THE BANZHAF VALUE AND GENERAL SEMIVALUES FOR DIFFERENTIABLE MIXED GAMES

Ori Haimanko

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Monaster Center for
Economic Research
Ben-Gurion University of the Negev
P.O. Box 653
Beer Sheva, Israel

Fax: 972-8-6472941
Tel: 972-8-6472286

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Abstract

We consider semivalues on pM_∞ – a vector space of games with a continuum of players (among which there may be atoms) that possess a robust differentiability feature. We introduce the notion of a derivative semivalue on pM_∞, and extend the standard Banzhaf value from the domain of finite games onto pM_∞ as a certain particularly simple derivative semivalue. Our main result shows that any semivalue on pM_∞ is a derivative semivalue. It is also shown that the Banzhaf value is the only semivalue on pM_∞ that satisfies a version of the composition property of Owen (1978) and that, in addition, is non-zero for all non-zero monotonic finite games.

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1 Introduction

The efficiency of the Shapley (1953) value is one of its central properties. The need for efficiency is obvious in most applications, where the Shapley value is viewed as a tool for distributing the proceeds of full cooperation in the game between its players.

*Department of Economics, Ben-Gurion University of the Negev, Beer Sheva 84105, Israel. E-mail: orih@exchange.bgu.ac.il

However, there are contexts in which the value is not (or, at least, not obviously) a sharing mechanism. As suggested by Roth (1977a, 1988), the value can be viewed as an evaluation of the prospect of playing the game by the various players, in which case possible inconsistency in subjective evaluations may stand in the way of efficiency. Indeed, the famous Banzhaf value,[1] which is non-efficient in general, may appropriately describe a case of strong "strategic risk aversion" (see Roth (1977b, 1988)), where all players have an unduly pessimistic view of their bargaining abilities. In the context of simple (voting) games, the restriction of the Shapley and Banzhaf values to that domain (which leads, respectively, to the Shapley and Shubik (1954) power index and a version of the Banzhaf (1965, 1966, 1968) power index) intends to capture an even more vaguely defined "voting power" of the various players, in which case the a priori assumption of efficiency is also strongly questionable. It was for these reasons that Dubey et al. (1981) commenced the first systematic study of *semivalues* – solution concepts with value-like properties (namely, linearity, symmetry, positivity, and being a projection on additive games) with efficiency excluded.

Dubey et al. (1981) fully characterized all semivalues on the space of finite games (with a finite or infinite universe of players), and all semivalues on pNA – a space of games with a nonatomic continuum of players that was introduced in Aumann and Shapley (1974) (which they discovered to be sufficiently rich and also amenable to mathematical analysis for producing insightful and sharp results). In the context of finite simple games, an identical characterization of all semivalues was later obtained by Einy (1987).

Among the non-efficient semivalues, the Banzhaf value has received the most attention. Nevertheless, it has only been formally defined and treated in the context of finite games. The definition of the Banzhaf value of a player in a finite game is simple and clean, and is expressed as the expected marginal contribution of that player to a random coalition which each other player joins – independently of the rest – with probability $\frac{1}{2}$. In a simple game, this yields one of the classical definitions of voting power (see Banzhaf (1965), Dubey and Shapley (1979)) as the probability that the given voter "swings" the vote in a random coalition of Yes-voters, under the

[1] Defined in Owen (1975), based on the rendering in Dubey and Shapley (1978) of the earlier definitions (Penroe (1946), Banzhaf (1965), Coleman (1971)) of voting power in simple games.

assumption that all voters cast their votes independently, with probability $\frac{1}{2}$ for Yes and for No.

An attempt to extend the Banzhaf value to games with infinitely many players immediately runs into two difficulties, both of which might explain why it has never been carried out. First, the natural counterparts of marginal contributions in finite games are directional derivatives of the (appropriately extended) characteristic functions when games with a continuum of players are concerned, but these directional derivatives need not always exist even for games in pNA, which is the closure in the bounded variation norm of the vector space of polynomial functions of nonatomic measures.[2] Second, there is no natural way to extend the existing axiomatizations of the Banzhaf value for finite games into the framework with a nonatomic continuum of players, as at least one central axiom seems to be meaningful only for finite games. This particularly stands out in the famous axiomatization of Lehrer (1988). The central axiom, *superadditivity*, states that no two players will be harmed if they "merge" and act as a single player; this axiom does not apply to sets, but only to singletons. As no player is individually significant in the nonatomic continuum setting, the game in which two players have merged into one is identical to the original game, and thus the superadditivity axiom loses all its bite.

In one of the earliest axiomatizations of the Banzhaf index, by Owen (1978), the main axiom would also become inadequate in the nonatomic continuum scenario. Owen's main axiom, the *composition property*, uses a compounding of games played on two tiers, and is defined as follows. Consider a game v with player set $N = \{1, 2, ..., n\}$, and, additionally, let there be n other games $w_1, ..., w_n$ with disjoint player sets $S_1, ..., S_n$, such that $w_i(S) \in [0, 1]$ and $w_i(S_i) = 1$ for every i. Player i is viewed as the delegate of S_i into the first-tier game v that describes the payoffs to all possible coalitions of delegates in N. In the second-tier game w_i, $w_i(S)$ is the probability that a coalition $S \subset S_i$ "controls" S_i's delegate i in N, i.e., forces him to act entirely on S's behalf (in particular, $S = S_i$ controls i with certainty). The

[2]The existence of a directional derivative was established by Aumann and Shapley (1974) for *almost every* point of the "diagonal" (i.e., the set of all "perfect samples" of the grand coalition), which is sufficient for the definition of the Aumann-Shapley value on pNA. However, defining the Banzhaf value requires differentiability at the exact midpoint of the "diagonal" (corresponding to the perfect $\frac{1}{2}$-sample). Games in pNA do not necessarily possess this property.

compound game $v[w_1, ..., w_n]$ has $\cup_{i=1}^n S_i$ as its player set. Given $S \subset \cup_{i=1}^n S_i$, each $i \in N$ is controlled by S with probability $w_i(S \cap S_i)$, and it is assumed that the events of taking control of different delegates are independent.[3] The payoff to S in $v[w_1, ..., w_n]$ is then defined as the expected worth (according to v) of the random coalition composed of all S-controlled delegates. The composition property of Owen (1978) is stated for all compound games $v[w_1, ..., w_n]$ where, in addition, the second-tier games $w_1, ..., w_n$ are constant-sum.[4] This property requires the prospect of a player $j \in S_i$ in $v[w_1, ..., w_n]$ to be determined based on his prospect in his second-tier game w_i and the prospect of his delegate i in the first-tier game v. Specifically, j obtains the prospect of i in v with probability equal to his own prospect in the "control game" w_i.

The idea of compounding appears to be much less suitable in the framework with a nonatomic continuum of players. When there are no atoms, the first-tier game v must have a continuum of individually negligible but collectively powerful delegates as its player set. Putting aside the question of whether a *continuum of delegates* (and hence, a continuum of disjoint delegate-selecting groups S_i, each of which is itself a continuum) may be a reasonable approximation of any realistic scenario, an immediate problem is that the expected worth of a coalition obtained by randomly and independently selecting delegates from a continuum cannot be defined in a natural manner.[5]

In this paper we will use a framework that overcomes the aforementioned difficulties that are inherent in extending the domain of the Banzhaf value to games with a continuum of players. As in Aumann and Shapley (1973), we shall consider games that are polynomial functions of measures, but, in passing to limits of such functions, we will replace the bounded variation norm by a stronger $\|\cdot\|_\infty$ norm, first defined in Monderer and Neyman (1988). It was shown in Monderer and Hart (1997)

[3] When $w_1, ..., w_n$ are simple games, the setting becomes deterministic: delegate $i \in N$ is being controlled by S if and only if $S \cap S_i$ is a winning coalition in the game w_i.

[4] This means that the probabilities of i being controlled by S $(\subset S_i)$ and $S_i \backslash S$ are complementary.

[5] Assuming that the player ("delegate") set of v is the interval $[0, 1]$ together with the σ-algebra B of Borel sets (i.e., a *standard measurable space* as in Aumann and Shapley (1974)), the event that the random process of independently choosing delegates generates a coalition $S \in B$ (which ensures that $v(S)$ is defined) will typically be nonmeasurable w.r.t. the product σ-algebra that is behind the random process; see, e.g., Judd (1985).

that $\|\cdot\|_\infty$-limits preserve differentiability, and hence the $\|\cdot\|_\infty$-closure of the space of polynomials in measures (that are differentiable) will consist of *differentiable games*, paving the way for defining the Banzhaf value along with a host of other derivative-based semivalues. Furthermore, we will not confine ourselves to nonatomic measures as those works did, but consider the space of polynomials in *mixed* measures which have a nonatomic and a purely atomic part (as in Hart (1973)). The $\|\cdot\|_\infty$-closure of that space, denoted pM_∞, can thus be referred to as the space of *differentiable mixed games*.[6] It contains both pNA_∞ (i.e., the $\|\cdot\|_\infty$-closure of the space of polynomials in nonatomic measures) and the space of all finite games (i.e., games with a finite support in the continuum). The inclusion of all finite games in pM_∞ is of great utility since this fact allows the composition property to be stated for games in pM_∞; indeed, when the first-tier game v, for which the finiteness of the player set is a natural assumption, is indeed assumed to be finite (and hence, in particular, in pM_∞), its compounding with second-tier games in pM_∞ gives rise to a well-defined game in pM_∞.

The first contribution of this work is to introduce the notion of a *derivative semivalue* on pM_∞. To describe derivative semivalues, first fix a point $(t_1, t_2) \in [0,1]^2$. Any game $v \in pM_\infty$ has a well-defined (at most) countable set $A(v)$ of atoms – individually significant players – in the player continuum I. Now envision an *ideal coalition*[7] $I(t_1, t_2)$ that is attended by every atom in $A(v)$ with probability t_2 (independently of other atoms), and contains fraction t_1 of the nonatomic continuum $I \setminus A(v)$. The (t_1, t_2)-induced semivalue $\varphi_{(t_1,t_2)}$ will attribute to every atom $a \in A(v)$ its (expected) marginal contribution to $I(t_1, t_2)$. Determining the prospect of a nonatomic coalition S is less straightforward, and requires a unifying derivative-based approach (that suits nonatomic coalitions as well as atoms). In brief, the prospect attributed to any given S by $\varphi_{(t_1,t_2)}$ will be given by the derivative of an appropriate extension[8] of v

[6] The space pM_∞ was introduced in Haimanko (2000) who characterized partially symmetric (and, in particular, fully symmetric) values on pM_∞ and on pM (the closure in the bounded variation norm of the space of polynomials in mixed measures).

