Coping with multiple catastrophic threats:
An intertemporal approach

Yacov Tsur (Hebrew University)

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Coping with multiple catastrophic threats: An intertemporal approach

Yacov Tsur[*]   Amos Zemel[◇]

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Abstract

Economic modeling of catastrophic threats has largely dealt with each threat in isolation. Recently, Martin and Pindyck (2015) forcefully argued that such an approach can be misleading in the all too common situation of multiple threats, and offered a regulation framework for such situations. The present work extends this framework in two ways: first, it allows for endogenous catastrophic threats; second, it considers intertemporal policies, allowing, inter alia, for gradual threat-mitigation actions to accumulate over time. The long run properties of optimal policies are characterized in terms of key parameters. Possible effects of background threats on the optimal response to a potential catastrophe are illustrated numerically.

Keywords: multiple catastrophes, endogenous hazards, dynamic policy, long-run properties.

JEL classification: C61, Q54

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[*]Department of Environmental Economics and Management and the Center for Agricultural Economic Research, The Hebrew University of Jerusalem, POB 12, Rehovot 7610001, Israel (yacov.tsur@mail.huji.ac.il).

[◇]Department of Solar Energy and Environmental Physics, The Jacob Blaustein Institutes for Desert Research, Ben Gurion University of the Negev, Sede Boker Campus 8499000, Israel (amos@bgu.ac.il).

1 Introduction

Catastrophic threats refer to abrupt events that, upon occurrence at an unpredictable date, inflict substantial (non-marginal) damage. Economic modeling of such situations owes much to the earlier works of Kemp (1976), Long (1975) and Cropper (1976). These works and the literature they spawned have dealt primarily with an isolated catastrophic threat. Early examples include nuclear accidents (Cropper 1976, Aronsson et al. 1998), ecological collapse or species extinction (Reed and Heras 1992, Clarke and Reed 1994, Tsur and Zemel 1994), forest fire (Reed 1984, Yin and Newman 1996), seawater intrusion into coastal aquifers (Tsur and Zemel 1995) and climate change induced calamities (Tsur and Zemel 1996, Gjerde et al. 1999, Nævdal 2006).[1]

In actual practice, however, society often faces simultaneous catastrophic threats from a variety of sources. This situation has recently been addressed by Martin and Pindyck (2015), who showed that dealing with any catastrophic threat in isolation can be misleading when other threats lurk in the background. To make this point loud and clear, these authors consider a framework in which an exogenously growing consumption process is threatened by several types of catastrophic occurrences, each associated with a mean arrival (or hazard) rate that can be eliminated or reduced by an upfront (once-and-for-all) investment but is otherwise exogenous. Occurrence of any type of event reduces consumption by a certain (random) fraction from the occurrence time onward. The simple policy framework of Martin and Pindyck (2015) illuminates how the non-marginal nature of both the damage inflicted by each catastrophe and the cost required to avert it magnifies the effects of

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[1]More recent examples can be found in Polasky et al. (2011) and Tsur and Zemel (2014b).

background threats on the optimal response to a given catastrophe, thereby stressing the need for a comprehensive approach rather than dealing with each threat in isolation.

A policy response to catastrophic threats, however, is essentially intertemporal, hence should be evaluated within a dynamic framework.[2] In such a framework, mitigation efforts can be spread over a long time period, such that the instantaneous investment decisions consider marginal variations of risk reduction activities over time, even if the cumulative effort is non-marginal. Our aim in this work is to incorporate Martin and Pindyck's (2015) message within a dynamic framework that accounts for such intertemporal considerations. In so doing, we extend Martin and Pindyck's (2015) model in two ways: first, we allow for endogenous catastrophic threats; second, we consider intertemporal policies, allowing the risk reduction actions to evolve smoothly over time.

As in Martin and Pindyck (2015), society in our model faces several sources of catastrophic threat, each with a hazard rate that governs its occurrence probability. Unlike Martin and Pindyck (2015), these hazard rates are endogenous and can be regulated continuously over time by efforts that consume resources. The regulation problem, then, is to allocate income between consumption and hazard mitigating activities over a long period of time. The problem is formulated in terms of a multidimensional dynamic model, where a policy determines the level of threat society faces at each point of time for each catastrophic source. Averting a particular catastrophe means reducing its associated hazard to zero, while ignoring it entails allowing this hazard to grow unregulated. In between these extremes lies a wide spectrum of intertemporal

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[2]A case in point is the debate between advocates of an early vigorous climate policy (Stern 2007) and those supporting a more gradual approach (Nordhaus 2008).

policies. We provide a complete characterization of the optimal policy in the long run, including the optimal levels of threats and the resources allocated to mitigate each source of threat.

It is convenient to begin, in the next section, with a single source of threat and use this simpler case to "set the stage" and, in particular, show where our model deviates from Martin and Pindyck's (2015) framework. For a meaningful comparison, we follow Martin and Pindyck (2015) as closely as possible (e.g., we assume CRRA utility with $\eta = 2$, random occurrence times, recurrent events and each time an event strikes reduces effective consumption by some factor) with the additional features of policy-dependent hazards and a dynamic policy allowing, inter alia, to gradually regulate the hazards over time.

The situation of multiple sources of catastrophic threats is considered in Section 3, which distinguishes between the case of identical (but independent) sources and the more realistic case of heterogenous sources. We show that the optimal policy under several identical threats is derived much like that under a single threat, although the ensuing optimal policies may differ markedly. The derivation of the optimal policy under heterogenous sources of catastrophic threats turns out to be quite different from the derivation in the previous two cases. These differences are illustrated in Section 4 by means of numerical examples of the three cases (an isolated recurrent catastrophe, two identical recurrent catastrophes and two heterogenous recurrent catastrophes). Section 5 concludes.

2 An isolated catastrophic threat

The catastrophic threat refers to a damaging event that reduces utility each time it strikes from the occurrence time onward. Before the first occurrence, society derives utility from consumption $c \leq 1$ according to the iso-elastic form $u(c) = c^{1-\eta}/(1 - \eta)$, with the relative risk aversion coefficient $\eta = 2$, where consumption is constrained above by a constant income stream normalized at unity. After an event occurrence, each consumption unit is rendered equivalent to $\psi < 1$ units. Thus, prior to the first occurrence the utility is $u(c) = -1/c$; between the first and second occurrence, utility is $-1/(\psi c) = \varphi(-1/c) = \varphi u(c)$, where $\varphi \equiv 1/\psi > 1$, and so on: between the n'th and $n+1$'st occurrence the utility is $\varphi^n u(c) < u(c)$.[3] The damage associated with an event occurrence, as measured by the difference between the post- and pre-event utility, is thus proportional to $\varphi - 1$. A catastrophic event, then, corresponds to $\varphi >> 1$. The damage cannot be reversed, and the utility loss continues indefinitely.

The above utility and damage specifications follow those of Martin and Pindyck (2015), who also assumed that the mean arrival rate of the event is exogenous and can be changed, once and for all, by giving up consumption. In contrast, we allow the mean arrival rate, i.e., the hazard rate, to depend on a state variable, denoted P, that can be intertemporally regulated at a cost. For example, in the context of climate catastrophes, P stands for the atmospheric GHG concentration that can be regulated by emission abatement efforts, or in the context of a disease outburst, P represents the medical state of knowledge that can vary with investments in pharmaceutical R&D and in other prevention activities.

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[3]Recall that $u(c)$ is negative.

Notice that a pollution state and a know-how state have opposite effects on catastrophic threats (the former increases while the latter decreases occurrence risk). For clarity of presentation it helps to consider a concrete situation and we choose the case in which $P(t)$ represents a pollution stock (e.g., GHG in the atmosphere, radioactive waste, or other types of pollution in the air, water or soil) and $q(t)$ stands for emission abatement. Consequently, $P(t)$ evolves in time according to

$$\dot{P}(t) = e - q(t) - \delta P(t), \ P(t) \geq 0, \tag{2.1}$$

where e is an unabated emission rate and δ is a natural pollution decay rate. Without mitigation, the emission flow is fixed at $e \leq 1$. Mitigating at the rate $q(t)$ reduces emission to $e - q(t) = e - 1 + c(t)$. The constraint $c(t) \leq 1$ implies that $P(t)$ is bounded above by

$$\bar{P} = e/\delta. \tag{2.2}$$

In such a context, the hazard rate increases with P and is specified as[4]

$$h(P) = \alpha P, \ \ \alpha > 0. \tag{2.3}$$

Note that the pollution stock affects utility indirectly, via its effect on the hazard rate.

