ALL-PAY AUCTIONS WITH ASYMMETRIC EFFORT CONSTRAINTS

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Discussion Paper No. 17-06

November 2017

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All-Pay Auctions with Asymmetric Effort Constraints

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July 31, 2017

Abstract

We study all-pay auctions with discrete strategy sets and analyze the equilibrium strategies when players have asymmetric values of winning as well as asymmetric effort constraints. We show that for any number of players if one of them has the highest effort constraint then, independent of the players' values of winning, he is the only player with a positive expected payoff. In a case that two players have the same highest effort constraint then they do not necessarily have the highest expected payoffs. Our results show a significant distinction of the equilibrium strategies between two players and a larger number of players, particularly when the player with the highest effort constraint is not unique.

Keywords: All-pay auctions, asymmetric effort constraints, asymmetric players, weakly asymmetric players.

JEL classification: D44.

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1 Introduction

The all-pay auction is one of the main contest forms in the literature on contest theory. Numerous applications have been made to rent-seeking and lobbying in organizations, R&D races, political contests, promotions in labor markets, trade wars and military and biological wars of attrition. In all-pay auctions, all players, including those who do not win the prize, incur costs as a result of their efforts, but only the player with the highest effort receives the prize. Hillman and Samet (1987), Hillman and Riley (1989) and Bay et al. (1996) characterized the equilibrium strategies of the all-pay auction under complete information. Che and Gale (1996) showed that the all-pay auction dominates the first-price auction with respect to the players' total effort when the players are effort constrained and these constraints are private information to the players. When the players have the same effort constraint which is commonly known, Che and Gale (1998) calculated the bidding equilibrium of the all-pay auction with two players having different values for a prize and linear cost functions, and demonstrated that if the effort constraint is smaller than or equal to half of the players' smaller winning value, the expected total effort might be higher than in the same contest where the players do not have any effort constraint.[1] On the other hand, if the players have the same effort constraint and that it is larger than half of the players' smaller winning value, the players' expected payoffs as well as their expected total effort are the same as in the standard all-pay auction without any effort constraint.[2] Later, Hart (2016) showed that

[1] Maskin (2000) showed that the all-pay auction is constrained efficient, namely, it maximizes expected welfare subject to incentive-compatibility and budget constraints.

[2] All-pay auctions under incomplete information have been studied by Gavious, Moldovanu and Sela (2003) who showed that, regardless of the number of bidders, if agents have linear or concave cost functions then setting a bid cap (effort constraint) is not profitable for a designer who wishes to maximize the average bid. On the other hand, if agents have convex cost functions, then effectively capping the bids is profitable for a designer facing a sufficiently large number of bidders.

the all-pay auction is an application of the Captain Lotto game which is a Lotto game with caps. Using this equivalence he characterized the equilibrium strategies in the two-player all-pay auction with different effort constraints. The analysis of the all-pay auction with more than two effort-constrained players has not as yet been done, and therefore the aim of this paper is to shed light on this contest when there are more than two players with asymmetric effort constraints, and particularly to show that the behavior of these players might be completely different than in the all-pay auction with only two players. Furthermore, our results will show that the behavior of the players in all-pay auctions with multiple players and asymmetric effort constraints breaks some well-known conventions about the model of the all-pay auction with and without effort constraints.

We consider all-pay auctions with discrete strategy sets, namely, the players have effort constraints and finite strategy sets. The main difference between our model with discrete strategies and the standard all-pay auction with continuous strategies is that in the standard model the probability of a tie (the players exert the same effort) is zero, while in our model there is a positive probability for a tie. Baye, Kovenock and de Vries (1994) and Cohen and Sela (2007) studied the symmetric and asymmetric two-player all-pay auction with discrete strategies but without effort constraints, and showed that the equilibrium strategies are similar to those of the standard all-pay auction with continuous strategies. However, Dechenaux et al. (2012) showed that when there is a symmetric budget constraint, the all-pay auction with discrete strategies and the standard all-pay auction with continuous strategies are not necessarily similar. Given that we do not know much about the standard multiple-player all-pay auction with asymmetric effort constraints, we can only conjecture that the distinction between the standard all-pay auction and the all-pay auction with discrete strategies is even stronger when there are more than two players with asymmetric effort constraints.

