CONSENSUS AND SINGLE-PEAKEDNESS

Muhammad Mahajne and Oscar Volij

Discussion Paper No. 17-02

June 2017

Monaster Center for
Economic Research
Ben-Gurion University of the Negev
P.O. Box 653
Beer Sheva, Israel

Fax: 972-8-6472941
Tel: 972-8-6472286

Consensus and single-peakedness *

Muhammad Mahajne[†] Oscar Volij[†]

June 19, 2017

Abstract

Consider a set of K alternatives. We say that an alternative is socially acceptable if the number of individuals that rank it among their most preferred half of the alternatives is at least as large as the number of individuals that rank it among the least preferred half. A Condorcet winner is not necessarily socially acceptable. We propose a concept of consensus as a condition on preference profiles that guarantees that majority rule yields a transitive preference relation and furthermore, that the resulting Condorcet winner is a socially acceptable alternative. This new condition is weaker than single-peakedness of preferences.

1 Introduction

It can be said that a main goal of social choice theory is to find a reasonable way to aggregate individual preferences into one social preference. Arrow's [1] impossibility theorem, however, has dealt a fatal blow to such an aspiration by showing that any social welfare function defined over an unrestricted domain, that satisfies the unanimity and the independence of irrelevant alternatives axioms must be dictatorial. Unanimity is a weak property requiring

*We thank Shmuel Nitzan for useful comments, suggestions and support.

[†]Ben-Gurion University of the Negev

that if all individuals share a particular preference relation, this common relation must be the social preference relation. Not only is unanimity a weak requirement, but also it is so natural that its violation would render any preference aggregation function unacceptable. As a result, the search for reasonable preference aggregation functions has focused on domain restrictions or on weakenings of the independence axiom. A notable form of domain restriction is single-peakedness of preference relations (introduced by Black [2] and Inada [5]). See Gaertner [4] and Nitzan [8] for surveys of this literature. Profiles of single-peaked preferences have interesting properties. For instance, when preferences are single-peaked there is always a Condorcet winner. Furthermore, if the number of individuals is odd the majority rule delivers a transitive preference relation.

Although the assumption of single-peaked preferences is very powerful, it is also somewhat restrictive. In this paper we propose a new condition on a preference profile that guarantees the existence of a Condorcet winner and the transitivity of the relation induced by the majority rule. We call this condition consensus and it is a weakening of the concept of level r consensus proposed in Mahajne et. al. [7]. Roughly speaking, a preference profile exhibits consensus around a given preference relation if whenever one (convex or homogeneous) subset of $K!/2$ preference relations (K being the number of alternatives over which preferences are defined) is more similar to that preference than another disjoint (convex or homogeneous) subset of the same size, there are at least as many individuals with preferences in the former than in the latter. According to this notion, a preference profile may exhibit consensus around some preference relation even if not all individuals share the same preference. It turns out that our consensus condition is weaker than the single-peakedness condition. Indeed, one of our main results states that if a preference profile consists of an odd number of single-peaked preference relations, then we can identify one preference relation around which there is consensus. This preference relation coincides with the one induced by the majority rule. Furthermore, the resulting Condorcet winner turns out to be socially acceptable in the sense that a majority of voters rank it among their most preferred half of the alternatives. In some sense our result is an extension of the median voter theorem. According to the median voter theorem (see Black[2] and Downs [3]), when a profile consists of an odd number of single-peaked preferences one can identify an alternative, the median voter's peak, which is a Condorcet winner, and furthermore the majority relation turns out to be transitive. Under the same circumstances our theorem identifies a preference relation, which turns out to be the majority relation, around which there is consensus.

2 Definitions

Let $A = \{a_1, \dots, a_K\}$ be a set of K alternatives and let $N = \{1, \dots, n\}$ be a set of individuals. Also, let $\mathcal{R}$ be the set of binary relations on A, and $\mathcal{P}$ be the subset of complete, transitive and antisymmetric binary relations on A. We will refer to the elements of $\mathcal{P}$ as preference relations or simply as preferences. A *preference profile* is a mapping $\pi = (\succ_1, \dots, \succ_n)$ of preference relations on A to the individuals in N. For each individual $i \in N$, $\succ_i$ represents i's preferences over the alternatives in A. The inverse of a preference relation $\succ$ is the preference relation $\succ^{-1}$ defined by $a \succ^{-1} b \Leftrightarrow b \succ a$. We denote by $\mathcal{P}^n$ the set of preference profiles. Strict linear orders on A will sometimes be denoted by $<$.

Let $\pi = (\succ_1, \dots, \succ_n)$ be a preference profile. For each preference relation $\succ \in \mathcal{P}$, $\mu_\pi(\succ) = |\{i \in N : \succ_i = \succ\}|$ is the number of individuals whose preference relation is $\succ$. More generally, for any subset $C \subseteq \mathcal{P}$ of preference relations, $\mu_\pi(C) = |\{i \in N : \succ_i \in C\}|$ is the number of individuals whose preference relations are in C. Also, $\pi(N) = \{\succ \in \mathcal{P} : \exists i \in N \text{ s.t. } \succ_i = \succ\}$ is the set of preferences that are present in the profile π. An *aggregation rule* is a function $f : \mathcal{P}^n \to \mathcal{R}$ that assigns to each preference profile a social binary relation. An aggregation rule is a *Social Welfare Function* if its range is a subset of transitive binary relations on A.

An important example of an aggregation rule is the *majority rule*, which we define next. Let $a, a' \in A$ be two alternatives. Denote by $C(a \to a') = \{\succ \in \mathcal{P} : a \succ a'\}$ the set of preference relations according to which a is strictly preferred to a'. The majority rule assigns to each preference profile $\pi \in \mathcal{P}$ the binary relation M_π on A defined by

$$aM_\pi a' \Leftrightarrow \mu_\pi(C(a \to a')) \geq \mu_\pi(C(a' \to a)).$$

It is well known that the majority rule does not deliver a transitive binary relation for each preference profile, and thus it is not a social welfare function.

Let $d : \mathcal{P}^2 \to \mathbb{R}$ be a metric on $\mathcal{P}$. That is, for every $\succ, \succ', \succ'' \in \mathcal{P}$, d satisfies

- $d(\succ, \succ') \geq 0$
- $d(\succ, \succ') = 0 \;\Leftrightarrow \succ = \succ'$
- $d(\succ, \succ') = d(\succ', \succ)$
- $d(\succ, \succ'') \leq d(\succ, \succ') + d(\succ', \succ'')$

In this paper we will restrict to the *inversion metric* only, which is defined as follows:[1] $d(\succ, \succ')$ is the minimum number of pairwise adjacent transpositions needed to obtain $\succ'$ from $\succ$. Alternatively, $d(\succ, \succ')$ is the number of pairs of alternatives in A that are ranked differently by $\succ$ and $\succ'$. Formally, the inversion metric is defined by

$$d(\succ, \succ') = |\succ \setminus \succ'|.$$

A metric d defined on $\mathcal{P}$ allows us to determine for any two preference relations $\succ$ and $\succ'$ which one is closer to a third one $\succ_0$. We are interested in extending this kind of comparison to equal-sized sets of preferences as well. The following definition identifies circumstances under which a given set of preferences $C \subseteq \mathcal{P}$ can be said to be closer to $\succ_0$ than is an alternative set $C' \subseteq \mathcal{P}$.