Apart from multiplying utility by the factor φ, an event occurrence does not change any of the stock variables or constraints and the process proceeds as before, including the risk of yet another occurrence which will introduce another factor φ to the ensuing utility, as described above.[5] The distribution

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[4]The linear specification in (2.3) is made in the interest of simplicity and can be generalized without affecting the nature of our results.

[5]Martin and Pindyck (2015) assume that φ is random, allowing for a different realization upon each occurrence. This distinction does not affect the model, when φ is interpreted as the expected value of the random factor.

of the next event occurrence date is governed by the same hazard function, resembling the recurrent events of Tsur and Zemel (1998).

Let T denote the first (random) occurrence time. The hazard rate is related to the the survival probability $S(t) \equiv Pr(T > t)$ according to

$$S(t) = \exp\left(-\int_0^t h(P(\tau))d\tau\right), \tag{2.4}$$

which implies

$$\dot{S}(t) = -h(P(t))S(t). \tag{2.5}$$

Note that the emission abatement process $q(t)$ determines, via (2.1) and (2.3), the dynamics of the hazard process, hence also the probability of future occurrences and the capacity of the economy to derive utility from future consumption.

Let $v(P(t))$ represent the value function at time t when the pollution stock is $P(t)$, i.e., the present value at time t of the stream of utilities from time t onward under the optimal policy (assumed to exist). At the first occurrence time T, the value is promptly changed to $\varphi v(P(T))$ (recall that $v(\cdot)$ is negative). The payoff at $t = 0$ can thus be expressed as

$$\int_0^T u(c(t))e^{-\rho t}dt + e^{-\rho T}\varphi v(P(T)),$$

where ρ is the time rate of discount. Taking expectation with respect to T, using its distribution $1 - S(t)$ and density $h(P(t))S(t)$, gives the expected payoff

$$\int_0^\infty \left[u(1 - q(t)) + h(P(t))\varphi v(P(t))\right] S(t)e^{-\rho t}dt. \tag{2.6}$$

The optimal abatement policy is the feasible $q(\cdot)$ process that maximizes (2.6) subject to (2.1) and (2.5), given $P(0)$ and $S(0) = 1$. The value at $t = 0$ is the

expected payoff evaluated along the optimal policy:

$$v(P(0)) = \int_0^\infty \left[u(1 - q^*(t)) + h(P^*(t))\varphi v(P^*(t))\right] S^*(t)e^{-\rho t}dt, \qquad (2.7)$$

where the asterisk signifies optimal processes. As the value function $v(\cdot)$ appears on both sides of (2.7), it is only implicitly defined. While bearing on the derivation of the optimal policy, for the purpose of finding optimal steady states this complication is easily circumvented.

As noted above, for recurrent events of the type considered here, occurrence does not entail a change in the mitigation policy. Moreover, the optimal stock process must evolve monotonically in time and since $P \in [0, \bar{P}]$, this process must converge to a steady state in that interval. The location of the steady state reveals the optimal policy in the long run: corner steady states ($P = \bar{P}$ or $P = 0$) imply the extreme policies of either ignoring the risk or eliminating it completely, while interior steady states entail partial mitigation.

Suppose that at some state P the (not necessarily optimal) steady state policy $\hat{q}(P) = e - \delta P$ is adopted, leaving the stock fixed at P indefinitely. The steady state consumption $\hat{c}(P) = 1 - \hat{q}(P)$ can be expressed as $\hat{c}(P) = \delta(P + \pi)$, where

$$\pi \equiv (1 - e)/\delta \geq 0 \qquad (2.8)$$

(the inequality follows from $e \leq 1$). Using (2.6), the expected payoff under the steady state policy is given by

$$W(P) = \left[u(\hat{c}(P)) + h(P)\varphi W(P)\right]/\left[\rho + h(P)\right]. \qquad (2.9)$$

Solving for $W(P)$, recalling $u(\hat{c}(P)) = -1/\hat{c}(P) = -1/(\delta(P + \pi))$, gives

$$W(P) = \frac{-1}{\delta(P + \pi)[\rho - h(P)(\varphi - 1)]}, \qquad (2.10)$$

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provided $\rho > h(P)(\varphi - 1)$ or (using (2.3)) $P < P^W$, where

$$P^W \equiv \rho/(\alpha(\varphi - 1)). \qquad (2.11)$$

For pollution stocks $P \geq P^W$, $W(P)$ diverges to $-\infty$ and P cannot be an optimal steady state.[6]

To better understand this result, write (2.10) as

$$W(P) = \frac{-1}{\rho\delta(P+\pi)} \sum_{n=0}^{\infty} \left[\frac{h(P)(\varphi-1)}{\rho} \right]^n = \frac{-1}{\rho\delta(P+\pi)} \sum_{n=0}^{\infty} \left[\frac{P}{P^W} \right]^n. \qquad (2.12)$$

Thus, $W(P)$ represents the sum of expected damages from an infinite series of Poisson occurrences, each discounted at the factor corresponding to the respective random occurrence time. When $P > P^W$, the quotient in the geometric series (2.12) exceeds unity and the series diverges.

It is expedient to write the threshold state P^W in terms of the damage parameter

$$D \equiv \alpha(\varphi - 1) \qquad (2.13)$$

as $P^W = \rho/D$. We see that a large damage $\varphi - 1$ or high hazard sensitivity to pollution α imply a low threshold pollution state P^W, hence optimal steady states must be correspondingly cleaner.

To identify optimal steady states, we use the L-method (Tsur and Zemel 1998, 2001, 2014a) as follows. Write the state dynamics equation (2.1) as $\dot{P} = g(P, q)$, where $g(P, q) \equiv e - q - \delta P$, and let $f(P, q) \equiv u(1-q) + h(P)v(P)$.[7] Define (see Tsur and Zemel 2016, equation (3.4a))

$$L(P) \equiv [\rho + h(P)][f_q(P, \hat{q}(P))/g_q(P, \hat{q}(P)) + W_P(P)], \qquad (2.14)$$

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[6]An analogous result is derived in Martin and Pindyck (2015, p. 2955). It is typical of models of recurrent events where the hazard must be bounded to avoid infinite damage.

[7]Noting (2.4), the expected payoff (2.6) can be expressed as $\int_0^\infty f(P(t), q(t))e^{-\int_0^t [\rho+h(P(\tau))]d\tau} dt$. With $\rho + h(P(t))$ representing the "effective discount rate," $f(P, q)$ can be interpreted as the "effective utility".

where $f_q \equiv \partial f/\partial q$, $g_q \equiv \partial g/\partial q$, $W_P \equiv \partial W/\partial P$ and it is recalled that $\hat{q}(P) = e - \delta P$ is the steady state abatement. Let $\hat{P}^*$ denote an *optimal* steady state. Then:

Property 1 (location). *(i) If $\hat{P}^* \in (0, \bar{P})$, then $L(\hat{P}^*) = 0$. (ii) if $\hat{P}^* = 0$, then $L(0) \leq 0$. (iii) if $\hat{P}^* = \bar{P}$, then $L(\bar{P}) \geq 0$.*

Property 2 (stability). *If $\hat{P}^* = \hat{P} \in (0, \bar{P})$ is (locally) stable, then $L'(\hat{P}^*) < 0$.*

In the present context, the general expression (2.14) specializes (after some algebraic manipulations) to

$$L(P) = \frac{\rho + \alpha P}{\delta^2 (P + \pi)^2} \left[1 + \frac{\delta P^W (P^W - 2P - \pi)}{\rho (P^W - P)^2} \right], \qquad (2.15)$$

which bears the same sign as the quadratic

$$\tilde{L}(P) = \rho P^2 - 2P^W (\delta + \rho) P + (P^W)^2 (\delta + \rho - \delta \pi / P^W). \qquad (2.16)$$

The condition $L(P) = \tilde{L}(P) = 0$ admits the two real solutions (roots)

$$\frac{\rho + \delta \pm \sqrt{\delta(\rho + \delta) + \rho \delta \pi / P^W}}{\rho} P^W .$$

The larger root exceeds P^W, thus corresponds to unbounded welfare and cannot be an optimal steady state. Recalling $\rho \delta \pi / P^W = (1 - e)D \geq 0$, the smaller root can be expressed as

$$\hat{P} = \frac{\rho + \delta - \sqrt{\delta(\rho + \delta) + (1 - e)D}}{\rho} P^W, \qquad (2.17)$$

which lies below P^W, hence qualifies (i.e., satisfies the necessary condition of Property 1) for an optimal steady state if it lies in $[0, \bar{P}]$. Moreover, $L'(\hat{P}) < 0$ (since $L(\cdot)$ decreases at the smaller root), which is consistent with local stability of $\hat{P}$ (Property 2).