[7] Formally, as in Aumann and Shapley (1974), an ideal coalition is a measurable function $f : I \to [0,1]$, as opposed to a real coalition that can be viewed, in this context, as an indicator function $f : I \to \{0,1\}$. For an atom $a \in A(v)$, $f(a)$ can be thought of as its probability of being active, while $f|_{I \setminus A(v)}$ describes the "fractional participation" of the nonatomic continuum $I \setminus A(v)$.

[8] Constructed in Mertens (1988).

onto the domain of all ideal coalitions, evaluated at $I(t_1, t_2)$ in the direction S. In the particular case of $t_1 = t_2 = \frac{1}{2}$, when the participation rate of nonatomic players and the participation probability of atoms are identical and equal to $\frac{1}{2}$, a semivalue $\beta = \varphi_{\left(\frac{1}{2},\frac{1}{2}\right)}$ is obtained, which naturally extends the Banzhaf value for finite games onto the domain pM_∞. We will refer to β as the Banzhaf value on pM_∞.

When $t_1 \neq t_2$, the semivalue $\varphi_{(t_1,t_2)}$ does not violate the symmetry assumption (i.e., covariance under all automorphisms of the player set) despite that the atomic and the nonatomic part of the game are treated differentially. This is because the automorphisms move atoms into atoms and nonatomic players into nonatomic players, creating no link between the prospects of the two parts. Furthermore, the following generalization of $\varphi_{(t_1,t_2)}$ will also preserve symmetry. Let us allow the nonatomic and the atomic players to have (possibly inconsistent) probabilistic assessments of the participation rate t_1 of the nonatomic continuum / participation probability t_2 of atoms, and denote by ξ (respectively, η) the probability distribution over all possible pairs $(t_1, t_2) \in [0,1]^2$ that represents the assessment held by the nonatomic players (respectively, by atoms). The expectation of $\varphi_{(t_1,t_2)}$, taken w.r.t. ξ for nonatomic coalitions and w.r.t. η for atoms, gives rise to a semivalue $\varphi_{\xi,\eta}$, which is the general form of a derivative semivalue.

Our main result is that derivative semivalues exhaust the set of all semivalues on pM_∞, namely, any semivalue on pM_∞ turns out to have the form $\varphi_{\xi,\eta}$ for some distributions ξ, η on $[0,1]^2$ (see Theorem 1). The method of proof is partially borrowed from Haimanko (2000), who characterized the values (i.e., efficient semivalues) on pM_∞ and pM with type-restricted symmetry by constructing an isomorphism between these values (on pM_∞ or pM) and partially symmetric values of finite games with an infinite universe of players, based on the fact that multilinear games[9] are dense in these spaces. Here, too, for any semivalue φ on pM_∞ we will construct an operator ϕ on the domain of finite games in pM_∞ that is uniquely identifiable with φ and that is a semivalue except for a restricted symmetry feature – the covariance of ϕ will be limited to automorphisms that preserve some partition of the player space into two infinite sets (the restricted symmetry of ϕ is an expression of the disconnect between nonatomic and atomic players that may be exhibited by the original semivalue φ

[9] Defined in Monderer and Neyman (1988).

on pM_∞ despite its symmetry). An auxiliary result that characterizes such a ϕ by means of two distributions ξ, η on $[0,1]^2$, which extends Dubey's et al. (1981) characterization of (fully symmetric) semivalues on finite games,[10] will then be used to show that the semivalue φ (to which ϕ corresponds) has the form $\varphi_{\xi,\eta}$.

We will also provide a counterpart to Owen's (1978) result for finite games, and show that the Banzhaf value is the only semivalue on pM_∞ that satisfies a version of the composition property (see Proposition 1 and Theorem 2). As was mentioned earlier, our notion of compounding requires a first-tier game v to be a *finite* game in pM_∞ (whose atoms will be the "delegates" of the second-tier games), but places no restriction on the second-tier games in pM_∞ other than those dictated by the nature of compounding and the premises of the composition property.[11] Our result requires an additional, though minor, assumption that the semivalue is non-zero for all non-zero monotonic finite games.

The paper is organized as follows. Section 2 recalls the central concepts and notation pertaining to games with a continuum of players, combined with a synopsis of relevant basic facts. In particular, the section formally defines the space pM_∞, the extension of a game in pM_∞ to ideal coalitions, and the directional derivatives of the game. Section 3 introduces our new concept of derivative semivalues on pM_∞. It then defines the Banzhaf value on pM_∞, states the composition property for semivalues on pM_∞, and verifies that the Banzhaf value has that property. It is also shown that Hart's (1973) values on pM_∞ are representable as derivative semivalues. Section 4 contains our main result, Theorem 1, that characterizes a general semivalue on pM_∞ as a derivative semivalue. Section 5 is devoted to the characterization of the Banzhaf value as a semivalue that satisfies the composition property, which is the content of Theorem 2. The concluding Appendix contains the proofs of two auxiliary lemmas.

[10] Specifically, it will be shown that each player is assigned his expected marginal contribution to a random coalition that the other players are joining in a conditionally independent manner, with the same conditional probability for the members of the same "type" (i.e., a set in the partition). ξ, η will be the mixing distributions of the parameters (t_1, t_2) of the two-type i.i.d. process of forming the random coalition, that describe the (possibly different) probabilistic assessments held by the two player types.

[11] The possibility to include nonatomic and purely atomic probability measures among the second-tier games is instrumental in our proof.

2 Preliminaries

The concepts and notation introduced in Aumann and Shapley (1974) form the basis of our setting. It also contains some subsequent additions that are rooted in the works of Monderer and Neyman (1988), Monderer (1990), and Haimanko (2000).

2.1 Games with a continuum of players

Let (I, C) be a standard measurable space (i.e., one that isomorphic to $([0,1], B)$, where B is the σ-field of Borel subsets of $[0,1]$). The elements of I are called *players*, the elements of C – *coalitions*, and real-valued functions on C that vanish at $\varnothing$ – *games*. Given a game v and $T \in C$, we denote by v_T the restriction of v to T, namely, the game given by $v_T(S) = v(S \cap T)$ for every $S \in C$. Coalition T is a *support* of v if $v = v_T$. Player $a \in I$ is an *atom* of v if it belongs to every support of v. If a game v possesses a minimal support, we denote it by $Supp\,(v)$. For any $U \subset I$, the vector space of all games v that have a finite support and $Supp(v) \subset U$, referred to as *finite games* on U, will be denoted by G_U.

A game v is *monotonic* if $v\,(S) \le v\,(T)$ whenever $S \subset T \in C$. The vector space of all games that are differences of two monotonic games is denoted by BV. The subspace of BV containing all finitely additive games (i.e., measures) is denoted by FA. We will be primarily interested in the following subspaces of $FA : M$, which contains all countably additive measures; NA, which contains all nonatomic measures in M; and A, which contains purely atomic measures in M. M^1 (resp., NA^1), will stand for the sets of probability measures in M (resp., NA). Any $\lambda \in M$ possesses a unique decomposition $\lambda = \lambda^{NA} + \lambda^A$, where $\lambda^{NA} \in NA$ and $\lambda^A \in A$. If, more generally, $\lambda = (\lambda_1, ..., \lambda_n) \in (M)^n$, we will denote $\lambda^{NA} = \left(\lambda_i^{NA}\right)_{i=1}^n, \lambda^A = \left(\lambda_i^A\right)_{i=1}^n$, and $A\,(\lambda) = \cup_{i=1}^n Supp(\lambda_i^A)$.

The set of all (bi-measurable) automorphisms of (I, C) will be denoted by Θ. Each $\theta \in \Theta$ induces an operator $\theta : BV \to BV$, where θv is given by $(\theta v)(S) = v(\theta S)$ for every $v \in BV$. A subspace Q of BV will be called symmetric if $\theta\,(Q) = Q$ for every $\theta \in \Theta$.

The bounded variation norm on BV is defined by

$$\|v\|_{BV} = \inf\{u\,(I) + w\,(I) \mid u, w \in BV \text{ are monotonic and } v = u - w\}.$$

The space BV is a Banach algebra w.r.t. the norm $\|\cdot\|_{BV}$.

Given games v and w, v is said to be a Lipschitz game w.r.t. w (written $v \preceq w$) if $w - v$ is monotonic. Denote by $AC_\infty \subset BV$ the set of all games v satisfying $-\mu \preceq v \preceq \mu$ for some positive $\mu \in M$. The norm

$$\|v\|_\infty = \sup\{v(S) \mid S \in C\} + \inf\{\mu(I) \mid \mu \in M \text{ is positive and } -\mu \preceq v \preceq \mu\}$$

turns $(AC_\infty, \|\cdot\|_\infty)$ into a Banach algebra, and $\|v\|_{BV} \le \|v\|_\infty$ for every $v \in AC_\infty$. Denote by pM (resp., pM_∞) the closure w.r.t. the norm $\|\cdot\|_{BV}$ (resp., $\|\cdot\|_\infty$) of the linear span of all games of the form $q \circ \mu$, where q is a polynomial vanishing at 0, and $\mu \in M$. The spaces pNA and pNA_∞ are defined likewise using measures $\mu \in NA$. Clearly, $G_I \subset pM_\infty \subset pM$, and the spaces are symmetric. If $f \in C^1([0,1]^n)$ and $\lambda \in (M^1)^n$ then $v = f \circ \lambda \in pM_\infty$ (and hence also $f \circ \lambda \in pM$), as follows from the proof of Proposition 7.1 in Aumann and Shapley (1974).

2.2 Extension of games to ideal coalitions

Let $B(I,C)$ be the space of all bounded, measurable, real-valued functions on (I,C), and consider $B_+^1(I,C) = \{f \in B(I,C) \mid 0 \le f \le 1\}$ – the set of *ideal coalitions*. Each coalition $S \in C$ can viewed as the ideal coalition χ_S (the indicator function of the set S), and we will identify S with χ_S whenever convenient. A real-valued function on $B_+^1(I,C)$ that vanishes at 0 is called an ideal game. An ideal game v is called monotonic if for each $f, g \in B_+^1(I,C)$ such that $f \ge g$ pointwise, $v(f) \ge v(g)$. The space IBV of ideal games of bounded variation is defined as the space of all differences of two monotonic ideal games.