Definition 1 Let C and C' be two subsets of $\mathcal{P}$ with the same cardinality. We say that C *is closer than* C' *to* $\succ_0$, denoted $C \gg_{\succ_0} C'$, if there is a one to one function $\phi : C \to C'$ such that for all $\succ \in C$, $d(\succ, \succ_0) < d(\phi(\succ), \succ_0)$.

In other words, C is closer than C' to some given preference relation $\succ_0 \in \mathcal{P}$ if each preference relation $\succ'$ in C' can be paired with a preference relation $\succ$ in C that is closer to $\succ_0$, according to d, as $\succ'$ is.

[1] See Kemeny and Snell [6] for a characterization of this metric.

An alternative way to check whether $C \gg_{\succ_0} C'$ is as follows. Let $d(C, \succ_0)$ be the list of distances $(d(\succ, \succ_0))_{\succ \in C}$ arranged in a non-decreasing order. Similarly, let $d(C', \succ_0)$ be the list $(d(\succ, \succ_0))_{\succ \in C'}$ also arranged in a non-decreasing order. Then $C \gg_{\succ_0} C' \Leftrightarrow d(C, \succ_0) < d(C', \succ_0)$.

Example 1 Let the set of alternatives be $A = \{a, b, c\}$. The set $\mathcal{P}$ contains six preference relations, given by

$$\begin{array}{rclcrcl} \succ_1 & = & a, b, c & \quad & \succ_2 & = & a, c, b \\ \succ_3 & = & b, a, c & & \succ_4 & = & c, a, b \\ \succ_5 & = & b, c, a & & \succ_6 & = & c, b, a \end{array}$$

There are 10 ways to partition $\mathcal{P}$ into two subsets with three preference relations each. One such partition is $\{C_1, \overline{C}_1\}$ where

$$C_1 = \{\succ_1, \succ_2, \succ_3\} \text{ and } \overline{C}_1 = \{\succ_4, \succ_5, \succ_6\}.$$

Consider the preference relation $\succ_1$. It can be checked that according to the inversion metric, the distances of each preference relation in $\mathcal{P}$ to $\succ_1$ are given by

$$\begin{aligned} d(\succ_1, \succ_1) &= 0 \\ d(\succ_2, \succ_1) = d(\succ_3, \succ_1) &= 1 \\ d(\succ_4, \succ_1) = d(\succ_5, \succ_1) &= 2 \\ d(\succ_6, \succ_1) &= 3 \end{aligned}$$

It can also be checked that $C_1 \gg_{\succ_1} \overline{C_1}$. Indeed, $d(\succ_1, \succ_1) < d(\succ_4, \succ_1)$, $d(\succ_2, \succ_1) < d(\succ_5, \succ_1)$ and $d(\succ_3, \succ_1) < d(\succ_6, \succ_1)$.

We next define when a preference relation is between two other preferences.

Definition 2 Let $\succ_i, \succ_j, \succ_k$ be three distinct preference relations. We shall say that $\succ_j$ is *between* $\succ_i$ and $\succ_k$ if

$$d(\succ_i, \succ_k) = d(\succ_i, \succ_j) + d(\succ_j, \succ_k).$$

The next Claim shows that if two preference relations agree on the ranking of two alternatives, a and b, then any preference relation that is between them also agrees on this ranking.

Claim 1 Let $\succ_1$, $\succ_2$ and $\succ_3$ be three preference relations. If $\succ_2$ is between $\succ_1$ and $\succ_3$ then, for all $a, b \in A$,

$$a \succ_1 b \text{ and } a \succ_3 b \quad \Rightarrow \quad a \succ_2 b$$

Proof : Let A_1 be the set of pairs of alternatives on which $\succ_1$ disagrees with both $\succ_2$ and $\succ_3$. Similarly, let A_2 be the set of pairs of alternatives on which $\succ_2$ disagrees with both $\succ_1$ and $\succ_3$, and let A_3 be the set of pairs of alternatives on which $\succ_3$ disagrees with both $\succ_1$ and $\succ_2$. Note that these sets are pairwise disjoint. By definition $d(\succ_1, \succ_2) = |A_1| + |A_2|$, $d(\succ_2, \succ_3) = |A_2| + |A_3|$, and $d(\succ_1, \succ_3) = |A_1| + |A_3|$. Since $\succ_2$ is between $\succ_1$ and $\succ_3$ we have that

$$d(\succ_1, \succ_3) = d(\succ_1, \succ_2) + d(\succ_2, \succ_3)$$

or,

$$|A_1| + |A_3| = |A_1| + |A_2| + |A_2| + |A_3|$$

which implies that $|A_2| = 0$. Namely, there is no pair of alternatives about whose ranking $\succ_2$ disagrees with both $\succ_1$ and $\succ_3$. $\square$

We now define the concept of a convex set of preferences. A set is convex if whenever two preference relations belong to it, so do the preferences that are between them.

Definition 3 Let C be a subset of $\mathcal{P}$. We say that C is *convex* if for all three preference relations $\succ_1$, $\succ_2$ and $\succ_3$, such that $\succ_2$ is between $\succ_1$ and $\succ_3$, we have that

$$\succ_1, \succ_3 \in C \quad \Rightarrow \quad \succ_2 \in C.$$

For any preference relation $\succ$ and for any alternative $a \in A$, the rank of a in $\succ$, denoted by $\text{rank}_{\succ}(a)$, is 1 + the number of alternatives that are strictly preferred to a according to

$\succ$. Formally, $\text{rank}_{\succ}(a) = K - |\{a' \in A : a \succ a'\}|$. Alternatives whose ranks are less than $(K+1)/2$ are said to be *above the line* and those whose ranks are greater than $(K+1)/2$ are said to be *below the line*.

For any $a \in A$ denote

$$\begin{aligned} U(a) &= \{\succ \in \mathcal{P} : \text{rank}_{\succ}(a) < \frac{K+1}{2}\} \\ I(a) &= \{\succ \in \mathcal{P} : \text{rank}_{\succ}(a) = \frac{K+1}{2}\} \\ L(a) &= \{\succ \in \mathcal{P} : \text{rank}_{\succ}(a) > \frac{K+1}{2}\} \end{aligned}$$

$U(a)$ is the set of preference relations that locate alternative a above the line. The sets $L(a)$ and $I(a)$ have analogous interpretations.

We now define what it means for an alternative to be socially acceptable.

Definition 4 Let π be a preference profile, and let $a \in A$ be an alternative. We say that a is *socially acceptable* if the number of individuals for whom a is above the line is at least as large as the number of individuals for whom a is below the line.

The next example show that a Condorcet winner may not be socially acceptable.

Example 2 Consider the following profile of 29 preference relations on $A = \{a, b, c, d\}$.

Preference	# of indiv.	Preference	# of indiv.
abcd	1	bcad	2
abdc	2	bcda	2
acbd	3	bdac	1
bacd	1	cbad	1
acdb	3	bdca	2
adbc	2	cbda	1
badc	0	cdab	1
cabd	0	dbac	1
adcb	2	cdba	1
cadb	0	dbca	1
dabc	0	dcab	1
dacb	0	dcba	1
	14		15

It can be seen that alternative a is a Condorcet winner. Indeed, out of the twenty nine individuals, fifteen prefer a to b, sixteen prefer a to c, and seventeen prefer a to d. On the other hand, only fourteen individuals rank a above the line as opposed to fifteen individuals who rank it below the line. Therefore a is not socially acceptable.

Next we define the concept of a homogeneous set of preferences.