We first analyze the equilibrium strategies in the two-player all-pay auction. Since the equilibrium in this model with asymmetric effort constraints is not unique even when the smallest money unit converges to zero, the equilibrium in our model and the standard all-pay auction is not necessarily similar. However, independent of the size of the smallest money unit, we characterize common properties that provide a uniform framework to all the equilibrium points. We show that the probability of every pure strategy to be chosen in equilibrium, except the lowest (zero) and the highest (minimal level of the players' budget constraints) possible efforts, converges to zero when the smallest money unit also converges to zero. Furthermore, the maximal distance between two adjacent strategies that are chosen by a player with a positive probability is twice than the smallest money unit. Although the equilibrium is generally not unique, we show that when the smallest money unit converges to zero in any equilibrium point, independent of the players' values of winning, the expected payoff of the player with the lower effort constraint converges to zero, while the expected payoff of the player with the higher effort constraint converges to the difference of this player's value of winning and his opponent's effort constraint. Last, we demonstrate that by imposing an effort cap on the player with the higher value of winning, a contest designer can attaint an expected total effort that is larger than in the all-pay auction with either discrete or continuous strategies when both players do not face any constraint.

In contrast to the two-player all-pay auction, when there are more than two players with asymmetric effort constraints, all the players may be active when each of them has a completely different strategy as well as a different expected payoff and a different probability to win the contest. However, for each all-pay auction with multiple asymmetric effort-constrained players, by mathematical methods, we are able to numerically calculate systems of non-linear equations and derive the equilibrium strategies . Using these equilibrium calculations we find that several well-known facts about the two-player all-pay auction with and without effort constraints no longer hold. For instance, when there are more than two players the probability of every pure strategy to be chosen in equilibrium does not necessarily converge to zero when the smallest money unit converges to zero, and the maximal distance between two adjacent strategies that are chosen by a player with a positive probability might be significantly higher than twice the size of the money unit. We also show that in all-pay auctions with more than two effort-constrained players, a player may have a completely different expected payoff for different equilibrium strategies. Moreover, given an all-pay auction, in one equilibrium player i is the only player with a positive expected payoff, and in another player j is the only player with a positive expected payoff.

In the all-pay auction with multiple asymmetric effort-constrained players it is not clear which parameters either the players' values of winning or their effort constraints have a higher effect on the players' expected payoffs. Although their equilibrium strategies are quite complex we provide a clear answer to this last question by showing that when the smallest money unit converges to zero, independent of the players' values of winning, the player with the highest effort constraint is the only one with a positive expected payoff. However, when players are weakly asymmetric such that more than one player has the highest effort constraint, there is a major difference between the all-pay auction with more than two players and the one with either two symmetric or asymmetric players: In the two-player contest the values of the players do not have any effect on who the player with a positive expected payoff is but only on the level of this player's expected payoff. The parameters that affect which player has the higher expected payoff in the two-player contest are the effort constraints, in that the player with the higher effort constraint is the only one with a positive expected payoff. On the other hand, in the all-pay auction with more than two players the value of a player as well as his effort constraint affect whether this player has a positive expected payoff or not. In other words, if players are weakly asymmetric and a player has a sufficiently high value of winning, even if he does not have the highest effort constraint, namely, there are at least two players with higher effort constraints, he might have a positive expected payoff.

The rest of this paper is organized as follows: In Section 2 we introduce the all-pay auction with discrete strategies. In Section 3 we analyze the all-pay auction with two asymmetric effort-constrained players. In Sections 3 and 4 we analyze the all-pay auction with a larger number of effort-constrained players who are asymmetric and weakly asymmetric. Section 5 concludes. The proofs appear in the Appendix.

2 The model

Consider n players competing for a single prize in an all-pay auction. The value of winning in the contest for player i is $v_i, i = 1, 2, ..., n$. Valuations are common knowledge. Each player exerts an effort $x_i \in \{n\epsilon : n = 0, 1, 2, 3...\}$ where ϵ denotes the smallest money unit and satisfies $\epsilon = \frac{1}{k}$ for some integer $k = 1, 2, 3, ...,$ and then bears the cost of his effort. The player with the highest effort wins. For simplicity, we postulate a deterministic relation between effort and output, and assume them to be equal. In the case of a tie in which some players exert the same effort, we assume that each of the players with the highest effort wins with the same probability. Player i has a commonly-known effort constraint $d_i = m_i\epsilon < v_i$ where $m_i, i = 1, 2, ..., n$ are some integers. Player i's effort is smaller than or equal to his effort constraint, namely, $x_i \leq d_i$. If the players exert efforts $x_i \leq d_i, i = 1, 2, ..., n$ then the payoff for player i is given by

$$u_i(x_1, ..., x_n) = \begin{cases} -x_i & \text{if } x_i < \max_j x_j \\ \frac{1}{m(x)}v_i - x_i & \text{if } x_i = \max_{j \neq i} x_j \\ v_i - x_i & \text{if } x_i > \max_j x_j \end{cases}$$

where $m(x)$ denotes the number of players who exert the highest effort, namely,

$$m(x) = \left| s \in \{1, 2, ..., n\} : x_s = \max\{x_j\}_{j=1}^n \right|$$

3 Asymmetric two-player contests

We begin with the analysis of the two-player all-pay auction with discrete strategies. It should be noted that such an analysis is not necessarily similar to that of the all-pay auction with continuous strategies (Hart 2016). The following example illustrates a mixed strategy equilibrium in an all-pay auction with asymmetric effort constrained players.