For any $\lambda \in M$ and $g \in B(I,C)$, denote $\lambda(g) = \int_I g(a)\, d\lambda(a)$. Mertens (1988) constructed a linear operator on a large subspace of BV, containing pM, that extends every game from the domain C (identified with the set of indicator functions of coalitions in C) onto the entire set $B_+^1(I,C)$ of ideal coalitions. For any vector measure game $v = f \circ \lambda$ ($\in pM_\infty$) with $f \in C^1([0,1]^n)$ and $\lambda \in (M^1)^n$ for some $n \ge 1$, its well-defined *Mertens extension* $\overline{v}$ is given as follows. Consider a family $\{Z_a\}_{a \in A(\lambda)}$ of random variables that are uniform i.i.d. on $[0,1]$. Then

$$\overline{v}(g) = E\left[f\left(\lambda^{NA}(g) + \sum_{a \in A(\lambda)} \lambda^A(a)\, \chi_{\{Z_a \le g(a)\}}\right)\right], \tag{1}$$

where $g \in B^1_+(I,C)$, χ_B is the indicator function of an event B, and E is the expectation operator. Intuitively (though by no means formally[12]), $\overline{v}(g)$ can be thought of as the expected worth of a random coalition $\overline{S}$ which each player $x \in I$ joins, independently of others, with probability $g(x)$. The players from the nonatomic continuum $I \setminus A(\lambda)$ that join $\overline{S}$ are regarded as a deterministic perfect g-sample of the continuum, while the atoms in $A(\lambda)$ that join $\overline{S}$ are a genuinely random coalition.

The Mertens extension is defined, in particular, on the space G_I of finite games on I. It is just the familiar multilinear extension of Owen (1972). Given $v \in G_I$ and a nonempty finite support T of v, define $f_{v,T} \in C^1\left([0,1]^T\right)$ by

$$f_{v,T}\left((x_a)_{a\in T}\right) = \sum_{S\subset T} v(S) \cdot \prod_{a\in S} x_a \cdot \prod_{a\in T\setminus S}(1-x_a); \tag{2}$$

then

$$\overline{v}(g) = f_{v,T}\left((g(a))_{a\in T}\right) \tag{3}$$

for any $g \in B^1_+(I,C)$. It is easy to see that if $T' \supset T$ is another finite support of v then

$$f_{v,T}\left((x_a)_{a\in T}\right) = f_{v,T'}\left((x_a)_{a\in T}, (y_a)_{a\in T'\setminus T}\right) \tag{4}$$

for any $(y_a)_{a\in T'\setminus T}$, and hence $\overline{v}(g)$ is independent of the choice of T.

The Mertens extension $v \mapsto \overline{v}$ is linear and positive (i.e., it maps monotonic games into monotonic ideal games). It is also symmetric, i.e., $\overline{\theta v}(g) = \overline{v}\left(g \circ \theta^{-1}\right)$ for every $v \in BV, \theta \in \Theta, g \in B^1_+(I,C)$. If $v \in pM$ is *constant-sum*, that is, $v(S)+v(I\setminus S) = v(I)$ for every $S \in C$, then $\overline{v}$ is also constant-sum: $\overline{v}(g) + \overline{v}(1-g) = \overline{v}(1)$ for every $g \in B^1_+(I,C)$ (see Section 1.2 in Mertens (1988)).

Remark 1. The Mertens extension is continuous, being of operator norm 1: for every $v \in pM$, $\|\overline{v}\|_{IBV} \leq \|v\|_{BV}$, where $\|\cdot\|_{IBV}$ is the bounded variation norm on IBV, defined for any ideal game in IBV as the supremum of its variation over all increasing finite sequences $0 \leq f_1 \leq f_2 \leq ... \leq 1$ in $B^1_+(I,C)$. In particular, if $v \in pM$ and $S \in C$ is a set disjoint from some support of v, then $\overline{v}(g)$ is independent of the values of the ideal coalition g for players in S. Indeed, since v can be $\|\cdot\|_{BV}$-approximated by polynomials in measures, it can also be $\|\cdot\|_{BV}$-approximated by polynomials in measures for which S is a null set. By (1), the extensions of approximating games are

[12] See Judd (1985).

independent of $g\mid_S$, and by the continuity of the Mertens extension the limit $\overline{v}(g)$ is also independent of $g\mid_S$.

2.3 Differentiability of games

Given any $v \in pM_\infty$ and any $g \in B^1_+(I, C)$, there exists a well-defined *directional derivative* $d\overline{v}(g, \cdot)$, satisfying

$$d\overline{v}(g, h) = \lim_{\varepsilon \to 0+} \frac{\overline{v}(g + \varepsilon h) - \overline{v}(g)}{\varepsilon} \tag{5}$$

for every nonnegative $h \in B(I, C)$ such that $g + \varepsilon h \in B^1_+(I, C)$ for some $\varepsilon > 0$, and

$$d\overline{v}(g, h) = \lim_{\varepsilon \to 0+} \frac{\overline{v}(g) - \overline{v}(g - \varepsilon h)}{\varepsilon} \tag{6}$$

for every nonnegative $h \in B(I, C)$ such that $g - \varepsilon h \in B^1_+(I, C)$ for some $\varepsilon > 0$; moreover, $d\overline{v}(g, h)$ is additive in h. This was established in Section 2 of Hart and Monderer (1997) for $v \in pNA_\infty$, but similar arguments apply for $v \in pM_\infty$.

For $v = f \circ \lambda$ ($\in pM_\infty$), where $f \in C^1([0, 1]^n)$ and $\lambda \in (M^1)^n$ for some $n \geq 1$, the directional derivative $d\overline{v}(g, h)$ is given by the following formula that easily follows from (1) and (5), (6). For a nonnegative $h \in B(I, C)$ with $h\mid_{A(\lambda)} \equiv 0$ such that either $g + \varepsilon h \in B^1_+(I, C)$ or $g - \varepsilon h \in B^1_+(I, C)$ for some $\varepsilon > 0$,

$$d\overline{v}(g, h) = E\left\langle \nabla f\left(\lambda^{NA}(g) + \sum_{a \in A(\lambda)} \lambda(a)\, \chi_{\{Z_a \leq g(a)\}}\right), \lambda(h)\right\rangle, \tag{7}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product on $\mathbb{R}^n$ and $\{Z_a\}_{a \in A(\lambda)}$ are as in the premise for (1), while for $h = c \cdot \chi_{\{b\}}$, where $b \in A(\lambda)$ and $c \geq 0$,

$$d\overline{v}(g, h) = c \cdot E\left[\begin{array}{c} f\left(\lambda^{NA}(g) + \sum_{a \in A(\lambda), a \neq b} \lambda(a)\, \chi_{\{Z_a \leq g(a)\}} + \lambda(b)\right) \\ -f\left(\lambda^{NA}(g) + \sum_{a \in A(\lambda), a \neq b} \lambda(a)\, \chi_{\{Z_a \leq g(a)\}}\right) \end{array}\right]. \tag{8}$$

In words, for a nonatomic ideal coalition h the directional derivative $d\overline{v}(g, h)$ is the expected derivative of f in the direction $\lambda(h)$, evaluated at a random coalition that includes a perfect g-sample of the nonatomic players and is joined by each atom a with probability $g(a)$. If h is supported on $A(\lambda)$, then $d\overline{v}(g, h)$ is the expected sum of h-weighted marginal contributions of the atoms to that random coalition.

2.4 Compound games

The notion of a compound game was introduced by Shapley (1964) in the context of simple finite games, and extended by Owen (1964) to general finite games. We will generalize the notion of compounding for games in pM_∞.

Let $v \in G_I$ be a game with some finite support T. Consider a collection $(\alpha(a))_{a\in T}$ of disjoint sets in C and a collection $\left(w_{\alpha(a)}\right)_{a\in T}$ of games in pM_∞ such that, for every $a \in T$, the game $w_{\alpha(a)} \geq 0$ is supported on $\alpha(a)$ and $w_{\alpha(a)}(I) = 1$. (As each game $w_{\alpha(a)}$ is indexed by a set $\alpha(a)$, the games $\left(w_{\alpha(a)}\right)_{a\in T}$ can be viewed, in accord with our notation, as restrictions of some $w \in pM_\infty$ to disjoint sets $(\alpha(a))_{a\in T}$.) The game $u \in pM_\infty$ is said to be the *compounding* of v with $\left(w_{\alpha(a)}\right)_{a\in T}$ if

$$u(S) = f_{v,T}(\left(w_{\alpha(a)}(S)\right)_{a\in T}) \tag{9}$$

for every $S \in C$, where $f_{v,T}$ is the multilinear extension of v defined in (2). Notice that $u \in pM_\infty$ since pM_∞ is a Banach algebra.

When $\left(w_{\alpha(a)}\right)_{a\in T}$ are finite *simple* games (i.e., each $w_{\alpha(a)}$ is, additionally, monotonic and $\{0,1\}$-valued) and $(\alpha(a))_{a\in T}$ are finite sets, one can think of u as describing a two-tier voting process in which only the players in $\cup_{a\in T}\alpha(a)$ are entitled to take part. First, there is a simultaneous vote among the players within each of the groups $(\alpha(a))_{a\in T}$. The outcome of the vote in $\alpha(a)$ is determined via $w_{\alpha(a)}$. The vote then moves to the council of delegates. The delegate of each group $\alpha(a)$ is $a \in T$, and his vote must concur with the plebiscite in $\alpha(a)$. Outcomes of the council vote among the delegates in T are given by v. Clearly,

$$u(S) = f_{v,T}(\left(w_{\alpha(a)}(S)\right)_{a\in T}) = v(\{a \in T \mid w_{\alpha(a)}(S) = 1)$$

describes the outcome of the compound voting.

The compound game $f_{v,T}(\left(w_{\alpha(a)}\right)_{a\in T})$ has meaning, however, even in the general scenario, when the supports $(\alpha(a))_{a\in T}$ are infinite sets, possibly containing nonatomic parts and atoms, and the (not necessarily finite) games $\left(w_{\alpha(a)}\right)_{a\in T}$ obtain general values in $[0,1]$. Each game $w_{\alpha(a)}$ can be thought of as determining the *probability* that a subcoalition of the group $\alpha(a)$ "controls" its delegate a in the first-tier game v (e.g., this may mean that the subcoalition successfully accomplishes a project a that was assigned to $\alpha(a)$, at which point that project begins its "interaction" with projects in T accomplished by other groups, determining the final payoff via v).

The following fact, stated as a lemma and proved in the Appendix, will be of use in the sequel.

