-p} = \left(\frac{\sqrt{v_{iH}}(v_j + v_{iL}) - v_j\sqrt{v_{iL}}}{v_j + v_{iL}}\right)^2 - v_{iH}\left(\frac{v_{iL}\cdot v_{iH} + p_L \cdot v_j(v_{iH} - \sqrt{v_{iL}\cdot v_{iH}})}{v_{iL}\cdot v_{iH} + p_H\cdot v_{iL}\cdot v_j + p_L\cdot v_{iH}\cdot v_j}\right)^2$$

Thus, we need to show that

$$\begin{aligned} &(\sqrt{v_{iH}}(v_j + v_{iL}) - v_j\sqrt{v_{iL}})(v_{iL}\cdot v_{iH} + p_H\cdot v_{iL}\cdot v_j + p_L\cdot v_{iH}\cdot v_j) \\ &-(v_{iL}\cdot v_{iH} + p_L\cdot v_j(v_{iH} - \sqrt{v_{iL}\cdot v_{iH}}))(v_j + v_{iL})\sqrt{v_{iH}} \\ \geq\ & 0 \end{aligned}$$

The last inequality holds iff

$$v_j(\sqrt{v_{iH}} - \sqrt{v_{iL)}}) - \sqrt{v_{iL}\cdot v_{iH}}(\sqrt{v_{iH}} - \sqrt{v_{iL)}}) \geq 0$$

Thus, $\widetilde{\Delta}^H_{s-p} \geq 0$ iff $v_j \geq \sqrt{v_{iH}\cdot v_{iL}}$. *Q.E.D.*

6.1 Proof of Theorem 5

By Proposition 3, if $v_j < \sqrt{v_{iH} \cdot v_{iL}}$, player i with type H has a higher expected payoff under the separating equilibrium than under the pooling equilibrium, and player i with type L has a lower expected payoff under the separating equilibrium than under the pooling equilibrium.

1. If $\Delta^H_{s-p} < -\Delta^L_{s-p}$, type H is willing to pay less for the separating equilibrium than what type L is willing to pay for the pooling equilibrium. In that case, by Definition 2, for any signal s_{iH} there will be the same pooling equilibrium as without this signal. Thus, player i with type H has no incentive to send a costly signal and therefore the pooling equilibrium occurs.

2. If $\Delta^H_{s-p} > -\Delta^L_{s-p}$, type H is willing to pay more for the separating equilibrium than what type L is willing to pay for the pooling equilibrium. In that case, if there is a separating equilibrium, by Proposition 4 player i with type L prefers that player j will believe that he has type H. However, since player i with type H sends a signal of $s_{iH} = \overline{\Delta}^L_{s-s} > 0$, type L will not have an incentive to send this signal as well. Thus, since only type H sends a signal, by Definition 2 player j can distinguish between player i's types according to the signal. If player i with type L will send a signal $s_{iL} < \overline{\Delta}^L_{s-s}$, by Definition 2 there will be the same pooling equilibrium as without this signal. Thus, player i with type L has no incentive to send a costly signal. In order to complete the proof we need to show that a signal of $\overline{\Delta}^L_{s-s}$ is not too expensive for type H, namely, he prefers the separating equilibrium with the signal payment $\overline{\Delta}^L_{s-s}$ over the pooling equilibrium without any signal payment. Below, we show that

$$\lim_{p_H \to 0} (\Delta^H_{s-p} - \overline{\Delta}^L_{s-s}) > 0$$

By (5) and (8),

$$\lim_{p_H \to 0} \Delta^H_{s-p} = v_{iH} \left(\frac{v_{iH}^2}{(v_{iH} + v_j)^2} - \left(\frac{v_{iL} \cdot v_{iH} + v_j(v_{iH} - \sqrt{v_{iL} \cdot v_{iH}})}{v_{iH}(v_{iL} + v_j)} \right)^2 \right)$$

and by (5), (8) and (11),

$$\overline{\Delta}^L_{s-s} = \left(\frac{\sqrt{v_{iL}}(v_{iH} + v_j) - v_j \sqrt{v_{iH}}}{v_{iH} + v_j} \right)^2 - \frac{v_{iL}^3}{(v_{iL} + v_j)^2}$$