Example 1 *Consider a two-player all-pay auction where players have the same value of winning* $v = 8$*, player 1's effort constraint is* $d_1 = 5$*, and player 2's effort constraint is* $d_2 = 4$*. Let the smallest money unit be* $\epsilon = 1$*. Then, it can be verified that there is an equilibrium where player 1's strategy is:*

$x_1:$	0	1	2	3	4	5
$p_{x_1}:$	0	$\frac{1}{4}$	0	$\frac{1}{4}$	0	$\frac{1}{2}$

That is, player 1 chooses every $x_1 \in \{1, 3\}$ *with probability* $p_{x_1} = \frac{1}{4}$*, and* $x_1 = 5$ *with probability* $p_{x_1} = \frac{1}{2}$*. Player 2's equilibrium strategy is:*

$x_2:$	0	1	2	3	4
$p_{x_2}:$	$\frac{1}{2}$	0	$\frac{1}{4}$	0	$\frac{1}{4}$

That is, player 2 chooses $x_2 = 0$ *with probability* $p_{x_2} = \frac{1}{2}$*, and every* $x_2 \in \{2, 4\}$ *with probability* $p_{x_2} = \frac{1}{4}$ *. Then, player 2 has an expected payoff of zero and player 1 has an expected payoff of* 3.

Similarly to the above example, it can be verified that in an all-pay auction with two players and asymmetric effort constraints $d_1 > d_2 \geq \epsilon$, there is no pure strategy equilibrium. Moreover, in this model the equilibrium strategies are generally not unique. Nevertheless, below we characterize common properties that provide a uniform framework for all the equilibria. The first result demonstrates that, when ϵ approaches zero, there are no mass points on the internal points of the support of the players' mixed strategies.

Proposition 1 *Consider a two-player all-pay auction where players have the values of winning v_1 and v_2 and effort constraints $d_1 \geq d_2 > \epsilon$. Then, in any equilibrium, the probability that player 2 chooses the effort $0 < x_2 < d_2$ satisfies*

$$p_{x_2} \leq \frac{2\epsilon}{v_1}$$

and the probability that player1 chooses the effort $0 \leq x_1 < d_2$ satisfies

$$p_{x_1} \leq \frac{2\epsilon}{v_2}$$

Thus, for all $0 < x_2 < d_2$ and $0 < x_1 < d_1$ we have

$$\lim_{\epsilon \longrightarrow 0} p_{x_2}(\epsilon) = \lim_{\epsilon \longrightarrow 0} p_{x_1}(\epsilon) = 0$$

Proof. See Appendix. ■

In Proposition 1 we showed that the probability of every effort (except the lowest and the highest possible efforts) to be chosen in equilibrium converges to zero when the smallest money unit ϵ approaches zero. Note that this result does not indicate that each of the player's efforts is uniformly distributed over the support of the players' mixed strategies. However, the following result shows that the distance between two adjacent strategies x_i, y_i that are chosen with positive probability by player i is not larger than twice the smallest money unit ϵ.

Proposition 2 *Consider a two-player all-pay auction where players have the values of winning v_1 and v_2 and the effort constraints $d_1 \geq d_2 > \epsilon$. Then, in any equilibrium, if player i assigns a positive probability to the efforts x_i and $y_i = \min\{z_i : z_i = n\epsilon, n \in N, d_i > z_i > x_i\}$, the difference $y_i - x_i$ is equal to either ϵ or 2ϵ.*

Proof. See Appendix. ■

In the following, we demonstrate that in any equilibrium point when the smallest money unit ϵ converges to zero, independent of the players' values of winning, the expected payoff of the player with the lower effort constraint approaches zero, and the expected payoff of the player with the higher effort constraint approaches the difference of his value of winning and the level of his opponent's effort constraint.

Proposition 3 *Consider a two-player all-pay auction where the players have the values of winning* v_1 *and* v_2 *and the effort constraints* $d_1 > d_2 > \epsilon$*. Then in any equilibrium, the expected payoff of player 2 satisfies*

$$\lim_{\epsilon \to 0} R_2(\epsilon) = 0$$

and the expected payoff of player 1 satisfies

$$\lim_{\epsilon \to 0} R_1(\epsilon) = v_1 - d_2$$

Proof. See Appendix. ■

Below we show that imposing asymmetric effort caps (constraints) may improve the players' expected total effort with respect to the same contest with and without any symmetric effort cap.