Lemma 1. Let v, T, $(\alpha(a))_{a\in T}$ and $\left(w_{\alpha(a)}\right)_{a\in T}$ be as in the premise for (9), and consider the compound game $u = f_{v,T}(\left(w_{\alpha(a)}\right)_{a\in T})$. For any $g \in B^1_+(I, C)$ that obtains finitely many values, and any $S \in C$ which is a subset of $\alpha(b)$ for some $b \in T$,

$$d\overline{u}(g, S) = \frac{df_{v,T}}{dx_b}(\left(\overline{w_{\alpha(a)}}(g)\right)_{a\in T}) \cdot d\overline{w_{\alpha(b)}}(g, S).$$

2.5 Multilinear Games

In treating multilinear games we will adopt the definitions in Monderer and Neyman (1988) and Haimanko (2000) with some minor modifications, and will use the (conveniently available) notation of the previous section pertaining to compounding. Let $\lambda \in M^1$ be a measure for which $\lambda^{NA}(I) > 0$ and the set $A(\lambda)$ $(= Supp(\lambda^A))$ is infinite. A game u will be called λ-multilinear if[13] $u = f_{v,T}\left(\left(\frac{\lambda_{\alpha(a)}}{\lambda(\alpha(a))}\right)_{a\in T}\right)$, where $v \in G_I$, T is a finite support of v, and $(\alpha(a))_{a\in T}$ is a collection of disjoint sets in C, such that $\alpha(a) = \{a\}$ if $a \in A(\lambda)$, and $\lambda^{NA}(\alpha(a)) > 0$ and $\alpha(a) \subset I \setminus A(\lambda)$ if $a \in I \setminus A(\lambda)$. The set[14] $ML(\lambda)$ of λ-multilinear games is dense in $pM(\lambda)_\infty$ (=the space defined in the same way as pM_∞ but with an additional requirement that the measure μ in a polynomial generator $q \circ \mu$ is absolutely continuous w.r.t. λ); see Proposition 8 in Haimanko (2000).[15]

Remark 2. Any semivalue on φ on pM_∞ is uniquely determined by its restriction $\varphi\mid_{ML(\lambda)}$ to games in $ML(\lambda)$ for $\lambda \in M^1$ as above. Indeed, since any semivalue on pM_∞ is continuous and $ML(\lambda)$ is dense in $pM(\lambda)_\infty$, $\varphi\mid_{ML(\lambda)}$ uniquely determines $\varphi\mid_{pM(\lambda)_\infty}$. Moreover, as $pM_\infty = \bigcup_{\theta\in\Theta} \theta\left(pM(\lambda)_\infty\right)$, φ is uniquely determined by $\varphi\mid_{pM(\lambda)_\infty}$ due to its symmetry.

[13] Recall that, for any $S \in C$, λ_S stands for the restriction of λ to the set S, given by $\lambda_S(S') = \lambda(S' \cap S)$ for every $S' \in C$.

[14] It can easily be shown that $ML(\lambda)$ is, in fact, a vector space.

[15] Although Proposition 8 of Haimanko (2000), and Theorem 4 of Monderer and Neyman (1988) upon which it is based, are stated for convergence in $\|\cdot\|_{BV}$, both results hold (with the same proofs) for convergence in the norm $\|\cdot\|_\infty$ on $pM(\lambda)_\infty$.

3 Semivalues

3.1 The general definition

Given a symmetric subspace Q of BV, we say that a linear operator $\varphi : Q \to FA$ is:

(i) *symmetric* if $\theta\varphi(v) = \varphi(\theta v)$ for every $v \in Q$ and $\theta \in \Theta$;

(ii) *positive* if $\varphi(v)$ is nonnegative whenever v is monotonic;

(iii) a *projection* if the restriction $\varphi\mid_{FA\cap Q}$ is an identity function on $FA \cap Q$.

A linear projection $\varphi : Q \to FA$ that is symmetric and positive is called a *semivalue* on Q.

Our main interest will be in semivalues on pM_∞. Any positive linear projection operator φ on pM_∞ is continuous of $\|\cdot\|_\infty$-norm 1 (see the proof of the Lemma in Section 9 of Monderer and Neyman (1988)). Moreover, $\varphi(v) \in M$ for every $v \in pM_\infty$, as for every positive $\mu \in M$ such that $-\mu \preceq v \preceq \mu$ the positivity and projection properties of φ yield $-\mu \preceq \varphi(v) \preceq \mu$, implying that $\varphi(v)$ is countably additive. Hence, in particular, any semivalue φ on pM_∞ is continuous and $\varphi(pM_\infty) = M$.

3.2 Derivative semivalues

In what follows we introduce a class of semivalues on pM_∞ (that, as will be made clear in Section 4, contains *all* possible semivalues on pM_∞). Denote by $M([0,1]^2)$ the set of Borel probability measures on $[0,1]^2$. Given $\xi, \eta \in M([0,1]^2)$, define an operator $\varphi_{\xi,\eta}$ as follows. For every $v \in pM_\infty$, denote by $A(v)$ the (at most countable) set of atoms of v. Then, for every $S \in C$, let

$$\varphi_{\xi,\eta}(v)(S) = \int_{[0,1]^2} d\overline{v}\left(t_1\left(I \setminus A(v)\right) + t_2 A(v), S \setminus A(v)\right) d\xi(t_1,t_2) \tag{10}$$

$$+ \int_{[0,1]^2} d\overline{v}\left(t_1(I \setminus A(v)) + t_2 A(v), S \cap A(v)\right) d\eta(t_1,t_2) \tag{11}$$

As each $d\overline{v}\left(t_1\left(I \setminus A(v)\right) + t_2 A(v), \cdot\right)$ is additive, $\varphi_{\xi,\eta}(v) \in FA$, and hence (10) defines an operator $\varphi_{\xi,\eta} : pM_\infty \to FA$.

Note that the integrals in (10)-(11) remains unchanged if $A(v)$ is replaced by any countable superset of $A(v)$, since $d\overline{v}(g,h)$ does not depend on the values of g and h on any given countable sets of nonatomic players in v (see Remark 1). Thus, due to the linearity of the Mertens extension in v, and the linearity of the directional derivative in $\overline{v}$, $\varphi_{\xi,\eta}$ is linear in v. The operator $\varphi_{\xi,,\eta}$ is symmetric since so is the Mertens extension and $A(\theta v) = \theta^{-1}(A(v))$ for any $v \in pM_\infty$ and $\theta \in \Theta$. The operator $\varphi_{\xi,\eta}$ is also positive since so is the Mertens extension, and the directional derivatives in a direction $S \in C$ are nonnegative for any monotonic ideal game. Finally, $\varphi_{\xi,\eta}$ is a projection on $FA \cap pM_\infty = M$, as $d\overline{\mu}(t_1(I \setminus A(\mu)) + t_2 A(\mu), S) = \mu(S)$ for any $\mu \in M$, $(t_1, t_2) \in [0,1]^2$, and $S \in C$.

Consequently, $\varphi_{\xi,\eta}$ is a semivalue on pM_∞ for any pair $\xi, \eta \in M([0,1]^2)$. We shall call it ξ,η-*induced derivative semivalue.* If both ξ and η are the Dirac measure concentrated on some $(t_1, t_2) \in [0,1]^2$, the corresponding derivative value will be denoted $\varphi_{(t_1,t_2)}$.

3.3 The Banzhaf value on pM_∞

The Banzhaf value, which is one of the most prominent semivalues in the context of finite games, can be extended to the entire pM_∞ as a particular derivative semivalue. Let us denote $\beta = \varphi_{\left(\frac{1}{2},\frac{1}{2}\right)}$ and call it *the Banzhaf value on* pM_∞. From (10)-(11), applied to the measures ξ and η that are concentrated on $\left(\frac{1}{2},\frac{1}{2}\right)$, we obtain

$$\beta(v)(S) = d\overline{v}\left(\frac{1}{2}, S\right) \tag{12}$$

for every $v \in pM_\infty$ and $S \in C$.

For $v = f \circ \lambda$ $(\in pM_\infty)$, where $f \in C^1([0,1]^n)$ and $\lambda \in (M^1)^n$ for some $n \geq 1$, the combination of (12) with (7) and (8) yields the following explicit formula: if $S \subset I \setminus A(v)$,

$$\beta(v)(S) = E\left\langle \nabla f\left(\frac{1}{2}\lambda^{NA}(I) + \sum_{a \in A(\lambda)} \lambda(a) Y_a\right), \lambda(S)\right\rangle,$$

where $\{Y_a\}_{a \in A(\lambda)}$ are $\{0,1\}$-valued i.i.d. random variables with mean $\frac{1}{2}$, and if $a' \in A(v)$,

$$\beta(v)(\{a'\}) = E\left[\begin{array}{c} f\left(\frac{1}{2}\lambda^{NA}(I) + \sum_{a \in A(\lambda), a \neq a'} \lambda(a) Y_a + \lambda(a')\right) \\ -f\left(\frac{1}{2}\lambda^{NA}(I) + \sum_{a \in A(\lambda), a \neq a'} \lambda(a) Y_a\right) \end{array}\right].$$

This boils down to the standard definition of the Banzhaf value for finite games: if $v \in G_I$ and $a \in Supp(v)$ $(= A(v))$, then

$$\beta(v)(\{a\}) = \sum_{S \subset Supp(v) \setminus \{a\}} \frac{1}{2^{|Supp(v)|-1}} \left[v(S \cup \{a\}) - v(S)\right], \tag{13}$$

and $\beta(v)(S) = 0$ for any $S \in C$ with $S \cap Supp(v) = \varnothing$.

It has been shown by Owen (1978) that the Banzhaf value for compound finite games has a strikingly simple multiplicative decomposition, which translates in our general context into the following definition. We say that a semivalue φ on pM_∞ has the *composition property* if, given any $v \in G_I$ with some finite support T, a collection $(\alpha(a))_{a \in T}$ of disjoint sets in C and a collection $\left(w_{\alpha(a)}\right)_{a \in T}$ of games in pM_∞ such that, for every $a \in T$, $w_{\alpha(a)} \geq 0$ is supported on $\alpha(a)$ and is constant-sum with $w_{\alpha(a)}(I) = 1$, the following holds for the compound game $u = f_{v,T}(\left(w_{\alpha(a)}\right)_{a \in T})$:

$$\varphi(u)(S) = \varphi(v)(\{a\}) \cdot \varphi\left(w_{\alpha(a)}\right)(S) \tag{14}$$

for every $a \in T$ and $S \in C$ that is a subset of $\alpha(a)$.

The composition property provides a simple computational link between the semivalue of the compound game and the semivalues of the first- and second-tier games. Namely, a subgroup of $\alpha(a)$ is assigned the semivalue of its delegate a in the first-tier game with a certain probability. That probability is equal to the semivalue evaluating the subgroup's contribution in the second-tier game to the success of controlling its delegate.

Proposition 1. The Banzhaf value β satisfies the composition property.

