An Approximate Law of One Price in Random Assignment Games

(Job Talk)

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Schedule: **December 4-2014, 11:15-12:30**

Place: **Bld. 72, room 465, Dept. of Economics, BGU**

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November 23, 2014

Abstract

Assignment games represent a tractable model of two-sided markets with transfers. We study the likely properties of the core of randomly generated assignment games. When the joint productivity of every firm and worker has a noise element with a bounded distribution, with high probability all workers who have approximately the same human capital level are paid roughly equal wages, and all firms of similar quality make similar profits. This implies that core allocations vary significantly in balanced markets, but that there is core convergence in even slightly unbalanced markets. The same phenomenon occurs when firms' quality and workers' human capital level are complementary factors in productivity. When the noise element is unbounded, there may be a large variation in payoffs.

*The authors thank Itai Ashlagi, Elchanan Ben-Porath, Drew Fudenberg, Jerry Green, Sergiu Hart, Johannes Hörner, Sangram Kadam, Scott Kominers, Jacob Leshno, Déborah Marciano-Romm, Eric Maskin, Motty Perry, Alvin Roth, Ran Shorrer, and seminar and conference participants at Bar-Ilan University, Harvard University, the Hebrew University, the Technion, and Tel Aviv University for helpful discussions and suggestions. The first author also thanks the Hebrew University department of economics and the Center for the Study of Rationality for their hospitality. e-mail for correspondence: aromm@hbs.edu

1 Introduction

The "law of one price" asserts that homogeneous goods must sell for the same price across locations and vendors. This basic postulate is assumed in much of the economic literature, and its origins can be traced to Adam Smith's discussion on arbitrage (Smith, 1776, e.g., Book I, Chapter V). While many (sometimes consistent) deviations from this "law" have been observed and documented in the real world (see, for example, Lamont and Thaler, 2003, and references therein), it remains an interesting and useful building block in economic theory, and serves as a benchmark for empirical studies. A crucial underlying assumption used in arguing for the validity of the law of one price is the homogeneity of goods and buyers: buyers do not care which of the goods they end up buying, or which seller they are buying it from, nor do sellers care about the identity of the buyers. In other words, any two instances of the good are perfect (or at least near-perfect) substitutes for the buyers, as are any two buyers from any seller's point of view.

However, there are many markets in which the assumption of homogeneity is highly implausible. For example, in labor markets there are some workers are skilled and some unskilled, and similarly some firms are generally considered better places to work. In addition to these measurable quality differences, workers may exhibit heterogeneous preferences over being employed by different firms, due to personal likes and dislikes, location, values, and a variety of other individually determined factors. Firms may also have diverse preferences over workers, and may, for example, favor workers who seem to share their vision or fit well within their corporate culture. Similarly, in markets where buyers and sellers have heterogeneous preferences over trading with the other side, the law of one price generally should not hold.

This paper makes the formal claim that even in the presence of heterogeneous preferences, an approximate version of the law remains valid, and the approximation improves as the market grows large. We focus on labor markets as our leading example, and argue that a likely outcome of the market is that workers who are roughly equally skilled receive similar wages, and firms of similar quality garner similar profits. Because of the inherent heterogeneity in firms' and workers' preferences, the law of one price holds only approximately, with some workers being paid more than their peers with identical levels of human capital.

To prove this result we use the assignment game model of Shapley and

Shubik (1971) in which there is a finite set of firms and a finite set of workers, and each firm is looking to hire exactly one worker in exchange for a negotiable salary. Each firm has a (possibly different) value for hiring each of the workers, and each worker has a (possibly different) reservation value for working for each of the firms, and utilities are assumed to be linear in money. Since transfers are freely allowed, we can describe the net productivity of each firm-worker pair by a single number, and we assume that this productivity is separable in the firm's quality, the worker's human capital level, and an idiosyncratic noise element that is independently and identically distributed according to some bounded distribution.[1] We then provide a probabilistic analysis of the core of the game, and show that with high probability the differences in the payoffs of agents on the same side of the market behave like $\frac{\log n}{n}$, where n is the size of the market (Theorem 1). We also prove that this bound is tight (Theorem 2).

The fact that there are heterogeneous preferences in the market also implies that there are good and bad matchings between firms and workers, and that there is a surplus that is created by matching the right worker to the right firm.[2] Our approximate law of one price helps us to analyze the distribution of this surplus between firms and workers in balanced and unbalanced markets. In an unbalanced market with more workers than firms, at least one worker will be left out, and that worker will be willing to transact with any matched firm even for a minuscule gain. This constrains the profit of the worker matched to any firm that has good idiosyncratic fit with the unmatched worker, and by the approximate law of one price, the rest of the agents on the long side will necessarily make very small profits as well (Corollary 4). This argument shows why most of the surplus goes to market participants on the short side, despite the assumed idiosyncratic nature of pairwise productivities. In a balanced market we show that the surplus can be distributed in a variety of ways (Corollary 3).

[1]This assumption is similar in spirit to the one made in many papers in auction theory, where bidders' valuations are assumed to be heterogeneous and determined according to some random distribution. However, unlike most of the literature on auction theory, we do not wish to study the effects of the random generation on agents' beliefs and equilibrium behavior. Instead we take a different approach and characterize the likely outcomes in a typical complete information matching market created in that manner.

[2]One interpretation of the productivities appearing in our model is to think of them as actual output of workers, which is likely to be affected by heterogeneous person-organization fit. See Kristof-Brown and Guay (2011) for a recent survey of most of the important contributions to the literature on this issue.

These two results extend our economic intuitions about competition and surplus distribution in markets for homogeneous goods. If there are 10 farmers trying to sell 10 bushels of wheat to 9 identical buyers, and each of the buyers is interested in buying exactly one bushel of wheat and is willing to pay up to $100 for it, then the price of wheat will be $0, and each buyer's welfare is $100. In a market with 10 farmers and 10 buyers, the price of wheat can be as high as the buyers' willingness to pay.

As mentioned earlier, some of our results rely heavily on two assumptions: separability of production factors and boundedness of the idiosyncratic noise factor. We relax the first assumption by considering a model with a Cobb–Douglas productivity function, in which the firm's quality and the worker's human capital level are complements. We prove that in this model the efficient assignment is with high probability approximately assortative (Lemma 6), and recover the approximate law of one price (Theorem 7). This analysis reveals that the argument for an approximate law of one price is at least to some extent robust to other forces in the market, such as efficiently matching good workers with good firms (and vice versa).

We conclude by focusing on the boundedness assumption and show that it cannot be dispensed with. We consider a model with exponential noise and show that the differences in workers' payoffs do not vanish as the market grows (Proposition 9). Nevertheless, we do present computer simulations and a partial argument for why surplus distribution under exponential noise may present similar properties to surplus distribution under bounded noise (Theorem 11).

The rest of the paper is organized as follows. Section 2 reviews the literature related to our paper. Section 3 introduces the model and the formal notation. Section 4 contains the statement and the proof of the main result, as well as the tightness result, and an analysis of surplus distribution. Section 5 discusses the extension of the main result to a market with interaction terms in the joint productivity of firms and workers. Section 6 presents some results related to unbounded noise distributions. Section 7 provides simulation results, and Section 8 concludes.

2 Related Literature

Assignment games were first introduced by Shapley (1955). Shapley and Shubik (1971) thoroughly analyze them and show that the core can be described as the set of solutions to a linear program dual to the optimal assignment problem, and that it is therefore nonempty, compact, and convex. They also prove that it contains two special allocations: a firm-optimal and a worker-optimal core allocation. Demange and Gale (1985) extend the analysis and show, among other things, that the core has a lattice structure. They also point to the nonmanipulability by workers of the worker-optimal core allocation. Assignment games bear a great resemblance to the very familiar assortative matching model of Becker (1981), with the main difference being the lack of agreement of agents on one side over the ranking of agents on the other side in the more general assignment game model. In a slightly different interpretation, Demange et al. (1986) use the assignment game framework to describe auctions of heterogeneous items with unit demand bidders (with this interpretation in mind, core allocations are equivalent to Walrasian equilibria, and therefore our results provide insight into revenue acquired by multiple auctioneers under different market conditions).

Within the literature that focuses on assignment games, a paper related to ours is Kanoria et al. (2014). They too study a random version of the assignment game and show core convergence in the sense of agents getting similar payoffs across different core allocations. The most striking difference between the models is that in theirs each agent has a type (out of a finite set of fixed types), and agents' preferences depend only on the type of the agent to which they are matched, whereas in our model each agent may have a ranking over individual agents on the other side of the market. Other relevant papers within this literature are those that study the size of the core (in deterministic assignment games) such as Quint (1987) who defines two measures for core elongation and shows the relation between them, and Núñez and Rafels (2008) who investigate the dimension of the core based on the entries in the productivity matrix.

Several recent empirical works estimate a model similar to ours (and even more closely related to Kanoria et al., 2014), with the caveat of using an extreme value distribution for the idiosyncratic component. Choo and Siow (2006) consider marital behavior in the United States and estimate a model in which each agent has a type, and idiosyncratic preferences over being matched with any type of agent on the other side of the market. Similarly, Botticini and Siow (2008) study whether there are increasing returns to scale in marriage markets, and Chiappori et al. (2011) study the marital college premium.

From a broader point of view, this paper belongs to the theoretical literature on matching in two-sided markets. This literature gained prominence in the 1960s and early 1970s following the publication of the seminal papers by Gale and Shapley (1962) and Shapley and Shubik (1971), and research remained mostly divided (with some notable exceptions) into two parallel strands: with and without transferable utility (i.e., money). The bulk of the literature on matching markets without transfers, also known as the marriage market model (in the one-to-one case) and the college admissions model (in the many-to-one case), is focused on studying theory related to markets with fixed preferences, often under the additional assumption of complete information. Within this realm, two important papers for our discussion are Crawford and Knoer (1981) and Kelso and Crawford (1982). These papers describe the detailed connection between marriage markets and assignment games, and point to an auction process similar to the deferred-acceptance algorithm that produces an approximation to a side-optimal core allocation.[3] We employ a similar auction process in the proof of our lower bound of variation in workers' salaries (Theorem 2).

The past two decades have seen the emergence of more models that allow for stochastic markets and incomplete information. This new focus has revealed to market designers that some of the subtleties related to small markets may very well become negligible once we consider large "likely" markets. Yet the works on large markets most relevant to our present study were already written in the 1970s by Wilson (1972) and Knuth (1976), and were extensively developed by Pittel (1989, 1992). These papers analyze marriage markets with preferences that are determined uniformly at random and show that in a situation in which the number of men is equal to the number of women, with high probability the proposing side's (in a deferred acceptance algorithm) mean rank of partners behaves like $\log n$, whereas the other side's mean rank of partners behaves like $\frac{n}{\log n}$. This particular strand of the literature remained dormant for almost three decades, but several papers have recently used similar methods. Ashlagi et al. (2013) show that in unbalanced random marriage markets with high probability under any stable matching the short side's mean rank of partners behaves like $\log n$, whereas the long side's mean rank of partners behaves like $\frac{n}{\log n}$. Coles et al. (2014) and Coles and Shorrer (2014) employ these results to study aspects of strategic behavior

[3]For further generalizations of the marriage market model and the assignment game model see, for example, the works by Hatfield and Milgrom (2005), Ostrovsky (2008), Hatfield et al. (2013), and references therein.

in marriage markets with incomplete information. Lee (2014) and Lee and Yariv (2014) assume that preferences are derived from underlying cardinal utilities and study the issues of core convergence and efficiency, respectively.

Using somewhat different methods, but still trying to explain core convergence using different modes of competition, Immorlica and Mahdian (2005) explain in a breakthrough paper why in a large random marriage market with one of the sides having rank-ordered lists of bounded length and with incomplete information, truth-telling become an approximately dominant strategy. Kojima and Pathak (2009) extend this result to the college admissions model, and Storms (2013) extends it to many-to-one markets with substitutable preferences.[4] Kojima et al. (2013) use a similar strategy to prove that in a market with "not too many" couples, a stable matching exists despite the complementarities imposed by couples' preferences. Ashlagi et al. (forthcoming) further improve this result, show that stability is also implied for groups that can contain more than two members, and provide a counterexample to the case of a similar number of singles and couples.