Definition 5 Let $C \subseteq \mathcal{P}$ be a subset of preferences and let (a, b) be a pair of alternatives. We say that C is upper homogeneous with respect to (a, b) if for all $\succ \in C$, $\succ$ places a above or on the line, and if it places a exactly on the line then it places b above the line. Formally, if for all $\succ \in C$,

$$\text{rank}_{\succ}(a) \quad < \quad (K+1)/2$$

$$\text{or}$$

$$\text{rank}_{\succ}(a) = (K+1)/2 \quad \text{and} \quad \text{rank}_{\succ}(b) < (K+1)/2,$$

We say that C is lower homogeneous with respect to (a, b) if for all $\succ \in C$, $\succ$ places a below or on the line, and if it places a on the line then it places b below the line. Formally, if for all $\succ \in C$,

$$\text{rank}_{\succ}(a) \quad > \quad (K+1)/2$$

$$\text{or}$$

$$\text{rank}_{\succ}(a) = (K+1)/2 \quad \text{and} \quad \text{rank}_{\succ}(b) > (K+1)/2.$$

We also say that C is homogeneous with respect to (a, b) if it is either upper or lower homogeneous. If C is homogeneous with respect to some pair of alternatives, we say that C is homogeneous.

A set of preferences C is upper homogeneous with respect to (a, b) if for all preference relations $\succ \in C$, a is ranked at least as high as half of the alternatives and whenever the number of alternatives that are ranked above a equals the number of alternatives that rank below a, alternative b is ranked above more than half of the alternatives. Formally, C is upper homogeneous with respect to (a, b) if $C \subseteq U(a) \cup (I(a) \cap U(b))$. The interpretation of C being lower homogeneous is analogous.

Example 3 Let $A = \{a, b, c, d\}$ be the set of alternatives and consider the following set of preferences on A:

$$C = \{abcd, abdc, acbd, acdb, adbc, adcb, bacd, badc, cabd, cadb, dabc, dacb\}.$$

It can be checked that for all preferences in this set, a is in the first or second place. Therefore, this set is homogeneous. However, it is not a convex set. Indeed, it can be checked that $bcad$ is between $cadb$ and $bacd$, but while $cadb, bacd \in C$, $bcad \notin C$. On the other hand, the set $C(a \to b)$ is convex but not homogeneous. Indeed, for any alternative $x \neq a, b$ there are preference relations in $C(a \to b)$ in which x is ranked at the top and there are other preference relations in $C(a \to b)$ in which x is rank at the bottom. Also, this set contains a preference relation in which a and b are ranked first and second, and it also contains a preference relation that they are ranked in the $K-1$th and Kth place.

If $\{C, C'\}$ is a partition of $\mathcal{P}$ into convex sets, we say that $\{C, C'\}$ is a *convex partition.* If it is a partition into homogeneous sets we say that $\{C, C'\}$ is a *homogeneous partition.*

Note that if $\{C, C'\}$ is a partition of $\mathcal{P}$ into sets of equal cardinality and C is upper-homogeneous with respect to (a, b) then C' is lower-homogeneous with respect to (a, b).

Also note that if (C, C') is either a homogenous or a convex partition into sets of the same cardinality, then $\succ \in C$ implies that $\succ^{-1} \in C'$. Homogeneous partitions with caridnality $K!/2$ play a central role in this paper. The following auxiliary result, allows us to express them in a simple and useful way.

Claim 2 Let C be a set of preferences with cardinality $K!/2$. If it is upper-homogeneous with respect to (a, b) then

$$C = \{\succ : \text{rank}_\succ(a) < (K+1)/2\} \cup \{\text{rank}_\succ(a) = (K+1)/2 \text{ and } \text{rank}_\succ(b) < (K+1)/2\}.$$

If it is lower-homogeneous with respect to (a, b) then

$$C = \{\succ : \text{rank}_\succ(a) > (K+1)/2\} \cup \{\text{rank}_\succ(a) = (K+1)/2 \text{ and } \text{rank}_\succ(b) > (K+1)/2\}.$$

Proof : See Appendix. □

We now concentrate on single-peaked preferences. Recall that a set of preference relations are single-peaked with respect to a given linear order of the alternatives if each preference has a peak such that for any two alternatives on the same side of the peak, one is preferred over the other if it is closer to the peak.

Definition 6 Let A be a set of K alternatives. Let $<$ be a linear order on A. We say that the preference relation $\succ$ is single-peaked with respect to $>$ if there is an alternative $p \in A$ such that

$$(a < b \leq p \text{ or } p \leq b < a) \Rightarrow b \succ a.$$

If $a < b$ we say that a is to the left of b or that b is to the right of a. Alternatives that are to the left of at least $\lceil K/2 \rceil$ alternatives are said to be on the left side of the spectrum, and alternatives that are to the right of at least $\lceil K/2 \rceil$ alternatives are said to be on the right side of the spectrum.

We now define the notion of consensus which plays a central role in our results.

Definition 7 Let $\succ_0 \in \mathcal{P}$. A preference profile $\pi \in \mathcal{P}^n$ exhibits *consensus around* $\succ_0$ if for all convex or homogeneous partitions $\{C, C'\}$ of $\mathcal{P}$ into two sets with the same cardinality we have that $C \gg_{\succ_0} C' \Rightarrow \mu_\pi(C) > \mu_\pi(C')$.

In words, a preference profile π exhibits consensus around preference relation $\succ_0$ if whenever $\{C, C'\}$ is a convex or homogeneous partition of $\mathcal{P}$ with the same cardinality and subset C is closer to $\succ_0$ than C', the number of preference relations in π that belong to C is greater than the number of preferences relations in π that belong to C'.

The following proposition states a powerful implication of consensus.

Proposition 1 Let $\pi = \{\succ_1, \ldots, \succ_n\} \in \mathcal{P}$ be a preference profile. If π exhibits consensus around $\succ_0 \in \mathcal{P}$, then the binary relation assigned by the majority rule to π, M_π coincides with $\succ_0$. In particular, M_π is transitive.

Proof : Let $a, b \in A$ be two distinct alternatives. We need to show that $aM_\pi b$ if and only if $a \succ_0 b$. Assume without loss of generality that $a \succ_0 b$. Partition $\mathcal{P}$ into the two sets $C(a \to b)$ and $C(b \to a)$. These two sets constitute a convex partition. To see this, let $\succ_1, \succ_3 \in C(a \to b)$ and let $\succ_2$ be a preference relation that is between them. By Claim 1, $\succ_2 \in C(a \to b)$. Furthermore, these sets contain $K!/2$ elements each. Consider the one-to-one function $\varphi : C(a \to b) \to C(b \to a)$ defined as follows: for each $\succ \in C(a \to b)$, let $\varphi(\succ) \in \mathcal{P}$ be the preference relation that is obtained from $\succ$ by switching a and b in the ranking. Since $a \succ_0 b$, $d(\succ, \succ_0) < d(\phi(\succ), \succ_0)$ for all $\succ \in C(a \to b)$. In other words, according to the inversion metric, $C(a \to b) \gg_{\succ_0} C(b \to a)$. Since there is consensus around $\succ_0$, we have that $\mu_\pi(C(a \to b)) > \mu_\pi(C(b \to a))$, which means that $aM_\pi b$. Conversely, if $aM_\pi b$ we must have $\mu_\pi(C(a \to b)) \geq \mu_\pi(C(b \to a))$. It follows that $a \succ_0 b$ since otherwise, by the previous argument we would have that $\mu_\pi(C(b \to a)) > \mu_\pi(C(a \to b))$. Given that $\succ_0$ is transitive we obtain that M_π, is transitive. $\square$

The fact that majority rule induces a transitive preference relation does not imply that there is consensus around the preference relation induced by it. To see this consider Example 2. It can be checked that the preference relation induced by majority rule is $\succ_0 = abcd$. Furhtermore, $\{U(a), L(a)\}$ constitutes a homogeneous partition of $\mathcal{P}$ such that $U(a) \gg_{\succ_0} L(a)$. On the other hand, $\mu_\pi(U(a)) = 14 < 15 = \mu_\pi(L(a))$. Namley, π does not exhibit consensus around $\succ_0$.