Proof. For each $a \in T$ the game $w_{\alpha(a)}$ is constant-sum by assumption, and the Mertens extension preserves this constancy. Therefore $\overline{w_{\alpha(a)}}\left(\frac{1}{2}\right) + \overline{w_{\alpha(a)}}\left(\frac{1}{2}\right) = \overline{w_{\alpha(a)}}(1) = w_{\alpha(a)}(I) = 1$, and hence $\overline{w_{\alpha(a)}}\left(\frac{1}{2}\right) = \frac{1}{2}$. Thus, for each $b \in T$ and $S \subset \alpha(b)$,

$$\begin{aligned}
\beta(u)(S) &= d\overline{u}\left(\frac{1}{2}, S\right) \\
\text{[by Lemma 1]} &= \frac{df_{v,T}}{dx_b}\left(\left(\overline{w_{\alpha(a)}}\left(\frac{1}{2}\right)\right)_{a \in T}\right) \cdot d\overline{w_{\alpha(b)}}\left(\frac{1}{2}, S\right) \\
&= \frac{df_{v,T}}{dx_b}\left(\frac{1}{2}, ..., \frac{1}{2}\right) \cdot d\overline{w_{\alpha(b)}}\left(\frac{1}{2}, S\right) \\
\text{[by (3)]} &= d\overline{v}\left(\frac{1}{2}, \{b\}\right) \cdot d\overline{w_{\alpha(b)}}\left(\frac{1}{2}, S\right) = \beta(v)(\{b\}) \cdot \beta\left(w_{\alpha(b)}\right)(S). \blacksquare
\end{aligned}$$

3.4 Values on pM_∞ as a particular case of derivative semivalues

Given a symmetric subspace Q of BV, an operator $\varphi : Q \to BV$ is *efficient* if $\varphi(v)(I) = v(I)$ for every $v \in Q$. An efficient semivalue on Q is called *value.* It is well known that there is a unique value on G_I and on pNA. These are the Shapley value and the Aumann-Shapley value, respectively; see Dubey et al. (1981), Aumann and Shapley (1974). However, when mixed games, which have both nonatomic and atomic parts, are allowed, which is the case with pM_∞, the uniqueness phenomenon is lost. Indeed, Hart (1973) constructed a large family of distinct values on pM_∞. (Haimanko (2000) showed that Hart values are the basic building blocks of the values on pM and pM_∞, as any such value must be a probabilistic mixture of Hart values.) In what follows we shall note how to represent Hart values as derivative semivalues.

In describing Hart values we follow the "path approach" of Haimanko (2000). Given two continuous and monotonic functions $F_1, F_2 : [0,1] \to [0,1]$ satisfying $F_1(0) = F_2(0) = 0$, $F_1(1) = F_2(1) = 1$, consider two measures $\xi = \xi(F_1, F_2) \in M([0,1]^2)$ and $\eta = \eta(F_1, F_2) \in M([0,1]^2)$ that are supported on the set $\{(F_1(s), F_2(s)) \mid 0 \le s \le 1\}$ (i.e., the image set of the path $s \mapsto F_1(s), F_2(s))$, and are determined by the equalities

$$\xi([F_1(s_1), F_1(s_2)] \times [F_2(s_1), F_2(s_2)]) = F_1(s_2) - F_1(s_1)$$

and

$$\eta([F_1(s_1), F_1(s_2)] \times [F_2(s_1), F_2(s_2)]) = F_2(s_2) - F_2(s_1)$$

for every $0 \le s_1 \le s_2 \le 1$. The induced derivative semivalue, $\varphi_{\xi(F_1,F_2),\eta(F_1,F_2)}$, is then given by

$$\begin{aligned}\varphi_{\xi(F_1,F_2),\eta(F_1,F_2)}(v)(S) &= \int_0^1 d\overline{v}\left(F_1(s)(I \setminus A(v)) + F_2(s)A(v), S \setminus A(v)\right) dF_1'(s) \\ &+ \int_0^1 d\overline{v}\left(F_1(s)(I \setminus A(v)) + F_2(s)A(v), S \cap A(v)\right) dF_2'(s)\end{aligned}$$

for every $v \in pM_\infty$ and $S \in C$. This is precisely the definition of the *symmetric* (F_1, F_2)*-path value* of Haimanko (2000). It is shown in Haimanko (2000) that the set of symmetric path-values coincides with the set of Hart values, and that their probabilistic mixtures generate all values on pM_∞.

4 Characterization of semivalues on pM_∞

Our first theorem will show that derivative semivalues exhaust the set of all semivalues on pM_∞. Its proof draws on the ideas of Haimanko (2000), who constructed an isomorphism between the sets of (partially symmetric) values on certain subspaces of pM_∞ and (partially symmetric) values of finite games. The construction that we will present is simpler, and more explicit and detailed. Given a semivalue φ on pM_∞, we will show that it induces an operator ϕ on G_I that, despite not being a true semivalue, satisfies a restricted form of symmetry, namely, covariance under automorphisms that preserve a certain partition of I into two types of players. (Intuitively, the two types of players in G_I will represent the nonatomic and atomic players of games in pM_∞, that may be treated differentially by φ despite its symmetry.) It will then be shown that ϕ has a measure-based representation akin to some ξ, η-induced derivative semivalue. Importantly, the connection between φ and ϕ is such that the measure-based representation of ϕ can be translated back to obtain the ξ, η-induced derivative semivalue form for φ.

Theorem 1. For any semivalue φ on pM_∞ there exists a uniquely determined pair of measures $\xi, \eta \in M([0,1]^2)$ such that $\varphi = \varphi_{\xi,\eta}$.

Proof. Fix $\lambda \in M^1$ for which $\lambda^{NA}(I) > 0$ and the set $A(\lambda)$ $(= Supp(\lambda^A))$ is infinite. Let $\overline{\Pi} = \{I \setminus A(\lambda), A(\lambda)\}$. An operator $\psi : G_I \to FA$ will be called $\overline{\Pi}$-symmetric if $\theta\psi(v) = \psi(\theta v)$ for every $v \in G_I$ and every $\theta \in \Theta$ that preserves the partition $\overline{\Pi}$, i.e., that satisfies $\theta\left(A(\lambda)\right) = A(\lambda)$. A positive $\overline{\Pi}$-symmetric linear projection $\psi : G_I \to FA \cap G_I$ will called a $\overline{\Pi}$-*symmetric semivalue* on G_I.

Now consider a semivalue φ on pM_∞. It induces a $\overline{\Pi}$-symmetric semivalue ϕ on G_I in the following fashion. Given a game $v \in G_I$ and its finite support T that satisfies $T \setminus A(\lambda) \neq \varnothing$, choose a countable measurable partition Π of I that refines[16] $\overline{\Pi}$ and, in addition, satisfies the following: $|\{\pi \in \Pi \mid \pi \subset I \setminus A(\lambda)\}| = |T \setminus A(\lambda)|$, $\{\pi \in \Pi \mid \pi \subset A(\lambda)\} = \{\{a\} \mid a \in A(\lambda)\}$, and $\lambda^{NA}(\pi) > 0$ for all $\pi \in \Pi$ with $\pi \subset I \setminus A(\lambda)$. Take any one-to-one and onto mapping $\tau : T \cup A(\lambda) \to \Pi$ such that $\tau(a) \subset I \setminus A(\lambda)$ if $a \in T \setminus A(\lambda)$ and $\tau(a) = \{a\}$ if $a \in A(\lambda)$. We will define $\phi(v) \in FA \cap G_T$ by involving an appropriate λ-multilinear game: let

[16] Namely, $\pi \subset I \setminus A(\lambda)$ or $\pi \subset A(\lambda)$ for every $\pi \in \Pi$.

$$\phi(v)(\{b\}) = \varphi\left(f_{v,T}\left(\left(\frac{\lambda_{\tau(a)}}{\lambda(\tau(a))}\right)_{a\in T}\right)\right)(\tau(b)) \tag{15}$$

for every $b \in T$.

We shall first show that ϕ is well-defined. Indeed, consider another support T' for the game v (w.l.o.g., $T \subset T'$), and let Π', τ' be some appropriate counterparts of Π, τ above, leading to $\phi'(v) \in FA \cap G_{T'}$ that is given by

$$\phi'(v)(\{b\}) = \varphi\left(f_{v,T'}\left(\left(\frac{\lambda_{\tau'(a)}}{\lambda(\tau'(a))}\right)_{a\in T'}\right)\right)(\tau'(b))$$

for every $b \in T'$. Since $f_{v,T'}\left(\left(\frac{\lambda_{\tau'(a)}}{\lambda(\tau'(a))}\right)_{a\in T'}\right) = f_{v,T}\left(\left(\frac{\lambda_{\tau'(a)}}{\lambda(\tau'(a))}\right)_{a\in T}\right)$ by (4), in fact

$$\phi'(v)(\{b\}) = \varphi\left(f_{v,T}\left(\left(\frac{\lambda_{\tau'(a)}}{\lambda(\tau'(a))}\right)_{a\in T}\right)\right)(\tau'(b)) \tag{16}$$

for every $b \in T'$.

Let a_0 be some dedicated atom in $T \setminus A(\lambda) \neq \varnothing$. Since each uncountable $\pi \in \Pi \cup \Pi'$ together with the corresponding measurable subsets is a standard measurable space, for any $a \in T \setminus A(\lambda), a \neq a_0$, there exists[17] a bi-measurable isomorphism $\theta_a : \tau(a) \to \tau'(a)$ such that $\frac{\lambda_{\tau(a)}}{\lambda(\tau(a))} = \frac{\theta_a \lambda_{\tau'(a)}}{\lambda(\tau'(a))}$ on $\tau(a)$, and there also exists a bi-measurable isomorphism $\theta_{a_0} : \tau(a_0) \to \tau'(a_0) \cup \left(\cup_{a\in T'\setminus T}\tau'(a)\right)$ such that $\frac{\lambda_{\tau(a_0)}}{\lambda(\tau(a_0))} = \frac{\theta_{a_0}\lambda_{\tau'(a_0)}}{\lambda(\tau'(a_0))}$ on $\tau(a_0)$. By letting $\theta(x) = \theta_a(x)$ whenever $x \in \tau(a)$ for some $a \in T \setminus A(\lambda)$, and $\theta(x) = x$ for every $x \in A(\lambda)$, an automorphism $\theta \in \Theta$ is constructed; it satisfies $\frac{\lambda_{\tau(a)}}{\lambda(\tau(a))} = \frac{\theta\lambda_{\tau'(a)}}{\lambda(\tau'(a))}$ for every $a \in T$. Thus, when $b \in T$, $b \neq a_0$,

$$\begin{aligned}
\varphi\left(f_{v,T}\left(\left(\frac{\lambda_{\tau(a)}}{\lambda(\tau(a))}\right)_{a\in T}\right)\right)(\tau(b)) &= \varphi\left(f_{v,T}\left(\left(\frac{\theta\lambda_{\tau'(a)}}{\lambda(\tau'(a))}\right)_{a\in T}\right)\right)(\tau(b)) \\
&= \varphi\left(\theta f_{v,T}\left(\left(\frac{\lambda_{\tau'(a)}}{\lambda(\tau'(a))}\right)_{a\in T}\right)\right)(\tau(b)) \\
\text{[by the symmetry of } \varphi\text{]} \quad &= \varphi\left(f_{v,T}\left(\left(\frac{\lambda_{\tau'(a)}}{\lambda(\tau'(a))}\right)_{a\in T}\right)\right)(\theta(\tau(b))) \\
&= \varphi\left(f_{v,T}\left(\left(\frac{\lambda_{\tau'(a)}}{\lambda(\tau'(a))}\right)_{a\in T}\right)\right)(\tau'(b)),
\end{aligned}$$

The right-hand terms in (15) and (16) are therefore identical when $b \in T$, $b \neq a_0$, and hence $\phi(v)(\{b\}) = \phi'(v)(\{b\})$ in this case.

