THE SOCIALLY ACCEPTABLE SCORING RULE

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Abstract

We say that an alternative is socially acceptable if the number of individuals that rank it among their most preferred half of the alternatives is at least as large as the number of individuals that rank it among the least preferred half. We show that there exists a unique scoring rule that always selects a subset of socially acceptable alternatives.

1 Introduction

Consider a set A of K alternatives. A social choice rule selects a subset of alternatives for every preference profile. A scoring rule is a special class of social choice rule that asks voters to match a fixed set of K scores to the set of alternatives, and selects those alternatives that maximize the sum of their scores.

We say that a voter places a given alternative *above the line* if he prefers it to at least half of the alternatives, and that he places it *below the line* if at least half of the alternatives are preferred to it. We further say that an alternative is *socially acceptable* if it is placed above the line by at least as many voters as those who place it below the line. In this paper we are interested in those scoring rules that always select a subset of socially acceptable

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alternatives. It turns out that there is only one such scoring rule. This rule chooses the alternatives that maximize the difference between the number of voters that place them above the line and the number of voters that place them below the line. We call it the half accepted-half rejected rule. It is similar to some voting rules recently discussed in the literature, e.g., the 1-best 1-worst voting rule characterized by García-Lapresta et al. [4], the dis&aproval rule characterized by Alcantud and Laruelle [1], and the single-approval multiple-rejection (SAMR) rules characterized by Baharad and Nitzan [3]. All of them share with the half accepted-half rejected rule the attribute of being simple rules in the sense that they do not require from the voters to report their whole preference relations. For instance, the half accepted-half rejected rule rule only asks voters to report the sets of alternatives that they place above and below the line. Similarly, the 1-best 1-worst rule asks voters to report the best and the worst alternatives in their preference orderings, and the dis&approval rule asks them to report the sets of alternatives that they approve and disapprove.

It may be worth noting that whereas both the half accepted-half rejected rule and the 1-best 1-worst rules are standard scoring rules, the dis&approval and the SAMR rules are not. They are what is known as flexible scoring rules since voters are asked to map a set of scores to the set of alternatives but the map needs not necesarily be a matching.

The paper is organized as follows. Section 2 lays out the basic definitions and introduces the concept of socially acceptable alternatives. Section 3 states and proves the main result, and Section 4 concludes.

2 Scoring Rules

Let $A = \{a_1, \ldots, a_K\}$ be a set of $K > 2$ alternatives. Also, let $\mathcal{P}$ be the set of complete, transitive and antisymmetric binary relations on A. We will refer to the elements of $\mathcal{P}$ as preference relations. Let $I\!N$ be the set of non-negative integers, which represents the names of the potential voters. For any finite set $V \subseteq I\!N$ of voters, a preference profile is an assignment of a preference relation to each voter in V. A social choice rule is a function that assigns to each preference profile a nonempty subset of alternatives. A social choice rule is anonymous if it does not depend on the names of the voters. When we restrict attention to anonymous social choice rules, a preference profile can be summarized by a list $\pi = (\succ_1, \ldots, \succ_n)$ of preference relations where n is the number of voters.

A special class of anonymous social choice rules consists of *scoring rules*. A scoring rule is characterized by a list $S = \{S_1, S_2, \cdots, S_K\}$ of K non-negative scores with $S_1 \geq S_2 \geq \cdots \geq S_K$ and $S_1 > S_K$. Given a preference profile $\pi = (\succ_1, \ldots, \succ_n)$, each individual $i = 1, \ldots, n$ assigns S_k points to the alternative that is ranked k-th in his preference relation, for $k = 1, \ldots, K$. That is, each agent assigns S_1 points to his most preferred alternative, S_2 points to the second best alternative and so on. The scoring rule associated with the scores in S, denoted by F_S, chooses the alternatives with the maximum total score. It is easy to see that, for any $\alpha > 0$ and $\beta \in \mathbb{R}$, the scoring rules associated with the scores S_i and with the scores $\alpha S_i + \beta$, for $i = 1, \ldots, K$ are one and the same rule. Therefore it is sometimes convenient to restrict attention to scores $\{S_1, \ldots, S_K\}$ where $S_1 = 1$ and $S_K = 0$.