Technically, our analysis is also related to what is known in the operations research and computer science literature as the random linear sum assignment problem. Specifically, two results that are used repeatedly in our proofs are the calculation of the limit value of a large random assignment game (Aldous, 2001), and the bounding of the minimal productivity in the optimal assignment (Frieze and Sorkin, 2007). For a more exhaustive survey of the random linear sum assignment problem (and closely related problems) see Krokhmal and Pardalos (2009).

3 Model and Notation

Consider a sequence of markets $\{M^n\}_{n=1}^{\infty}$, such that each market can be described as $M^n = (F^n, W^n, q^n, h^n, \alpha^n)$, where F^n is a set of firms of size n, W^n is a set of workers of size $n + k(n)$, with $k(n) \in \mathbb{N}$ and $k(n) = O(n)$,[5] q^n is a vector of qualities related to firms in F^n, h^n is a vector of human capital levels related to workers in W^n, and α^n is an $|F^n| \times |W^n|$ real matrix representing the value of pairs of firms and workers. We assume throughout

[4]Related analysis was also applied by Manea (2009), Che and Kojima (2010) and Kojima and Manea (2010) to the problem of optimal object assignments.

[5]The latter assumption is introduced for mathematical convenience.

that each element of α^n can be described as

$$\alpha^n_{ij} = u\left(q^n_i, h^n_j\right) + \varepsilon^n_{ij},$$

where u is the part of the production function that depends only on the firm's quality and the worker's human capital level, and ε^n_{ij} is idiosyncratic noise representing the productivity related to the identities of the firm and the worker. ε^n_{ij} is independently and identically distributed according to the cumulative distribution function G which has a continuous and strictly positive probability density function g.[6]

For technical purposes we will assume (unless otherwise noted) that the elements of the vectors h^n are identically and independently distributed on the interval $\left[\underline{h}, \overline{h}\right]$ according to the cumulative distribution function H. If $\underline{h} \neq \overline{h}$ we will also require H to have positive and continuous density on this interval. This assumption can easily be relaxed, but it is kept for clarity. Note that it does not hold for the specific distribution we use in Appendix C.

- **The separable case:** $u(q, h) = q + h$.
- **The interactive case:** $u(q, h) = q^\gamma h^{1-\gamma}$.

Note that while q and h appear without a transformation in both cases, any continuous transformation can be applied directly to their distributions. Therefore, the word "separable" accurately describes the domain of the first case. We also distinguish between several possible assumptions on G:

- **Bounded noise:** G is bounded on the interval $[0, 1]$ ($G(1) = 1$).
- **Unbounded noise:** There exists no $c \in \mathbb{R}$ such that $G(c) = 1$.
- **Exponential noise:** $G = \mathrm{Exp}(1)$ (special case of unbounded noise).

We prove our main result for the separable case with bounded noise, and extend it (under a certain technical assumption to be mentioned later) to the interactive case with bounded noise. We show that an approximate law of one price (properly formulated) does not hold in general for unbounded noise. Nevertheless, we explain why we believe some of our surplus distribution results do hold (in a weak form), at least for the case of exponential noise.

[6] In fact for our results to hold we only need that the density be continuous near its supremum.

In market M^n, the value of a coalition of firms and workers S is given by

$$v(S) = \max \left\{ \alpha^n_{i_1 j_1} + \alpha^n_{i_2 j_2} + \cdots + \alpha^n_{i_l j_l} \right\},$$

where the maximum is taken over all arrangements of $2l$ distinct agents, $f^n_{i_1}, \ldots, f^n_{i_l} \in S \cap F^n$, $w^n_{j_1}, \ldots, w^n_{j_l} \in S \cap W^n$, $l \leq \min\{|S \cap F^n|, |S \cap W^n|\}$. An **allocation** is denoted by (μ, u, v) with μ being a matching of firms to workers and vice versa, and u and v being payoff vectors for the firms and workers, respectively. We refer to u as firms' "profits," and to v as workers' "salaries." Formally, $\mu: F^n \cup W^n \to F^n \cup W^n \cup \{\emptyset\}$, and satisfies

1. $\forall f \in F^n : \mu(f) \in W^n \cup \{\emptyset\}$,
2. $\forall w \in W^n : \mu(w) \in F^n \cup \{\emptyset\}$, and
3. $\forall f \in F^n, w \in W^n : \mu(f) = w \iff \mu(w) = f$.

An allocation is a **core allocation** if no coalition can deviate and split the resulting value between its members such that each member of the coalition becomes strictly better off. We denote the set of core allocations of M^n by $C(M^n)$. As mentioned above, Shapley and Shubik (1971) show that the core is a nonempty compact and convex set, and that it is elongated in the sense that there is a firm-optimal core allocation in which salaries are at their lowest level among all core allocations, and a worker-optimal core allocation in which salaries are at their highest level among all core allocations.

Most of our results are going to hold for "most" realizations of some stochastic matrices and vectors. We often use the technical term *with high probability* (or *whp* for short) to mean that some result holds for the sequences of markets M^n with probability $1 - O\left(\frac{1}{n}\right)$. Whenever it is not mentioned, the term refers to realizations of the stochastic matrices α^n as well as the quality vectors q^n and h^n. However, in some places we explicitly mention that the term refers only to α^n or only to the quality vectors.

4 An approximate law of one price

This section presents our main result, which shows that in the separable case with bounded noise there cannot be too much variation in the payoffs of the agents on either side of the market. We then proceed to improve our upper bound on this variation for the special case of side-optimal core allocations, and establish a lower bound. These two proofs use a different method that relies on the salary adjustment procedure described by Crawford and Knoer (1981) and Kelso and Crawford (1982). The following subsection employs these results to characterize surplus distribution in these markets, and argues that the range of potential outcomes (i.e., payoffs in the core) crucially depends on whether the market is exactly balanced or not.[7] If it is not exactly balanced, the short side keeps most of the created surplus (or at least the surplus due to the idiosyncratic noise).

4.1 The main result

In order to gain some intuition into the mechanics of the proof and the argument behind it, let us first assume that the market is balanced, that all firms have the same quality, and that all workers have the same level of human capital. In this specific scenario our result implies that whp all workers (for example) should earn a very similar salary.

Suppose that worker w_1 is employed by firm f_1 and earns a salary of s_1 and worker w_2 is employed by f_2 and earns a salary of s_2. Suppose further that $s_2 > s_1$. If workers and firms were homogeneous goods, firm f_2 could offer worker w_2's job to worker w_1 for any salary strictly between s_1 and s_2. That is the usual argument for the law of one price in a two-sided market. However, it may well be that the combination of f_2 and w_2 has much higher productivity than f_2 and w_1, and therefore there is no mutually beneficial opportunity for f_2 and w_1. Nevertheless, we do know that there are about $n^{\frac{2}{3}}$ workers in the market such that their productivity with firm f_2 is no less than $1 - \frac{1}{n^{\frac{1}{3}}}$. For each of those workers the original argument works perfectly, and so none of these workers can be paid less than $s_2 - \frac{1}{n^{\frac{1}{3}}}$ because otherwise she and firm f_2 might deviate. Now we have a set of size $n^{\frac{2}{3}}$, each getting a salary of at least $s_2 - \frac{1}{n^{\frac{1}{3}}}$. Consequently there are about $n^{\frac{2}{3}}$ firms paying a salary of at least $s_2 - \frac{1}{n^{\frac{1}{3}}}$, and whp one of these firms, say f', is a good match with worker w_1, in the sense that their joint productivity is more than $1 - \frac{1}{n^{\frac{1}{3}}}$. By considering the possibility of deviation by f' and w_1 we reach the conclusion that $s_1 \geq s_2 - \frac{2}{n^{\frac{1}{3}}}$.

The argument used above is not quite accurate, since we do not account

[7] The term "balanced" is also used in the context of cooperative game theory to describe games with a nonempty core. This meaning is not used anywhere in this paper.

for the fact that firms in the intermediate set are not random, but are rather chosen in a specific way (i.e., they are matched to workers who are also productive when matched with firm f_1). The formal proof handles this issue by considering the likely expansion properties of the directed graph induced by the random productivity matrix and showing that a path must exist between f_2 and w_1.

As implied, the other difference from the informal argument above is that the proof uses the smallest possible expansion that still results in the necessary paths between all pairs of agents, i.e., a strongly connected digraph. This minimality is also formally established in our derivation of a lower bound for the variation in agents' payoffs. The technical element of the proof that allows for constructing high-probability paths is based on the result of Frieze and Sorkin (2007), which we extend here to deal with unbalanced markets as well as bounded distributions other than the uniform distribution.

The intuition behind proving the result for unbalanced markets is pretty straightforward given our understanding of how to utilize improvement paths, as previously described. We first show that whp all workers above a certain level of human capital are matched. Otherwise, one could replace a low-quality worker with a high-quality worker, and then reshuffle the matched workers such that the impact on the efficiency coming from idiosyncratic noise component will not be too substantial. We next show that the same logic that was used in the balanced case can be applied to the unbalanced case, if we focus only on agents above a certain level of human capital.

Theorem 1. *In the separable case with bounded noise, there exists* $c \in \mathbb{R}_+$ *such that whp for any* $(\mu^n, u^n, v^n) \in C(M^n)$ *we have*

1. $\forall i, j \in \{1, \ldots |F^n|\} : u_i^n - u_j^n \leq \left(q_i^n - q_j^n\right) + \frac{c \log n}{n}$, *and*
2. $\forall i, j \in \{1, \ldots |W^n|\}, \mu^n(w_i^n), \mu^n(w_j^n) \in F^n : v_i^n - v_j^n \leq \left(h_i^n - h_j^n\right) + \frac{c \log n}{n}$.

Proof. See Appendix A. □

Theorem 1 demonstrates that in a large random assignment game, all firms make approximately the same profits, and all matched workers earn approximately the same salary. In a sense, this theorem states that the core is not only elongated, as implied in Shapley and Shubik (1971) and Demange and Gale (1985), but that it is also narrow.

The bounds already provided do not leave much room for further improvements (let alone the constants used in the proof), but we still wish to verify that they are tight, at least in terms of order of magnitude. The following theorem shows that they are. We focus on balanced markets with all firms having the same quality and all workers having the same human capital level, governed by a specific core allocation, namely, the firm-optimal core allocation. We know that we can find the firm-optimal core allocation via the auction-like algorithm proposed by Crawford and Knoer (1981). When firms propose to workers, the auction process ends when all workers have received an offer. We can compute the probability that at each stage a worker who has not received an offer so far receives an offer, and then calculate the number of discrete steps required to reach the last worker. The approximation is possible thanks to our bounds from Theorem 1. This gives us a lower bound for the expected sum of workers' salaries, which implies a lower bound on what the top earner gets with high probability. Since we know the lowest earner gets zero, we are done. We note that the same procedure can also be used to provide better constants in Theorem 1 for the specific cases of the side-optimal core allocations.

Theorem 2. *In the separable case with bounded noise, if $k(n) \equiv 0$, $q^n \equiv \underline{0}$ and $h^n \equiv 0$, there exists $c \in \mathbb{R}_+$ such that whp there exist $(\mu^n, u^n, v^n) \in C(M^n)$ and $i, j \in \{1, \ldots |W^n|\}$ for which $v_i^n - v_j^n \geq \frac{c \log n}{n}$.*

Proof. See Appendix B □

We conclude this subsection by suggesting an interpretation of our results in terms of the shape of the core. As mentioned above, Shapley and Shubik (1971) already noticed that the core is compact and convex, and that it is shaped like a nut, in the sense that it contains firm-optimal and worker-optimal core allocations. Our results suggest that in large markets the core tends to be almost one-dimensional in the sense that one parameter defines it up to very small perturbations. In balanced markets, once we know what is the average profits of firms, we also approximately know the average salaries of workers, and what every firm and worker makes under that core allocation. The same holds for unbalanced markets. However, as we will see in the next section, workers' salaries in unbalanced markets are in fact determined by the human capital levels of those workers who are left unmatched, and therefore the core actually has no real variation and resembles a point more than a line. An interesting exercise would be to calculate the elongation measures suggested by Quint (1987) for large unbalanced markets and show that they indeed converge to zero.