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In view of Property 1, we distinguish between the three cases depicted in Figure 1. To check if $\hat{P} \in [0, \bar{P}]$, use (2.8), (2.11), (2.13) to obtain

$$\begin{cases} \hat{P} \le 0 & \text{if } (1-e)D \ge \rho(\rho+\delta) \\ \hat{P} > 0 & \text{otherwise} \end{cases} \qquad (2.18a)$$

and, noting (2.2), (2.8), (2.11) and (2.17),

$$\begin{cases} \hat{P} < \bar{P} & \text{if } \delta(\rho+\delta) - \delta\sqrt{\delta(\rho+\delta) + (1-e)D} < eD \\ \hat{P} \ge \bar{P} & \text{otherwise} \end{cases} \qquad (2.18b)$$
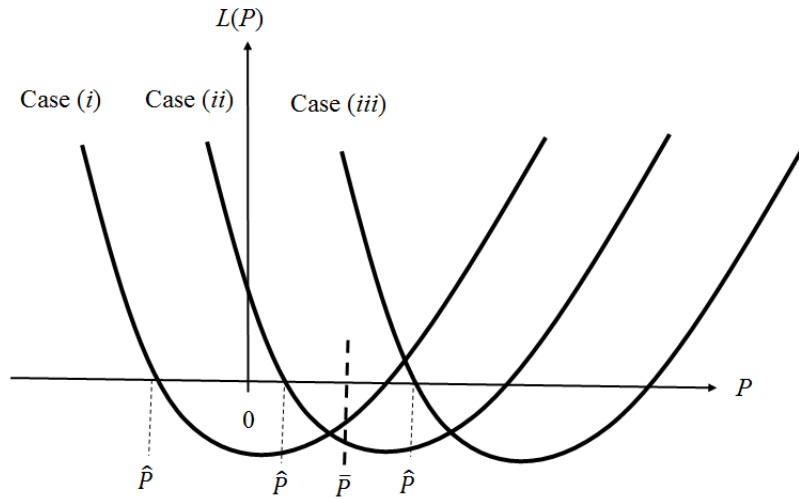

Figure 1: $\tilde{L}(\cdot)$ vs. the state P. Case (i): $\hat{P} \le 0$. Case (ii): $\hat{P} \in (0, \bar{P})$. Case (iii) $\hat{P} \ge \bar{P}$.

The three cases of Figure 1 can be classified in terms of the parameters as follows:

Case (i): $(1-e)D \ge \rho(\rho+\delta)$, so $\hat{P} \le 0$ is not feasible. The upper bound $\bar{P}$ cannot be an optimal steady state because if it is at or above the upper root it must also exceed P^W, hence $W(\bar{P}) = -\infty$, and if it lies below the upper root it falls between the two roots, where $L(P)$ is negative (see Figure 1), disqualifying $\bar{P}$ as an optimal steady state (by virtue of Property 1(iii)). The lower bound $P = 0$ also falls between the two roots (see Figure 1), hence

$L(0) \leq 0$ (equality holding if $\hat{P} = 0$) and Property 1(ii) allows the lower bound to be an optimal steady state. We conclude that $\hat{P}^* = 0$ is the unique optimal steady state in this case.

Case (ii): $(1-e)D < \rho(\rho+\delta)$ and $\delta(\rho+\delta) - \delta\sqrt{\delta(\rho+\delta) + (1-e)D} < eD$, so $\hat{P} \in (0, \bar{P})$. In this case $\hat{P}$ is the only state that qualifies as an optimal steady state. To see this note that $\hat{P} > 0$ implies $L(0) > 0$ (see Figure 1), excluding the lower bound as an optimal steady state (Property 1(ii)). Noting $\hat{P} < \bar{P}$, the explanation of Case (i) can be repeated to exclude $\bar{P}$ from the list of optimal steady states. We conclude that $\hat{P}^* = \hat{P} \in (0, \bar{P})$ is the unique optimal steady state in this case. Note that $L'(\hat{P}) < 0$, consistent with the stability condition (Property 2).

Case (iii): $\delta(\rho+\delta) - \delta\sqrt{\delta(\rho+\delta) + (1-e)D} \geq eD$, so $\bar{P} \leq \hat{P}$. Because $L(P) > 0$ for all $P \in [0, \bar{P}]$ (see Figure 1), only $\bar{P}$ satisfies the conditions for an optimal steady state (Property 1) and we conclude that $\hat{P}^* = \bar{P}$ is the unique optimal steady state in this case.

The implications for the long-run abatement policy are summarized in:

Proposition 1. (i) *If* $(1-e)D \geq \rho(\rho+\delta)$, *then in the long run it is optimal to eliminate the catastrophic threat altogether by abating at the rate* $\hat{q} = e$ *and reducing the pollution stock, and the ensuing hazard rate, to zero* $(\hat{P}^* = 0)$.
(ii) *If* $(1-e)D < \rho(\rho+\delta)$ *and* $\delta(\rho+\delta) - \delta\sqrt{\delta(\rho+\delta) + (1-e)D} < eD$, *then* $\hat{P}^* \in (0, \bar{P})$ *and partial abatement* $\hat{q} = e - \delta\hat{P}^*$ *is optimal in the long run.*
(iii) *If* $\delta(\rho+\delta) - \delta\sqrt{\delta(\rho+\delta) + (1-e)D} \geq eD$, *then in the long run it is optimal to ignore the catastrophic threat by avoiding abatement altogether* $(\hat{q} = 0)$ *and allowing pollution to reach the maximal level* $\bar{P}$.

It is seen how increasing the discount and depreciation rates, ρ and δ, vis-

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à-vis the emission and damage parameters e and D, shifts the optimal long run mitigation policy from full abatement via partial abatement to no abatement at all.

3 Multiple catastrophic threats

Suppose society faces $n > 1$ sources of catastrophic threat, each with the hazard

$$h_i(P_i) = \alpha_i P_i, \ \alpha_i > 0, \ i = 1, 2, \ldots n, \tag{3.1}$$

where P_i is the pollution stock affecting the occurrence of event i. The accumulation of P_i can be regulated by abatement activities q_i according to

$$\dot{P}_i(t) = e_i - q_i(t) - \delta_i P_i(t), \ i = 1, 2, \ldots n, \tag{3.2}$$

where e_i is the unmitigated emission rate associated with P_i with $\sum_i e_i \leq 1$, and δ_i is the natural removal rate. As above, the constant income flow is normalized at unity and the residual income $c = 1 - \sum_i q_i \geq 0$ is consumed to generate the utility $u(c) = -1/c$. It follows that $P_i(t) \leq \bar{P}_i \equiv e_i/\delta_i, \ i = 1, 2, \ldots n$.

Occurrence of event i reduces the capacity to derive utility from consumption by multiplying $u(c)$ by the factor $\varphi_i > 1$. The first occurrence time of event i is denoted T_i. The events are assumed to be independent, hence the survival probability corresponding to the first overall occurrence, $T \equiv \min(T_1, T_2, \ldots, T_n)$, is

$$S(t) \equiv Pr\{T > t\} = \prod_i Pr\{T_i > t\} = \prod_i S_i(t)$$

$$= \exp\left(-\sum_i \int_0^t h_i(P_i(\tau))d\tau\right) \tag{3.3}$$

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where $S_i(t) \equiv Pr\{T_i > t\} = \exp\left(-\int_0^t h_i(P_i(\tau))d\tau\right)$ is the survival probability associated with event i. From (3.3),

$$\dot{S}(t) = -S(t)\sum_i h_i(P_i(t)). \tag{3.4}$$

As a matter of notation, the n-dimensional vector with elements x_i, $i = 1, 2, \ldots, n$, is denoted x where x stands for the vectors $P, q, \alpha, h, \delta, D$ and φ. A center-dot between two vectors denotes their scalar product, e.g., $h \cdot \varphi \equiv \sum_i h_i \varphi_i$.