Many well-known social choice rules are instances of scoring rules. For example, the *plurality* rule is the scoring rule associated with the scores $\{1, 0, \ldots, 0\}$. The *inverse plurality* rule is the scoring rule associated with scores $\{1, \ldots, 1, 0\}$.[1] More generally, for $1 \leq t \leq K-1$, the *t-approval voting method* is the scoring rule associated with the scores $\{\underbrace{1, \ldots, 1}_{t}, \underbrace{0, \ldots, 0}_{K-t}\}$. Lastly, the Borda social choice rule is the scoring rule associated with the scores $\{K-1, K-2, \ldots, 0\}$.

As mentioned in the introduction, we say that a voter places a given alternative *above the line* if he prefers it to at least half of the alternatives, and *below the line* if at least half of the alternatives are preferred to it. For instance, if $K = 5$ and a voter's preference relation is given by $a_1 \succ a_2 \succ a_3 \succ a_4 \succ a_5$, then he places alternatives a_1 and a_2 above the line and alternatives a_4 and a_5 below the line. In this paper we focus on the scoring rule that assigns a score of 1 to the alternatives placed above the line, a score of -1 to alternatives below the line, and a score of 0 to the alternative (if there is one) that is neither above nor below the

[1] See Baharad and Nitzan [2] for an axiomatization of this rule.

line. Formally, the half accepted-half rejected (HAHR) rule is defined by the scores

$$S_j = \begin{cases} 1 & j < \frac{K+1}{2} \\ 0 & j = \frac{K+1}{2} \\ -1 & j > \frac{K+1}{2} \end{cases} \tag{1}$$

We now single out some alternatives for being above the line for a sufficient number of voters.

Definition 1 Let π be a preference profile, and let $a \in A$ be an alternative. We say that a is *socially acceptable with respect to* π if the number of individuals that place it above the line is at least as large as the number of individuals that place it below the line.

In principle, one would like to attain the ideal of unanimity and look for an alternative that is ranked first by all individuals. Since such an alternative may not exist, we may look for alternatives that are ranked first in the preference order of a majority of individuals. Or, more generally, we may look for alternatives that are ranked in the first k by most individuals. The concept of socially acceptable alternatives singles out those alternatives that most individuals rank above at least half of the alternatives in their preference relations. As we will see later, for any preference profile we can always find socially acceptable alternatives. Furthermore, if we strengthened the requirement on socially acceptable alternatives and ask that most individuals rank them one level higher than the mid-point, then there would be profiles with no correspondingly defined socially acceptable alternatives.

The next example shows that a Condorcet winner may not be socially acceptable.

Example 1 Assume $A = \{a, b, c, d\}$ and consider the preference profile $(abcd, acbd, cdab, cbad, bdac)$. It can be seen that whereas alternative a is a Condorcet winner, it is not socially acceptable. The only socially acceptable alternatives are b and c. Furthermore, it can be checked that the HAHR rule chooses precisely these two alternatives.

The next proposition shows that all the alternatives chosen by the HAHR rule are socially acceptable.

Proposition 1 Let H be the HAHR scoring rule and let π be a preference profile. If $a \in H(\pi)$ then a is socially acceptable.

Proof : Recall that HAHR is the scoring rule defined by the list of scores given in (1). Since the sum of the scores in S is 0, the total number of scores that are distributed among the alternatives is 0. As a result, the average score obtained by the alternatives is 0, and any alternative that gets the maximum score gets a score of at least 0. Consider an alternative a that is not socially acceptable. Each individual that places a above the line, assigns it a score of 1. And each individual that places it below the line, assigns it a score of -1. Since the number of individuals that place a below the line is greater than the number of individuals that place it above the line, alternative a gets a strictly negative score. This means that it is not chosen by the rule. $\square$

Corollary 1 For any preference profile, there exists at least one socially acceptable alternative with respect to it.

Example 1 above suggests the following property of social choice rules.

Definition 2 The social choice rule F satisfies social acceptability if for any profile π, $F(\pi)$ consists of socially acceptable alternatives with respect to π.

3 The result

We can now state our main result.

Theorem 1 A scoring rule satisfies social acceptability if and only if it is the HAHR rule.

Proof : Proposition 1 already showed that HAHR satisfies social acceptability. We now show that it is the only scoring rule that satisfies this property. Let F be a scoring rule that satisfies social acceptability and assume that it can be represented by a list of normalized scores $S = \{S_1, \ldots, S_K\}$ where $S_1 = 1$ and $S_K = 0$. We shall show that unless F is HAHR, that is, unless the scores are given by (1), there is a preference profile π such that $F(\pi)$ contains an alternative that is not socially acceptable.

Recall that $A = \{a_1, \ldots, a_K\}$.