4.2 Surplus distribution

With the results from the previous subsection at hand, we are now ready to explore their implications for surplus distribution. However, before doing so it is important to understand how much surplus is created in a large market. Aldous (2001) proved that in a large balanced random market with all firms having a quality of zero, and all workers having a human capital level of zero, and noise being distributed according to the uniform distribution on $[0,1]$, the expected surplus created is $n - \frac{\pi^2}{6}$. This result can be easily extended both to general bounded distributions (with positive and continuous density) and to unbalanced markets, and in general we know that the surplus to be divided between firms and workers is $\Omega(n)$. As for qualities and human capital, our analysis suggests that with high probability the workers who will take part in the optimal assignments are all those above a certain human capital level (see Lemma 16 in Appendix A), and so we can tell from the distribution of qualities and human capital levels what is going to be the surplus created due to those factors.

Our main result in this subsection is that when the market is exactly balanced (i.e., $k(n) \equiv 0$) the surplus that is created from the idiosyncractic matching between firms and workers can be divided in very different ways. However, in the presence of even a slight imbalance, most of the surplus related to the noise goes to the short side (the firms). This indicates that a large core is a knife-edge case that is not likely to be found in any real applications. This result is the assignment games parallel to Ashlagi et al. (2013), who prove that in the realm of matching without transfers a large core is only possible if the number of men and women is exactly equal, and that in unbalanced markets the short side has a big advantage in determining the resulting matching.

Corollary 3. *In the separable case with bounded noise, let* $k(n) \equiv 0$ *and let* $\left(\mu^n, u^{n,F}, v^{n,F}\right)$ *be the firm-optimal core allocation. Then there exist* $c \in \mathbb{R}_+$

such that whp

$$\forall w_j^n \in W^n : v_j^{n,F} \in \left(\left(h_j^n - \underline{h}\right) - \frac{c \log n}{n}, \left(h_j^n - \underline{h}\right) + \frac{c \log n}{n} \right).$$

Intuition for the proof. Under the firm-optimal core allocation there is at least one worker who gets a salary of exactly zero; otherwise we could reduce all salaries by a small constant without violating any of the inequalities defining the core. This worker's human capital level cannot be too high (otherwise, by the approximate law of one price, others with lower human capital levels would get negative salaries). Then, by the approximate law of one price, all workers must get only the difference between their human capital level and that worker's human capital level. For the full proof see Appendix B. □

A similar argument to the one we used for balanced markets can be applied to unbalanced markets. In this case, a worker who is left unmatched gets a salary of zero, and this constrains at least some of the salaries of the workers who are matched. Then, by the approximate law of one price, we get bounds on the salaries of all workers.

Corollary 4. *In the separable case with bounded noise, let* $k(n) > 0$ *for all* n*. Then there exist* $c \in \mathbb{R}_+$ *such that whp for all* $(\mu^n, u^n, v^n) \in C(M^n)$ *and for all* $w_j^n \in W^n$ *such that* $\mu(w_j^n) \in F^n$*,*

$$v_j^n \in \left(\left(h_j^n - h^n[n]\right) - \frac{c \log n}{n}, \left(h_j^n - h^n[n]\right) + \frac{c \log n}{n} \right),$$

where $h^n[n]$ *signifies the* n*-th highest element in the vector* h^n*.*

Proof. See Appendix B. □

Corollary 3 implies in particular that in a balanced market the expected division of surplus is such that the workers get the contribution of their excess human capital (above $\underline{h}$) and then only $O\left(\frac{\log n}{n}\right)$ out of the part of the surplus that is related to the noise distribution. Note that while Corollary 3 is put in terms of the firm-optimal core allocation, it is completely symmetric, and therefore the same applies to the opposite case of the worker-optimal core allocation. The convexity property of the core ensures that any compromise distribution is also possible in a core allocation. Unlike the long (and narrow) core characterization in balanced markets, Corollary 4 shows that in unbalanced markets the core quickly converges to almost a point. The resulting surplus division is such that under any core allocation, the agents on the long side (the workers) get the contribution of their excess quality (not above the lower bound of the distribution, but rather above the highest quality of an unmatched agent) plus a $O\left(\frac{\log n}{n}\right)$ fraction of the surplus created by the idiosyncratic matching.

5 Extension to Cobb–Douglas productivities

In the previous section we showed that an approximate law of one price holds for markets in which both firms' quality and workers' human capital affect the productivity of each matched pair, but we did not allow for any interaction between those two properties. In other words, good workers provided the same output regardless of whether they were working in a good firm or in a bad firm. While mathematically convenient, it is not a very plausible assumption. In this section we wish to relax our previous separability assumption and consider also the family of productivity functions suggested by Cobb and Douglas (1928).

Our main concern when considering interaction is that workers and firms will tend to ignore their idiosyncratic productivity noise and will match solely on the basis of their respective qualities. This is known in the economics literature as "assortative matching," and within the matching literature it is most identified with the work of Becker (1981). If firms and workers match assortatively, there will not be any chance of having an approximate version of the law of one price, since the idiosyncratic productivities can tilt the profits of matching pairs.

We find that as the market grows large (and under certain technical assumptions on the qualities of firms and workers), there is a trade-off between matching assortatively on the quality dimension and matching efficiently on the noise dimension. We define the concept of "approximately assortative matching," which means that all firms are matched to workers who have approximately the same level of human capital as the firms' quality. The fact that the matching is only *approximately* assortative and not completely assortative allows for more efficient matching in terms of idiosyncratic noise.

Definition 5. *A model exhibits approximately assortative matching if there exist* $c \in \mathbb{R}_+$ *and* $a \in (0,1)$ *such that whp for any* $(\mu^n, u^n, v^n) \in C(M^n)$ *and for any* i, j *such that* $\mu^n(f_i^n) = w_j^n$ *we have* $\left| q_i^n - h_j^n \right| \leq cn^{-a}$.

We now turn to a specific model, which we refer to as *the Cobb–Douglas benchmark model*. The Cobb–Douglas benchmark model consists of a balanced market ($k(n) \equiv 0$) in which productivities are given by $\alpha_{ij}^n = 2\sqrt{q_i^n h_j^n} + \varepsilon_{ij}^n$, and $q_k^n = h_k^n = \frac{k}{n}$, i.e., qualities of firms and human capital levels of workers are evenly spaced.

Lemma 6. *The Cobb–Douglas benchmark model exhibits approximately assortative matching.*

Proof. See Appendix C. □

Having established an approximately assortative matching, we can prove the approximate law of one price using the tools developed for the separable case, but not quite the same ones since we need to make sure that we limit the paths used in those proofs so that they do not go through firms or workers that have very different qualities. Even then a direct comparison between firms or between workers of different qualities is not straightforward, and so we restate our main result in terms of agents that have similar qualities.

Theorem 7. *In the Cobb–Douglas benchmark model there exist* $c_1, c_2 \in \mathbb{R}_+$ *and* $a, b \in (0,1)$ *such that whp for any* $(\mu^n, u^n, v^n) \in C(M^n)$*:*

- $\forall i, j \in \{1, \ldots, n\}$ *such that* $\left| q_i^n - q_j^n \right| \leq c_1 n^{-b}$: $u_i^n - u_j^n \leq c_2 n^{-a}$, *and*
- $\forall i, j \in \{1, \ldots, n\}$ *such that* $\left| h_i^n - h_j^n \right| \leq c_1 n^{-b}$: $v_i^n - v_j^n \leq c_2 n^{-a}$.

Proof. See Appendix C. □

5.1 Surplus distribution

Still focusing on the Cobb–Douglas benchmark model, it is quite clear that while the analysis of surplus distribution is not as straightforward as the separable case, it is still not much different. The rough intuition for the next result is that we can compare the salary of any worker with that of a worker who has a slightly lower or slightly higher human capital level, if both workers have a relatively high joint productivity with the firm that employs one of them. This allows us to build paths from any worker to one of the workers with the lowest human capital levels and deduct that the former can only make a salary that is the sum of the differences between productivities of workers along the path. In other words, the salary of a worker with human capital level h_j^n is roughly the integral from 0 to h_j^n of the marginal productivities of workers. Since we know that there is approximately assortative matching, we also know the quality of firms matched to workers along the path.

Corollary 8. *In the Cobb–Douglas benchmark model let* $\left(\mu^n, u^{n,F}, v^{n,F}\right)$ *be the firm-optimal core allocation. Then there exist* $c, a \in \mathbb{R}_+$ *such that*

$$\forall j \in \{1, \ldots, n\} : v_j^{n,F} \in \left(\frac{j}{n} - cn^{-a}, \frac{j}{n} + cn^{-a}\right).$$

Proof. Omitted. □

We note that the surplus created by worker w_j^n is approximately $\frac{2j}{n} + 1$, and so we learn that the workers get only the share of the surplus related to their own contribution to the correlated component, and none of the surplus related to the idiosyncratic component under the firm-optimal core allocation.

We conclude this section by noting that none of the technical steps we took seem to require balancedness. We therefore conjecture that in unbalanced markets any worker's salary under any core allocation will be bounded above by the integral of the marginal productivity from the highest human capital level of any unemployed worker to her own human capital level, plus an expression that behaves like $O\left(\frac{1}{n^a}\right)$ for some $a \in (0, 1)$. Simulation results presented in Section 7 also indicate that this conjecture holds.

6 Unbounded noise

Up until now we have established that an approximate version of the law of one price holds in two-sided economies with heterogeneous preferences. However, one of the more restrictive assumptions that we used was the boundedness of the noise distribution, which obviously leads to a relatively high concentration of "good enough" matches, and in particular allows an assignment so efficient that it misses a potential first-best only by a constant (Aldous, 2001). In this subsection we relax this assumption for the first time and try to understand what happens when the noise is unbounded. Apart from the mathematical elegance and conceptual difference of unbounded noise, understanding the implications of this concept is also important for comparing our work with some of the empirical papers on two-sided matching markets with transfers, which are based on models with unbounded noise (e.g., Choo and Siow, 2006).

When we discuss unbounded noise it is important to understand what it means to have "one price" in the market, since the average productivity may tend to infinity as the market grows large. Our interpretation is that an approximate law of one price holds if the variation among agents' profits is a vanishing fraction of the average productivity. In the bounded case, the average productivity approaches a constant, and therefore any sub-constant differences in profits are considered as an approximate law of one price. In what follows we focus on the exponential distribution, under which the average productivity behaves like $\log n$, and we show that whp there are two workers in the market whose salaries differ by $\Theta(\log n)$. Hence, we conclude that in the presence of unbounded noise the law of one price might not hold.

The intuition for our "counterexample" is that unbounded distributions with a heavy tail may create "good" outliers, i.e., agents that are highly productive compared to others, and such that agents from the other side fiercely compete to be matched with them. These agents share a significant portion of the surplus they help to create, and if they are common enough, they may offset other forces that would otherwise squeeze the surplus from their side (such as an adversarial core allocation, or a slight imbalance in favor of the other side of the market). Our example is based precisely on the existence of such agents.

Proposition 9. *In the separable case with exponential noise, let the market be balanced ($k(n) \equiv 0$), with all firms having the same qualities ($q^n \equiv \underline{0}$), and with all workers having the same human capital level ($h^n \equiv \underline{0}$). Let $\left(\mu^{n,F}, u^{n,F}, v^{n,F}\right)$ denote the firm-optimal core allocation of M^n. Then average productivity is $\Theta(\log n)$, and there exists $c \in \mathbb{R}_+$ such that whp there are two workers w^n_i and w^n_j with $\left| v^{n,F}_i - v^{n,F}_j \right| > c \log n$.*

Proof. See Appendix B. □

6.1 Surplus distribution under exponential noise

Despite the fact that the law of one price does not apply in general to unbounded noise, we would like to argue that at least some of the main conclusions, i.e., the convergence of the share of the surplus that each side gets, continues to hold to some extent. By studying simulation data carefully (see Figure 11 in Section 7), one suspects that the behavior of the workers' expected share of the surplus in a balanced market under the firm-optimal core allocation is $\Theta\left(\frac{\log\log n}{\log n}\right)$. In what follows we assume the following mathematical conjecture is true, and show that indeed the share of the surplus behaves in that manner.