Our first result states that if there exists consensus around some preference relation, then the Condorcet winner is socially acceptable.

Theorem 1 Let π be a preference profile and let $\succ_0$ be a preference relation. If π exhibits consensus around $\succ_0$ then there exists a Condorcet winner which is socially acceptable.

Proof : Let π be a preference profile that exhibits consensus around $\succ_0$. Assume without loss of generality that alternative a is on top of $\succ_0$. Then, by Proposition 1 a is a Condorcet winner. We will now show that a is socially acceptable. For this purpose, we observe the following:

Lemma 1 Let a be the alternative that is on top of the preference relation $\succ_0$, and let b be any other alternative. Let C, C' be a partition of $\mathcal{P}$ into two sets of the same cardinality where C is upper homogeneous with respect to (a, b) and therefore C' is lower homogeneous with respect to (a, b). Then, $C \gg_{\succ_0} C'$.

Proof : Case 1: K is even. In this case, $C = \{\succ \in \mathcal{P} : \text{rank}_\succ(a) < (K+1)/2\}$. Let $\succ \in C$. We map $\succ$ to the preference relation $\succ'$ that is obtained from it by moving alternative a $K/2$ places down. This can be done since a is above the line. By construction, this is a one-to-one map from C onto C'. Since a is on top of $\succ_0$, we have that $d(\succ, \succ_0) < d(\succ', \succ_0)$ and conclude that $C \gg_{\succ_0} C'$.

Case 2: K is odd. In this case, $C = \{\succ \in \mathcal{P} : \text{rank}_{\succ}(a) < (K+1)/2\} \cup \{\succ \in \mathcal{P} : \text{rank}_{\succ}(a) = (K+1)/2, \text{and } \text{rank}_{\succ}(b) < (K+1)/2\}$. Let $\succ \in C$. If $\succ \in \{\succ \in \mathcal{P} : \text{rank}_{\succ}(a) < (K+1)/2\}$ we map $\succ$ to the preference relation $\succ'$ that is obtained from it by moving alternative a $(K+1)/2$ places down. If $\succ \in \{\succ \in \mathcal{P} : \text{rank}_{\succ}(a) = (K+1)/2\}$ we map $\succ$ to the preference relation $\succ'$ that is obtained from it by moving alternative b $(K+1)/2$ places down. It is clear that this is a one-to-one map from C onto C'. By construction, $d(\succ, \succ_0) < d(\succ', \succ_0)$ and we conclude that $C \gg_{\succ_0} C'$. $\square$

We now show that a is socially acceptable. When K is even this is simple. In this case, the set $C = \{\succ \in \mathcal{P} : \text{rank}_{\succ}(a) < (K+1)/2\}$, has cardinality $K!/2$ and is upper homogeneous with respect to (a, b) for any b. By Lemma 1, $C \gg_{\succ_0} C'$, where C' is the complement of C. Since π exhibits consensus around $\succ_0$, we have that $\mu_\pi(C) > \mu_\pi(C')$, namely a is socially acceptable.

Assume now that K is odd. For each $\alpha \in A \setminus \{a\}$ let $C(a, \alpha)$ be the subset of preferences with cardinality $K!/2$ that is upper homogeneous with respect to (a, α), and let $C'(a, \alpha)$ be its complement. By Lemma 1, $C(a, \alpha) \gg_{\succ_0} C'(a, \alpha)$. Since π exhibits consensus around $\succ_0$, we have that $\mu_\pi(C(a, \alpha)) > \mu_\pi(C'(a, \alpha))$. Denote

$$\Delta(a, \alpha) = \mu_\pi(C(a, \alpha)) - \mu_\pi(C'(a, \alpha)) > 0. \tag{1}$$

We need to show that $\mu_\pi(U(a)) - \mu_\pi(L(a)) > 0$. By equation 1, it is enough to show that $\sum_{\alpha \neq a} \Delta(a, \alpha) = (K-1)(\mu_\pi(U(a)) - \mu_\pi(L(a)))$.

We now compute $\sum_{\alpha \neq a} \Delta(a, \alpha)$. Define $\mathcal{F} = \{F \subseteq A \setminus \{a\} : |F| = (K-1)/2\}$, and for $\alpha \in A \setminus \{a\}$ let $\mathcal{F}(\alpha) = \{F \in \mathcal{F} : \alpha \in F\}$. For $F \in \mathcal{F}$, let

$$T(F) = \cap_{x \in F}(I(a) \cap U(x))$$

$T(F)$ is the set of all preference relations where all the members of F are above the line and a is on the line.

Similarly, let

$$B(F) = \cap_{x \in F}(I(a) \cap L(x)).$$

$B(F)$ is the set of all preference relations where all the members of F are below the line and a is on the line. Then, we can write,

$$\begin{aligned} C(a,\alpha) &= \{\succ: \text{rank}_{\succ}(a) < \frac{K+1}{2}\} \cup \{\succ: \text{rank}_{\succ}(a) = \frac{K+1}{2} \text{ and } \text{rank}_{\succ}(\alpha) < \frac{K+1}{2}\} \\ &= U(a) \cup \left(\cup_{F\in\mathcal{F}(\alpha)} T(F)\right). \end{aligned}$$

Similarly.

$$C'(a,\alpha) = L(A) \cup \left(\cup_{F\in\mathcal{F}(\alpha)} B(F)\right).$$

Therefore,

$$\mu_\pi(C(a,\alpha)) = \mu_\pi(U(a)) + \sum_{F\in\mathcal{F}(\alpha)} \mu_\pi\left(T(F)\right)$$

$$\mu_\pi(C'(a,\alpha)) = \mu_\pi(L(a)) + \sum_{F\in\mathcal{F}(\alpha)} \mu_\pi\left(B(F)\right)$$

$$\begin{aligned} \sum_{\alpha\neq a} \Delta(a,\alpha) &= \sum_{\alpha\neq a} \mu(U(a)) - \mu(L(a)) + \sum_{\alpha\neq a}\sum_{F\in\mathcal{F}(\alpha)} \left(\mu\left(T(F)\right) - \mu\left(B(F)\right)\right) \\ &= \sum_{\alpha\neq a} \mu(U(a)) - \mu(L(a)) + \sum_{F\in\mathcal{F}}\sum_{\alpha\in F} \left(\mu\left(T(F)\right) - \mu\left(B(F)\right)\right) \\ &= (K-1)\left(\mu(U(a)) - \mu(L(a))\right) + \sum_{F\in\mathcal{F}} (K-1)\left(\mu\left(T(F)\right) - \mu\left(B(F)\right)\right) \end{aligned}$$

For $F \in \mathcal{F}$, denote $\bar{F} = (A \setminus a) \setminus F$. Note that for any such F we have $T(F) = B(\bar{F})$. Therefore,

$$\begin{aligned} \sum_{F\in\mathcal{F}} \left(\mu\left(T(F)\right) - \mu\left(B(F)\right)\right) &= \sum_{F\in\mathcal{F}} \mu\left(T(F)\right) - \sum_{F\in\mathcal{F}} \mu\left(B(F)\right) \\ &= \sum_{F\in\mathcal{F}} \mu\left(T(F)\right) - \sum_{F\in\mathcal{F}} \mu\left(T(\bar{F})\right) \\ &= \sum_{F\in\mathcal{F}} \mu\left(T(F)\right) - \sum_{F\in\mathcal{F}} \mu\left(T(F)\right) \\ &= 0. \end{aligned}$$

As a result,

$$\sum_{\alpha \neq a} \Delta(a,\alpha) \quad = \quad (K-1)\left(\mu(U(a)) - \mu(L(a))\right)$$

which is what we wanted to show. Therefore, a is socially acceptable. $\square$

We now turn to the implications of consensus for profiles of single-peaked preference relations. The next theorem states that if a preference profile consists of an odd number of single-peaked preference relations, then we can identify one preference relation around which there is consensus. This preference relation is the one induced by the majority rule.