With $v(P(t))$ representing the value at time t when the pollution state is $P(t)$, the payoff at $t = 0$ is expressed as

$$\int_0^T u(c(t))e^{-\rho t}dt + e^{-\rho T}\left(\sum_i \varphi_i \mathbf{I}(T = T_i)\right)v(P(T)),$$

where $\mathbf{I}(\cdot)$ is the index function that assumes the value 1 if its argument is true and 0 otherwise. Taking expectation, recalling that the distribution and density of T_i are, respectively, $1 - S_i(t)$ and $h_i(P_i(t))S_i(t)$, and the distribution and density of T are, respectively, $1 - S(t)$ and $S(t)\sum_i h_i(P_i(t))$, the first term gives $\int_0^\infty u(c(t))S(t)e^{-\rho t}dt$ and the second term becomes

$$\int_0^\infty e^{-\rho t}\left(\sum_i \varphi_i h_i(P_i(t))S(t)\right)v(P(t))dt$$

$$= \int_0^\infty [h(P(t)) \cdot \varphi]S(t)v(P(t))e^{-\rho t}dt.$$

The expected payoff is therefore

$$\int_0^\infty \left[u\left(1 - \sum_i q_i(t)\right) + [h(P(t)) \cdot \varphi]v(P(t))\right]S(t)e^{-\rho t}dt. \tag{3.5}$$

The optimal abatement policy $\{q^*(t), t \geq 0\}$ maximizes the expected payoff subject to (3.2), (3.4), $S(0) = 1$, $q(t) \geq 0$ and $\sum_i q_i(t) \leq 1$, giving rise to the

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value

$$v(P(0)) = \int_0^\infty \left[u \left(1 - \sum_i q_i^*(t) \right) + [h(P^*(t)) \cdot \varphi] v(P^*(t)) \right] S^*(t) e^{-\rho t} dt,$$

(3.6)

where, as before, the asterisk denotes evaluation along the optimal policy.

The (not necessarily optimal) steady state policy $\hat{q}_i(P_i) = e_i - \delta_i P_i$, $i = 1, 2, \ldots, n$, entails the steady state consumption $\hat{c}(P) = 1 - \sum_i e_i + \delta \cdot P \geq 0$. Letting

$$\pi \equiv \left(1 - \sum_i e_i \right) / \sum_i \delta_i \quad \text{and} \quad \Pi \equiv \pi \mathbf{1},$$

(3.7)

where $\mathbf{1}$ is an n-dimensional vector of ones, the steady state consumption is expressed as

$$\hat{c}(P) = \delta \cdot (P + \Pi).$$

(3.8)

Noting (3.5), the expected payoff under the steady state policy is

$$W(P) = \frac{u(\delta \cdot (P + \Pi)) + \left[\sum_i h_i(P_i) \varphi_i \right] W(P)}{\rho + \sum_i h_i(P)}$$

$$= \frac{-1/(\delta \cdot (P + \Pi)) + [h(P) \cdot \varphi] W(P)}{\rho + \alpha \cdot P}. \quad (3.9)$$

Solving (3.9) for $W(P)$, using $D_i = \alpha_i(\varphi_i - 1)$, gives

$$W(P) = \frac{-1}{\delta \cdot (P + \Pi)(\rho - D \cdot P)},$$

(3.10)

provided $\rho - D \cdot P > 0$ (otherwise $W(P)$ diverges to $-\infty$, see (2.10) - (2.11)).

The multidimensional L-method (Tsur and Zemel 2014c) makes use of an n-dimensional L function, extending (2.14) along the following steps. First, the state dynamics equations (3.2) are expressed as $\dot{P}(t) = G(P, q)$, where

$$G(P, q) \equiv \begin{pmatrix} g_1(P, q) \\ \vdots \\ g_n(P, q) \end{pmatrix} = \begin{pmatrix} e_1 - q_1 - \alpha_1 P_1 \\ \vdots \\ e_n - q_n - \alpha_n P_n \end{pmatrix},$$

(3.11)

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and $J_q^G(P,q)$ denotes the Jacobian matrix of $G(P,q)$ with respect to q (with $\partial g_i/\partial q_j$ as the i,j element). Second, $f(P,q)$, defined above equation (2.14), becomes $f(P,q) = u(1-\sum_i q_i)+[h(P)\cdot\varphi]v(P)$ and $f_q(P,q)$ is its n-dimensional gradient vector (with $\partial f/\partial q_i$ as the i'th element).[8] Finally, $W_P(P)$ is now the n-dimensional gradient vector of $W(P)$, with $\partial W/\partial P_i$ as the i's element. The n-dimensional L function is defined as (Tsur and Zemel 2014c)

$$L(P) \equiv \begin{pmatrix} L_1(P) \\ \vdots \\ L_n(P) \end{pmatrix} = \left[\rho + \sum_{i=1}^n h_i(P_i)\right]\left(\left[J_q^G(P,\hat{q}(P))\right]'^{-1} f_q(P,\hat{q}(P)) + W_P(P)\right)$$

(3.12)

(a prime over a matrix denotes transpose).

Let the feasible vector $\hat{P}^*$ denote an optimal steady state. The multidimensional extension of Property 1 states (see Tsur and Zemel 2014c):

Property 3. (i) *If* $\hat{P}_i^* \in (0, \bar{P}_i)$ *for some* i, *then* $L_i(\hat{P}^*) = 0$. (ii) *If* $\hat{P}_i^* = 0$ *for some* i, *then* $L_i(\hat{P}^*) \leq 0$. (iii) *If* $\hat{P}_i^* = \bar{P}_i$ *for some* i, *then* $L_i(\hat{P}^*) \geq 0$.

For the sake of completeness, the proof is given in the appendix.

In the present context we obtain, using (3.8),

$$f_q(P,\hat{q}(P)) = \frac{-du(\hat{c}(P))}{dc}\mathbf{1} = \frac{-1}{[\delta\cdot(P+\Pi)]^2}\mathbf{1},$$

(3.13)

where it is recalled that $\mathbf{1}$ is the n-dimensional vector of ones and

$$u(\hat{c}(P)) = u(1 - \sum_i \hat{q}_i(P)) = -1/(1 - \sum_i \hat{q}_i(P)).$$

From (3.11) we find that the Jacobian J_q^G equals the negative of the n-dimensional identity matrix, and (3.10) gives

$$W_P(P) = W(P)^2\left([\rho - D\cdot P]\delta - [\delta\cdot(P+\Pi)]D\right).$$

(3.14)

---

[8]As in the case of an isolated catastrophe, $f(P,q)$ is interpreted as the "effective utility" corresponding to the expected payoff (3.5).

Thus, (3.12) specializes in our model to

$$L(P) = \frac{\rho + \alpha \cdot P}{[\delta \cdot (P + \Pi)]^2} \left[\mathbf{1} + \frac{[(\rho - D \cdot P)]\delta - [\delta \cdot (P + \Pi)]D}{(\rho - D \cdot P)^2} \right]. \qquad (3.15)$$

Let

$$\tilde{L}_i(P) \equiv (\rho - D \cdot P)^2 + (\rho - D \cdot P)\delta_i - \delta \cdot (P + \Pi)D_i, \ \ i = 1, 2, \ldots, n. \ \ (3.16)$$

It is readily verified that $\tilde{L}_i(P)$ and $L_i(P)$ have the same sign, hence $\tilde{L}_i(P)$ can be used instead of $L_i(P)$ in Property 3 to locate optimal steady states. In particular, an internal steady state $\hat{P}$, at which all sources of catastrophic threats are partially regulated, satisfies

$$\tilde{L}_i(\hat{P}) = 0, \ \ i = 1, 2, \ldots, n. \qquad (3.17)$$

To compare the consequences of multiple catastrophic threats vis-à-vis an isolated threat, it helps to distinguish between the case in which all sources of threats are identical and the more general case of heterogenous sources. We consider each case in turn.

3.1   Identical sources of catastrophic threats

Suppose all threats are identical, taking the same values for the parameters D_i, δ_i and e_i. Let a subscript n indicate the common value of a parameter, e.g., δ_n represents δ_i for all i, and $\delta = \delta_n \mathbf{1}$, where it is recalled that $\mathbf{1}$ is the n-dimensional vector of ones. The n equations of (3.16) are identical, each being equal to

$$\tilde{L}_n(P) = \left(\rho - D_n \sum_i P_i \right)^2 + \left(\rho - D_n \sum_i P_i \right) \delta_n - D_n \delta_n \left(\sum_i P_i + \tilde{\pi}(n) \right),$$
$$(3.18)$$

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where

$$\tilde{\pi}(n) \equiv (1 - ne_n)/\delta_n \geq 0 \qquad (3.19)$$

(the inequality follows from $\sum_i e_i \leq 1$). Thus, if $\hat{P}$ satisfies the L-conditions for a steady state (Property 3), then any feasible vector P satisfying $\sum_i P_i = \sum_i \hat{P}_i$ also meets these conditions.