[17] Use, e.g., Proposition 1.1 and Lemma 6.2 in Aumann and Shapley (1974).

For $b = a_0$, the symmetry of φ similarly implies that

$$\varphi\left(f_{v,T}\left(\left(\frac{\lambda_{\tau(a)}}{\lambda\left(\tau\left(a\right)\right)}\right)_{a\in T}\right)\right)(\tau(a_0)) = \varphi\left(\left(f_{v,T}\left(\frac{\lambda_{\tau'(a)}}{\lambda\left(\tau'\left(a\right)\right)}\right)_{a\in T}\right)\right)\left(\tau'\left(a_0\right)\cup\left(\cup_{a\in T'\setminus T}\tau'\left(a\right)\right)\right). \tag{17}$$

Notice that, for some $A > 0$,

$$-A\sum_{a\in T}\lambda_{\tau'(a)} \preceq f_{v,T}\left(\left(\frac{\lambda_{\tau'(a)}}{\lambda\left(\tau'\left(a\right)\right)}\right)_{a\in T}\right) \preceq A\sum_{a\in T}\lambda_{\tau'(a)},$$

and hence

$$-A\sum_{a\in T}\lambda_{\tau'(a)} \preceq \varphi\left(f_{v,T}\left(\frac{\lambda_{\tau'(a)}}{\lambda\left(\tau'\left(a\right)\right)}\right)_{a\in T}\right) \preceq A\sum_{a\in T}\lambda_{\tau'(a)} \tag{18}$$

because φ is a positive projection. When $b \in T' \setminus T$, (18) implies that

$$\varphi\left(f_{v,T}\left(\left(\frac{\lambda_{\tau'(a)}}{\lambda\left(\tau'\left(a\right)\right)}\right)_{a\in T}\right)\right)(\tau'(b)) = 0. \tag{19}$$

It follows that $\varphi\left(f_{v,T}\left(\left(\frac{\lambda_{\tau'(a)}}{\lambda(\tau'(a))}\right)_{a\in T}\right)\right)\left(\cup_{a\in T'\setminus T}\tau'\left(a\right)\right) = 0$, and thus (17) leads to

$$\varphi\left(f_{v,T}\left(\left(\frac{\lambda_{\tau(a)}}{\lambda\left(\tau\left(a\right)\right)}\right)_{a\in T}\right)\right)(\tau(a_0)) = \varphi\left(f_{v,T}\left(\left(\frac{\lambda_{\tau'(a)}}{\lambda\left(\tau'\left(a\right)\right)}\right)_{a\in T}\right)\right)\left(\tau'\left(a_0\right)\right),$$

which shows (via (15) and (16)) that $\phi\left(v\right)\left(\{a_0\}\right) = \phi'\left(v\right)\left(\{a_0\}\right)$. Therefore $\phi\left(v\right)\left(\{b\}\right) = \phi'\left(v\right)\left(\{b\}\right)$ for every $b \in T$.

When $b \in T'\setminus T$, $\phi'\left(v\right)\left(\{b\}\right) = \phi\left(v\right)\left(\{b\}\right)$, as these are, respectively, the left-hand and the right-hand sides of (19). Hence $\phi'\left(v\right)\left(\{b\}\right) = \phi\left(v\right)\left(\{b\}\right)$ for *every* $b \in T'$, and we have thereby shown that ϕ is well defined.

It is obvious that $\phi : G_I \to FA \cap G_I$ is linear, as φ is linear and $f_{v,T}$ is linear in $v \in G_T$ for any finite $T \subset I$. The operator ϕ is also a positive projection, since $f_{v,T}\left(\left(\frac{\lambda_{\tau(a)}}{\lambda(\tau(a))}\right)_{a\in T}\right)$ is a monotonic game for a monotonic $v \in G_I$ and is a measure for an additive $v \in G_I$, and φ is a positive projection.

Finally, we will show that ϕ is $\overline{\Pi}$-symmetric. Indeed, let $v \in G_I$ and consider any $\theta \in \Theta$ such that $\theta\left(A(\lambda)\right) = A(\lambda)$. Pick a finite $T \subset I$ such that $Supp(v) \subset T$, $Supp\left(\theta v\right) = \theta^{-1}\left(Supp(v)\right) \subset T$, and $T \setminus A(\lambda) \neq \varnothing$, and let Π and τ be as in the definition of ϕ in (15). For any $a \in \theta^{-1}\left(Supp(v)\right)\setminus A(\lambda)$ there exists a bi-measurable isomorphism $\theta_a : \tau\left(a\right) \to \tau\left(\theta\left(a\right)\right)$ such that $\frac{\lambda_{\tau(a)}}{\lambda(\tau(a))} = \frac{\theta_a\lambda_{\tau(\theta(a))}}{\lambda(\tau(\theta(a)))}$ on $\tau\left(a\right)$; now take some $\widehat{\theta} \in \Theta$ that coincides with θ_a on the set $\tau(a)$ for every $a \in \theta^{-1}(Supp(v)) \setminus A(\lambda)$, and with θ on the set $A(\lambda)$. Thus

$$\left(\frac{\widehat{\theta}\lambda_{\tau(\theta(a))}}{\lambda(\tau(\theta(a)))}\right)_{a\in\theta^{-1}(Supp(v))} = \left(\frac{\lambda_{\tau(a)}}{\lambda(\tau(a))}\right)_{a\in\theta^{-1}(Supp(v))}. \tag{20}$$

It is also easy to verify that

$$f_{v,T}\left(\left(\frac{\lambda_{\tau(a)}}{\lambda(\tau(a))}\right)_{a\in T}\right) = f_{\theta v,T}\left(\left(\frac{\lambda_{\tau(\theta(a))}}{\lambda(\tau(\theta(a)))}\right)_{a\in T}\right). \tag{21}$$

Hence, due to (4),

$$\begin{aligned}
f_{\theta v,T}\left(\left(\frac{\lambda_{\tau(a)}}{\lambda(\tau(a))}\right)_{a\in T}\right) &= f_{\theta v,T}\left(\left(\frac{\lambda_{\tau(a)}}{\lambda(\tau(a))}\right)_{a\in\theta^{-1}(Supp(v))}\right) \\
\text{[by (20)]} &= f_{\theta v,T}\left(\left(\frac{\widehat{\theta}\lambda_{\tau(\theta(a))}}{\lambda(\tau(\theta(a)))}\right)_{a\in\theta^{-1}(Supp(v))}\right) \\
&= \widehat{\theta} f_{\theta v,T}\left(\left(\frac{\lambda_{\tau(\theta(a))}}{\lambda(\tau(\theta(a)))}\right)_{a\in\theta^{-1}(Supp(v))}\right) \\
&= \widehat{\theta} f_{\theta v,T}\left(\left(\frac{\lambda_{\tau(\theta(a))}}{\lambda(\tau(\theta(a)))}\right)_{a\in T}\right) \\
\text{[by (21)]} &= \widehat{\theta} f_{v,T}\left(\left(\frac{\lambda_{\tau(a)}}{\lambda(\tau(a))}\right)_{a\in T}\right),
\end{aligned}$$

and we conclude that

$$f_{\theta v,T}\left(\left(\frac{\lambda_{\tau(a)}}{\lambda(\tau(a))}\right)_{a\in T}\right) = \widehat{\theta} f_{v,T}\left(\left(\frac{\lambda_{\tau(a)}}{\lambda(\tau(a))}\right)_{a\in T}\right). \tag{22}$$

Thus, for every $b \in \theta^{-1}(Supp(v))$,

$$\begin{aligned}
\phi(\theta v)(\{b\}) &= \varphi\left(f_{\theta v,T}\left(\left(\frac{\lambda_{\tau(a)}}{\lambda(\tau(a))}\right)_{a\in T}\right)\right)(\tau(b)) \\
\text{[by (22)]} &= \varphi\left(\widehat{\theta} f_{v,T}\left(\left(\frac{\lambda_{\tau(a)}}{\lambda(\tau(a))}\right)_{a\in T}\right)\right)(\tau(b)) \\
\text{[by the symmetry of } \varphi\text{]} &= \varphi\left(f_{v,T}\left(\left(\frac{\lambda_{\tau(a)}}{\lambda(\tau(a))}\right)_{a\in T}\right)\right)\left(\widehat{\theta}(\tau(b))\right) \\
&= \varphi\left(f_{v,T}\left(\left(\frac{\lambda_{\tau(a)}}{\lambda(\tau(a))}\right)_{a\in T}\right)\right)(\tau(\theta(b))) \\
&= \phi(v)(\{\theta(b)\}).
\end{aligned}$$

As $\phi(\theta v)(\{b\}) = 0 = \phi(v)(\{\theta(b)\})$ for any $b \notin \theta^{-1}(Supp(v))$ by the definition of ϕ, we have established that ϕ is $\overline{\Pi}$-symmetric.

To summarize, it has been shown that ϕ is a $\overline{\Pi}$-symmetric semivalue on G_I. The following lemma, in the spirit of Theorem 1(a) of Dubey et al. (1981), provides a complete characterization of such semivalues. It is proved in the Appendix.

Lemma 2. Any $\overline{\Pi}$-symmetric semivalue on G_I has the form $\phi_{\xi,\eta}$ for a uniquely determined pair $\xi, \eta \in M([0,1]^2)$, where

$$\phi_{\xi,\eta}(v)(\{a\}) = \int_{[0,1]^2} d\overline{v}\left(t_1 (I \setminus A(\lambda)) + t_2 A(\lambda), \{a\}\right) d\xi(t_1, t_2) \tag{23}$$

if $a \in I \setminus A(\lambda)$, and

$$\phi_{\xi,\eta}(v)(\{a\}) = \int_{[0,1]^2} d\overline{v}\left(t_1 (I \setminus A(\lambda)) + t_2 A(\lambda), \{a\}\right) d\eta(t_1, t_2) \tag{24}$$

if $a \in A(\lambda)$.