Theorem 2 Let π be a profile of n preference relations that are single-peaked with respect to some linear order $<$ on A, and assume that n is odd. Also let M_π be the binary relation induced by the majority rule on π. Then, there is consensus around M_π.

Note that under the assumptions of the theorem, M_π is transitive and therefore it is a preference relation.

In some sense, Theorem 1 is an extension of the median voter theorem. According to this theorem, when a preference profile consists of an odd number of single-peaked preferences, one can identify a Condorcet winner and furhtermore, the majority rule is transitive. Under the same circumstances our theorem identifies a preference relation which turns out to be the one derived from the majority rule, around which there is consensus. Note that it follows from our theorem that the assumption of single-peakednes of the preference profile is stronger than the assumption of the existence of consensus. In fact, as the following example shows, there are actually non-single-peaked preference profiles that exhibit consensus around some preference relation.

Example 4 Consider the following profile of 10 preference relations on $A = \{a, b, c\}$.

Preference	# of indiv.	Preference	# of indiv.
abc	3	bca	1
acb	2	cab	1
bac	2	cba	1
	7		3

We can see that there is no linear order on A with respect to which all the preferences in the profile are single-peaked. Nonetheless, this profile exhibits consensus around $\succ_0 = abc$. Indeed, it can be checked that in this example a partition is convex if and only if it is homogeneous. It turns out that there are three such partitions: $(C(a \to b), C(b \to a))$, $(C(a \to c), C(c \to a))$, and $(C(c \to b), C(b \to c))$. As can be seen from the following table, for all such partitions the condition for consensus is satisfied.

C	C'	$\mu_\pi(C)$	$\mu_\pi(C)$
$C(a \to c)$	$C(c \to a)$	7	3
$C(a \to b)$	$C(b \to a)$	6	4
$C(b \to c)$	$C(c \to b)$	6	4

We now turn to the proof of the theorem.

3 Proof of Theorem 2

We first show that if $\{C, C'\}$ is a convex partition of $\mathcal{P}$ where both C and C' have the same cardinality and such that $C \gg_{M_\pi} C'$, then $\mu_\pi(C) > \mu_\pi(C')$. We later show that the same is true when $\{C, C'\}$ is homogeneous.

Let $\{C, C'\}$ be a convex partition of $\mathcal{P}$ where both C and C' have the same cardinality. Assume that $C \gg_{M_\pi} C'$. We need to show that $\mu_\pi(C) > \mu_\pi(C')$.

The next claim states that there is a path from any preference relation to any other preference relation such that the distance between any two adjacent preferences in the path is one.

Claim 3 Let $\succ_1, \succ_3 \in \mathcal{P}$ be two distinct preference relations. Then, there is a preference relation $\succ_2$ such that

$$d(\succ_1, \succ_2) = 1 \text{ and } d(\succ_1, \succ_3) = d(\succ_1, \succ_2) + d(\succ_2, \succ_3).$$

Proof : Let $\succ_1$ rank the alternatives as follows: $a_1 \succ_1 \cdots \succ_1 a_K$. There must be two consecutive alternatives a_i, a_{i+1} such that $a_{i+1} \succ_3 a_i$. Otherwise $\succ_1 = \succ_3$, Let $\succ_2$ be the preference relation that is obtained from $\succ_1$ by reversing the ranking between a_i and a_{i+1}. The preference $\succ_2$ is the one we are looking for. $\square$

Now let $\succ_0 \in C$ and $\succ_0' \in C'$ such that $d(\succ_0, \succ_0') = 1$. There is such a pair because, by Claim 3, from any $\succ \in C$ and any $\succ' \in C'$ there is a path $(\succ_1, \ldots, \succ_k)$ of preferences such that $\succ_1 = \succ$, $\succ_k = \succ'$, and $d(\succ_i, \succ_{i+1}) = 1$. Therefore, the only difference between $\succ_0$ and $\succ_0'$ is that there are $a, b \in A$ such that $a \succ_0 b$ and $b \succ_0' a$. Let $\succ_1$ be a preference relation such that $b \succ_1 a$ and let $\succ_0^{-1}$ be the inverse of $\succ_0$. Note that $\succ_0^{-1} \in C'$. Otherwise, both $\succ_0, \succ_0^{-1} \in C$ and in that case, since every $\succ \in \mathcal{P}$ is between $\succ_0$ and $\succ_0^{-1}$ and C is convex, we would have that $C = \mathcal{P}$ which we know not to be true. Note also that $\succ_1$ is between $\succ_0'$ and $\succ_0^{-1}$. To see this, let A' be the set of pairs of alternatives on which $\succ_0'$ disagrees with both $\succ_1$ and $\succ_0^{-1}$. Similarly, let A_1 be the set of pairs of alternatives on which $\succ_1$ disagrees with both $\succ_0'$ and $\succ_0^{-1}$, and let A_{-1} be the set of pairs of alternatives on which $\succ_0^{-1}$ disagrees with both $\succ_0'$ and $\succ_1$. Note that $A_1 = \emptyset$. By definition $d(\succ_0', \succ_1) = |A'| + |A_1|$, $d(\succ_1, \succ_0^{-1}) = |A_1| + |A^{-1}|$, and $d(\succ_0', \succ_0^{-1}) = |A'| + |A^{-1}|$. Since $|A_1| = 0$, $d(\succ_0', \succ_1) + d(\succ_1, \succ_0^{-1}) = d(\succ_0', \succ_0^{-1})$, which means that $\succ_1$ is between $\succ_0'$ and $\succ_0^{-1}$, and since C' is convex, we obtain that $\succ_1 \in C'$. This proves that $C' = C(b \to a)$, and consequently that $C = C(a \to b)$. That is, there are two alternatives $a, b \in A$ such that $C = C(a \to b)$ and $C' = C(b \to a)$. Since $C \gg_{M_\pi} C'$, we must have that $M_\pi \in C$. Consequently, $aM_\pi b$. Namely, $|i \in N : a \succ_i b| \geq |i \in N : b \succ_i a|$. Since $\mu_\pi(C) = |i \in N : a \succ_i b|$, we obtain that $\mu_\pi(C) \geq n/2$. Since n is odd, this implies that $\mu_\pi(C) > \mu_\pi(C')$. This shows that there is consensus around M_π.

We have shown that when $\{C, C'\}$ is a convex partition into sets of the same cardinality,

$C \gg_{M_\pi} C' \Rightarrow \mu_\pi(C) > \mu_\pi(C')$. We now turn to the case of homogeneous partitions. We begin with three propositions each of which identify for each alternative an alternative that is in some sense its mirror image.