Case 1: K is even. In this case we can subdivide the set of scoring rules that are not HAHR into two classes.

Case 1.1: The scores are given by $S = (1, \ldots, S_{K/2}, 0, \ldots, 0)$, where $S_{K/2} < 1$. There are $(K/2)!$ permutations of $(a_{K/2+1}, \ldots, a_K)$. Denote them by $\sigma_i(a_{K/2+1}, \ldots, a_K)$, for $i = 1, \ldots, (K/2)!$ and consider the following preference profile with $2n + 1$ individuals:

Preference			# of voters
$\succ_1$	$=$	$(\sigma_1(a_{K/2+1}, \ldots, a_K), a_1, \ldots, a_{K/2})$	$\frac{n+1}{(K/2)!}$
$\succ_2$	$=$	$(\sigma_2(a_{K/2+1}, \ldots, a_K), a_1, \ldots, a_{K/2})$	$\frac{n+1}{(K/2)!}$
$\vdots$	$\vdots$		$\vdots$
$\succ_{(K/2)!}$	$=$	$(\sigma_{(K/2)!}(a_{K/2+1}, \ldots, a_K), a_1, \ldots, a_{K/2})$	$\frac{n+1}{(K/2)!}$
$\succ_{(K/2)!+1}$	$=$	$(a_1, \ldots, a_{K/2}, a_{K/2+1}, \ldots, a_K)$	n

It can be seen that a_1 is not socially acceptable. Indeed, $n+1$ individuals put it below the line and only n individuals put it above the line. For any alternative a, let $Sc(a)$ stand for the total score attained by a in the above profile. We now show that a_1 is an alternative with maximum total score. Since a_1 is preferred by all individuals to any alternative in $\{a_1, \ldots, a_{K/2}\}$, we have that $Sc(a_1) \geq Sc(a_i)$ for $i = 1, \ldots, K/2$. By construction, $Sc(a_{K/2+1}) \geq Sc(a_j)$ for $j > K/2$. Therefore it is enough to show that $Sc(a_1) \geq Sc(a_{K/2+1})$. By direct computation we have that

$$Sc(a_1) = n.$$

Also, by direct computation

$$Sc(a_{K/2+1}) \leq (n+1)\big(\frac{1}{K/2}S_{K/2} + \frac{K/2-1}{K/2}1\big).$$

Routine calculations show that a sufficient condition for $Sc(a_1) > Sc(a_{K/2+1})$ is that n be chosen so that

$$n > \frac{K}{2(1-S_{K/2})} - 1.$$

Case 1.2: $S = (1, \ldots, S_{K/2+1}, \ldots, 0)$, where, $S_{K/2+1} > 0$.

Consider the following preference profile with $2n+1$ individuals:

		Preference	# of voters
$\succ_1$	$=$	$(a_1, \ldots, a_{K/2}, \sigma_1(a_{K/2+1}, \ldots, a_K))$	$\frac{n}{(K/2)!}$
$\succ_2$	$=$	$(a_1, \ldots, a_{K/2}, \sigma_2(a_{K/2+1}, \ldots, a_K))$	$\frac{n}{(K/2)!}$
$\vdots$		$\vdots$	$\vdots$
$\succ_{(K/2)!}$	$=$	$(a_1, \ldots, a_{K/2}, \sigma_{(K/2)!}(a_{K/2+1}, \ldots, a_K))$	$\frac{n}{(K/2)!}$
$\succ_{(K/2)!+1}$	$=$	$(a_{K/2+1}, \ldots, a_K, a_1, \ldots, a_{K/2})$	$n+1$

It can be seen that a_1 is not socially acceptable. Indeed, $n+1$ individuals put it below the line and only n individuals put it above the line. We now show that a_1 is an alternative with maximum score. Since a_1 is preferred by all individuals to any alternative in $\{a_1, \ldots, a_{K/2}\}$, we have that $Sc(a_1) \geq S(a_i)$ for $i = 1, \ldots, K/2$. By construction, $Sc(a_{K/2+1}) \geq Sc(a_j)$ for $j > K/2$. Therefore it is enough to show that $Sc(a_1) \geq Sc(a_{K/2+1})$. By direct computation we have that

$$Sc(a_1) = n + (n+1)S_{K/2+1}.$$

Also, by direct computation

$$Sc(a_{K/2+1}) \leq n(\frac{K/2-1}{K/2})S_{K/2+1} + (n+1).$$

It follows that a sufficient condition for $S(a_1) > S(a_{K/2+1})$ is that n be chosen so that

$$n > \frac{K(1-S_{K/2+1})}{2\underbrace{S_{K/2+1}}_{>0}}$$

Case 2: K is odd. Denote by $M = (K+1)/2$ the median number of alternatives.