Following the derivation that led to equation (2.16), $\tilde{L}_n(P)$ can be expressed as

$$\tilde{L}_n(P) = \rho\left(\sum_i P_i\right)^2 - 2P_n^W(\delta_n + \rho)\sum_i P_i + (P_n^W)^2(\delta_n + \rho - \delta_n\tilde{\pi}(n)/P_n^W), \qquad (3.20)$$

where, similar to (2.11), $P_n^W \equiv \rho/D_n$. The roots of $\tilde{L}_n(\cdot)$ are vectors P with

$$\sum_i P_i = \frac{\rho + \delta_n \pm \sqrt{\delta_n(\rho + \delta_n) + \rho\delta_n\tilde{\pi}(n)/P_n^W}}{\rho} P_n^W.$$

As in the case of a single catastrophe, an optimal steady state must also satisfy $\rho - D \cdot P = \rho - D_n \sum_i P_i > 0$ hence

$$\sum_i P_i < \rho/D_n = P_n^W, \qquad (3.21)$$

excluding vectors corresponding to the larger root of $\tilde{L}_n$ from the list of optimal steady state candidates. Thus, any internal steady state $\hat{P}$ must satisfy

$$\sum_i \hat{P}_i = \frac{\rho + \delta_n - \sqrt{\delta_n(\rho + \delta_n) + (1 - ne_n)D_n}}{\rho} P_n^W, \qquad (3.22)$$

where $(1 - ne_n)D_n = \rho\delta_n\tilde{\pi}(n)/P_n^W$ follows from (3.19) and (3.21).

Comparing (2.17) and (3.22), we see that the roots of the L functions corresponding to an isolated catastrophe and to several identical catastrophes differ in the terms $(1 - e)D$ and $(1 - ne_n)D_n$, respectively. Increasing the number of catastrophes n shifts the functions $\tilde{L}_n(\cdot)$ upward, thereby shifting the lower

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root to the right and, in turn, increasing the overall long-run pollution. The effect of the background threat on the individual pollution state is best seen when the abatement efforts are divided symmetrically among all threats, so that each pollution state obtains the average long-run value $\sum_i \hat{P}_i / n$. Comparing this value to the pollution state of an isolated catastrophe with the same parameters shows that depending on the damage parameter, the effect can be either positive or negative. This result is illustrated and discussed using a numerical example in the following section.

Following the steps leading to Proposition 1, the optimal steady states for several identical sources of independent catastrophic threats are characterized in:

Remark 1. *(i) If $(1 - ne_n)D_n \geq \rho(\rho + \delta_n)$, then in the long run it is optimal to eliminate all catastrophic threats (i.e., $\hat{P}_i^* = 0$ for all i) by abating at the rate $\hat{q}_i = e_i = e_n$.*

(ii) If the conditions $\delta_n(\rho + \delta_n) - \delta_n\sqrt{\delta_n(\rho + \delta_n) + (1 - ne_n)D_n} < ne_nD_n$ and $(1 - ne_n)D_n < \rho(\rho + \delta_n)$ hold, then $\sum_i \hat{P}_i^* = \sum_i \hat{P}_i \in (0, n\bar{P}_n)$. In this case, partial abatement of all P_i can hold at a steady state.

(iii) If $\delta_n(\rho + \delta_n) - \delta_n\sqrt{\delta_n(\rho + \delta_n) + (1 - ne_n)D_n} \geq ne_nD_n$, then $\hat{P}_i^* = \bar{P}_i$ for all i: in the long run it is optimal to ignore the all catastrophic threats by avoiding abatement altogether.

Total income allocated to abate all threats in the long run is given by $\sum_i \hat{q}_i^* = ne_n - \delta_n \sum_i \hat{P}_i^*$ and the average abatement effort per catastrophic threat is $e_n - \delta_n \sum_i \hat{P}_i^* / n$.

3.2 Heterogenous sources of catastrophic threats

We consider heterogenous sources of catastrophic threats that differ in their damage parameters D_i (and possibly in other parameters as well). Any pair i, j gives, noting (3.16),

$$\tilde{L}_i(P)D_j - \tilde{L}_j(P)D_i = (\rho - D \cdot P)[(\rho - D \cdot P)(D_j - D_i) + \delta_i D_j - \delta_j D_i].$$

Thus, (3.17) can hold with $D_i \neq D_j$ only when $\rho - D \cdot \hat{P} = 0$ or when $\rho - D \cdot \hat{P} = (\delta_i D_j - \delta_j D_i)/(D_i - D_j)$. As the former condition cannot hold at an optimal steady state (cf. (3.10)), we conclude that the conditions

$$\rho - D \cdot \hat{P} = (\delta_i D_j - \delta_j D_i)/(D_i - D_j) > 0 \ \forall D_i \neq D_j, \ i, j = 1, 2, \ldots, n, \quad (3.23)$$

are necessary for $\hat{P}$ to be a legitimate candidate for an internal optimal steady state. Feasibility requires, in addition, that $\hat{P}_i \in [0, \bar{P}_i]$.

Consider the simplest case of $n = 2$, under which (3.23) specializes to

$$\rho - D \cdot \hat{P} = (\delta_2 D_1 - \delta_1 D_2)/(D_2 - D_1) \equiv R > 0, \quad (3.24)$$

where the inequality ensures that the steady state value does not diverge to minus infinity (cf. (3.10)). Condition (3.24) together with $D \cdot \hat{P} \geq 0$ (due to the feasibility condition $\hat{P} \geq 0$) imply

$$0 < (\delta_2 D_1 - \delta_1 D_2)/(D_2 - D_1) \leq \rho, \quad (3.25)$$

In addition, the steady state consumption $\hat{c}(\hat{P}) = \delta \cdot (\hat{P} + \Pi)$ (cf. (3.8)) must lie in $(0, 1]$, hence

$$e_1 + e_2 - 1 < \delta_1 \hat{P}_1 + \delta_2 \hat{P}_2 \leq e_1 + e_2. \quad (3.26)$$

Conditions (3.25) and (3.26) are both necessary for a feasible root with $\hat{P}_i \in (0, \bar{P}_i)$ to be a legitimate candidate for an optimal internal steady state.

Indeed, for the two-catastrophes case it is simple to derive this root explicitly: Equations (3.24) and (3.17) give $D \cdot \hat{P} = R - \rho$ and $\delta \cdot (\hat{P} + \Pi) D_1 = R^2 + R\delta_1$, respectively, which can be written as $A\hat{P} = b$, where

$$A \equiv \left(\begin{array}{cc} D_1 & D_2 \\ \delta_1 & \delta_2 \end{array} \right) \text{ and } b \equiv \left(\begin{array}{c} \rho - R \\ R(\delta_1 - \delta_2)/(D_1 - D_2) - \pi(\delta_1 + \delta_2) \end{array} \right).$$

The condition $R > 0$ implies that A is nonsingular, hence

$$\hat{P} = A^{-1}b \tag{3.27}$$

is the unique root of $L(\cdot)$. When $\hat{P}$ is feasible and conditions (3.25)-(3.26) are satisfied, then Property 3 implies that $\hat{P}$ can be an optimal internal steady state.

When no such feasible vector $\hat{P}$ exists, at least one component of the optimal steady state $\hat{P}^*$ falls on a corner, i.e., $\hat{P}_i^* = 0$ or $\bar{P}_i$ for either $i = 1$ or $i = 2$. The corresponding hazard, then, is either eliminated completely or left unregulated. In this case, Property 3 can be used to sort out the optimal steady states. Table 1 lists the possible corner steady states and the corresponding necessary conditions. These conditions, together with

$$\rho - D \cdot \hat{P} > 0 \tag{3.28}$$

and (3.26), which must hold in any optimal steady state (either corner or internal), limit the number of legitimate candidates for an optimal steady state.

Table 1: Necessary conditions on $L(\hat{P})$ for corner steady states according to Property 3. In the first four entries both states lie on a corner ($\hat{P}_i = 0$ or $\bar{P}_i$, $i = 1, 2$). In the remaining four entries, the steady state occurs with one state at a corner and the other internal.