Let A be a set of an even number of alternatives and fix a linear order $<$ on A. The next proposition states that for every alternative a there is another alternative a' such that for any preference relation that is single-peaked with respect to $<$, if one of them is above the line, the other one is below the line.

Proposition 2 Let $A = \{a_1, \ldots, a_K\}$ be a set of an even number of alternatives. Let $<$ be the linear order on A and assume without loss of generality that $a_1 < \ldots, < a_K$. Also let $\succ$ be a preference relation that is single-peaked with respect to $<$. Then, for all $i = 1, \ldots, K/2$, $\text{rank}_\succ(a_i) < (K+1)/2$ if and only if $\text{rank}_\succ(a_{K/2+i}) \geq (K+1)/2$.

Proof : If both $\text{rank}_\succ(a_i) < (K+1)/2$ and $\text{rank}_\succ(a_{K/2+i}) < (K+1)/2$ then we must have that $\text{rank}_\succ(c) < (K+1)/2$ for all c such that $a_i \leq c \leq a_{K/2+i}$. In this case we would have that more than $K/2$ alternatives have a rank lower than $K/2$ which is impossible. Similarly, if both $\text{rank}_\succ(a_i) > (K+1)/2$ and $\text{rank}_\succ(a_{K/2+i}) > (K+1)/2$ then we must have that $\text{rank}_\succ(c) > (K+1)/2$ for all c such that $c \leq a_i$ and for all c such that $a_{K/2+i} \leq c$. In this case we would have that less than $K/2$ alternatives have a rank lower than $K/2$ which is also impossible. $\square$

Let A be a set of an odd number of alternatives and fix a linear order $<$ on A. The next proposition states that for every alternative a that belongs to the $(K-1)/2$ leftmost alternatives, there is another alternative a' that belongs to the $(K+1)/2$ rightmost alternatives, such that for any preference relation that is single-peaked with respect to $<$, if one of them is above the line, the other one is below or on the line.

Proposition 3 Let $A = \{a_1, \ldots, a_K\}$ be a set of an odd number of alternatives. Let $<$ be the linear order on A and assume without loss of generality that $a_1 < \ldots, < a_K$. Let $\succ$ be a preference relation that is single-peaked with respect to $<$. For all $i = 1, \ldots, (K-1)/2$,

1. $\text{rank}_{\succ}(a_i) < (K+1)/2 \quad \Rightarrow \text{rank}_{\succ}(a_{(K-1)/2+i}) \geq (K+1)/2$

2. $\text{rank}_{\succ}(a_i) = (K+1)/2 \quad \Rightarrow \text{rank}_{\succ}(a_{(K-1)/2+i}) < (K+1)/2$

3. $\text{rank}_{\succ}(a_i) > (K+1)/2 \quad \Rightarrow \text{rank}_{\succ}(a_{(K-1)/2+i}) \leq (K+1)/2$

Proof :

1. If both $\text{rank}_{\succ}(a_i) < (K+1)/2$ and $\text{rank}_{\succ}(a_{(K-1)/2+i}) < (K+1)/2$ then we must have that $\text{rank}_{\succ}(c) < (K+1)/2$ for all c such that $a_i \leq c \leq a_{(K-1)/2+i}$. In this case we would have that more than half of the alternatives have a rank lower than $K/2$ which is impossible.

2. If both $\text{rank}_{\succ}(a_i) = (K+1)/2$ and $\text{rank}_{\succ}(a_{(K-1)/2+i}) > (K+1)/2$, then we must have that $\text{rank}_{\succ}(c) > (K+1)/2$ for all c such that $c < a_i$, and for all c such that $a_{(K-1)/2+i} \leq c$. In this case we would have that more than half of the alternatives have a rank higher than $K/2$ which is impossible.

3. If both $\text{rank}_{\succ}(a_i) > (K+1)/2$ and $\text{rank}_{\succ}(a_{(K-1)/2+i}) > (K+1)/2$, then we must have that $\text{rank}_{\succ}(c) > (K+1)/2$ for all c such that $c \leq a_i$ and for all c such that $a_{(K-1)/2+i} \leq c$. In this case we would have that more than half of the alternatives have rank higher than $K/2$ which is impossible.

□

Let A be a set of an odd number of alternatives and fix a linear order $<$ on A. The next proposition states that for every alternative a that belongs to the $(K-1)/2$ leftmost alternatives, there is another alternative a' that belongs to the $(K+1)/2$ rightmost alternatives, such that for any preference relation that is single-peaked with respect to $<$, if one of them is above or on the line, the other one is below the line.

Proposition 4 Let $A = \{a_1, \ldots, a_K\}$ be a set of an odd number of alternatives. Let $<$ be the linear order on A such that $i < j \Rightarrow a_i < a_j$. Also let $\succ$ be a preference relation that is single peaked with respect to $<$. For all $i = 1, \ldots, (K-1)/2$, $\text{rank}_{\succ}(a_i) \leq (K+1)/2$ if and only if $\text{rank}_{\succ}(a_{(K+1)/2+i}) > (K+1)/2$.

Proof : If both $\text{rank}_{\succ}(a_i) \leq (K+1)/2$ and $\text{rank}_{\succ}(a_{(K+1)/2+i}) \leq (K+1)/2$ then we must have that $\text{rank}_{\succ}(c) \leq (K+1)/2$ for all c such that $a_i \leq c \leq a_{(K+1)/2+i}$. In this case we would have that more than half of the alternatives have a rank lower than $K/2$ which is impossible. Similarly, if both $\text{rank}_{\succ}(a_i) > (K+1)/2$ and $\text{rank}_{\succ}(a_{(K-1)/2+i}) > (K+1)/2$ then we must have that $\text{rank}_{\succ}(c) > (K+1)/2$ for all c such that $c \leq a_i$ and for all c such that $a_{(K+1)/2+i} \leq c$. In this case we would have that more than half of the alternatives have a rank higher than $K/2$ which is also impossible. $\square$

Fix a linear order on the set of alternatives A. The next corollary has three parts. The first part states that for any alternative a that belongs to either the left or the right side of the spectrum, there is another alternative b such that for any preference relation $\succ$ that is single-peaked with respect to $<$, a is above the line if and only if $a \succ b$. The second one states that for any alternative a that belongs to either the left or the right side of the spectrum, there is another alternative b' such that for any preference relation $\succ$ that is single-peaked with respect to $<$, a is above or on the line if and only if $a \succ b$. The third part states that if a is the central alternative, then for any preference relation $\succ$ that is single-peaked with respect to $<$, if a is above the line then it is strictly preferred to both the rightmost and leftmost alternatives.

Corollary 1 Let $<$ be a linear order on $A = \{a_1, \ldots, a_K\}$ and assume without loss of generality that $a_1 < \ldots, < a_K$.

1. For all $a \in A$ such that $a \neq a_{(K+1)/2}$ there is $b \in A$ such that for all preferences $\succ$ that are single-peaked with respect to $<$,

$$\text{rank}_{\succ}(a) < (K+1)/2 \Leftrightarrow a \succ b.$$

2. For all $a \in A$ such that $a \neq a_{(K+1)/2}$ there is $b' \in A$ such that for all preferences $\succ$ that are single-peaked with respect to $<$,

$$\text{rank}_{\succ}(a) \leq (K+1)/2 \Leftrightarrow a \succ b'.$$

3. For $a = a_{(K+1)/2}$ and for all preference relations $\succ$ that are single peaked with respect to $<$, if $\text{rank}_{\succ}(a) < (K+1)/2$ then $a \succ a_1$ and $a \succ a_K$.