Proposition 4 *In a two-player all-pay auction where the players have values of winning* $v_1 > v_2 = 2s\epsilon$*, for some integer* s*, if an effort cap of* $d_1 = v_2 - 2\epsilon$ *is imposed on player 1 only, the players' expected total effort satisfies*

$$\lim_{\epsilon \to 0} TB(\epsilon) = \frac{3v_2}{2} - \frac{(v_2)^2}{2v_1} \geq v_2$$

Proof. See Appendix. ■

By Proposition 4, we can see that the upper limit of the players' expected total effort is $\frac{3v_2}{2}$ which occurs when $\frac{v_2}{v_1}$ converges to zero. Moreover, independent of the players' values of winning, when an effort cap of $d = v_2 - 2\epsilon$ is imposed on the player with the higher value of winning (player 1), the players' expected total effort is larger than or equal to v_2

which is the players' highest expected total effort when a symmetric effort cap is imposed on both players. Interestingly, Szech (2012) showed that the designer can achieve such a high expected total effort also in the standard all-pay auction with a symmetric effort cap by imposing an asymmetric tie-breaking rule that favors the weaker player. Similarly to Proposition 4, Hart (2016) showed that the designer can achieve such a high expected total effort also in the standard all-pay auction with continuous strategies and without an asymmetric tie-breaking rule.

4 Asymmetric multi-player contests

We now consider the case of the all-pay auction with more than two effort-constrained players. While the generalization of the standard two-player all-pay auction without effort constraints to the case with more than two players is quite simple (see Baye Kovenock and de Vries 1996) the generalization of our all-pay auction model with asymmetric budget constraints is rather complex. In order to shed some light on this issue, we consider three players with the values of winning $v_1 \geq v_2 \geq v_3$ who have different budget constraints d_1, d_2, d_3 respectively. As such, we have six possible cases, for three of which the equilibrium strategies are immediately derived from the two-player model as follows:

1. $d_1 \geq d_2 \geq d_3$: In this case, we have the equilibrium where players 1 and 2 participate in the contest and player 3 stays out. The expected payoff of player 1 is positive while the expected payoff of player 2 is zero.

2. $d_2 \geq d_1 \geq d_3$: In this case, we have the equilibrium where players 1 and 2 participate in the contest and player 3 stays out. The expected payoff of player 2 is positive while the expected payoff of player 1 is zero.

3. $d_3 \geq d_1 \geq d_2$: In this case, we have the equilibrium where players 1 and 3 participate in the contest and player 2 stays out. The expected payoff of player 3 is positive while the

expected payoff of player 1 is zero.

For the other three cases: $d_1 \geq d_3 \geq d_2$, $d_2 \geq d_3 \geq d_1$ and $d_3 \geq d_2 \geq d_1$, the analysis of the equilibrium is more complex since all the three players might take part in the contest. Indeed, in the following example where $d_3 \geq d_2 \geq d_1$ we show that when there are three players who have different values of winning as well as different effort constraints all the players may be active.

Example 2 *Consider an all-pay auction with three players where the players' values of winning are* $v_1 = 10$ *,*$v_2 = 8, v_3 = 7$ *and they face effort constraints of* $d_1 = 3, d_2 = 4, d_3 = 5$*. Let the smallest money unit be* $\epsilon = 1$*. Then, it can be verified that there is an equilibrium where player 1 has the strategy:*

$x_1:$	0	1	2	3	4	5
p_{x_1}	$\frac{3}{5}$	0	$\frac{2}{5}$	0	0	0

That is, player 1 chooses $x_1 = 0$ *with probability* $p_{x_1} = \frac{3}{5}$ *and* $x_1 = 2$ *with probability* $p_{x_1} = \frac{2}{5}$*. Player 2 has the strategy:*

$x_2:$	0	1	2	3	4	5
p_{x2}	$\frac{5}{7}$	0	0	0	$\frac{2}{7}$	0

That is, player 2 chooses $x_2 = 0$ *with probability* $p_{x_2} = \frac{5}{7}$*, and* $x_2 = 4$ *with probability* $p_{x_2} = \frac{2}{7}$*. Player 3 has the strategy:*

$x_3:$	0	1	2	3	4	5
p_{x_3}	0	$\frac{7}{25}$	0	0	$\frac{11}{25}$	$\frac{7}{25}$

That is, player 3 chooses $x_3 = 1$ *with probability* $p_{x_3} = \frac{7}{25}$*,* $x_3 = 4$ *with probability* $p_{x_3} = \frac{11}{25}$ *and* $x_3 = 5$ *with probability* $p_{x_3} = \frac{7}{25}$*. In that case, player 3's expected payoff is 2 and the other players have an expected payoff of zero.*

An all-pay auction with multiple players might have several equilibria points where in each of them the set of active players may be different. Nevertheless, similarly to the all-pay auction with only two players, we can demonstrate that if there is a single player who has the highest effort constraint, in any equilibrium point when the smallest money unit ϵ converges to zero, the expected payoffs of all the other players with lower effort constraints approach zero.