Conjecture 10. *In the separable case with exponential noise, let* $k(n) \equiv 0$*,* $q^n \equiv \underline{0}$ *and* $h^n \equiv \underline{0}$*. Then there exists* $c \in \mathbb{R}_+$ *such that whp under the maximal assignment each firm is matched to one of the* $c \log n$ *workers who have the highest joint productivity with that firm.*

We note that Conjecture 10 parallels Theorem 2 of Frieze and Sorkin (2007), in the sense that it bounds the lowest possible element in the optimal assignment. While computer simulations suggest that it holds (see Section 7), we are not familiar with any work within the computer science literature or the operations research literature that tackles the problem of unbounded distributions.[8]

Theorem 11. *In the separable case with exponential noise, let* $k(n) \equiv 0$*,* $q^n \equiv \underline{0}$*, and* $h^n \equiv \underline{0}$*. Assume Conjecture 10 holds, and let* $\psi^F(M^n) = \left(\mu^n, u^{n,F}, v^{n,F}\right)$ *be the firm-optimal core allocation. Then there exists* $c \in \mathbb{R}_+$ *such that*

$$E\left[\frac{\sum_j v_j^{n,F}}{\sum_i u_i^{n,F} + \sum_j v_j^{n,F}}\right] \leq \frac{c \log\log n}{\log n}$$

Intuition for the proof. In a balanced market governed by the firm-optimal core allocation, a worker cannot make more than the value she creates together with the firm that employs her minus the lowest value that any other worker creates (Lemma 20). Given the assumption and the above claim, it remains to show that with high probability the lowest value created by any

[8] Those two literatures focus on minimizing the sum of costs, and not maximizing productivity, and therefore unbounded distributions are less intuitive.

worker behaves like $\log n - c \log\log n$ (Lemma 21). The full proof appears in Appendix B. □

It is worth mentioning that by observing simulation results for unbalanced markets (Figure 11), one may arrive at the following conjecture.

Conjecture 12. *In the separable case with exponential noise, let* $k(n) > 0$*,* $q^n \equiv \underline{0}$*, and* $h^n \equiv \underline{0}$*, and let* $\psi^W(M^n) = \left(\mu^n, u^{n,W}, v^{n,W}\right)$ *be the worker-optimal core allocation. Then there exists* $c \in \mathbb{R}_+$ *such that*

$$E\left[\frac{\sum_j v_j^{n,W}}{\sum_i u_i^{n,W} + \sum_j v_j^{n,W}}\right] \leq \frac{c \log\log n}{\log n}.$$

In particular, this implies that for any core mechanism (that is, any function from markets to core allocations) the expected surplus of the workers is $O\left(\frac{\log\log n}{\log n}\right)$*.*

We conclude this section by suggesting that although we were focused on the study of the exponential distribution, much can be inferred about other unbounded distributions. Proposition 9 provided a counterexample to a theorem that held for the bounded case. The conjectures we discussed in this subsection were strictly about the exponential distribution, but it is our belief that other distributions that have similar tail behavior will exhibit the same phenomena (see also Figure 12 and Figure 13 in Section 7).

7 Simulations

In this section we present results of computerized simulations that demonstrate how quickly the dispersion of payoffs contracts, and how this affects the market. Unless explicitly noted, figures are based on averaging 400 trials for each market size, where the size of balanced markets ranges from $(10, 10)$ to $(300, 300)$ with jumps of 5 agents on each side, and the size of unbalanced markets ranges from $(5, 6)$ to $(300, 301)$ with jumps of 5 agents on each side.

7.1 The separable case with bounded noise

We first focus on the benchmark case of uniform $[0, 1]$ distribution with all firms having the same quality ($q^n \equiv \underline{0}$) and all workers having the same human capital level ($h^n \equiv \underline{0}$), and study wage dispersion in balanced markets under the firm-optimal core allocation. Figure 1 shows that indeed in a balanced market the maximal difference between the profits of any two firms in any core allocation behaves like $\frac{\log n}{n}$, as proved by Theorem 1 and Theorem 2.[9]

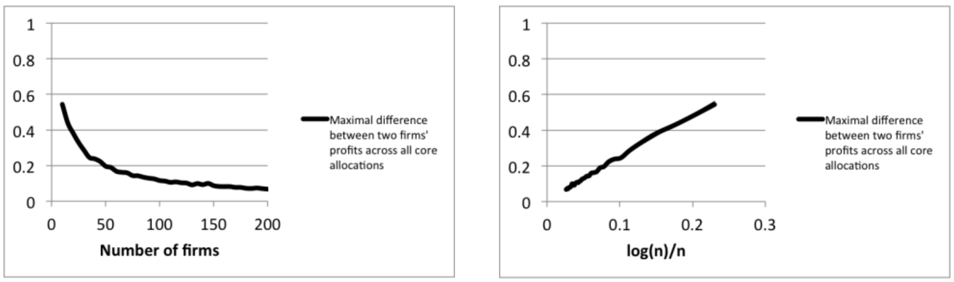

Figure 1: Approximate law of one price in balanced markets

The left panel of Figure 2 shows that in this case the maximum salary any worker gets under the firm-optimal core allocation also behaves like $\frac{\log n}{n}$, and the right panel of the same figure exemplifies the fact that the core in balanced markets is long, as suggested by Corollary 3.

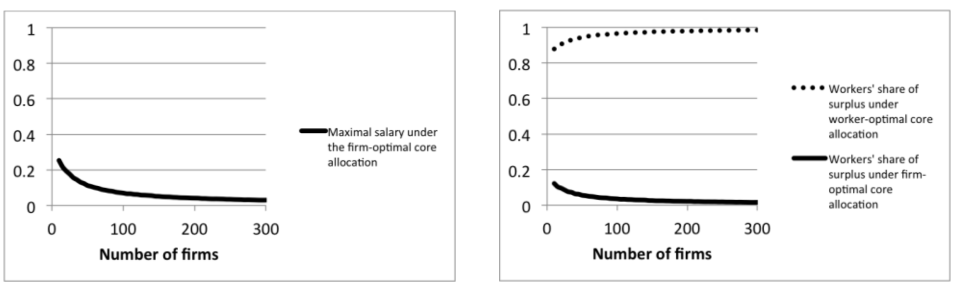

Figure 2: Surplus distribution in balanced markets

In unbalanced markets we expect the core to be much more narrow, per Corollary 4. The left panel of Figure 3 shows that even when the number of workers is only one more than the number of firms, the maximal salary any worker gets approaches zero rapidly, even under the worker-optimal core allocation. Furthermore, as the right panel demonstrates, in this case the

[9]Figure 1 is based on only 25 trials for every market size, since finding the maximal difference across all core allocations requires solving $n(n-1)$ linear-programming problems.

workers' share in the surplus approaches 0, even under the worker-optimal core allocation. Figure 4 parallels Figure 4 of Ashlagi et al. (2013), and depicts the workers' share of the surplus when the number of workers is constant at 50, and the number of firms varies from 20 to 80.

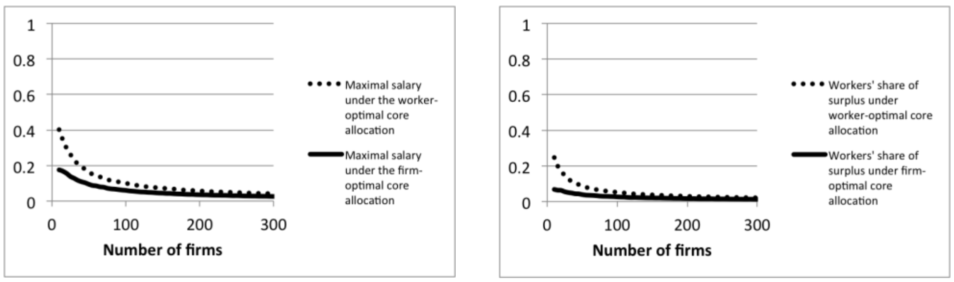

Figure 3: Surplus distribution in unbalanced markets

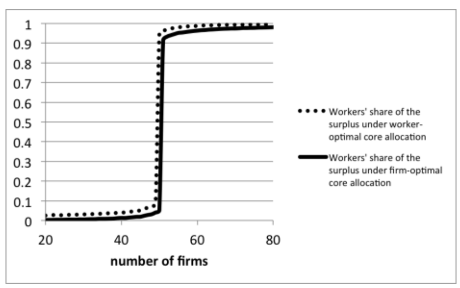

Figure 4: Surplus distribution with 50 workers

We now wish to verify that adding qualities to the mix does not substantially change any of these results. We let $q_i^n \sim U[0,1]$ for every i, and $h_j^n \sim U[0,1]$ for every j. In a balanced market we expect each worker to get roughly her human capital level, and for all workers to take 25% of the surplus. Under the worker-optimal core allocation we expect workers to take about 75% of the surplus. This is indeed shown in Figure 5. In an even slightly unbalanced market, we expect each worker to get her human capital level under any core allocation, and for the whole population of workers to take 25% of the surplus. This is demonstrated in Figure 6.

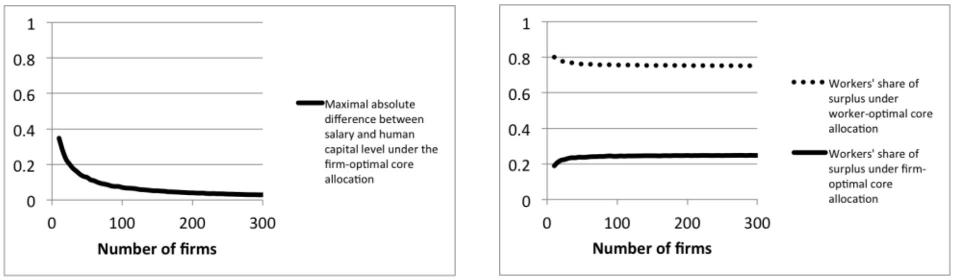

Figure 5: Surplus distribution in balanced markets with qualities

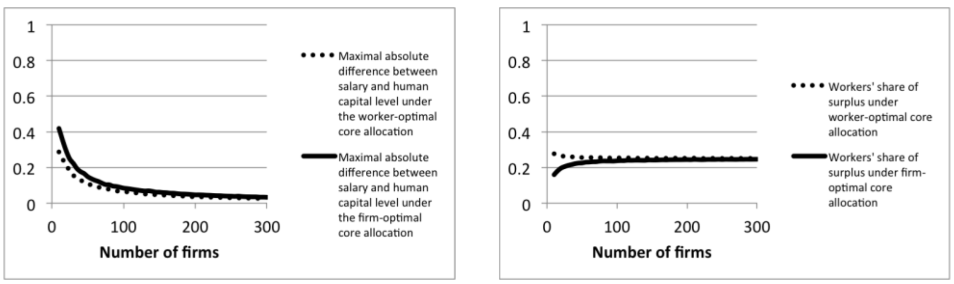

Figure 6: Surplus distribution in unbalanced markets with qualities

7.2 Cobb–Douglas productivity with bounded noise

We first try to demonstrate that assortative matching takes place in the model mentioned in Appendix C; i.e., each side of the market is characterized by evenly spaced qualities on the interval $[0, 1]$, and the idiosyncratic noise is distributed according to $U[0, 1]$. In Section 5 we proved just one aspect of assortative matching, namely, whp no firm is matched to a worker whose human capital level is substantially different from the firm's own quality (Lemma 6). The left panel of Figure 7 depicts the average and the maximal absolute quality difference between firms and the workers they employ under the optimal assignment. It is easy to see that these differences shrink as market size grows, and by looking at the logarithms of both axes (right panel) we can see that indeed these differences behave like a negative power of n.