	Corner steady state		Necessary conditions	
	$\hat{P}_1$	$\hat{P}_2$	$L_1(\hat{P})$	$L_2(\hat{P})$
1.	0	$\bar{P}_2$	≤ 0	≥ 0
2.	0	0	≤ 0	≤ 0
3.	$\bar{P}_1$	0	≥ 0	≤ 0
4.	$\bar{P}_1$	$\bar{P}_2$	≥ 0	≥ 0
5.	0	$(0, \bar{P}_2)$	≤ 0	$= 0$
6.	$\bar{P}_1$	$(0, \bar{P}_2)$	≤ 0	$= 0$
7.	$(0, \bar{P}_1)$	$\bar{P}_2$	$= 0$	≥ 0
8.	$(0, \bar{P}_1)$	0	$= 0$	≤ 0

We summarize the discussion concerning two heterogenous sources of catastrophic threats in:

Proposition 2. (*i*) *Condition (3.24) is necessary for the existence of a root* $\hat{P}$ *satisfying* $L(\hat{P}) = 0$; *when this condition holds,* $\hat{P}$ *can be found by (3.27).* *If* $\hat{P}$ *is feasible and satisfies conditions (3.26) and (3.28), it is a legitimate candidate for an optimal internal steady state.*

(*ii*) *If condition (3.24) fails or* $\hat{P}$ *is not feasible or fails to satisfy conditions (3.26) and (3.28), then* $\hat{P}^*$ *must fall on a corner, where at least one of its elements falls on either its lower or upper bound.* *In this case,* $\hat{P}^*$ *must satisfy (3.26), (3.28) and the conditions of Table 1.*

In the numerical examples presented below, the optimal steady states are identified uniquely for the cases of a single catastrophic threat, two identical sources of catastrophic threats and two heterogenous sources of catastrophic threats.

4   Numerical illustrations

We apply the analysis to three cases: a single catastrophic threat, two identical threats, and two heterogenous threats. The parameter values used for the single threat and for two identical threats are presented in Table 2.

Table 2: Parameter values used in the examples of a single catastrophic threat and two identical catastrophic threats.

Parameter	Expression	Value	Description
ρ		0.03	time rate of discount
$e = e_n$		0.5	unabated emission rate
$\delta = \delta_n$		0.01	pollution removal rate
$D = D_n$	$\alpha(\varphi - 1)$	0.001, 0.01	damage parameter
P^W	ρ/D	30, 3	upper bound on $\hat{P}^*$
P_n^W	ρ/D_n	30, 3	upper bound on $\sum_i \hat{P}_i^*$
π	$(1-e)/\delta$	50	
$\tilde{\pi}(2)$	$(1-2e_n)/\delta_n$	0	

Figure 2 shows the $\tilde{L}$ functions corresponding to a single catastrophic threat $(n = 1)$ and two identical threats $(n = 2)$ under the high damage scenario $(D = D_n = 0.01)$. Because the lower root of the $\tilde{L}$ function of the single threat case is negative, it follows that $\hat{P}^* = 0$ (Proposition 1) and the optimal policy is to eliminate pollution altogether in the long run. The $\tilde{L}$ function corresponding to two identical threats admits a lower root at $\sum_i \hat{P}_i = 2$, implying, noting Remark 1, that all feasible states $\hat{P}^*$ with $\sum_i \hat{P}_i^* = 2$ are optimal.

The low damage scenario of $D = D_n = 0.001$ is shown in Figure 3. The $\tilde{L}$ functions corresponding to $n = 1$ and $n = 2$ admit lower roots at $\hat{P} = 10$ and $\sum_i \hat{P}_i = 20$, respectively, where both roots lie below the upper bound $P^W = P_n^W = 30$. In view of Proposition 1 and Remark 1, the optimal long-
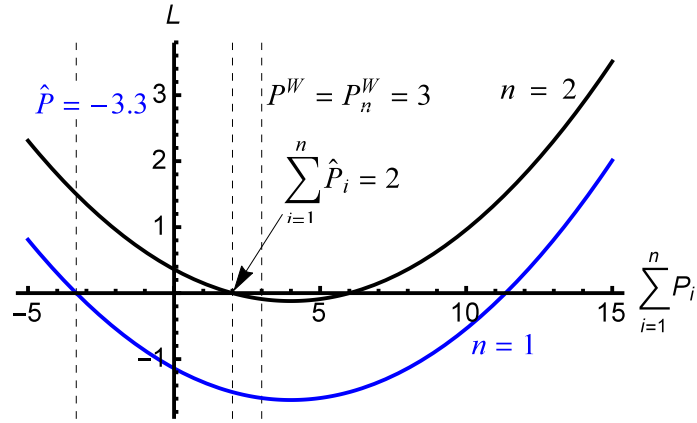
Figure 2: The $\tilde{L}$ functions for a single threat ($n = 1$) and two identical threats ($n = 2$) under the high damage scenario $D = 0.01$.

run pollution levels under the single catastrophic threat and under the two identical threats are $\hat{P}^* = 10$ and $\sum_{i=1}^{2} \hat{P}_i^* = 20$, respectively.
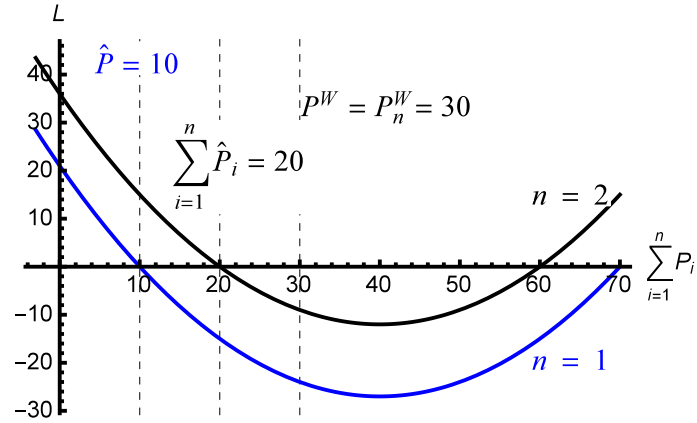

Figure 3: $\tilde{L}$ functions for a single threat ($n = 1$) and two identical threats ($n = 2$) vs. total pollution under the low damage scenario $D = D_n = 0.001$.

The steady state abatements under a single catastrophic threat for the low and high damage scenarios are

$$\hat{q}^* = e - \delta \hat{P}^* = \begin{cases} 0.4 & \text{in the low damage scenario} \\ 0.5 & \text{in the high damage scenario} \end{cases}.$$

Facing two sources of catastrophic threats, each identical to the above single source, induces society to increase the long-run income share allocated to

abatement to

$$\sum_{i=1}^{2} \hat{q}_i^* = 2e_n - \delta_n \sum_{i=1}^{2} \hat{P}_i^* = \begin{cases} 0.8 & \text{in the low damage scenario} \\ 0.98 & \text{in the high damage scenario} \end{cases}.$$

Figure 4 presents steady-state pollution per source of catastrophic threat under a single source ($n = 1$) and two sources ($n = 2$) for different damage scenarios (values of D). It reveals an interesting pattern regarding the "background risk" effect, as measured by the effect on long-run abatement per source due to the addition of an (identical) source of risk. The corresponding abatement per source is presented in Figure 5. At $D = 0.001$ the background risk effect disappears, which means that adding a second source of catastrophic threat does not affect the long-run abatement (or pollution) per source (Figures 4 and 5). At lower damage scenarios ($D < 0.001$), the effect is positive (abatement per source increases when a second source is introduced) and at higher damage scenarios ($D > 0.001$) the effect is negative.
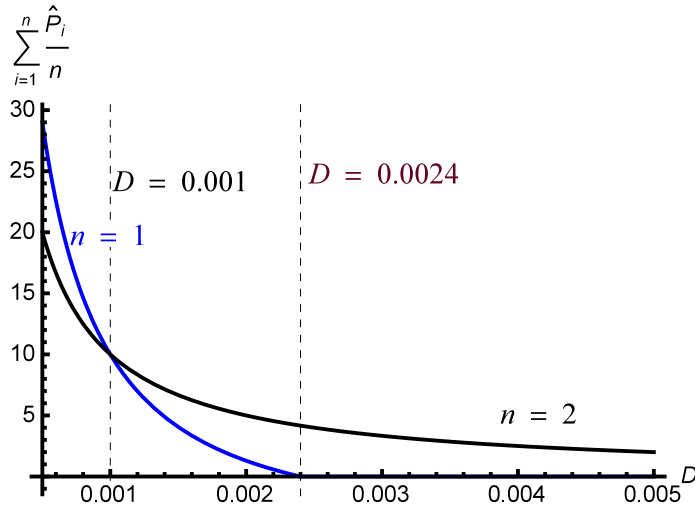