Theorem 1 *Consider an all-pay auction where players have the values of winning* $v_1, v_2, ..., v_n$ *and the effort constraints* $d_1, ..., d_n$ *that satisfy* $d_1 > d_j$ *for all* $j = 2, 3, ..., n$. *Then, in any equilibrium, the expected payoff of player* $j, j \neq 1$ *satisfies*

$$\lim_{\epsilon \to 0} R_j(\epsilon) = 0$$

Proof. See Appendix. ■

5 Weakly asymmetric multi-player contests

Interestingly, the most complex case in multi-player all-pay auctions with asymmetric effort constraints is when the players are weakly asymmetric such that there are more than one player with the highest effort constraint. In such a case it is hard to provide general properties of the equilibrium strategies in the all-pay auction with asymmetric effort-constrained players. One of the reasons is that the properties of the equilibrium strategies of the two-player all-pay auction do not hold for the equilibrium strategies of the all-pay auction with more than two players. In the next example, we show that Proposition 2 does not hold when there are more than two players, and in particular, there are adjacent strategies which are not chosen in equilibrium by any of the players.

Example 3 *Consider an all-pay auction with three players where the players' values of winning are* $v_1 = 15$ *,*$v_2 = v_3 = 8$ *and they face effort constraints of* $d_1 = 3, d_2 = d_3 = 5$. *Let the smallest money unit be* $\epsilon = 1$. *In that case, it can be verified that there is an equilibrium* *where player 1 has the following strategy:*

$x_1:$	0	1	2	3	4	5
p_{x_1}	1	0	0	0	0	0

That is, player 1 chooses $x_1 = 0$ *with probability* 1. *Players 2 and 3 have the following strategy:*

$x_j:$	0	1	2	3	4	5
p_{x_j}	0	$\frac{1}{4}$	0	0	0	$\frac{3}{4}$

That is, player $j, j = 2, 3$ *chooses* $x_j = 1$ *with probability* $p_{x_j} = \frac{1}{4}$ *and* $x_j = 5$ *with probability* $p_{x_j} = \frac{3}{4}$. *In that case, all the players have an expected payoff of zero and none of them chooses the efforts* $x \in \{2, 3, 4\}$.

The following example shows that different players have positive expected payoffs at different equilibrium points, namely, given an all-pay auction, player i may be the only player with a positive expected payoff in an equilibrium, and player j may be the only player with a positive expected payoff in a different one. Moreover, it also demonstrates that in contrast to Proposition 3 and Theorem 1 a player without the highest effort constraint may have a positive expected payoff.[3]

Example 4 *Consider an all-pay auction with three players where the players' values of winning are* $v_1 = 100$,$v_2 = v_3 = 6$ *and they face effort constraints of* $d_1 = 3, d_2 = d_3 = 4$. *Let the smallest money unit be* $\epsilon = 1$. *In that case, it can be verified that there is an equilibrium where player 1 has the following strategy:*

$x_1:$	0	1	2	3	4
$p_{x_1}:$	0	1	0	0	0

[3]Numerical calculations point out that the player without the highest effort constraint in the following example has a positive expected payoff even when ϵ approaches zero.

That is, player 1 chooses $x_1 = 1$ *with probability* 1. *Player* $j, j = 2, 3$ *has the following strategy:*

$x_j :$	0	1	2	3	4
$p_{x_j} :$	$\frac{1}{3}$	0	0	0	$\frac{2}{3}$

That is, player $j, j = 2, 3$ *chooses* $x_j = 0$ *with probability* $p_{x_j} = \frac{1}{3}$ *and* $x_j = 4$ *with probability* $p_{x_j} = \frac{2}{3}$. *Then, player 1's expected payoff is* 10.1, *and each of the other players' expected payoff is zero. On the other hand, we have another equilibrium where player 1 has the following strategy:*

$x_1 :$	0	1	2	3	4
$p_{x_1} :$	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	0

That is, player 1 chooses each $x_1 \in \{0, 1, 3\}$ *with probability* $p_{x_1} = \frac{1}{3}$. *Player 2 has the strategy:*

$x_2 :$	0	1	2	3	4
$p_{x_2} :$	0	$\frac{2}{100}$	0	$\frac{2}{100}$	$\frac{96}{100}$

That is, player 2 chooses each $x_2 \in \{1, 3\}$ *with probability* $p_{x_2} = \frac{2}{100}$, *and* $x_2 = 4$ *with probability* $p_{x_2} = \frac{96}{100}$. *Player 3 has the strategy:*