According to Lemma 2, the $\overline{\Pi}$-symmetric semivalue ϕ on G_I that is induced by the given φ is equal to $\phi_{\xi,\eta}$ for some uniquely determined $\xi, \eta \in M([0,1]^2)$.

Now consider any $v \in G_I$, together with any set T, partition Π and mapping τ defined as in the premise of (15), and denote $u = f_{v,T}\left(\left(\frac{\lambda_{\tau(a)}}{\lambda(\tau(a))}\right)_{a\in T}\right)$. Notice that for every $g \in B^1_+(I, C)$ that is constant on $I \setminus A(\lambda)$ and on $A(\lambda)$, and for every $b \in T$,

$$\begin{aligned} d\overline{u}(g, \tau(b)) &= \frac{df_{v,T}}{dx_b}\left(\left(\frac{\overline{\lambda_{\tau(a)}}(g)}{\lambda(\tau(a))}\right)_{a\in T}\right) \\ &= \frac{df_{v,T}}{dx_b}\left(\left(\frac{\lambda_{\tau(a)}(g)}{\lambda(\tau(a))}\right)_{a\in T}\right) = \frac{df_{v,T}}{dx_b}\left((g(a))_{a\in T}\right) = d\overline{v}(g, \{b\}), \end{aligned}$$

where the first equality follows from Lemma 1 (and the obvious fact that $d\left(\frac{\overline{\lambda_{\tau(b)}}}{\lambda(\tau(b))}\right)(g, \tau(b)) = 1$), and the last equality holds by (3). It follows that

$$d\overline{u}(g, \pi) = d\overline{v}\left(g, \{\tau^{-1}(\pi)\}\right) \tag{25}$$

for every $\pi \in \tau(T)$ and every g as above; if $\pi \in \Pi \setminus \tau(T)$, (25) still holds as both derivatives in it are then equal to 0.

If $\pi \in \Pi$ and $\pi \subset I \setminus A(\lambda)$, then

$$\begin{aligned} \varphi(u)(\pi) &= \phi(v)\left(\{\tau^{-1}(\pi)\}\right) = \phi_{\xi,\eta}(v)\left(\{\tau^{-1}(\pi)\}\right) = \\ &= \int_{[0,1]^2} d\overline{v}\left(t_1 (I \setminus A(\lambda)) + t_2 A(\lambda), \{\tau^{-1}(\pi)\}\right) d\xi(t_1, t_2) \\ \text{[by (25)]} &= \int_{[0,1]^2} d\overline{u}\left(t_1 (I \setminus A(\lambda)) + t_2 A(\lambda), \pi\right) d\xi(t_1, t_2) \\ \text{[by Remark 1]} &= \int_{[0,1]^2} d\overline{u}\left(t_1 (I \setminus A(u)) + t_2 A(u), \pi\right) d\xi(t_1, t_2) = \varphi_{\xi,\eta}(u)(\pi), \end{aligned}$$

and, similarly, $\varphi(u)(\pi) = \varphi_{\xi,\eta}(u)(\pi)$ if $\pi \in \Pi$ and $\pi \subset A(\lambda)$ (i.e., $\pi = \{a\}$ for some $a \in A(\lambda)$). Thus

$$\varphi(u)(\pi) = \varphi_{\xi,\eta}(u)(\pi) \tag{26}$$

for every $\pi \in \Pi$. Note that $\theta u = u$ for every $\theta \in \Theta$ that preserves the measure λ^{NA} and satisfies $\theta(\pi) = \pi$ for every $\pi \in \Pi$. Therefore, by a slight change in the proof of Proposition 6.1 in Aumann and Shapley (1974), using the symmetry of φ and $\varphi_{\xi,\eta}$ we obtain that $\varphi(u)(S) = \frac{\lambda_\pi(S)}{\lambda(\pi)} \cdot \varphi(u)(\pi)$ and $\varphi_{\xi,\eta}(u)(S) = \frac{\lambda_\pi(S)}{\lambda(\pi)} \cdot \varphi_{\xi,\eta}(u)(\pi)$ for any nonatomic $\pi \in \Pi$ and any $S \in C$, $S \subset \pi$. From this and (26) it follows that $\varphi(u) = \varphi_{\xi,\eta}(u)$. But, clearly, *any* λ-multilinear game can be represented as the above u for some v, T, Π and τ, and hence $\varphi|_{ML(\lambda)} = \varphi_{\xi,\eta}|_{ML(\lambda)}$. By Remark 2, φ and $\varphi_{\xi,\eta}$ coincide on the entire pM_∞. ■

Remark 3 (*Semivalues on pNA_∞ and pNA*). If the domain of a semivalue is restricted to be pNA_∞, a much simple characterization is obtained. As $v \in pNA_\infty$ has no atomic part, any ξ,η-induced derivative semivalue $\varphi_{\xi,\eta}$ on pNA_∞ is identical to $\varphi_{\overline{\xi}}$, where $\overline{\xi} \in M([0,1])$ is the marginal distribution of t_1 induced by ξ, and $\varphi_{\overline{\xi}}$ is given by

$$\varphi_{\overline{\xi}}(v)(S) = \int_{[0,1]} d\overline{v}(tI, S)\, d\overline{\xi}(t)$$

for any $v \in pNA_\infty$ and $S \in C$. Any semivalue on pNA_∞ is equal to $\varphi_{\overline{\xi}}$ for some $\overline{\xi} \in M([0,1])$. To prove this fact, one may repeat the arguments in the proof of Theorem 1, dropping all reference to the atomic part of the measure λ, and using Theorem 1(a) of Dubey et al. (1981) instead of Lemma 2.[18] If semivalues on pNA are considered instead, $\varphi_{\overline{\xi}}$ is still well-defined if $\overline{\xi}$ has a bounded Radon-Nikodym derivative w.r.t. the Lebesgue measure on $[0,1]$, and any semivalue on pNA is of this form (see Theorem 2 in Dubey et al (1981)).

5 Characterization of the Banzhaf Value

Our second theorem singles out the Banzhaf value on pM_∞ based on two requirements: the composition property, and being non-zero for every non-zero monotonic finite game.

[18] Alternatively, this characterization is implied by a more general result in Theorem 22 of Haimanko (2000).

Theorem 2. The Banzhaf value β is the only semivalue φ on pM_∞ that has the composition property and satisfies $\varphi(v) \neq 0$ for every monotonic $v \in G_I$ with $v(I) > 0$.

Proof. The composition property of β was established in Proposition 1. If $v \in G_I$ is a monotonic game and $v(I) > 0$, then at least one $a \in \mathit{Supp}(v)$ has a positive marginal contribution to some $S \subset \mathit{Supp}(v) \setminus \{a\}$, and it follows from (13) that $\beta(v)(\{a\}) > 0$. Thus, it remains to show that any semivalue φ on pM_∞ that has the composition property and that transforms non-zero monotonic finite games into non-zero measures must coincide with β. Fix one such φ for the duration of the proof.

Denote by SG_I the set of simple games in G_I, i.e., the set of monotonic and $\{0,1\}$-valued games $v \in G$ that satisfy $v(I) = 1$. The restriction of φ to SG_I, $\varphi|_{SG_I}$, obviously satisfies the composition property on the restricted domain. As φ is linear, $\varphi|_{SG_I}$ satisfies the *transfer axiom* (see Dubey et al. (2005)). By the positivity of φ and the non-zero assumption, $\varphi|_{SG_I}$ is nonnegative and non-zero on SG_I. By the Theorem of Dubey et al. (2005), the mentioned properties of $\varphi|_{SG_I}$ ensure that $\varphi|_{SG_I} = \beta|_{SG_I}$. The linearity of both φ and β implies $\varphi|_{G_I} = \beta|_{G_I}$.[19]

Now fix $\lambda \in M^1$ for which $\lambda^{NA}(I) > 0$ and $A(\lambda)$ $(= \mathit{Supp}(\lambda^A))$ is infinite, and consider any λ-multilinear game $u = f_{v,T}\left(\left(\frac{\lambda_{\alpha(a)}}{\lambda(\alpha(a))}\right)_{a\in T}\right)$, for some $v \in G_I$, finite support T of v, and a collection $(\alpha(a))_{a\in T}$ of disjoint sets in C such that $\alpha(a) = \{a\}$ if $a \in A(\lambda)$, and $\lambda^{NA}(\alpha(a)) > 0$ and $\alpha(a) \subset I \setminus A(\lambda)$ if $a \in I \setminus A(\lambda)$. By the composition property of φ, for every $a \in T$ and every $S \in C$ that is a subset of $\alpha(a)$,

$$\varphi(u)(S) = \varphi(v)(\{a\}) \cdot \varphi\left(\frac{\lambda_{\alpha(a)}}{\lambda(\alpha(a))}\right)(S) = \beta(v)(\{a\}) \cdot \frac{\lambda(S)}{\lambda(\alpha(a))},$$

where the second equality holds since $\varphi|_{G_I} = \beta|_{G_I}$ and φ is a projection. Since, obviously, the equalities also hold for the Banzhaf value β instead of φ, we have shown that $\varphi(u) = \beta(u)$, and as u is an arbitrary λ-multilinear game, $\varphi|_{ML(\mu)} = \beta|_{ML(\mu)}$. By Remark 2, $\varphi = \beta$. ■

Remark 4 (*On the necessity of the non-zero assumption in Theorem 2*). Consider the semivalue $\varphi_{(0,0)}$ on pM_∞. It can be shown (just as in the proof of Proposition 1)

[19] An alternative way of showing $\varphi|_{G_I} = \beta|_{G_I}$ is by a direct appeal to the results of Owen (1978) (Theorems 4 and 8) that, in conjunction with our non-zero assumption and the projection property of the semivalue establish uniqueness of the Banzhaf value for finite games.

that $\varphi_{(0,0)}$ satisfies the composition property. As $\varphi_{(0,0)}(v)(\{a\}) = v(\{a\})$ for every $v \in G_I$ and $a \in I$, it is obvious that $\varphi_{(0,0)}(v) = 0$ if $v \in G_I$ is any game for which $v(\{a\}) = 0$ for every $a \in I$. Thus, the requirement that $\varphi(v) \neq 0$ for every monotonic $v \in G_I$ with $v(I) > 0$ cannot be dispensed with in the statement of Theorem 2. Note also that the requirement cannot be strengthened to include games $v \in pM_\infty \setminus G_I$, as the Banzhaf value β would not satisfy the strengthened version. Indeed, consider any nondecreasing and continuously differentiable function $f : [0,1] \to [0,1]$ that satisfies $f(0) = 0$, $f(1) = 1$, and $f\left(\left[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right]\right) = \left\{\frac{1}{2}\right\}$ for some $\varepsilon > 0$, and let $\mu \in NA^1$. By the definition of the Banzhaf value, $\beta(f \circ \mu) = 0$.