Proof :

1. Let $\succ$ be a preference relation that is single peaked with respect to $<$. Assume first that K is even. Let $i = 1, \ldots, K/2$. By Proposition 2,

$$\text{rank}_{\succ}(a_i) < (K+1)/2 \Leftrightarrow \text{rank}_{\succ}(a_{K/2+i}) \geq (K+1)/2.$$

   Therefore, if $a = a_i$ for some $i = 1, \ldots, K/2$, the alternative b we are looking for is $b = a_{K/2+i}$ and if $a = a_{K/2+i}$, the alternative b we are looking for is $b = a_i$.

   The case where K is odd is similar. Let $i = 1, \ldots, (K-1)/2$. It follows from Proposition 3 that

$$\text{rank}_{\succ}(a_i) < (K+1)/2 \Leftrightarrow a_i \succ a_{(K-1)/2+i},$$

   namely the alternative b we are looking for is $b = a_{(K-1)/2+i}$. And if $a = a_{(K-1)/2+(i+1)}$, the alternative b we are looking for is $b = a_{i+1}$.

2. If K is even, we are back to case 1. So assume that K is odd. Let $i = 1, \ldots, (K-1)/2$. By Proposition 4,

$$\text{rank}_{\succ}(a_i) \leq (K+1)/2 \Leftrightarrow \text{rank}_{\succ}(a_{(K+1)/2+i}) > (K+1)/2.$$

   Therefore, if $a = a_i$ for some $i = 1, \ldots, (K-1)/2$, the alternative b we are looking for is $b = a_{(K+1)/2+i}$, and if $a = a_{(K+1)/2+i}$, the alternative b we are looking for is $b = a_i$.

3. Assume that $a = a_{(K+1)/2}$ and let $\succ$ be a single-peaked preference with respect to $<$. If $\text{rank}_{\succ}(a) < (K+1)/2$ we must have that $a \succ a_1$. Otherwise, there would be at least $(K+1)/2$ alternatives with ranks lower than $(K+1)/2$ which is impossible. Similarly, if $\text{rank}_{\succ}(a) < (K+1)/2$ we must have that $a \succ a_K$. Because otherwise, there would be at least $(K+1)/2$ alternatives with ranks lower than $(K+1)/2$ which is impossible.

$\square$

Let A be a set with an odd number of alternatives and fix a linear order $<$ on A. Let a be the central alternative and b another alternative. Let C be a set of preferences that is homogeneous with respect to (a, b) and assume that $|C| = K!/2$. The next two claims characterize the subset of single-peaked preferences that belong to a given homogeneous set.

The next claim states that if b is on the left side of the spectrum then the set of single-peaked preferences that belong to C is none other than the set of single-peaked preferences according to which a is preferred to the rightmost alternative. And similarly, if b is on the right side of the spectrum then the set of single-peaked preferences that belong to C is none other than the set of single-peaked preferences according to which a is preferred to the leftmost alternative.

Claim 4 Let C be a upper homogeneous set with respect to (a, b). Assume that $|C| = K!/2$, where K is odd. Let $\mathcal{S}$ be the set of preferences that are single-peaked with respect to the order $a_1 < \ldots < a_K$. Assume that $a = a_{(K+1)/2}$ and $b < a$. Then

$$C \cap \mathcal{S} = \{\succ \in \mathcal{S} : a \succ a_K\}.$$

If $a < b$ then

$$C \cap \mathcal{S} = \{\succ \in \mathcal{S} : a \succ a_1\}.$$

Proof : Assume that $b < a$. The proof when $a < b$ is similar and is left to the reader. Let $M = \{\succ \in \mathcal{S} : a \succ a_K\}$. We want to show that $C \cap \mathcal{S} = M$.

Assume that $\succ \in C \cap \mathcal{S}$. There are two cases: either $\text{rank}_{\succ}(a) < (K+1)/2$ or $\text{rank}_{\succ}(a) = (K+1)/2$. In the first case, by Corollary 1.3, $a \succ a_K$, which implies that $\succ \in M$. In the second, since $b < a$ and $b \succ a$ (recall that C is homogeneous), we must have that $a \succ a_K$. This implies that $\succ \in M$.

Assume now that $\succ \in M$. Then $\succ \in \mathcal{S}$ and $a \succ a_K$. Since $a = a_{(K+1)/2}$, we have that $\text{rank}_{\succ}(a) \leq (K+1)/2$. If $\text{rank}_{\succ}(a) < (K+1)/2$, then, given that $|C| = K!/2$, by Claim 2, $\succ \in C$. Hence $\succ \in \mathcal{S} \cap C$. If $\text{rank}_{\succ}(a) = (K+1)/2$, then by homogeneity of C, $\text{rank}_{\succ}(b) < (K+1)/2$. Since $|C| = K!/2$, by Claim 2, $\succ \in C$. Hence $\succ \in \mathcal{S} \cap C$.

$\square$

Let A be a set with an odd number of alternatives and fix a linear order $<$ on A. Let a be an alternative that is not the the central one. Let C be a set of preferences that is upper homogeneous with respect to (a, b) and assume that $|C| = K!/2$. The next proposition states that the set of single-peaked preferences that are in C is either the set of single-peaked preferences that rank a above or on the line or the set of single-peaked preferences that rank a above the line.

Proposition 5 Let $A = \{a_1, \ldots, a_K\}$ be a set of an odd number of alternatives. Let $<$ be the linear order on A and assume without loss of generality that $a_1 < \ldots, < a_K$. Let C be a subset of $\mathcal{P}$ with cardinality $K!/2$ which is upper homogeneous with respect to (a, b). Let $\mathcal{S}$ be the set of single-peaked preferences with respect to the linear order $<$ and assume that $a \neq a_{(K+1)/2}$. Then either

$$C \cap \mathcal{S} = \{\succ \in \mathcal{S} : \text{rank}_{\succ}(a) \leq (K+1)/2\}$$

or

$$C \cap \mathcal{S} = \{\succ \in \mathcal{S} : \text{rank}_{\succ}(a) < (K+1)/2\}.$$

Proof : Asssume that K is odd. Since C is homogeneous with respect to (a, b), for all $\succ \in C$ we have $\text{rank}_{\succ}(a) \leq (K+1)/2$ and whenever $\text{rank}_{\succ}(a) = (K+1)/2$ we have $\text{rank}_{\succ}(b) < (K+1)/2$.

By Claim 2

$$C = \{\succ \in \mathcal{P} : \text{rank}_\succ(a) < (K{+}1)/2\} \cup \{\succ \in \mathcal{P} : \text{rank}_\succ(a) = (K{+}1)/2,\ \text{rank}_\succ(b) < (K{+}1)/2\}).$$

Therefore,

$$C \cap \mathcal{S} = \{\succ \in \mathcal{S} : \text{rank}_\succ(a) < (K{+}1)/2\} \cup \{\succ \in \mathcal{S} : \text{rank}_\succ(a) = (K{+}1)/2,\ \text{rank}_\succ(b) < (K{+}1)/2\}.$$

There are two cases to consider.

Case 1: $a = a_i \in \{a_1, \ldots, a_{(K-1)/2}\}$.

By Proposition 4 we have that for all $\succ \in \mathcal{S}$ such that $\text{rank}_\succ(a) = (K+1)/2$, $\text{rank}_\succ(a_{(K+1)/2+i}) > (K+1)/2$.