Case 2.1: $S = (1, \ldots, S_M, S_{M+1}, \ldots, 0)$ where $S_{M+1} > 0$. There are $(M-1)!$ permutations of $(a_{M+1}, \ldots, a_K)$. Denote each of these permutations by $\sigma_i(a_{M+1}, \ldots, a_K)$, for $i = 1, \ldots, (M-1)!$. Consider the following preference profile with $2n+2$ individuals:

Preference			# of voters
$\succ_1$	$=$	$((a_1, \ldots, a_{M-1}), a_M, \sigma_1(a_{M+1}, \ldots, a_K))$	$\frac{n}{(M-1)!}$
$\succ_2$	$=$	$((a_1, \ldots, a_{M-1}), a_M, \sigma_2(a_{M+1}, \ldots, a_K))$	$\frac{n}{(M-1)!}$
$\vdots$	$\vdots$		$\vdots$
$\succ_{(M-1)!}$	$=$	$((a_1, \ldots, a_{M-1}), a_M, \sigma_{(M-1)!}(a_{M+1}, \ldots, a_K))$	$\frac{n}{(M-1)!}$
$\succ_{(M-1)!+1}$	$=$	$((a_{M+1}, \ldots, a_K), a_M, (a_1, \ldots, a_{M-1}))$	$n+1$
$\succ_{(M-1)!+2}$	$=$	$((a_{M+1}, \ldots, a_K), a_1, (a_M, a_2, \ldots, a_{M-1}))$	1

It can be seen that alternative a_1 is not socially acceptable; while n individuals put it above the line, $n+1$ individuals place it below the line. Alternative a_M is not socially acceptable either; no voter places it above the line and one voter places it below the line. On the other hand, alternatives $a_{M+1}, \ldots, a_K$ are all socially acceptable; whereas n individuals put them below the line, $n+2$ individuals put them above the line. We will show that none of these alternatives is chosen by the social choice rule. For this purpose, as in the previous cases, it is enough to show that $Sc(a_1) > Sc(a_{M+1})$. By direct computation

$$Sc(a_1) \geq n + (n+2)S_{M+1}.$$

Also, by direct computation

$$Sc(a_{M+1}) \leq \frac{M-2}{M-1} n S_{M+1} + (n+2).$$

It follows that a sufficient condition for $S(a_1) > S(a_{M+1})$ is that

$$n \geq \frac{2(M-1)(1-S_{M+1})}{S_{M+1}}.$$

Case 2.2: $S = (1, \ldots, S_M, 0, \ldots, 0)$ where $S_M > 1/2$. There are $(K-1)!$ permutations of $(a_2, \ldots, a_K)$. For $i = 1, \ldots, (K-1)!$ let $\sigma_i(a_2, \ldots, a_K)$ denote each of these permutations.

Also let $\tau_i(a_2, \dots, a_M, a_1, a_{M+1}, \dots, a_K)$ be all the permutations of $(a_1, \dots, a_K)$ that place a_1 in the Mth place. There are $(K-1)!$ such permutations. Consider the following preference profile:

Preference			# of voters
$\succ_1$	$=$	$(a_1, \sigma_1(a_2, \dots, a_K))$	n
$\succ_2$	$=$	$(a_1, \sigma_2(a_2, \dots, a_K))$	n
$\vdots$	$\vdots$		$\vdots$
$\succ_{(K-1)!}$	$=$	$(a_1, \sigma_{(K-1)!}(a_2, \dots, a_K))$	n
$\succ_{(K-1)!+1}$	$=$	$\tau_1(a_2, \dots, a_M, a_1, a_{M+1}, \dots, a_K)$	$2n$
$\vdots$	$\vdots$		$\vdots$
$\succ_{2(K-1)!}$	$=$	$\tau_{(K-1)!}(a_2, \dots, a_M, a_1, a_{M+1}, \dots, a_K)$	$2n$
$\succ_{2(K-1)!+1}$	$=$	$(\sigma_1(a_2, \dots, a_K), a_1)$	$n+1$
$\vdots$	$\vdots$		$\vdots$
$\succ_{3(K-1)!}$	$=$	$(\sigma_{(K-1)!}(a_2, \dots, a_K), a_1)$	$n+1$