The surplus distribution described in Corollary 8 is depicted on the left panel of Figure 8. This panel shows the average absolute difference between workers' salaries and workers' human capital levels under the firm-optimal and the worker-optimal core allocations. The right panel supports the con-

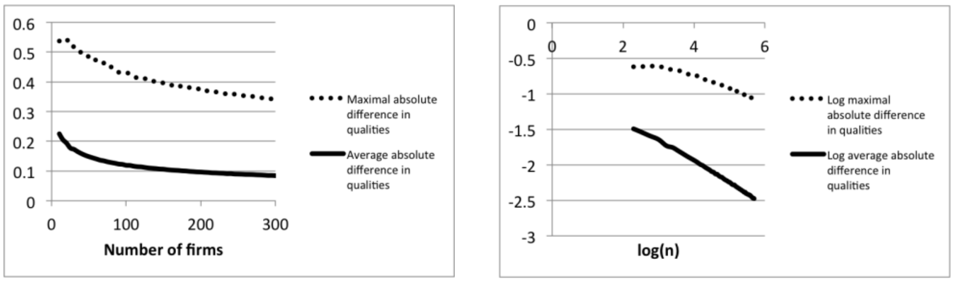

Figure 7: Assortative matching when production factors are complements

jecture we raised at the end of Section 5 by showing the same metric in unbalanced markets for both the firm-optimal and the worker-optimal core allocations.

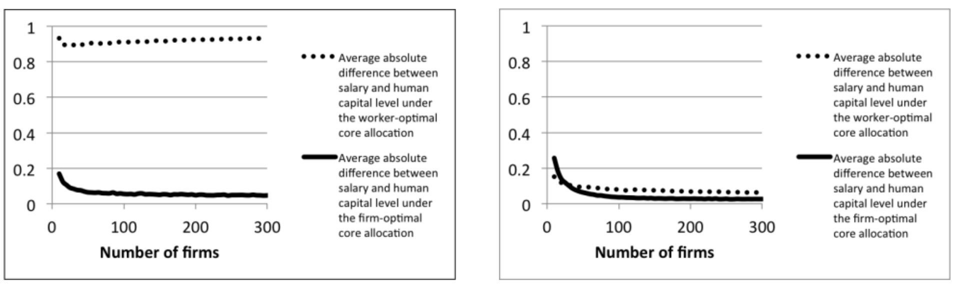

Figure 8: Surplus distribution when production factors are complements

7.3 Unbounded distributions

As mentioned in Section 6, unbounded noise distributions give rise to quite different phenomena than those mentioned with respect to bounded distributions. Figure 9 depicts the maximal difference between any two workers' salaries divided by the average surplus created under the optimal assignment, in a balanced market with exponential noise governed by the firm-optimal core allocation. As predicted by Proposition 9, the difference does not vanish as n gets large.

In Section 6 we also mentioned a conjecture about the behavior of the optimal assignment under the exponential distribution (Conjecture 10). Figure 10 shows that indeed it holds for medium-sized markets. The left panel of Figure 11 exemplifies how this conjecture translates into the conclusion of

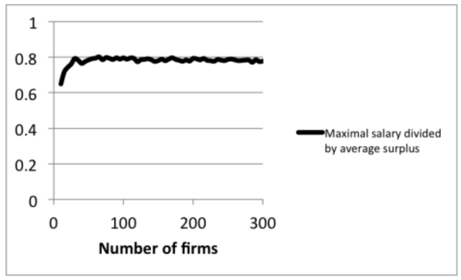

Figure 9: No law of one price under Exponential distribution

Theorem 11, and the right panel of that figure suggests that Conjecture 12 is true.

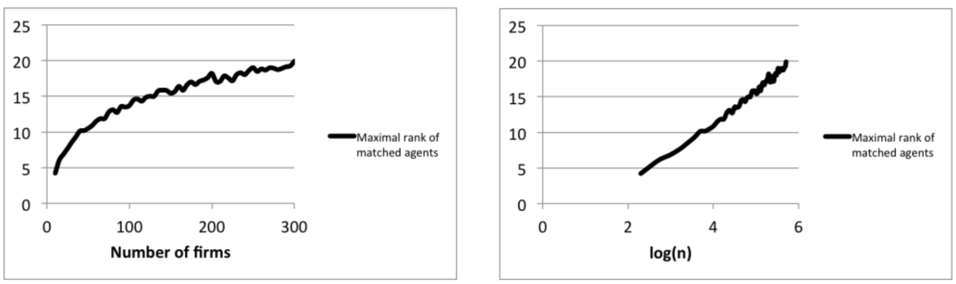

Figure 10: Maximal rank of matched agents under exponential distribution

We conclude this subsection by noting that while our discussion was mostly about the exponential distribution, there are many other distributions that have similar tail behavior, and therefore are likely to exhibit the same phenomena. In particular, the extreme value distribution used in some empirical papers seems to have similar effects. Figure 12 parallels Figure 10 and shows the maximal rank of any two matched agents in a balanced market with noise distributed according to an extreme value distribution, and Figure 13 shows surplus distribution for both balanced and slightly unbalanced markets.

8 Conclusion

During the 1980s, as it became clear that real-life centralized clearing houses could be immensely improved using intuitions gained in the study of mar-

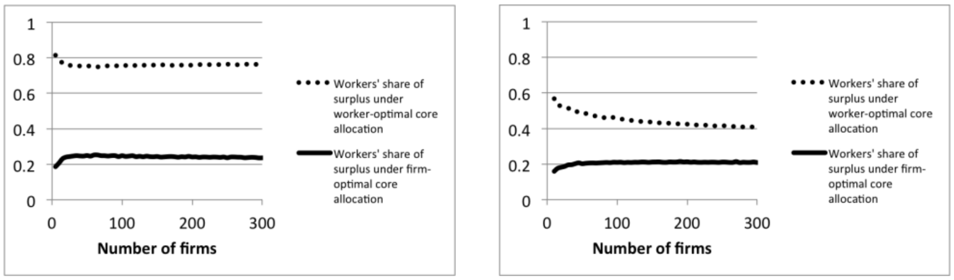

Figure 11: Surplus distribution under exponential distribution (balanced and unbalanced)

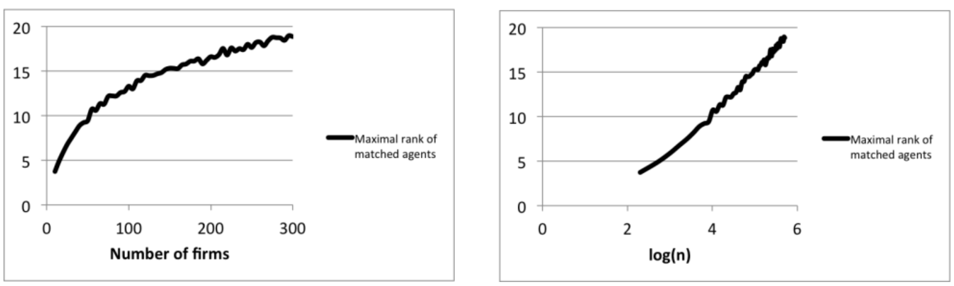

Figure 12: Maximal rank of matched agents under extreme value distribution

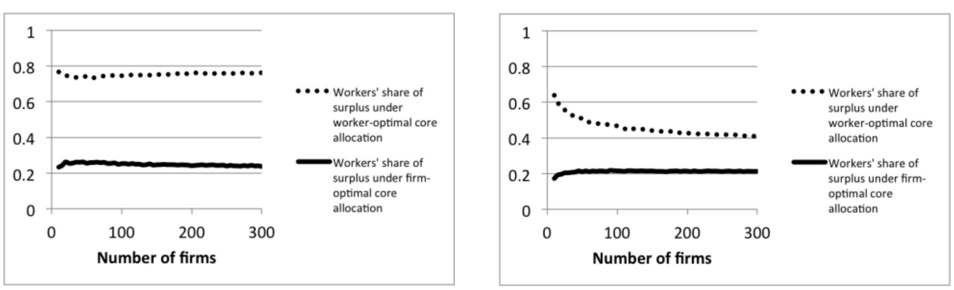

Figure 13: Surplus distribution under extreme value distribution (balanced and unbalanced)

riage markets, the transferable utility strand of the literature became slightly neglected compared to its glorified non-transferable utility half-sibling. We decided to focus our attention in this paper on assignment games because it is our belief that they provide an excellent way to model decentralized markets, and that both strands of the matching theory literature can benefit from the continuous cross-fertilization.

We have investigated the applicability of the law of one price in two-sided matching markets with transfers, when agents have heterogeneous preferences over matching with the other side of the market. We have shown that an approximate law of one price holds, and that it implies core convergence and sharp predictions about surplus distribution in unbalanced markets. We have explained why the same kind of forces continue to work in markets in which there is interaction between the production factors, and why they fail to hold in markets in which the idiosyncratic noise is unbounded. These results indicate that only in knife-edge cases, in which the markets are exactly balanced, can we expect to see any significant variation in core outcomes.

We conclude the paper by noting that many of our assumptions were for expositional clarity only. The fact that firms had unit demand and workers supplied one unit of work is of course not crucial to our results, nor is the fact that all agents can possibly work in all the firms. The same results will hold in markets with discrete and finite demand and supply, and in markets that are less thick (at least to some extent). Nevertheless, some of the assumptions were crucial, and weakening them could lead to further understanding of markets with heterogeneous preferences. Specifically, the mechanism through which markets with unbounded noise converge remains a mystery, and the extent to which these results hold for markets with general utility functions (not quasi-linear) can be further studied. Finally, generalizing our results and the results of Ashlagi et al. (2013) to markets with substitutable preferences (with or without transferable utility) is another very promising direction for future research.

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A Proof of Theorem 1

Proof. For simplicity, the proof uses results that were proven for the uniform noise distribution $G = U[0, 1]$. However, all claims hold for more general distributions. A complete proof for general distributions (albeit one that provides slightly less tight bounds and only deals with balanced markets) can be found in the working paper version of the present work (Hassidim and Romm, 2014).

The general structure of the proof is as follows.

1. Given an arbitrary vector of workers' human capital, show that whp (relevant to the distribution of $\{\varepsilon_{ij}^n\}$) there are only finitely many workers above a certain human capital level who are unemployed, and similarly finitely many workers below a different human capital level who are employed (Lemma 13).

2. Based on the previous step, show that a version of the result of Frieze and Sorkin (2007) holds, but with some restrictions on its applicability to workers (Lemma 14).

3. Show that in fact whp all workers above a certain human capital level are employed, and all workers below a certain human capital level are not employed (Lemma 15).

4. Improve the applicability of Lemma 14 to workers (Lemma 16).

5. Put everything together with the intuition presented in the main text to complete the proof.

For a given a vector of human capital levels h^n (of length $n + k(n)$), let us denote by $h^n[m]$ the m-th highest value.

Lemma 13. *For any $E > 0$ there exist $M \in \mathbb{N}$ such that*

1. *whp there are at most M workers with a human capital level greater than $h^n[n] + E$ who are unemployed under the optimal assignment for M^n;*
2. *whp there at most M workers with a human capital level less than $h^n[n] - E$ who are employed under the optimal assignment for M^n.*

Proof. Denote by V^n_{opt} the value resulting from the optimal assignment in M^n, and by V^n_{bound} the value resulting from optimally assigning the top n workers (in terms of human capital level) to the n available firms. From Aldous (2001) we know that

$$\lim_{n\to\infty} E[V^n_{\text{bound}}] = \sum_{i=1}^{n} q^n_i + \sum_{j=1}^{n} h^n[j] + \left(n - \frac{\pi^2}{6}\right). \tag{1}$$

Taking q^n and h^n as given, we know from Wästlund (2005) that

$$\text{Var}[V^n_{\text{bound}}] = \frac{4\xi(2) - 4\xi(3)}{n} + O\left(\frac{1}{n^2}\right) \approx \frac{1.7715}{n} + O\left(\frac{1}{n^2}\right). \tag{2}$$

By approximating the limit in (1), bounding the variance in (2), and using Markov inequality:

$$\Pr\left(V^n_{\text{bound}} \leq \sum_{i=1}^{n} q^n_i + \sum_{j=1}^{n} h^n[j] + (n-2)\right) \leq \frac{13.6}{n}.$$

This also implies that whp

$$V^n_{\text{opt}} \geq \sum_{i=1}^{n} q^n_i + \sum_{j=1}^{n} h^n[j] + (n-2).$$

Now assume that there are M workers with a human capital level greater than or equal to $h^n[n] + E$ who do not participate in the optimal assignment (or alternatively that there are M workers with a human capital level less than or equal to $h^n[n] - E$ who do participate in the optimal assignment). It must be that

$$V^n_{\text{opt}} \le \sum_{i=1}^{n} q_i^n + \sum_{j=1}^{n} h^n[j] - ME + n,$$

and therefore

$$M \le \frac{2}{E}.$$

■

Now, given some arbitrary matchings $\{\mu^n\}$, construct digraphs $G^n = (V^n, E^n)$, with $V^n = F^n \cup W^n$ and

$$\begin{aligned} E^n = & \left\{(w_j^n, f_i^n) \mid \mu^n(f_i^n) = w_j^n\right\} \\ & \cup \left\{(f_i^n, w_j^n) \mid w_j^n \in N^n_{h^n[n]+E,40+M}(f_i^n)\right\} \\ & \cup \left\{(f_i^n, w_j^n) \mid f_i^n \in N^n_{h^n[n]+E,40+M}(w_j^n)\right\}, \end{aligned}$$

where $N^n_{x,k}(f_i^n)$ represent the top k workers in terms of idiosyncratic fit to f_i^n (i.e., ε_{ij}) out of those workers who have a human capital level above x, and similarly for $N^n_{x,k}(w_j^n)$. We call the edges from F^n to W^n "forward edges" and the edges from W^n to F^n "backward edges." The weight on each forward edge (f_i^n, w_j^n) is ε^n_{ij} (and not α^n_{ij}).