6 Appendix

Proof of Lemma 1. We will first show that

$$\overline{u} = f_{v,T}(\left(\overline{w_{\alpha(a)}}\right)_{a \in T}). \tag{27}$$

Observe that, since $f_{v,T}$ in (2) is a sum of products, it suffices to show that $\overline{v_1 \cdot v_2} = \overline{v_1} \cdot \overline{v_2}$ for any $v_1, v_2 \in pM$ with disjoint supports S_1, S_2. It is easy to see, based on (1), that the equality holds when $v_1 = f \circ \lambda, v_2 = g \circ \mu$, where – for some $m,\ k \geq 1$ – $f \in C^1([0,1]^m)$, $g \in C^1\left([0,1]^k\right)$, and all components of the vector measures $\lambda \in (M^1)^m$ and $\mu \in (M^1)^k$ are supported on S_1 and S_2, respectively. But general $v_1, v_2 \in pM$ with disjoint supports S_1, S_2 are $\|\cdot\|_{BV}$-approximable by the vector measure games v_1, v_2 as above, and thus the equality $\overline{v_1 \cdot v_2} = \overline{v_1} \cdot \overline{v_2}$ holds in general, by the continuity of the Mertens extension and the continuity of the product function in BV and IBV.

By (27),

$$d\overline{u}(g,S) = df_{v,T}\left(\left(\overline{w_{\alpha(a)}}\right)_{a \in T}\right)(g,S), \tag{28}$$

and by an elementary chain rule argument[20]

$$df_{v,T}\left(\left(\overline{w_{\alpha(a)}}\right)_{a \in T}\right)(g,S) = \left\langle \nabla f_{v,T}(\left(\overline{w_{\alpha(a)}}(g)\right)_{a \in T}), \left(d\overline{w_{\alpha(a)}}(g,S)\right)_{a \in T} \right\rangle. \tag{29}$$

[20] In order for the chain rule argument to work, either (5) or (6) must hold, requiring that either $g + \varepsilon S$ or $g - \varepsilon S$ be in $B_+^1(I, C)$ for some $\varepsilon > 0$. As g obtains finitely many values, (29) can be proved separately for coalitions S on which the value of g is constant (and hence $g + \varepsilon S$ or $g - \varepsilon S$ are as required), and then extended by additivity of the directional derivative to any $S \in C$.

It follows from Remark 1 that $d\overline{w_{\alpha(a)}}\,(g, S) = 0$ for $a \neq b$ since $S \subset \alpha\,(b)$, and hence the required equality is immediate from (28) and (29). $\square$

Proof of Lemma 2. Let $N \subset I$ be a finite set with $|N \setminus A(\lambda)| = m > 0$, and $|N \cap A(\lambda)| = n > 0$. Generalizing the notion in the proof of Theorem 1, we will say that $\psi : G_N \to FA\cap\, G_N$ is a $\overline{\Pi}$-symmetric semivalue on G_N if it is a linear positive projection that in addition satisfies $\theta\psi(v) = \psi(\theta v)$, for every $v \in G_N$ and every $\theta \in \Theta$ for which $\theta\,(A(\lambda)) = A(\lambda)$ and $\theta(N) = N$. Let $p^{m,n} = (p^{m,n}_{s,t})_{0\le s\le m-1, 0\le t\le n}$ and $q^{m,n} = (q^{m,n}_{s,t})_{0\le s\le m, 0\le t\le n-1}$ be two nonnegative matrices such that

$$\sum_{s=1}^{m-1}\sum_{t=1}^{n}\binom{m-1}{s}\cdot\binom{n}{t}\cdot p^{m,n}_{s,t} = 1,$$

$$\sum_{s=1}^{m}\sum_{t=1}^{n-1}\binom{m}{s}\cdot\binom{n-1}{t}\cdot q^{m,n}_{s,t} = 1.$$

For any $S \subset N$, denote $m\,(S) = |S \setminus A(\lambda)|$ and $n\,(S) = |S \cap A(\lambda)|$. Define $\psi^N_{p^{m,n},q^{m,n}}$: $G_N \to FA\cap\, G_N$ by

$$\psi^N_{p^{m,n},q^{m,n}}\,(v)\,(\{a\}) = \sum_{S\subset N\setminus\{a\}} p^{m,n}_{m(S),n(S)}\,[v(S\cup\{a\} - v(S)] \tag{30}$$

if $a \in I \setminus A(\lambda)$, and

$$\psi^N_{p^{m,n},q^{m,n}}\,(v)\,(\{a\}) = \sum_{S\subset N\setminus\{a\}} q^{m,n}_{m(S),n(S)}\,[v(S\cup\{a\} - v(S)]$$

if $a \in A(\lambda)$. Similarly to the proof of the Lemma in Dubey at al. (1981) the following can be established:

Claim. For each $p^{m,n}, q^{m,n}$ as above, $\psi^N_{p^{m,n},q^{m,n}}$ is a $\overline{\Pi}$-symmetric semivalue on G_N. Moreover, every $\overline{\Pi}$-symmetric semivalue on G_N is of this form, and the mapping $(p^{m,n}, q^{m,n}) \mapsto \psi^N_{p^{m,n},q^{m,n}}$ is one-to-one.

We proceed by mainly adopting the arguments in the *Alternative Proof of Theorem 1(a)* in Dubey at al. (1981). Let ϕ be a $\overline{\Pi}$-symmetric semivalue on G_I. Fix $a \in I \setminus A(\lambda)$, and consider an increasing sequence $\left\{N^k\right\}_{k=1}^{\infty}$ of subsets of $I \setminus \{a\}$ with $\lim_{k\to\infty} m\left(N^k\right) = \lim_{k\to\infty} n\left(N^k\right) = \infty$. For each k, ϕ induces a $\overline{\Pi}$-symmetric semivalue $\phi\mid_{G_{N^k\cup\{a\}}}$ on $G_{N^k\cup\{a\}}$, and hence, by the Claim, it also induces a probability measure c_{N^k} on the subsets of N^k such that $c_{N^k}\,(S) = p^{m\left(N^k\right)+1, n\left(N^k\right)}_{m(S),n(S)}$ for every $S \subset N^k$ (where $p^{m(N^k)+1,n(N^k)}$ is the p-matrix corresponding to $\phi\ |_{G_{N^k\cup\{a\}}}$). By considering the natural embedding of $G_{N^k\cup\{a\}}$ into $G_{N^{k+1}\cup\{a\}}$, we have, for every $S \subset N^k$,

$$c_{N^k}(S) = \sum_{S\subset T\subset N^{k+1}, T\cap N^k = S} c_{N^{k+1}}(T).$$

The measures on the subsets of the various N^k are therefore "consistent", and thus by Kolmogorov's consistency theorem there exists a collection $\{Y_b\}_{b\in\cup_{k=1}^{\infty}N^k}$ of $\{0,1\}$-valued random variables such that

$$\Pr\left(\{b\in N^k \mid Y_b = 1\} = S\right) = c_{N^k}(S) = p_{m(S),n(S)}^{m(N^k)+1,n(N^k)}$$

for every $k \geq 1$ and $S \subset N^k$. Thus $\{Y_b\}_{b\in\cup_{k=1}^{\infty}N^k}$ is partially exchangeable: its (uniquely determined) distribution is invariant under all permutations of $\cup_{k=1}^{\infty}N^k$ that move finitely many elements of $\cup_{k=1}^{\infty}N^k$ and preserve the sets $\cup_{k=1}^{\infty}N^k \setminus A(\lambda)$ and $\cup_{k=1}^{\infty}N^k \cap A(\lambda)$. By the de-Finetti partial exchangeability principle for two types (see, e.g., Diaconis (1988)), the distribution of $\{Y_b\}_{b\in\cup_{k=1}^{\infty}N^k}$ is a unique mixture of sets $\{X_b\}_{b\in\cup_{k=1}^{\infty}N^k}$ of independent random variables such that both $\{X_b\}_{b\in\cup_{k=1}^{\infty}N^k\setminus A(\lambda)}$ and $\{X_b\}_{b\in\cup_{k=1}^{\infty}N^k\cap A(\lambda)}$ are i.i.d. As all variables are $\{0,1\}$-valued, this means that there exists a unique $\xi \in M\left([0,1]^2\right)$ such that for any $k \geq 1$ and $S \subset N^k$,

$$\begin{aligned} p_{m(S),n(S)}^{m(N^k)+1,n(N^k)} &= \Pr\left(\{b\in N^k \mid Y_b = 1\} = S\right) \\ &= \int_{[0,1]^2} t_1^{m(S)}\cdot(1-t_1)^{m(N^k)-m(S)}\cdot t_2^{n(S)}\cdot(1-t_2)^{n(N^k)-n(S)}d\xi(t_1,t_2). \end{aligned}$$

Substituting thus obtained formula for $p_{m(S),n(S)}^{m(N^k)+1,n(N^k)}$ into (30) shows with the aid of (8) that $\phi(v)(\{a\})$ is identical to $\phi_{\xi,\eta}(v)(\{a\})$ defined in (23), for every $k \geq 1$ and $v \in G_{N^k\cup\{a\}}$ (note that $\phi_{\xi,\eta}(v)(\{a\})$ *does not depend* on η, yet to be defined, as $a \in I \setminus A(\lambda)$). By the $\overline{\Pi}$-symmetry of ϕ, $\phi(v)(\{a\}) = \phi_{\xi,\eta}(v)(\{a\})$ for every $v \in G_I$ and every $a \in I \setminus A(\lambda)$. It is also obvious from the $\overline{\Pi}$-symmetry that the distribution of $\{Y_b\}_{b\in\cup_{k=1}^{\infty}N^k}$ would have been the same had a different sequence $\{N^k\}_{k=1}^{\infty}$ or a different $a \in I \setminus A(\lambda)$ been chosen, and thus ξ is determined uniquely by the given ϕ.

By fixing $a \in A(\lambda)$, analogous arguments can be made to show that there exists a unique $\eta \in M\left([0,1]^2\right)$ such that $\phi(v)(\{a\}) = \phi_{\xi,\eta}(v)(\{a\})$ for every $v \in G_I$ and $a \in A(\lambda)$. This establishes the lemma. $\square$

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