Note that $\overline{\Delta}^L_{s-s}$ does not depend on the value of p_H. Thus, we need to show that

$$\begin{aligned}\lim_{p_H \to 0} (\Delta^H_{s-p} - \overline{\Delta}^L_{s-s}) &= v_{iH}\left(\frac{v_{iH}^2}{(v_{iH}+v_j)^2} - \left(\frac{v_{iL}\cdot v_{iH} + v_j(v_{iH} - \sqrt{v_{iL}\cdot v_{iH}})}{v_{iH}(v_{iL}+v_j)}\right)^2\right) - \left(\frac{\sqrt{v_{iL}}(v_{iH}+v_j) - v_j\sqrt{v_{iH}})}{v_{iH}+v_j}\right)^2 - \frac{\quad}{(v_i} \\ &= = \frac{v_{iH}^3 - (v_{iH}\sqrt{v_{iL}} - v_j(\sqrt{v_{iH}} - \sqrt{v_{iL}}))^2}{(v_{iH}+v_j)^2} + \frac{v_{iL}^3 - (v_{iL}\sqrt{v_{iH}} + v_j(\sqrt{v_{iH}} - \sqrt{v_{iL}}))^2}{(v_{iL}+v_j)^2} \geq 0\end{aligned}$$

It can be verified that the last inequality is satisfied iff $v_j \leq \sqrt{v_{iH}\cdot v_{iL}}$ which is exactly our assumption here about the players' values. Therefore, in order to show that there is a separating perfect Bayesian equilibrium, it remains to show that

$$\lim_{p_H \to 0} (\Delta^H_{s-p} - (-\Delta^L_{s-p})) = \lim_{p_H \to 0} (\Delta^H_{s-p} + \Delta^L_{s-p}) > 0$$

By (5) and (8),

$$\lim_{p_H \to 0} \Delta^L_{s-p} = v_{iL}\left(\frac{v_{iL}^2}{(v_{iL}+v_j)^2} - \frac{v_{iL}^2}{(v_{iL}+v_j)^2}\right) = 0$$

and since $\lim_{p_H \to 0} \Delta^H_{s-p} > 0$, we obtain that $\lim_{p_H \to 0}(\Delta^H_{s-p} + \Delta^L_{s-p}) > 0$. *Q.E.D.*

6.2 Proof of Theorem 6

By Proposition 3, if $v_j > \sqrt{v_{iH}\cdot v_{iL}}$, player i with type L has a higher expected payoff under the separating equilibrium than under the pooling equilibrium and player i with type H has a lower expected payoff under the separating equilibrium than under the pooling equilibrium. Then we have two cases:

1. If $-\Delta^H_{s-p} > \Delta^L_{s-p}$, type L is willing to pay less for the separating equilibrium than what type H is willing to pay for the pooling equilibrium. In that case, by Definition 3, for any signal s_{iL} there will be the same pooling equilibrium as without this signal. Thus, player i with type L has no incentive to send a costly signal and the pooling equilibrium occurs.

2. If $-\Delta^H_{s-p} < \Delta^L_{s-p}$, type L is willing to pay more for the separating equilibrium than what type H is willing to pay for the pooling equilibrium. In that case, if there is a separating equilibrium, by Proposition 4 player i with type H prefers that player j will believe that he has type L. However, if player i with type H will send a signal of $s_{iH} \geq \overline{\Delta}^H_{s-s} > 0$, type L will not have an incentive to send this signal as well. Then, since only type L sends a signal, by Definition 3 player j could distinguish between player i's types according to the signal sent in the first stage. Any lower signal than $\overline{\Delta}^H_{s-s}$ will give type H the incentive to

send the same signal in order to pretend that he has type L. However, below we show that a signal of $\overline{\Delta}^{H}_{s-s}$ is too expensive for type L, namely, he prefers the pooling equilibrium without any signal payment over the separating equilibrium with the signal payment $\overline{\Delta}^{H}_{s-s}$. Thus, below we show that

$$\Delta^{L}_{s-p} - \overline{\Delta}^{H}_{s-s} < 0$$

By (5) and (8),

$$\Delta^{L}_{s-p} = v_{iL}\left(\frac{v_{iL}^2}{(v_{iL}+v_j)^2} - \left(\frac{v_{iL}\cdot v_{iH} + p_H\cdot v_j(v_{iL} - \sqrt{v_{iL}\cdot v_{iH}})}{v_{iL}\cdot v_{iH} + p_H\cdot v_{iL}\cdot v_j + p_L\cdot v_{iH}\cdot v_j}\right)^2\right)$$

and by (5), (8) and (11),

$$\overline{\Delta}^{H}_{s-s} = \left(\frac{\sqrt{v_{iH}}(v_{iL}+v_j) - v_j\sqrt{v_{iL}})}{v_{iL}+v_j}\right)^2 - \frac{v_{iH}^3}{(v_{iH}+v_j)^2}$$