Lemma 14. *If $\underline{h} \neq \bar{h}$,*[10] *there exists $c \in \mathbb{R}_+$ such that whp there is an alter- nating path between every two firms with the sum of weights on the forward edges being less than or equal to $\frac{c \log n}{n}$. Similarly, there is an alternating path from any matched worker to any worker with a human capital level above $h^n[n] + E$ with the sum of weights on the forward edges being less than or equal to $\frac{c \log n}{n}$.*

[10] This lemma also holds (with the proper adjustments) for the case where all workers have the same human capital level, but we omit the proof here since it can easily be recovered using the arguments presented in the more complicated case.

Proof. First, let us choose $E > 0$ such that whp the number of workers with a human capital level above $h^n[n] + E$ is no less than $0.99n$. To see that this is possible, let us denote by ν^n the fraction of workers who are unassigned in M^n, i.e., $\nu^n := \frac{k(n)}{n+k(n)}$, and let $\eta^n = H^{-1}(\nu^n)$. Let $E > 0$ be such that $\sup_{(x,y)\subseteq(\underline{h},\bar{h}),(y-x)<E} H(y) - H(y) < 0.0049$ (this is possible since we required the density to be continuous on $[\underline{h}, \bar{h}]$, and it is therefore bounded). By Hoeffding's inequality whp $h^n[n] \in (\eta^n - E, \eta^n + E)$. Then, using Hoeffding's inequality again, we know that whp $0.99n$ of the workers have a human capital level above $\eta^n + 2E \geq h^n[n] + E$.

Note that whp there exists c_1 such that there is a directed path of length less than $c_1 \log n$ between any two firms, using the same argument as Frieze and Sorkin (2007, Lemma 5). It is true that in our case some of the workers do not have related backward edges (since they are unmatched), but out of those workers who are connected to forward edges (with a human capital level above $\bar{h} - E$) at most M do not have backward edges. Therefore, by pointing to $M + 40$ workers we keep the expansion rate of at least 40. We also note that some of the constants have to be changed to account for the fact that only a constant fraction of the workers are connected by forward edges, and that the number of workers is not necessarily n but could rather be greater than that as long as it is $O(n)$. We remark that E must have been chosen such that a large majority of the firms will be matched to workers with human capital levels above $h^n[n] + E$; otherwise there would not necessarily be an overlap between the two "funnels" constructed in the proof.

We then use Lemma 7 of Frieze and Sorkin (2007) which works as is, except that the number 40 is replaced by $40 + M$ whenever it appears in the proof there. This completes the argument for the firms.

As for the workers, the same argument works, but we note that in order for a directed path to start from some worker, that worker must be matched, and in order for it to finish with some worker, that worker must have a human capital level above $h^n[n] + E$. ∎

Lemma 15. *If $\underline{h} \neq \bar{h}$, then there exist $c_1 \in \mathbb{R}_+$ such that*

1. *whp all workers with a human capital level greater than $h^n[n] + \frac{c_1 \log n}{n}$ are assigned under the optimal assignment for M^n;*
2. *whp no workers with a human capital level less than $h^n[n] - \frac{c_1 \log n}{n}$ are assigned under the optimal assignment for M^n.*

Proof. Let $c_1 \in \mathbb{R}_+$ be equal to $(c+2)$, where c is the constant recovered in the proof of Lemma 14. Assume on the contrary that there exists an unmatched worker w_1^n with human capital level $h_1^n > h^n[n] + \frac{c_1 \log n}{n}$. Let w_2^n be the worker with the lowest level of human capital in M^n that is matched. We want to argue that there exists a matching in which the set of matched workers is $\mu^n(F^n) \cup \{w_1^n\} \setminus \{w_2^n\}$ and that this matching gives a larger value. Replace the matching μ^n with the one in which $\mu^n(w_2^n)$ is matched with w_1^n. Note that this matching gives a value greater by $(h_1^n - h_2^n) \geq \frac{(c+2)\log n}{n}$ in human capital, but might provide us with less than optimal noise compatibility between $\mu^n(w_2^n)$ and w_1^n. Applying Lemma 14 to our new matching, find a directed path between w_1^n (which is now matched) and some worker who is also matched and who "likes" $\mu^n(w_2^n)$ (in the sense of having joint productivity greater than $1 - \frac{\log n}{n}$). Apply the directed path, in the sense that now each worker is going to be matched to the firm connected to her by a forward edge, and the last worker is connected to $\mu^n(w_2^n)$. The value of the resulting matching is at least $\text{val}(\mu^n) + \frac{(c+2)\log n}{n} - \frac{(c+1)\log n}{n} > \text{val}(\mu^n) + \frac{\log n}{n}$, a contradiction.

The exact same reasoning applies when a matched worker has a human capital level below $h^n[n] - \frac{c_1 \log n}{n}$, and is replaced by the best unmatched worker. •

Lemma 16. *If $\underline{h} \neq \bar{h}$, there exist $c, c_1 \in \mathbb{R}_+$ such that whp there is an alternating path from any matched worker to any worker with a human capital level above $h^n[n] + \frac{c_1 \log n}{n}$ with the sum of weights on the forward edges being less than or equal to $\frac{c \log n}{n}$.*

Proof. Use the same logic of Lemma 14 but replace E with $\frac{c_1 \log n}{n}$, which will work by virtue of Lemma 15. •

To complete the proof, let us first consider the firms. By Lemma 14 whp for every $i, j \in \{1, \ldots, |F^n|\}$ there exists an alternating path on G^n (induced by μ^n, the optimal assignment for M^n). Suppose one such path is $(f_i^n, w_1^n, f_1^n, w_2^n, f_2^n, \ldots, w_k^n, f_j^n)$. Since μ^n is a core allocation, it must be that $u_i^n + v_1^n \geq \alpha_{i1}$, and therefore

$$u_i^n \geq \alpha_{i1} - v_1^n \geq q_i^n + h_1^n + (1 - \varepsilon_{i1}) - (\alpha_{11} - u_1^n) \geq u_1^n + (q_i^n - q_1^n) - \varepsilon_{i1}.$$

Similarly we get

$$
\begin{aligned}
u_i^n &\geq u_1^n + (q_i^n - q_1^n) - \varepsilon_{i1}^n, \\
u_1^n &\geq u_2^n + (q_1^n - q_2^n) - \varepsilon_{12}^n, \\
&\dots \\
u_k^n &\geq u_j^n + \left(q_k^n - q_j^n\right) - \varepsilon_{kj}^n.
\end{aligned}
$$

Stacking all of those together we have

$$u_i^n \geq u_j^n + \left(q_i^n - q_j^n\right) - \sum \varepsilon_{xy}^n,$$

where the last sum goes over all the firms that alternate on the path, and therefore

$$u_i^n \geq u_j^n + \left(q_i^n - q_j^n\right) - \frac{c \log n}{n}.$$

Reordering terms we get

$$u_j^n - u_i^n \leq \left(q_j^n - q_i^n\right) + \frac{c \log n}{n},$$

which is exactly what we wanted.

As for the workers, we need to be slightly more careful. The same line of reasoning tells us that whp for any matched worker w_i^n and any worker w_j^n with a human capital level above $h^n[n] + \frac{c_1 \log n}{n}$ (as in Lemma 16) we have

$$v_i^n - v_j^n \leq \left(h_i^n - h_j^n\right) + \frac{c \log n}{n}.$$

However, we also want to account for matched workers with a human capital level in the interval $\left(h^n[n] - \frac{c_1 \log n}{n}, h^n[n] + \frac{c_1 \log n}{n}\right)$. Let w_i^n be some matched worker and let w_j^n be a matched worker in that interval. Since whp there are $\Theta(n)$ workers with human capital levels above $h^n[n] + E$ (for any constant E), then whp one of them, say w_k^n, is a good match for $\mu^n(w_j^n)$ in the sense that their joint idiosyncratic noise is above $1 - \frac{c_2 \log n}{n}$ for some constant c_2. Consider how a path that goes from w_i^n to w_k^n (whp such a path exists) and then continues to $\mu^n(w_j^n)$ and to w_j^n, and perform the same calculation as before. □

B Other proofs

B.1 Proof of Theorem 2

Lemma 17. *Let $Z = \sum_{k=1}^{n} X_k$ where each X_k is a geometric variable with stopping probability $p_k = \frac{ck}{n^3}$. Then whp $Z > \frac{1}{16c} n^3 \log n$.*

Proof. Let n' be the largest number smaller than n such that $\sqrt{n'}$ is an integer, i.e., $n' = (\lfloor \sqrt{n} \rfloor)^2$. Z dominates $Z' = \sum_{k=1}^{\sqrt{n'}} \sum_{l=1}^{\sqrt{n'}} X'_{kl}$ where each X'_{kl} is a geometric variable with stopping probability $p_{kl} = \frac{ck\sqrt{n'}}{n^3}$. Note that $X'_{kl} > \frac{1}{2p_{kl}}$ with probability $1-(1-p_{kl})^{\frac{1}{2p_{kl}}} \approx 1 - e^{-\frac{1}{2}} > 0.39$, and so using Hoeffding's inequality

$$\Pr\left[\sum_{l=1}^{\sqrt{n'}} X'_{kl} > \frac{1}{4}\sqrt{n'} \cdot \frac{1}{2p_{kl}}\right] > 1 - e^{-2(0.39-0.25)^2\sqrt{n'}} > 1 - e^{-0.03\sqrt{n'}}.$$

Therefore

$$\Pr\left[Z' > \sum_{k=1}^{\sqrt{n'}} \frac{\sqrt{n'}}{8p_{kl}}\right] \geq \left(1 - e^{-0.03\sqrt{n'}}\right)^{\sqrt{n'}} > 1 - n' e^{-0.03\sqrt{n'}}.$$

So with high probability

$$Z > \sum_{k=1}^{\sqrt{n'}} \frac{\sqrt{n'}}{8p_{kl}} = \frac{1}{8c} n^3 \sum_{k=1}^{\sqrt{n'}} \frac{1}{k} \approx \frac{1}{8c} n^3 \log\sqrt{n} = \frac{1}{16c} n^3 \log n$$

∎

Proof. For the sake of simplicity let us focus on the case of $G = U[0, 1]$. Let us take the variant of the approximation algorithm suggested by Crawford and Knoer (1981) to solve a generalized version of the assignment game, in which firms are ordered from f_1^n to f_n^n and at each round only the lowest-number firm that still wants to propose actually proposes. Take the step size to be $E = \frac{1}{n^3}$. We want to bound the minimal number of steps through the entire algorithm.