$x_3 :$	0	1	2	3	4
$p_{x_3} :$	1	0	0	0	0

That is, player 3 chooses $x_3 = 0$ *with probability* 1. *Then, player 2's expected payoff is 2 while the other players' expected payoffs are zero.*

6 Concluding remarks

The main goal of this paper was to show that the analysis of the two-player all-pay auction with asymmetric effort-constrained players is different than the analysis of the all-pay auction with more than two asymmetric effort-constrained players. We showed that in an all-pay auction with three asymmetric effort-constrained players all the players may be active when each of them has an asymmetric strategy. In that case, the results might be unexpected especially with respect to the same model with only two players. The main distinction between the models with two players and with a larger number of players is that while in the two-player all-pay auction, independent of the players' values of winning, the player with the highest effort constraint is necessarily the player with the only positive expected payoff, in the all-pay auction with more than two players this is not the case. In other words, in the all-pay auction with more than two effort-constrained players, a player's expected payoff depends on the players' values of winning and their effort constraints. Thus, in all-pay auctions with multiple asymmetric effort-constrained players, the players' strategies and their expected payoffs might be unpredictable.

7 Appendix

7.1 Proof of Proposition 1

We want to show that the probability that player 2 assigns for every possible positive effort $0 < x_2 < d_2$ is smaller than or equal to $\frac{2\epsilon}{v_1}$. Suppose that this is not true and that a strategy $0 < y < d_2$ is chosen with the probability of $q_y > \frac{2\epsilon}{v_1}$. Denote by R^1_y player 1's expected payoff if he chooses the pure strategy y. Since $q_y > \frac{2\epsilon}{v_1}$, we obtain that player 1 strictly prefers the effort $y + \epsilon$ over the effort y since

$$\begin{aligned} R^1_{y+\epsilon} - R^1_y &= [v_1(q_0 + q_1 + ... + q_y + \frac{q_{y+\epsilon}}{2}) - (y+\epsilon)] - [v_1(q_0 + q_1 + ... + \frac{q_y}{2}) - y] \\ &= v_1(\frac{q_y}{2} + \frac{q_{y+\epsilon}}{2}) - \epsilon > v_1\frac{q_{y+\epsilon}}{2} \geq 0 \end{aligned}$$

Moreover, when $q_y > \frac{2\epsilon}{v_1}$, we also obtain that player 1 strictly prefers the effort y over the effort $y - \epsilon$ since

$$\begin{aligned} R^1_y - R^1_{y-\epsilon} &= [v_1(q_0 + q_1 + ... + q_{y-\epsilon} + \frac{q_y}{2}) - y] - [v_1(q_0 + q_1 + ... + \frac{q_{y-\epsilon}}{2}) - (y - \epsilon)] \\ &= v_1(\frac{q_{y-\epsilon}}{2} + \frac{q_y}{2}) - \epsilon > v_1 \frac{q_{y-\epsilon}}{2} \geq 0 \end{aligned}$$

We obtained that player 1 prefers the effort $y + \epsilon$ over the effort y, and the effort y over the effort $y - \epsilon$. Thus, player 1 strictly prefers the effort $y + \epsilon$ over the efforts y and $y - \epsilon$, and therefore he does not choose the strategies y and $y - \epsilon$. However, this contradicts our assumption that $q_y > 0$. The reason is that if player 1 does not choose y and $y - \epsilon$, player 2 has no incentive to choose the effort y since if he chooses the effort $y - \epsilon$ he has the same probability to win and his cost is lower. Hence, we obtain that the probability that player 2 assigns to every effort $0 < x_2 < d_2$ is necessarily smaller than or equal to $\frac{2\epsilon}{v_1}$. A similar proof holds for player 1. Q.E.D.

7.2 Proof of Proposition 2

Suppose that player 2 assigns a positive probability to the effort y, i.e., $q_y > 0$, and that he does not choose the effort $y + \epsilon$, i.e., $q_{y+\epsilon} = 0$. We will now show that in that case player 2 necessarily assigns a positive probability to the effort $y + 2\epsilon$, i.e., $q_{y+2\epsilon} > 0$. Assume that $q_{y+2\epsilon} = 0$, and also that the smallest effort that is larger than y which is chosen with a positive probability by player 2 is $x < d_2$. By our assumption, $x - y \geq 3\epsilon$. But then we have a contradiction, since player 1 strictly prefers the effort $y + \epsilon$ over the effort x. To see that, remember that by Proposition 1 we have $q_x \leq \frac{2\epsilon}{v_1}$, which implies that

$$\begin{aligned} R^1_{y+\epsilon} - R^1_x &= [v_1(q_0 + q_1 + ... + q_y) - (y + \epsilon)] - [v_1(q_0 + q_1 + ...q_y + \frac{q_x}{2}) - x] \\ &= x - y - \epsilon - v_1 \frac{q_x}{2} \geq \epsilon \end{aligned}$$