Figure 4: Pollution level per source of catastrophic threat under a single ($n = 1$) source and two ($n = 2$) identical sources. The former exceeds or falls short of the latter in low damage scenarios ($D = D_n < 0.001$) or high damage scenarios ($D = D_n > 0.001$), respectively. For the single source, long-run pollution is eliminated completely at or above $D = 0.0024$. Under two identical sources, it is not desirable to eliminate pollution in the long run.
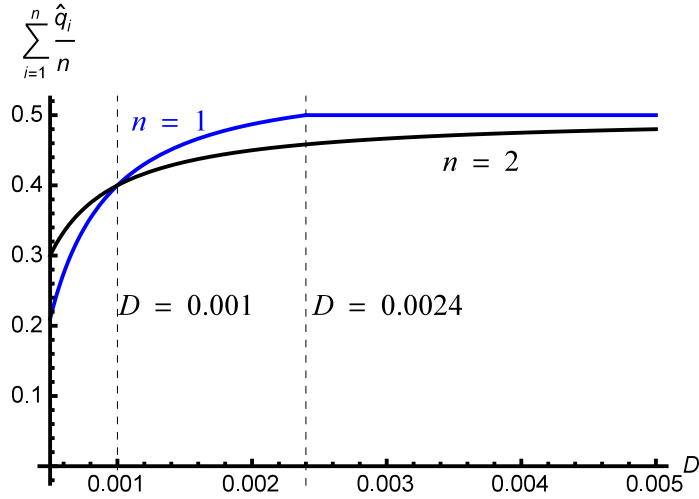
Figure 5: Abatement per source of catastrophic threat under a single ($n = 1$) and two ($n = 2$) identical sources vs. the damage parameter D. Abatement in the former case falls short or exceeds that of the latter case when $D = D_n < 0.001$ or $D = D_n > 0.001$, respectively.

To understand this observation, note that abatement investments affect welfare in two opposing ways: a positive hazard effect, due to the ensued reduction in hazard; and a negative consumption effect, due to the reduced resources available for consumption. Maintaining the same pollution level per source under one catastrophic threat as under two threats entails twice the occurrence risk (a double hazard) and requires twice as much abatement, hence also a smaller long-run consumption under two sources as compared to a single source (see Figure 6). At low damage scenarios ($D < 0.001$ in the example), pollution is relatively high, abatement is relatively low and consumption is relatively high (Figures 4-6). The higher steady state consumption leads to a weak (negative) consumption effect and as a result the (positive) hazard effect dominates, implying that abatement per source increases when a second source is added. At high damage scenarios ($D > 0.001$ in the example), long run pollution is lower, abatement is larger and consumption is smaller. The (negative) consumption effect dominates the (positive) hazard

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effect in such cases, hence long run abatement decreases when a second source of catastrophic threat is introduced.
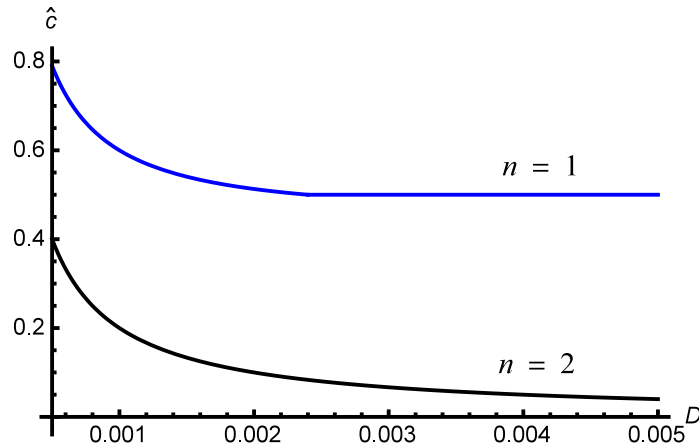

Figure 6: Steady state consumption under a single ($n = 1$) and two ($n = 2$) identical sources of catastrophic threats vs. the damage parameter D.

The third example considers two heterogenous sources of threat, with the parameter values of Table 3.

Table 3: Parameter values for the two heterogenous sources of catastrophic threats example.

Parameter	Expression	Value	Description
ρ		0.03	time rate of discount
e_1, e_2		0.5, 0.5	unabated emission rates
δ_1, δ_2		0.01, 0.01	pollution removal rates
D_1, D_2	$\alpha_i(\varphi_i - 1)$	0.001, 0.01	damage parameters
π	$(1 - e_1 - e_2)/(\delta_1 + \delta_2)$	0	
$\bar{P}_1, \bar{P}_2$	$e_1/\delta_1, e_2/\delta_2$	50, 50	pollution upper bounds

These values imply that the constant $R = (\delta_2 D_1 - \delta_1 D_2)/(D_2 - D_1)$ of (3.24) is negative, ruling out the possibility of an internal steady state (Proposition 2). We thus limit the search for an optimal steady state to corner states. Table 4 presents the candidates for corner steady states, along with their $\tilde{L}$-values, in the same order as they appear in Table 1. It also indicates, for each

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candidate, if it passes the necessary L-conditions (Property 3) and, for those that pass the L-test, whether condition (3.28), $\rho - D \cdot \hat{P} > 0$, is satisfied.

Table 4: Summary of necessary conditions for corner steady states in the example of heterogenous catastrophic threats. The "NA" in case #8 indicates that the equation $L_1(P_1, \bar{P}_2) = 0$ admits no solution for $P_1 \in [0, \bar{P}_1]$.

	$\hat{P}_1$	$\hat{P}_2$	$\tilde{L}_1(\hat{P})$	$\tilde{L}_2(\hat{P})$	Pass L conditions?	$\rho - D \cdot \hat{P} > 0$?
1.	0	$\bar{P}_2$	0.2157	0.2112	No	
2.	0	0	0.0012	0.0012	No	
3.	$\bar{P}_1$	0	-0.0003	-0.0048	No	
4.	$\bar{P}_1$	$\bar{P}_2$	0.2642	0.2552	Yes	No
5a.	0	2	0.00018	0	No	
5b.	0	6	0.00054	0	No	
6.	$\bar{P}_1$	6	0.005	0	Yes	No
7.	20	0	0	-0.0018	Yes	Yes
8.	$(0, \bar{P}_1)$	$\bar{P}_2$	NA		No	

The three candidates that satisfy the L-conditions are #4, #6 and #7. Of these, only candidate #7 satisfies the $\rho - D \cdot \hat{P} > 0$ condition. We are thus left with a unique candidate for an optimal steady state and conclude that the optimal state trajectory converges to a steady state in which both pollution stocks are abated: the low damage pollution P_1 (with $D_1 = 0.001$) is held at the interior state $\hat{P}_1^* = 20$ – below its maximal level $\bar{P}_1 = 50$ but above the steady state level of 10 which is obtained when this source is the only catastrophic threat (see Figure 3). The high damage pollution stock P_2 (with $D_2 = 0.01$) is eliminated altogether ($\hat{P}_2^* = 0$), as in the case when it is the only source of catastrophic threat (see Figure 2).

It is seen that when a high damage source of threat (with $D = 0.01$) is added to an already existing low damage source (with $D = 0.001$), long-run abatement rate of the existing source decreases from $q = 0.4$ (before the

introduction of the high damage source) to $q = 0.3$ (after the introduction) in order to allow allocating a sufficient income share (amounting to $q = 0.5$) to eliminate the (newly introduced) high-damage source. Alternatively, when a low-damage source (with $D = 0.001$) is introduced to an already existing high-damage source (with $D = 0.01$), the policy of eliminating the latter source remains unchanged even though additional resources are devoted to the abatement of the new, low-damage source. The significant effects of background threats suggest that both sources of catastrophic threat should be considered when deriving the optimal response.

5   Concluding remarks

We study the optimal long-run response to multiple catastrophic threats when the latter can be regulated continuously over time. The framework developed herein, thus, combines the existing catastrophic threat literature, which deals primarily with each threat in isolation, with Martin and Pindyck's (2015) recent contribution, extending the former to account for multiple threats and the latter to accommodate intertemporal policies.