We note that when it is firm f_i^n's to propose, and its previous aspiration level (i.e., the maximal utility it would get by giving some worker her current salary was $\underline{u}_i$, and if for all unmatched workers $w_j^n \in W^n$ we have $\varepsilon_{ij}^n \notin [\underline{u}_i - E, \underline{u}_i)$, then some worker's salary increases by E. The conditional probability of ε_{ij}^n not being in $[\overline{u}_i - E, \overline{u}_i)$ is $1 - \frac{E}{\overline{u}_i}$. We know that in the firm-optimal core allocation at least one worker gets a salary of zero, and from Theorem 1 we learn that all workers get no more than $\frac{c \log n}{n}$. Combining this with the results of Frieze and Sorkin (2007) gives us that whp $\underline{u}_i \geq 1 - \frac{c \log n}{n}$ for some constant $c \in \mathbb{R}_+$. Therefore the conditional probability mentioned before is at least $1 - \frac{E}{1 - \frac{c \log n}{n}} > 1 - 1.01E$.[11] This implies that when there are $n - k + 1$ ($k > 1$) still unemployed workers, the probability of raising the salary of one of the employed workers by E is at least

$$(1 - 1.01E)^{n-k+1} > 1 - 1.01(n - k + 1)E,$$

and the probability of employing a still unemployed worker is at most $1.01(n - k + 1)E = \frac{1.01(n-k+1)}{n^3}$. By Lemma 17, whp there are going to be at least $\frac{1}{16.16} n^3 \log n$ steps, and multiplying by E we get that whp the sum of workers' salaries is at least $\frac{1}{16.16} \log n$. This implies also that whp at least one of the workers has a salary that is at least $\frac{1}{16.06} \cdot \frac{\log n}{n}$. As mentioned before, in each realization one of the workers has a salary of zero. Together this means that whp there are two workers such that the difference between their salaries is $\frac{1}{16.06} \cdot \frac{\log n}{n}$, and we are done. □

B.2 Proof of Corollary 3

Proof. As mentioned in the intuition for the proof, there must be at least one worker whose salary is exactly zero. If $\underline{h} \neq h$, let $c_1 \in \mathbb{R}_+$ be such that for large enough n, $H\left(\underline{h} + \frac{c_1 \log n}{n}\right) > \frac{\log n}{n}$ (such c_1 exists since H has positive and continuous density at $\underline{h}$). It follows that the probability of having at least one worker with a human capital level below $\frac{c_1 \log n}{n}$ is at least

$$1 - \left(1 - \frac{\log n}{n}\right)^n \approx 1 - e^{-\log n} = 1 - \frac{1}{n}.$$

Let c_2 be the constant we arrived at in the proof of Theorem 1, if the worker who gets zero salary has a human capital level above $\frac{(c_1+c_2)\log n}{n}$; then Theorem 1 implies that any worker with a human capital level lower than $\frac{c_1 \log n}{n}$

[11] We assume that $G = U[0, 1]$. When $G \neq U[0, 1]$ we have to approximate the density near the upper bound, and rely on Theorem 1 to approximate the conditional probability of choosing a still unmatched worker.

gets a negative salary. Therefore, with high probability the worker getting a zero salary must have human capital level below $\frac{(c_1+c_2)\log n}{n}$. It follows from Theorem 1 that whp for every worker w_j^n

$$v_j^{n,F} \in \left(h_j^n - \frac{(c_1+2c_2)\log n}{n}, h_j^n + \frac{c_2 \log n}{n} \right).$$

By taking $c = c_1 + 2c_2$ we reach the desired conclusion. □

B.3 Proof of Corollary 4

Proof. We prove this corollary separately for the case of workers who have the same human capital level and for the case of workers with different human capital levels. In the first case ($h^n \equiv \underline{0}$) we recall that the same line of proof used in Lemma 14 could have shown us that in this case the approximate law of one price holds for *any* two workers (and not just two *matched* workers). The proof follows immediately from Theorem 1 by comparing any matched worker to one of the unmatched workers (whose salary is 0).

In the second case ($\underline{h} \neq \bar{h}$), note that there exists $c_1 \in \mathbb{R}_+$ such that whp all workers with a human capital level below $h^n[n] - \frac{c_1 \log n}{n}$ are unmatched (Lemma 15). Let $c_2 \in \mathbb{R}_+$ be such that whp there exists a worker w_j^n with a human capital level $h_j^n \in \left(h^n[n] - \frac{c_1 \log n}{n}, h^n[n] - \frac{(c_1+c_2)\log n}{n} \right)$. Note that there exists $c_3 \in \mathbb{R}_+$ such that whp this worker has a good match with one of the matched firms; i.e., there exists f_i^n such that $\varepsilon_{ij}^n > 1 - \frac{c_3 \log n}{n}$. It follows that the worker w_k^n employed by that firm gets no more than $\left(h_k^n - h_j^n\right) + \frac{c_3 \log n}{n} \le (h_k^n - h^n[n]) + \frac{(c_1+c_2+c_3)\log n}{n}$. Now set $c = c_1 + c_2 + c_3 + c_4$, where c_4 is the constant provided by Theorem 1, and we get the desired result using Theorem 1 (comparing matched workers to w_k^n). □

B.4 Proof of Proposition 9

Lemma 18.

$$E\left[\sum_{f_i^n = \mu^n(w_j^n)} \alpha_{ij}^n \right] \le n(\log n + 1)$$

Proof. We want to show that for every worker the expected value of the maximal element in the relevant column of the productivity matrix α^n equals $\log n$.

To see that, note first that the minimal element is distributed according to an exponential distribution with parameter n (think of the first arrival of one of n identical arrivals). Due to the memorylessness property of exponential random variables, the difference between the first minimal element and the second minimal element is distributed like an exponential distribution with parameter $n-1$, and so on. This implies that the expected value of the largest element is

$$\frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + \cdots + \frac{1}{2} + 1 \leq \log n + 1.$$

Lemma 19.

$$E\left[\sum_{f_i^n=\mu^n(w_j^n)} \alpha_{ij}^n\right] \geq 0.99 n \log n$$

Proof. Let μ be a matching that results from running a greedy algorithm: firm 1 picks the worker it likes best, then firm 2 picks a worker from those remaining, and so on. The expected value of μ is

$$E\left[\sum_{f_i=\mu(w_j)} \alpha_{ij}^n\right] = E\left[\max\{X_{1,1},\ldots,X_{1,n}\}\right] + E\left[\max\{X_{2,1},\ldots,X_{2,n-1}\}\right] + \cdots + E\left[X_{n,1}\right],$$

where $\{X_{i,j}\}$ are i.i.d. Exp(1). Therefore

$$E\left[\sum_{f_i^n=\mu(w_j^n)} \alpha_{ij}^n\right] = \sum_{i=0}^{n-1} \left[\log(n-i)+1\right] \approx n \log n.$$

The result then follows from the optimality of μ^n.

Proof. The first claim follows from Lemma 18 and Lemma 19. For the second claim, let p^n denote the probability that for a given firm $f_i^n \in F^n$ there exists a worker $w_j^n \in W^n$ such that $\alpha_{ij}^n > 1.1 \log n$ and $\max_{k \neq j} \alpha_{ik}^n < \log n$. Then

$$p^n = n \cdot e^{-1.1 \log n} \cdot \left(1 - e^{-\log n}\right)^{n-1} = \frac{1}{n^{0.1}} \cdot \left(1 - \frac{1}{n}\right)^{n-1} \approx \frac{1}{en^{0.1}}.$$

This specifically implies that for any $\epsilon > 0$ whp there are $\Omega(n^{0.9-\epsilon})$ firms that meet the above condition. If the same worker is the outlier in any two of these firms, then this worker must get paid at least $0.1 \log n$ under any core allocation. Since there are $\Omega(n^{1.8-2\epsilon})$ pairs, we get that there are many workers who get paid $\Theta(\log n)$. Finally, at least one worker's salary is 0 under the firm-optimal core allocation, and so we are done. □

B.5 Proof of Theorem 11

Lemma 20. *In an arbitrary balanced market with productivity matrix α^n, let $(\mu^n, u^{n,F}, v^{n,F})$ be a the firm-optimal core allocation. If $f_i^n = \mu^n(w_j^n)$ then*

$$v_j^{n,F} \leq \left(\alpha_{ij}^n - \min_{f_k^n = \mu^n(w_l^n)} \alpha_{kl}^n\right).$$

Proof. Let $\underline{\alpha} := \min_{f_k^n = \mu^n(w_l^n)}$. Consider a core allocation (μ', u', v') for a modified productivity matrix $\alpha' = \alpha^n - \underline{\alpha}$. It is trivial that $\mu' = \mu^n$. Since this is a core allocation it must be that $\forall i : u'_i \geq 0$, which means that $\forall f_i^n = \mu(w_j) : v'_j \leq \alpha'_{ij} = \alpha_{ij} - \bar{\alpha}$. Define $u_i = u'_i + \alpha$ and $v = v'$, and note that (μ, u, v) is a core allocation for α since all the constraints defining the core are preserved when we restore the constant. The result follows immediately from the worker-pessimality of the firm-optimal core allocation. ■

Lemma 21. *If Conjecture 10 holds, then there exists $c \in \mathbb{R}_+$ such that whp*

$$\min_{f_i^n = \mu^n(w_j^n)} \alpha_{ij}^n \geq \log n - \log\log n - \log c.$$

Proof. Let the constant used in Conjecture 10 be c_1, and let $c = c_1 + 3$. The probability that the $c_1 \log n$ highest element out of n exponential random variables will be lower than $\log n - \log\log n - \log c$ equals

$$
\begin{aligned}
P &= \sum_{m=n-c_1\log n+1}^{n} \binom{n}{m} \left(1 - e^{-\log n + \log(c\log n)}\right)^m \left(e^{-\log n+\log(c\log n)}\right)^{n-m} \\
&\le c_1 \log n \cdot \binom{n}{c_1\log n - 1} \left(1 - \frac{c\log n}{n}\right)^{n-c_1\log n+1} \left(\frac{c\log n}{n}\right)^{c_1\log n-1} \\
&\le c_1 \log n \left(\frac{en}{c_1\log n - 1}\right)^{c_1\log n-1} \left(1 - \frac{c\log n}{n}\right)^{n} \left(\frac{c\log n}{n}\right)^{c_1\log n-1} \\
&\le c_1 \log n \left(\frac{e}{c}\right)^{c_1\log n-1} e^{-c\log n} = c_1 \log n \left(\frac{e}{c}\right)^{c_1\log n-1} \frac{1}{n^c} \\
&\le \frac{c_1(c_1+3)\log n}{e\, n^3} \le \frac{1}{n^2},
\end{aligned}
$$

where the transition in the fourth line is by Stirling's approximation, and the one in the fifth line uses $c = c_1 + 3$. Therefore the probability that after taking the $c_1 \log n$ highest element out of n exponential random variables n times the minimal value is lower than $\log n - \log\log n - \log c$ is bounded above by

$$
1 - \left(1 - \frac{1}{n^2}\right)^n \approx 1 - \left(1 - \frac{1}{n} + O\left(\frac{1}{n^2}\right)\right) = \frac{1}{n} + O\left(\frac{1}{n^2}\right).
$$

Conjecture 10 ensures that whp μ^n does not assign any firm to a worker who is ranked below $c_1 \log n$, and therefore whp the claim holds. ∎

Proof of Theorem 11. Given Lemma 18, Lemma 20, and Lemma 21, we know that

$$
E\left[\sum_j v_j^{n,F}\right] \le n(\log n + 1) - \left(1 - \frac{c_1}{n}\right) \cdot n \cdot (\log n - c\log\log n),
$$

where c_1 is such that the statement in Lemma 21 holds with probability greater than $1 - \frac{c_1}{n}$. This implies that

$$
E\left[\sum_j v_j^{n,F}\right] \le cn\log\log n + n + c_1\log n - \frac{cc_1\log\log n}{n}.
$$

Finally, use Lemma 19 to complete the proof. □

C Analysis of the Cobb–Douglas benchmark model

This appendix demonstrates how one can get results similar to Theorem 1 in the presence of interaction between firms' quality and workers' human capital level.