Note that

$$\begin{aligned}\frac{d\Delta^{L}_{s-p}}{dp_L} &= -2v_{iL}\left(\frac{v_{iL}v_{iH} + (1-p_L)v_j(v_{iL}-\sqrt{v_{iH}v_{iL}})}{v_{iL}v_{iH} + (1-p_L)v_jv_{iL} + p_Lv_jv_{iH}}\right)\\ &\cdot\left(\frac{-v_j(v_{iL}-\sqrt{v_{iH}v_{iL}})(v_{iL}v_{iH} + (1-p_L)v_jv_{iL} + p_Lv_jv_{iH}) - v_j(v_{iH}-v_{iL})(v_{iL}v_{iH} + (1-p_L)v_j(v_{iL}-\sqrt{v_{iH}v_{iL}})}{(v_{iL}v_{iH} + (1-p_L)v_jv_{iL} + p_Lv_jv_{iH})^2}\right)\end{aligned}$$

Thus, $\frac{d\Delta^{L}_{s-p}}{dp_L} \leq 0$ iff

$$(v_{iL} - \sqrt{v_{iH}v_{iL}})(v_{iL}v_{iH} + (1-p_L)v_jv_{iL} + p_Lv_jv_{iH}) + (v_{iH}-v_{iL})(v_{iL}v_{iH} + (1-p_L)v_j(v_{iL}-\sqrt{v_{iH}v_{iL}}) \leq 0$$

It can verified that the last inequality holds iff $\sqrt{v_{iH}v_{iL}} < v_j$ which is exactly our assumption on the players' values.[4] Thus, Δ^{L}_{s-p} decreases in p_L and therefore it is sufficient to show that

$$\lim_{p_L\to 0}(\Delta^{L}_{s-p} - \overline{\Delta}^{H}_{s-s}) = v_{iL}\left(\frac{v_{iL}^2}{(v_{iL}+v_j)^2} - \left(\frac{v_{iL}\cdot v_{iH} + p_H\cdot v_j(v_{iL} - \sqrt{v_{iL}\cdot v_{iH}})}{v_{iL}\cdot v_{iH} + p_H\cdot v_{iL}\cdot v_j + p_L\cdot v_{iH}\cdot v_j}\right)^2\right) - \left(\frac{\sqrt{v_{iH}}(v_{iL}+v_j) - v_j\sqrt{v_{iL}})}{v_{iL}+v_j}\right)^2 - \qquad (29)$$

A comparison of equations (28) and (29) yields that

$$\lim_{p_L\to 0}(\Delta^{L}_{s-p} - \overline{\Delta}^{H}_{s-s}) = \lim_{p_H\to 0}(\Delta^{H}_{s-p} - \overline{\Delta}^{L}_{s-s})$$

Thus, the inequality in (29) is satisfied iff $v_j \geq \sqrt{v_{iH}\cdot v_{iL}}$ which is exactly our assumption here about the players' values. Therefore a separating equilibrium is not possible and we have only the pooling equilibrium in which no type of player i sends a signal. *Q.E.D.*

[4] The complete mathematical calculations are available upon request.

6.3 Proof of Theorem 7

By Proposition 3, if $v_j = \sqrt{v_{iH} \cdot v_{iL}}$, player i with either type L or type H has the same expected payoff under the separating equilibrium and under the pooling equilibrium. By Definition 4, any signal of player i will not change the prior beliefs of player j and therefore player i, independent of his type, has no incentive to send any signal. *Q.E.D.*

6.4 Proof of Proposition 5

We can see that the functions $F_{iH}(x)$, $F_j(x)$, given by (16) and (17), respectively, are well-defined, strictly increasing on $[0, v_{iH}]$, continuous, and that $F_j(0) = 0$, $F_{iH}(0) = \max\{0, \frac{p_H \cdot v_j - v_{iH}}{p_H \cdot v_j}\}$, $F_{iH}(v_{iH}) = F_j(v_{iH}) = 1$. Thus, $F_{iH}(x)$, $F_j(x)$ are cumulative distribution functions of continuous probability distributions supported on $[0, v_{iH}]$. In order to see that the above strategies are an equilibrium, note that when contestant j uses the mixed strategy $F_j(x)$, the expected payoff of contestant i with types L and H is zero for any effort $x \in [0, v_{iH}]$. Since it can be easily shown that efforts above v_{iH} would lead to a negative expected payoff for contestant i, any effort in $[0, v_{iH}]$ is a best response of contestant i with type H to $F_j(x)$. Likewise, $x = 0$ is the best response of contestant i with type L to $F_j(x)$. Similarly, when contestant i uses the mixed strategy $F_{iH}(x)$ and $F_{iL}(x)$, contestant j's expected payoff is $v_j - v_{iH}$ for any effort $x \in [0, v_{iH}]$. Since it can be easily shown that efforts above v_{iH} would result in a lower expected payoff for contestant j, any effort in $[0, v_{iH}]$ is a best response of contestant j to $F_{iH}(x)$ and $F_{iL}(x)$. Hence, $(F_{iH}(x), F_{iL}(x), F_j(x))$ are the equilibrium strategies in the second stage. *Q.E.D.*