Also, we have for all $\succ \in \mathcal{S}$ such that $\text{rank}_\succ(a) = (K+1)/2$, $\text{rank}_\succ(a') > (K+1)/2$ for all $a' < a = a_i$. Consequently, if $b \in \{a_1, \ldots, a_{i-1}\}$ then for all $\succ \in \mathcal{S}$, $\text{rank}_\succ(b) > (K+1)/2$ and therefore

$$C \cap \mathcal{S} = \{\succ \in \mathcal{S} : \text{rank}_\succ(a) \leq K/2\}.$$

If $b \in \{a_{(K+1)/2+i}, \ldots, b_K\}$ then for all $\succ \in \mathcal{S}$, $\text{rank}_\succ(b) > (K+1)/2$ and therefore

$$C \cap \mathcal{S} = \{\succ \in \mathcal{S} : \text{rank}_\succ(a) \leq K/2\}.$$

If $b \in \{a_{i+1}, \ldots, a_{(K-1)/2+i}\}$ then for all $\succ \in \mathcal{S}$, $\text{rank}_\succ(b) \leq (K+1)/2$ and therefore,

$$\begin{aligned} C \cap \mathcal{S} &= \{\succ \in \mathcal{S} : \text{rank}_\succ(a) \leq K/2\} \cup \{\succ \in \mathcal{S} : \text{rank}_\succ(a) = (K+1)/2\} \\ &= \{\succ \in \mathcal{S} : \text{rank}_\succ(a) \leq (K+1)/2\} \end{aligned}$$

Case 2: $a = a_i \in \{a_{(K+1)/2+i}, \ldots, a_K\}$.

The proof is analogous to te one of case 1, and is left to the reader. $\square$

After having proved all the above auxilliary propositions, we now continue with the proof of the theorem. Let $\{C, C'\}$ be a homogeneous partition of $\mathcal{P}$ with respect to (a, b) where both C and C' have the same cardinality. Assume that $C \gg_{M_\pi} C'$. We need to show that $\mu_\pi(C) > \mu_\pi(C')$.

Case 1: K is even. Since $\{C, C'\}$ is a homogeneous partition with respect to (a, b), by Claim 2 one of the two sets is $\{\succ: \text{rank}_\succ(a) < (K+1)/2\}$ and the other one is $\{\succ: \text{rank}_\succ(a) > (K+1)/2\}$. Assume without loss of generality that

$$C = \{\succ: \text{rank}_\succ(a) < (K+1)/2\}.$$

By Corollary 1.1, there is an alternative $b' \in A$ such that

$$\{\succ \in \mathcal{S} : \text{rank}_\succ(a) < (K+1)/2\} = C(a \to b') \cap \mathcal{S}.$$

Therefore,

$$\mu_\pi(C) = \mu_\pi(C \cap \mathcal{S}) = \mu_\pi(C(a \to b') \cap \mathcal{S}) = \mu_\pi(C(a \to b')).$$

Since $C \gg_{M_\pi} C'$, we must have that $M_\pi \in C$. Consequently, $aM_\pi b'$. Namely, $|\{i \in N : a \succ_i b'\}| \geq |\{i \in N : b' \succ_i a\}|$. Since $\mu_\pi(C) = |\{i \in N : a \succ_i b'\}|$, we obtain that $\mu_\pi(C) \geq n/2$. Since n is odd, this implies that $\mu_\pi(C) > \mu_\pi(C')$.

Case 2: K odd. Suppose that $a \neq a_{(K+1)/2}$. Then by Proposition 5 we can assume without loss of generality that either

$$C \cap S = \{\succ \in S : \text{rank}_\succ(a) \leq (K+1)/2\}$$

or

$$C \cap S = \{\succ \in S : \text{rank}_\succ(a) < (K+1)/2\}$$

regardless of which, by Corollary 1 there exists b' such that

$$C \cap S = \{\succ \in S : a \succ b'\} = C(a \to b') \cap S.$$

Therefore,

$$\mu_\pi(C) = \mu_\pi(C \cap \mathcal{S}) = \mu_\pi(C(a \to b') \cap \mathcal{S}) = \mu_\pi(C(a \to b')).$$

Since $C \gg_{M_\pi} C'$, we must have that $M_\pi \in C$. Consequently, $aM_\pi b'$. Namely, $|\{i \in N : a \succ_i b'\}| \geq |\{i \in N : b' \succ_i a\}|$. Since $\mu_\pi(C) = |\{i \in N : a \succ_i b'\}|$, we obtain that $\mu_\pi(C) \geq n/2$. Since n is odd, this implies that $\mu_\pi(C) > \mu_\pi(C')$.

Suppose now that $a = a_{(K+1)/2}$. By Claim 4 there is b' such that

$$C \cap \mathcal{S} = \{\succ \in \mathcal{S} : a \succ b'\} = C(a \to b') \cap \mathcal{S}.$$

Therefore

$$\mu_\pi(C) = \mu_\pi(C \cap \mathcal{S}) = \mu_\pi(C(a \to b) \cap \mathcal{S}) = \mu_\pi(C(a \to b)).$$

Since $C \gg_{M_\pi} C'$, we must have that $M_\pi \in C$. Consequently, $aM_\pi b'$. Namely, $|\{i \in N : a \succ_i b'\}| \geq |\{i \in N : b' \succ_i a\}|$. Since $\mu_\pi(C) = |\{i \in N : a \succ_i b'\}|$, we obtain that $\mu_\pi(C) \geq n/2$. Since n is odd, this implies that $\mu_\pi(C) > \mu_\pi(C')$. This shows that there is consensus around M_π, which concludes the proof of the theorem.

References

[1] Kenneth J Arrow. *Social choice and individual values*, volume 12. Yale University Press, 2012.

[2] Duncan Black. On the rationale of group decision-making. *The Journal of Political Economy*, pages 23–34, 1948.

[3] Anthony Downs. An economic theory of political action in a democracy. *Journal of Political Economy*, pages 135–150, 1957.

[4] Wulf Gaertner. *Domain conditions in social choice theory.* Cambridge University Press, 2001.

[5] Ken-ichi Inada. A note on the simple majority decision rule. *Econometrica*, pages 525–531, 1964.

[6] John G Kemeny and James Laurie Snell. *Mathematical models in the social sciences*, volume 9. Ginn Boston, 1962.

[7] Muhammad Mahajne, Shmuel Nitzan, and Oscar Volij. Level r consensus and stable social choice. *Social Choice and Welfare*, 45(4):805–817, 2015.

[8] Shmuel Nitzan. *Collective preference and choice.* Cambridge University Press, 2009.

A Appendix

Proof of Claim 2: Assume C is upper-homogeneous with respect to (a, b),

$$C \subseteq \{\succ : \mathrm{rank}_\succ(a) < (K+1)/2\} \cup \{\mathrm{rank}_\succ(a) = (K+1)/2 \text{ and } \mathrm{rank}_\succ(b) < (K+1)/2\}.$$

It can be checked that the cardinality of the above union is exactly $K!/2$. Indeed, if K is even,

$$|\{\succ : \mathrm{rank}_\succ(a) < (K+1)/2\}| = |\{\succ : \mathrm{rank}_\succ(a) \leq K/2\}| = K!/2.$$

If K is odd,

$$|\{\succ : \mathrm{rank}_\succ(a) < (K+1)/2\}| + |\{\mathrm{rank}_\succ(a) = (K+1)/2 \text{ and } \mathrm{rank}_\succ(b) < (K+1)/2\}| = \frac{K-1}{2}(K-1)! + \frac{K-1}{2}(K-2)! = K!/2.$$

The case where C is lower-homogeneous is analogous and left to the reader.