Hence, we obtain that our assumption $q_{y+2\epsilon} = 0$ is unfeasible. A similar proof holds for player 1. *Q.E.D.*

7.3 Proof of Proposition 3

We first show that the expected payoff of player 2 satisfies $\lim_{\epsilon \to 0} R_2(\epsilon) = 0$. Define $x_{1\min} = \min\{x_1 : p_{x_1} > 0\}$ and similarly $x_{2\min} = \min\{x_2 : p_{x_2} > 0\}$, namely, $x_{i\min}$ is the smallest effort for which player i assigns a positive probability. If $x_{1\min} < x_{2\min}$ player 1 has no positive expected payoff. But this is a contradiction, since if player 1 exerts a effort of $x_1 = d_2 + \epsilon$ he has an expected payoff of $v_1 - d_2 - \epsilon \geq v_1 - d_1 > 0$. Thus, let us instead assume that $x_{1\min} = x_{2\min}$. Then, if player 1 chooses $x_{1\min}$ his expected payoff is

$$v_1 \frac{p_{x_{2\min}}}{2} - x_{1\min}$$

where $p_{x_{2\min}}$ is the probability that player 2 assigns to the effort $x_{2\min}$. Given that player 1 chooses $x_{1\min}$ with a positive probability, he weakly prefers $x_{1\min}$ over $x_{1\min} + \epsilon$ such that

$$\begin{aligned} v_1 \frac{p_{x_{2\min}}}{2} - x_{1\min} &\geq v_1 p_{x_{2\min}} - (x_{1\min} + \epsilon) \\ &\Rightarrow v_1 \frac{p_{x_{2\min}}}{2} \leq \epsilon \end{aligned}$$

Therefore, if player 1 chooses $x_{1\min}$ his expected payoff is positive only if $x_{1\min} = 0$, and then it is equal to or smaller than ϵ. By the same argument, if player 2 chooses $x_{2\min} = 0$ his expected payoff is equal to or smaller than ϵ. Furthermore, if $v_1 - d_2 > 2\epsilon$ then by exerting an effort of $x_1 = d_2 + \epsilon$ player 1 has an expected payoff that is higher than ϵ, and we obtain that the equality $x_{1\min} = x_{2\min}$ is not possible. In such a case, we have that $x_{1\min} > x_{2\min}$ which implies that the expected payoff of player 2 is necessarily zero. In sum, we showed that player 2's expected payoff is smaller than or equal to ϵ.

We now show that the expected payoff of player 1 satisfies $\lim_{\epsilon \to 0} R_1(\epsilon) = v_1 - d_2$. Player 1 can always exert a effort of $x = d_2 + \epsilon$. Thus, his expected payoff is higher than or equal to $v_1 - (d_2 + \epsilon)$. On the other hand, the highest effort of player 1 is larger than or equal to $x_1 = d_2 - 2\epsilon$, since otherwise the expected payoff of player 2 is larger than ϵ. However, this contradicts the fact that $R_2 \leq \epsilon$. Thus, player 1's expected payoff is lower than or equal to $v_1 - (d_2 - 2\epsilon)$. We obtain therefore that the expected payoff of player 1 converges to $v_1 - d_2$ when ϵ converges to zero. *Q.E.D.*

7.4 Proof of Proposition 4

The players' equilibrium strategies are as follows: Player 1's strategy is

$$\begin{array}{cccccccc} x_1: & 0 & \epsilon & 2\epsilon & 3\epsilon & 4\epsilon & ... & v_2 - 2\epsilon \\ p_{x_1}: & \frac{2\epsilon}{v_2} & 0 & \frac{2\epsilon}{v_2} & 0 & \frac{2\epsilon}{v_2} & ... & \frac{2\epsilon}{v_2} \end{array}$$

That is, player 1 chooses every effort $x_1 \in \{0, 2\epsilon, 4\epsilon, ..., d_1\}$ with the same probability $p_{x_1} = \frac{2\epsilon}{v_2}$. Player 2's strategy is

$$\begin{array}{cccccccc} x_2: & 0 & \epsilon & 2\epsilon & 3\epsilon & 4\epsilon & ... & v_2 - \epsilon \\ p_{x_2}: & 0 & \frac{2\epsilon}{v_1} & 0 & \frac{2\epsilon}{v_1} & 0 & ... & \frac{v_1 - v_2 + 2\epsilon}{v_1} \end{array}$$