6.5 Proof of Proposition 6

We can see that the functions $F_{iL}, F_{iH}(x)$, $F_j(x)$, given by (19), (20) and (21), respectively, are well-defined, F_{iL} is strictly increasing on $[0, p_L \cdot v_j]$, $F_{iH}(x)$ strictly increasing on $[p_L \cdot v_j, v_j]$, and $F_j(x)$ is strictly increasing on $[0, v_j]$. They are all continuous, satisfy $F_{iL}(0) = F_j(0) = 0$, $F_{iL}(p_L \cdot v_j) = 1, F_{iH}(p_L \cdot v_j) = 0$, and that $F_j(v_j) = F_{iH}(v_j) = 1$. Thus, $F_{iL}, F_{iH}(x)$, $F_j(x)$ are cumulative distribution functions of continuous probability distributions supported on $[0, p_L \cdot v_j], [p_L \cdot v_j, v_j], [0, v_j]$, respectively. In order to see that the above strategies are an equilibrium, notice that when contestant i uses the mixed strategy $F_{iL}(x)$ or $F_{iH}(x)$, the expected payoff of contestant j is zero for any effort $x \in [0, v_j]$. Since it can be easily shown that efforts above v_j would lead to a negative expected payoff for contestant j, any effort in $[0, v_j]$ is a best response of contestant j. Likewise, when contestant j uses the mixed strategy $F_j(x)$, the expected payoff of contestant i with type H is $v_{iH} - v_j$ for any effort $x \in [p_L \cdot v_j, v_j]$, and the expected payoff of contestant i with type L is $v_{iL} - v_j(\frac{p_H \cdot v_{iL}}{v_{iH}} + p_L)$ for any effort $x \in [0, p_L \cdot v_j]$. Since it can be easily shown that efforts above $p_L \cdot v_j$ would result in a lower expected payoff for contestant i with type L, and efforts below $p_L \cdot v_j$ or above v_j would result in a lower expected payoff for contestant i with type H, any effort in $[0, p_L \cdot v_j]$ is a best response of contestant i with type L, and any effort in $[p_L \cdot v_j, v_j]$ is a best response of contestant i with type H to $F_j(x)$. Hence, $(F_{iH}(x), F_{iL}(x), F_j(x))$ are the equilibrium strategies in the second stage. *Q.E.D.*

6.6 Proof of Proposition 7

1. Assume first that $v_j > v_{iH} > v_{iL}$. Then, by (15), if there is a separating perfect Bayesian equilibrium, the players' expected payoffs in the second stage are

$$\begin{aligned} u_j &= v_j - v_i \\ u_i &= 0 \end{aligned}$$

where $v_i = v_{iH}$ if player i has type H and $v_i = v_{iL}$ if player i has type L. Since, independent of his type, player i has an expected payoff of zero in the second stage he has no incentive to send a costly signal in the first stage.

2. Assume that $v_j < v_{iL} < v_{iH}$. Then, by (15), if there is a separating perfect Bayesian equilibrium, the players' expected payoffs in the second stage are

$$\begin{aligned} u_j &= 0 \\ u_i &= v_i - v_j \end{aligned}$$

where $v_i = v_{iH}$ if player i has type H and $v_i = v_{iL}$ if player i has type L. If, on the other hand, there is a pooling perfect Bayesian equilibrium, by (22) the players' expected payoffs in the second stage are

$$\begin{aligned}
u_{iL} &= v_{iL} - v_j(\frac{p_H \cdot v_{iL}}{v_{iH}} + p_L) \\
u_{iH} &= v_{iH} - v_j \\
u_j &= 0
\end{aligned}$$

Since player i with type H has the same expected payoff in the second stage in both types of equilibrium he has no incentive to send a costly signal in the first stage. However, Player i with type L has an incentive to send a signal iff

$$v_{iL} - v_j > v_{iL} - v_j(\frac{p_H \cdot v_{iL}}{v_{iH}} + p_L)$$

or alternatively, iff

$$\frac{p_H \cdot v_{iL}}{v_{iH}} + p_L > 1$$

But since $v_{iH} > v_{iL}$, the last inequality does not hold. Thus, player i with either type H or L has no incentive to send a costly signal in the first stage. *Q.E.D.*

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