Long-run properties of optimal policies are characterized for the cases of identical and heterogenous sources of catastrophic threats and compared with the response to each threat in isolation. The optimal comprehensive policy can either ignore a given threat, eliminate it completely, or partially reduce its hazard via intermediate mitigation efforts. We find that Martin and Pindyck's (2015) message regarding the importance of a comprehensive treatment of all hovering threats remains valid also in a model in which the mitigation efforts are allocated smoothly over time and marginal variations may be applied

to these efforts at each point of time. Moreover, we find that the "background threats" effect can be either positive or negative, depending on the corresponding damage parameters, so that the presence of other threats may either increase or decrease the optimal long run mitigation efforts of a certain catastrophic threat relative to the policy adopted when this source of threat is considered in isolation.

The analysis is presented in the context of a simple model, trying to stay as close as possible to Martin and Pindyck's (2015) formulation. We note that the L-method – the main analytic tool in this study – is flexible enough to allow various extensions and modifications, including a general utility function, mitigation efforts that affect simultaneously several states, a nonlinear hazard-state dependence and events that positively affect welfare. We avoid such extensions for the sake of simplicity and clarity of presentation.

Appendix

Proof of Property 3: The following derivation presents the extension in Tsur and Zemel (2014c) of the single-state L-method to multi-state models. Let

$$\hat{f}(P, \hat{q}(P)) \equiv u\left(1 - \sum_i \hat{q}_i(P_i)\right) + \left(\sum_i h_i(P_i)\varphi_i\right)W(P)$$

denote the "effective utility" corresponding to the steady state policy $q = \hat{q}(P)$ which maintains the state vector fixed at P indefinitely. For any feasible P, we compare the payoff $W(P)$ obtained under the steady state policy with the payoff obtained from a small feasible variation of this policy. If the variation policy yields a payoff that exceeds $W(P)$, then the steady-state policy is not optimal at P and this state vector does not qualify as an optimal steady state.

For small $\varepsilon > 0$ and small vector Δ with the elements Δ_i, $i = 1, \ldots, n$, the variation policy is defined by[9]

$$q^{\varepsilon\Delta}(t) \equiv \begin{cases} \hat{q}(P) + [J_q^G(P, \hat{q}(P))]^{-1}\Delta & \text{if } t < \varepsilon \\ \hat{q}(P(\varepsilon)) & \text{if } t \geq \varepsilon \end{cases}.$$

While $t < \varepsilon$, $q^{\varepsilon\Delta}(t)$ deviates slightly from the steady-state policy $\hat{q}(P)$, then it enters a steady state at $P(\varepsilon)$. During the first period when $t < \varepsilon$,

$$\dot{P} = G(P, \hat{q}(P)) + J_q^G(P, \hat{q}(P))[J_q^G(P, \hat{q}(P))]^{-1}\Delta + o(\Delta) = \Delta + o(\Delta),$$

which brings the state at $t = \varepsilon$ to $P(\varepsilon) = P + \varepsilon\Delta + o(\varepsilon\Delta)$.

Let $\Gamma(t) \equiv \int_0^t [\rho + \alpha \cdot P(s)]ds$ denote the "effective" discount factor. The contribution to the payoff under the variation policy $q^{\varepsilon\Delta}(t)$ during $t < \varepsilon$ is

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[9]Although the simple form adopted here for the state equation $\dot{P} = G(P, q)$ reduces the Jacobian matrix J_q^G to (the negative of) the identity matrix, the formulation holds for more general specifications hence we refer to this Jacobian in its general form.

evaluated, up to $o(\varepsilon\Delta)$ terms, by

$$\int_0^\varepsilon \hat{f}\left(P(t), \hat{q}(P) + [J_q^G(P, \hat{q}(P))]^{-1}\Delta\right) e^{-\Gamma(t)}dt =$$

$$\int_0^\varepsilon \hat{f}\left(P(t), \hat{q}(P) + [J_q^G(P, \hat{q}(P))]^{-1}\Delta\right) e^{-[\rho+\alpha\cdot P]t}dt +$$

$$\int_0^\varepsilon \hat{f}\left(P(t), \hat{q}(P) + [J_q^G(P, \hat{q}(P))]^{-1}\Delta\right) [e^{-\Gamma(t)} - e^{-[\rho+\alpha\cdot P]t}]dt.$$

The first integral on the right can be expressed as

$$\int_0^\varepsilon \hat{f}(P, \hat{q}(P))e^{-[\rho+\alpha\cdot P]t}dt + [\hat{f}_q(P, \hat{q}(P))] \cdot [J_q^G(P, \hat{q}(P))]^{-1}[\varepsilon\Delta] + o(\varepsilon\Delta) =$$

$$W(P)\left[1 - e^{-[\rho+\alpha\cdot P]\varepsilon}\right] + [\hat{f}_q(P, \hat{q}(P))] \cdot [J_q^G(P, \hat{q}(P))]^{-1}[\varepsilon\Delta] + o(\varepsilon\Delta),$$

and the second integral is $o(\varepsilon\Delta)$.

The contribution of $q^{\varepsilon\Delta}$ during the infinite period $t \geq \varepsilon$ is evaluated, up to $o(\varepsilon\Delta)$ terms, by

$$\int_\varepsilon^\infty \hat{f}(P(\varepsilon), \hat{q}(P(\varepsilon)))e^{-[\rho+\alpha\cdot P(\varepsilon)]t}dt = \int_\varepsilon^\infty [\rho+\alpha\cdot P(\varepsilon)]W(P(\varepsilon))e^{-[\rho+\alpha\cdot P(\varepsilon)]t}dt =$$

$$\int_\varepsilon^\infty [\rho+\alpha\cdot P(\varepsilon)]W(P)e^{-[\rho+\alpha\cdot P(\varepsilon)]t}dt + \int_\varepsilon^\infty [\rho+\alpha\cdot P(\varepsilon)]W_P(P)\cdot[\varepsilon\Delta]e^{-[\rho+\alpha\cdot P(\varepsilon)]t}dt.$$

The first integral on the second line can be expressed as

$$W(P)\int_\varepsilon^\infty [\rho+\alpha\cdot P(\varepsilon)]e^{-[\rho+\alpha\cdot P(\varepsilon)]t}dt = W(P)e^{-[\rho+\alpha\cdot P(\varepsilon)]\varepsilon} = W(P)e^{-[\rho+\alpha\cdot P]\varepsilon}+o(\varepsilon\Delta)$$

and the second integral is approximated by $W_P(P)\cdot[\varepsilon\Delta]+o(\varepsilon\Delta)$.

Summing the contributions of the two periods gives the payoff $V^{\varepsilon\Delta}(P)$ obtained under the variation policy:

$$V^{\varepsilon\Delta}(P) = W(P) + \left[[J_q^G(P, \hat{q}(P))]'^{-1}\hat{f}_q(P, \hat{q}(P)) + W_P(P)\right] \cdot [\varepsilon\Delta] + o(\varepsilon\Delta).$$

Thus, noting (3.15) and $\hat{f}_q(\cdot, \cdot) = f_q(\cdot, \cdot)$,

$$V^{\varepsilon\Delta}(P) - W(P) = \frac{L(P) \cdot [\varepsilon\Delta]}{\rho + \alpha \cdot P} + o(\varepsilon\Delta).$$

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The signs of the elements of Δ can be freely chosen, while $\varepsilon > 0$. Now, if $L(P) \neq \mathbf{0}$ we can set $\Delta = dL(P)$, where d is a small positive constant, hence $L(P) \cdot \Delta > 0$. This implies $V^{\varepsilon\Delta}(P) > W(P)$ and P is not an optimal steady state. Thus, only the roots of $L(\cdot)$ qualify as legitimate candidates for an optimal steady state. The only possible exceptions are the feasibility bounds on P_i. Choosing $\Delta_i > 0$ is not feasible at $\bar{P}_i$ because this policy would drive the $P_i(\cdot)$ process outside the feasible domain. It follows that the state vector $P = (P_1, \ldots, \bar{P}_i, \ldots, P_n)'$ cannot be excluded as an optimal steady state if $L_i(P) > 0$. A similar argument implies that $P = (P_1, \ldots, 0, \ldots, P_n)'$ cannot be excluded as an optimal steady state if $L_i(P) < 0$. $\square$

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