It can be seen that alternative a_1 is not socially acceptable; while $(K-1)!\,n$ individuals put it above the line, $(K-1)!\,(n+1)$ individuals put it below the line. By Corollary 1 at least one of the other alternatives is socially acceptable. By symmetry, all of them are. We will show that none of them is chosen by the social choice rule. For this purpose it is enough to show that $Sc(a_1) > Sc(a_i)$, for $i \neq 1$. By direct computation

$$Sc(a_1) = (K-1)!\,(n + 2nS_M).$$

Also, by direct computation

$$Sc(a_i) \leq (K-1)!\,\big(n(\frac{\big(\frac{K-1}{2} - 1\big)}{K-1} + \frac{1}{K-1}S_M) + n + (n+1)(\frac{1}{2} + \frac{1}{K-1}S_M)\big). \qquad (2)$$

Indeed, for each of the preference relations $\succ_1, \dots, \succ_{(K-1)!}$, the $K-1$ alternatives $a_2, \dots, a_K$ are ranked in the 2nd to Kth place. Therefore, half of them get a score of 0, one of them gets a score of S_M, and $(K-1)/2 - 1$ of them get a score of at most 1. Therefore the sum of the scores assigned to any alternative $a_i \neq a_1$ by these preference relations is $(K-1)!\big(\frac{\big(\frac{K-1}{2} - 1\big)}{K-1} +$

$\frac{1}{K-1}S_M\big)n$. Similarly, for each of the preference relations $\succ_{(K-1)!+1},\ldots,\succ_{2(K-1)!}$, half of the $K-1$ alternatives $a_2,\ldots,a_K$ are ranked in the first $M-1$ places, and the other half are ranked in the last $M-1$ places. Therefore, half of them get a score of 0, and half of them get a score of at most 1. Therefore the sum of the scores assigned to any alternative $a_i \neq a_1$ by these preference relations is $(K-1)!n$. Finally for preference relations $\succ_{2(K-1)!+1},\ldots,\succ_{3(K-1)!}$, the $K-1$ alternatives $a_2,\ldots,a_K$ are ranked in the 1st to $K-1$th place. Consequently, half of them get a score of at most 1, one of them gets a score of S_M, and the rest get a score of 0. As a result, the sum of the scores assigned to any alternative $a_i \neq a_1$ by these preference relations is $(K-1)!\big(\frac{1}{2}+\frac{1}{K-1}S_M\big)(n+1)$. The sum of these three terms constitutes the bound that appears in equation 2.

It follows that a sufficient condition for $Sc(a_1) > Sc(a_{M+1})$ is that

$$n \geq \frac{K+2S_M-1}{2(K-2)(2S_M-1)}.$$

Case 2.3: $S=(1,\ldots,S_M,0,\ldots,0)$ where $S_M<1/2$. There are $(K-1)!$ permutations of $(a_2,\ldots,a_K)$. For $i=1,\ldots,(K-1)!$ let $\sigma_i(a_2,\ldots,a_K)$ denote each of these permutations. Consider the following preference profile:

Preference			# of voters
$\succ_1$	$=$	$(a_1,\sigma_1(a_2,\ldots,a_K))$	n
$\succ_2$	$=$	$(a_1,\sigma_2(a_2,\ldots,a_K))$	n
$\vdots$		$\vdots$	$\vdots$
$\succ_{(K-1)!}$	$=$	$(a_1,\sigma_{(K-1)!}(a_2,\ldots,a_K))$	n
$\succ_{(K-1)!+1}$	$=$	$(\sigma_1(a_2,\ldots,a_K),a_1)$	$n+1$
$\vdots$		$\vdots$	$\vdots$
$\succ_{2(K-1)!}$	$=$	$(\sigma_{(K-1)!}(a_2,\ldots,a_K),a_1)$	$n+1$

It can be seen that alternative a_1 is not socially acceptable; while $(K-1)!\,n$ individuals put it above the line, $(K-1)!\,(n+1)$ individuals put it below the line. All the other alternatives are socially acceptable. We will show that none of them is chosen by the social choice rule.