That is, player 2 chooses every effort $x_2 \in \{\epsilon, 3\epsilon, ..., v_2 - 2\epsilon\}$ with the same probability $p_{x_2} = \frac{2\epsilon}{v_1}$, and he chooses $x_2 = v_2$ with the probability $p_{x_2} = \frac{v_1 - v_2 + 2\epsilon}{v_1}$. The expected payoff of player 2 is then ϵ and the expected payoff of player 1 is zero. The expected effort of player 1 is

$$\begin{aligned} TB_1 &= \frac{2\epsilon}{v_2}(2\epsilon + 4\epsilon + ...v_2 - 2\epsilon) = \frac{v_2(v_2 - 2\epsilon)}{2v_2} \\ &\implies \lim_{\epsilon \to 0} TB_1(\epsilon) = \frac{v_2}{2} \end{aligned}$$

and the expected effort of player 2 is

$$\begin{aligned} TB_2 &= \frac{2\epsilon}{v_1}(\epsilon + 3\epsilon + ...v_2 - 3\epsilon) + (v_2 - \epsilon)\frac{v_1 - v_2 + 2\epsilon}{v_1} \\ &= \frac{(v_2 - 2\epsilon)^2 + 2(v_2 - \epsilon)(v_1 - v_2 + 2\epsilon)}{2v_1} \\ &\implies \lim_{\epsilon \to 0} TB_2(\epsilon) = v_2 - \frac{(v_2)^2}{2v_1} \end{aligned}$$

Hence, we obtain that the players' expected total effort is

$$TB(\epsilon) = \frac{v_2(v_2-2\epsilon)}{2v_2} + \frac{(v_2-2\epsilon)^2 + 2(v_2-\epsilon)(v_1-v_2+2\epsilon)}{2v_1} \quad (1)$$

$$\Longrightarrow \lim_{\epsilon\to 0} TB(\epsilon) = \frac{3v_2}{2} - \frac{(v_2)^2}{2v_1} \geq v_2$$

The equality $\lim_{\epsilon\to 0} TB(\epsilon) = v_2$ is obtained when $v_1 = v_2$; otherwise, if $v_1 > v_2$, the expected total effort is strictly larger than v_2. Q.E.D.

Q.E.D.

7.5 Proof of Theorem 1

We will show that the minimal effort of player 1 is larger than the minimal efforts of all the other players and therefore the other players' expected payoffs have to be equal to zero. Denote $x_{1\,\min} = \min\{x_1 : p_{x_1} > 0\}$ and similarly $x_{j\,\min} = \min\{x_j : p_{x_j} > 0\}$ namely, $x_{j\,\min}$ is the smallest effort for which player j assigns a positive probability. If there is $j \neq 1$ such that $x_{j\,\min} > x_{1\,\min}$, the expected payoff of player 1 is zero which contradicts the fact that by exerting an effort of $x_1 = \max_{j\neq 1} d_j + \epsilon$, player 1, who has the highest effort constraint, can be guaranteed a positive expected payoff, $v_1 - \max_{j\neq 1} d_j - \epsilon$, that is higher than or equal to ϵ.

Thus, we assume now that there is $j \neq 1$ such that $x_{j\,\min} = x_{1\,\min}$. In that case denote by $\widetilde{p}_{x_{1\,\min}}$ the probability of player 1 to win when he chooses the effort $x_{1\,\min}$. Then, if player 1 chooses $x_{1\,\min}$ his expected payoff is

$$v_1\widetilde{p}_{x_{1\,\min}} - x_{1\,\min}$$

Given that player 1 chooses $x_{1\,\min}$ with a positive probability, he weakly prefers $x_{1\,\min}$ over $x_{1\,\min} + \epsilon$ such that we have

$$v_1\widetilde{p}_{x_{1\,\min}} - x_{1\,\min} \geq v_1\widetilde{p}_{x_{1\,\min}+\epsilon} - (x_{1\,\min} + \epsilon)$$

where $\widetilde{p}_{x_{1\min}+\epsilon}$ is the probability of player 1 to win when he chooses the effort $x_{1\min}+\epsilon$. Since $x_{j\min} = x_{1\min}$ and there is at least another player k for which $x_{k\min} \leq x_{1\min}$, we obtain that $\widetilde{p}_{x_{1\min}+\epsilon} \geq 2\widetilde{p}_{x_{1\min}}$. Thus, we have

$$v_1\widetilde{p}_{x_{1\min}} \leq \epsilon$$

The last inequality contradicts the fact that player 1 can be guaranteed an expected payoff of at least ϵ by choosing the effort $x_1 = \max_{j\neq 1} d_j + \epsilon$. Thus, we obtain that $x_{i\min} < x_{1\min}$ for every $i \neq 1$. *Q.E.D.*

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