For this purpose, it is enough to show that $S(a_1) > S(a_i)$, for $i \neq 1$. By direct computation,

$$Sc(a_1) = (K-1)!\,n.$$

Also, by direct computation

$$Sc(a_i) \leq (K-1)!\big(n(\frac{\big(\frac{K-1}{2}-1\big)}{K-1} + \frac{1}{K-1}S_M) + (n+1)(\frac{1}{2} + \frac{1}{K-1}S_M)\big). \qquad (3)$$

Indeed, for preference relations $\succ_1, \ldots, \succ_{(K-1)!}$, the $K-1$ alternatives $a_2, \ldots, a_K$ are ranked in the 2nd to Kth place. Therefore, half of them get a score of 0, one of them gets a score of S_M, and $(K-1)/2-1$ of them get a score of at most 1. Therefore the sum of the scores assigned to any alternative $a_i \neq a_1$ by these preference relations is $(K-1)!\big(\frac{\left(\frac{K-1}{2}-1\right)}{K-1} + \frac{1}{K-1}S_M\big)n$. Similarly, for preference relations $\succ_{(K-1)!+1}, \ldots, \succ_{2(K-1)!}$, the $K-1$ alternatives $a_2, \ldots, a_K$ are ranked in the 1st to $K-1$th place. Therefore, half of them get a score of at most 1, one of them gets a score of S_M, and the rest get a score of 0. Therefore the sum of the scores assigned to any alternative $a_i \neq a_1$ by these preference relations is $(K-1)!\big(\frac{1}{2} + \frac{1}{K-1}S_M\big)(n+1)$. The sum of these three two constitutes the bound that appears in equation 3.

It follows that a sufficient condition for $Sc(a_1) > Sc(a_{M+1})$ is that

$$n \geq \frac{K-(1-2S_M)}{2(1-2S_M)}.$$

Case 2.4: $S = (1, \ldots, S_{M-1}, S_M, 0, \ldots, 0)$ where $S_{M-1} < 1$, and $S_M = 1/2$. There are $(M-1)!$ permutations of $(a_{M+1}, \ldots, a_K)$. Denote each of these permutations by $\sigma_i(a_{M+1}, \ldots, a_K)$, for $i = 1, \ldots, (M-1)!$. Consider the following preference profile with $(M-1)!(2n+2)$ individuals:

Preference			# of voters
$\succ_1$	$=$	$((a_1,\ldots,a_{M-1}),a_M,\sigma_1(a_{M+1},\ldots,a_K))$	n
$\succ_2$	$=$	$((a_1,\ldots,a_{M-1}),a_M,\sigma_2(a_{M+1},\ldots,a_K))$	n
$\vdots$		$\vdots$	$\vdots$
$\succ_{(M-1)!}$	$=$	$((a_1,\ldots,a_{M-1}),a_M,\sigma_{(M-1)!}(a_{M+1},\ldots,a_K))$	n
$\succ_{(M-1)!+1}$	$=$	$(\sigma_1(a_{M+1},\ldots,a_K),a_M,(a_1,\ldots,a_{M-1}))$	$n+1$
$\vdots$		$\vdots$	$\vdots$
$\succ_{2(M-1)!}$	$=$	$(\sigma_{(M-1)!}(a_{M+1},\ldots,a_K),a_M,(a_1,\ldots,a_{M-1}))$	$n+1$
$\succ_{2(M-1)!+1}$	$=$	$((a_{M+1},\ldots,a_K),a_1,(a_M,a_2\ldots,a_{M-1}))$	$(M-1)!$

It can be seen that alternative a_1 is not socially acceptable; while $(M-1)!\,n$ individuals place it above the line, $(M-1)!\,(n+1)$ individuals place it below the line. Alternative a_M is not socially acceptable either; no voter places it above the line and one voter places it below the line. On the other hand, alternatives $a_{M+1},\ldots,a_K$ are all above the line. We will show that none of them is chosen by the social choice rule. For this purpose, it is enough to show that $Sc(a_1) > Sc(a_{M+1})$. By direct computation,

$$Sc(a_1) = (M-1)!\,(n+1/2).$$

Also, by direct computation,

$$Sc(a_{M+1}) \leq (M-1)!\big((n+1)(\frac{(M-1)-1}{M-1} + \frac{1}{M-1}S_{M-1}) + 1\big).$$

It follows that a sufficient condition for $S(a_1) > S(a_{M+1})$ is that

$$n > \frac{3M + 2S_{M-1} - 5}{2(1 - S_{M-1})}.$$

□

4 Concluding remarks

We have shown that the only scoring rule that always selects socially acceptable alternatives is the HAHR rule. It is worth mentioning that in a celebrated paper, Young [5] has characterized the class of scoring rules as the only social choice rules that satisfy the axioms of anonymity, neutrality, reinforcement and continuity. Therefore, we obtain that a social choice rule satisfies anonymity, neutrality, reinforcement, continuity, and social acceptability if and only if it is the HAHR rule.

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