C.1 Sketch of proof of Lemma 6

Given any $n, d, m \in \mathbb{N}$, let

$$\mathrm{Sym}(n, d, m) := \{\sigma \in \mathrm{Sym}(n) \mid |\{i \mid \sigma(i) - i \geq d\}| \geq m\},$$

where $\mathrm{Sym}(n)$ is the symmetric group of size n. That is, $\mathrm{Sym}(n, d, m)$ is the set of all permutations σ of the set $\{1, \ldots, n\}$ such that there are at least m elements such that the difference between their images and themselves is equal to or larger than d. For any $\sigma \in \mathrm{Sym}(n)$ we let

$$\mathrm{val}(\sigma) := \frac{\sum_{i=1}^{n} \sqrt{i\sigma(i)}}{n}.$$

Lemma 22. *If $\sigma \in \mathrm{Sym}(n, d, m)$, and there exist $i < j$ such that $\sigma(i) > \sigma(j)$ and $\sigma(i) - i < d$, then there exists $\sigma' \in \mathrm{Sym}(n, d, m)$ such that $\mathrm{val}(\sigma') > \mathrm{val}(\sigma)$.*

Proof. Consider $\sigma' \in \mathrm{Sym}(n, d, m)$ defined by

$$\sigma'(k) = \begin{cases} \sigma(j) & \text{if } k = i, \\ \sigma(i) & \text{if } k = j, \\ \sigma(k) & \text{otherwise.} \end{cases}$$

We get that

$$\begin{aligned} \mathrm{val}(\sigma') - \mathrm{val}(\sigma) &= \frac{1}{n}\left(\sqrt{i\sigma(j)} + \sqrt{j\sigma(i)} - \sqrt{i\sigma(i)} - \sqrt{j\sigma(j)}\right) \\ &= \frac{1}{n}\left(\sqrt{i} - \sqrt{j}\right)\left(\sqrt{\sigma(j)} - \sqrt{\sigma(i)}\right) > 0 \end{aligned}$$

∎

Lemma 23. *If $\sigma \in \text{Sym}(n, d, m)$, and there exist $i < j$ such that $\sigma(i) > \sigma(j)$ and $\sigma(j) - j \geq d$, then there exists $\sigma' \in \text{Sym}(n, d, m)$ such that $\text{val}(\sigma') > \text{val}(\sigma)$.*

Proof. The proof is similar to the proof of Lemma 22. The only difference is that now $\sigma' \in \text{Sym}(n, d, m)$ because $\sigma'(i) - i = \sigma(j) - i > \sigma(j) - j \geq d$ and $\sigma'(j) - j = \sigma(i) - j > \sigma(j) - j \geq d$. ∎

Lemma 24. *If $\sigma \in \text{Sym}(n, d, m)$, $m > 0$, and $n - \sigma^{-1}(n) < d$, then there exists $\sigma' \in \text{Sym}(n, d, m)$ such that $\text{val}(\sigma') > \text{val}(\sigma)$.*

Proof. Let n' be the largest number such that $(n' - 1) - \sigma^{-1}(n' - 1) \geq d$ (such n' exists since $m > 1$). Denote $k := n' - \sigma^{-1}(n' - 1)$. If there exists i such that $i > k$ and $\sigma(i) - i > d$, then by Lemma 23 we are done. If $\sigma(n') \neq n'$, then by a simple counting argument there exist $i < j$ such that $\sigma(i) > \sigma(j)$ and $\sigma(i) - i < d$, and then by Lemma 22 we are done. Similarly, if $\sigma(n' - k + 1) > n' - k$, we can again find $i < j$ such that $\sigma(i) > \sigma(j)$ and $\sigma(i) - i < d$, and be done by Lemma 22. Define $\sigma' \in Sym(n)$ as

$$\sigma'(i) = \begin{cases} n' - 1 & \text{if } i = n', \\ n' & \text{if } i = n' - k + 1, \\ \sigma(n' - k + 1) & \text{if } i = n' - k, \\ \sigma(i) & \text{otherwise.} \end{cases}$$

We now have:

$$\operatorname{val}\sigma' - \operatorname{val}\sigma = \frac{1}{n}\left(\left((n'-k) + \sqrt{(n'-k+1)n'} + \sqrt{n'(n'-1)}\right) - \left(\sqrt{(n'-k)(n'-1)} + \sqrt{(n'-k+1)(n'-k)} + n'\right)\right)$$

$$= \frac{n'}{n}\left(\left(1 - \frac{k}{n'} + \sqrt{1-\frac{k-1}{n'}} + \sqrt{1-\frac{1}{n'}}\right) - \left(\sqrt{1-\frac{k}{n'}}\sqrt{1-\frac{1}{n'}} + \sqrt{1-\frac{k-1}{n'}}\sqrt{1-\frac{k}{n'}} + 1\right)\right)$$

$$= \frac{n'}{n}\left(-\frac{k}{n'} + \left(\sqrt{1-\frac{1}{n'}} + \sqrt{1-\frac{k-1}{n'}}\right)\left(1 - \sqrt{1-\frac{k}{n'}}\right)\right)$$

$$= \frac{k}{n}\left(\frac{\sqrt{1-\frac{1}{n'}} + \sqrt{1-\frac{k-1}{n'}}}{1+\sqrt{1-\frac{k}{n'}}} - 1\right)$$

$$= \frac{k}{n}\left(\frac{1}{1+\sqrt{1-\frac{k}{n'}}}\right)\left(F(k-1,n') - F(0,n)\right),$$

where $F(t,n') = \sqrt{1-\frac{t}{n'}} - \sqrt{1-\frac{t+1}{n'}}$. Note that

$$\frac{\partial F(t,n')}{\partial t} = -\frac{1}{2n'\sqrt{1-\frac{t}{n'}}} + \frac{1}{2n'\sqrt{1-\frac{t+1}{n'}}} = \frac{\sqrt{1-\frac{t}{n'}} - \sqrt{1-\frac{t+1}{n'}}}{2n'\sqrt{1-\frac{t}{n'}}\sqrt{1-\frac{t+1}{n'}}},$$

and therefore $\frac{\partial F}{\partial t} > 0$ for all $t \in (0, n'-1)$. This implies that $\operatorname{val}\sigma' > \operatorname{val}\sigma$ as required. ∎

Lemma 25. *For all* $\sigma \in \mathrm{Sym}(n,d,m)$, $\mathrm{val}(\sigma) \leq \frac{n+1}{2} - \frac{md^2}{8n(n+d+1)}$.

Proof. Let $\overline{\mathrm{val}}(n,d,m) := \max_{\sigma\in\mathrm{Sym}(n,d,m)} \mathrm{val}(\sigma)$. Given $\sigma_1 \in \mathrm{Sym}(n,d,m)$

such that $\text{val}(\sigma_1) = \overline{\text{val}}(n, d, m)$, we can define $\sigma_2 \in \text{Sym}(n + d + 1, d, m)$ by

$$\sigma_2(i) = \begin{cases} \sigma_1(i) & \text{if } i \leq n, \\ i & \text{if } i > n. \end{cases}$$

Following the same logic used in the proof of Lemma 24, there exists $\sigma_3 \in \text{Sym}(n + d + 1, d, m)$ such that $\sigma_3(n + 1) = n + d + 1$, $\sigma_3(i) = i - 1$ for $i \in \{n + 2, \ldots, n + d + 1\}$, and $\text{val}(\sigma_3) > \text{val}(\sigma_2)$. However, this also implies that there exists $\sigma_4 \in \text{Sym}(n + d + 1, d, m)$ such that σ_4 is identical to σ_3 for inputs larger than n, and is identical to a permutation $\sigma_4 \in \text{Sym}(n, d, m - 1)$ that achieves $\text{val}(n, d, m - 1)$ for inputs smaller than or equal to n. It follows that

$$\overline{\text{val}}(n, d, m) = \text{val}(\sigma_1) = \frac{1}{n}\left((n + d + 1)\,\text{val}(\sigma_2) - \sum_{i=n+1}^{n+d+1} i\right) \leq$$

$$\frac{1}{n}\left((n + d + 1)\,\text{val}(\sigma_4) - \sum_{i=n+1}^{n+d+1} i\right) =$$

$$\overline{\text{val}}(n, d, m - 1)\,\frac{1}{n}\left(\sum_{i=n+1}^{n+d+1} i - \sum_{i=n+1}^{n+d} \frac{}{i(i + 1)} - \frac{}{(n + 1)(n + d + 1)}\right).$$

Now note that

$$\sum_{i=n+1}^{n+d+1} i - \sum_{i=n+1}^{n+d} \sqrt{i(i+1)} - \sqrt{(n+1)(n+d+1)} =$$

$$n+d+1 - \sqrt{(n+1)(n+d+1)} - \sum_{i=n+1}^{n+d} \left(\sqrt{i(i+1)} - i \right) =$$

$$\frac{(n+d+1)^2 - (n+1)(n+d+1)}{n+d+1+\sqrt{(n+1)(n+d+1)}} - \sum_{i=n+1}^{n+d} \frac{i}{\sqrt{i(i+1)}+i} \geq$$

$$\frac{(n+d+1)d}{n+d+1+\sqrt{(n+1)(n+d+1)}} - \frac{d}{2} =$$

$$\frac{d}{2}\left(\frac{(n+d+1) - \sqrt{(n+1)(n+d+1)}}{(n+d+1)+\sqrt{(n+1)(n+d+1)}} \right) =$$

$$\frac{d}{2}\left(\frac{d(n+d+1)}{\left((n+d+1)+\sqrt{(n+1)(n+d+1)}\right)^2} \right) \geq \frac{d^2}{8(n+d+1)}.$$

Therefore

$$\overline{\text{val}}(n,d,m) \leq \overline{\text{val}}(n,d,m-1) - \frac{d^2}{8n(n+d+1)},$$

and

$$\overline{\text{val}}(n,d,m) \leq \overline{\text{val}}(n,d,0) - \frac{md^2}{8n(n+d+1)}.$$

To complete the proof, note that for $m = 0$ we know by Lemma 22 that $\overline{\text{val}}(n,d,0) = \sum_{i=1}^{n} \frac{i}{n} = \frac{n+1}{2}$. ■

Let μ^n be an assignment for a certain market M^n; we denote

$$\text{val}(\mu^n, M^n) := \sum_{\mu^n(f_i^n)=w_j^n} \sqrt{q_i^n h_j^n} + \varepsilon_{ij}^n.$$

Lemma 26. *Let μ^n be an assignment for M^n such that*

$$\left| \left\{ i : \left| \frac{1}{q_i^n} - \frac{1}{h^n_{\mu^n(i)}} \right| > n^{b-1} \right\} \right| \geq n^a,$$

for some $a, b \in (0,1)$*; then there exists* $c \in \mathbb{R}_+$ *such that*

$$\mathrm{val}\,(\mu^n, M^n) \leq 2n + 1 - cn^{a+2b-2}$$

Proof. Without loss of generality, assume that more than half the firms in the set $\{i : |q_i^n - h^n_{\mu^n(i)}| > n^{b-1}\}$ are matched with workers whose human capital level exceeds the firms' quality. Now the maximal value is given when all the firms fit workers perfectly in terms of the idiosyncratic component (i.e., $\varepsilon_{ij}^n = 1$), and then Lemma 25 bounds the sum of the interactive components, and we get

$$\mathrm{val}\,(\mu^n, M^n) \leq 2\left(\frac{n+1}{2} - \frac{\left(\frac{1}{2}n^a\right)\left(n^b\right)^2}{8n(n+n^{b-1}+1)}\right) + n \leq 2n + 1 - \frac{1}{8.01}n^{a+2b-2}.$$

∎

Lemma 27. *Let* $\{\mu^n\}$ *be a sequence of optimal assignments for* M^n*. Then there exists* $c \in \mathbb{R}_+$ *and* $\gamma \in (0,1)$ *such that whp* $\mathrm{val}\,(\mu^n, M^n) \geq n - cn^{\gamma}$*.*

Proof sketch. Use a greedy algorithm that divides the firms and workers into layers according to their quality/human capital level, where each layer contains $n^{1/3}$ firms/workers. Then perform an optimal assignment within each layer based only on the noise dimension. The result approximates the efficiency on both dimensions, and gives a lower bound on the efficient assignment. ∎

Sketch of proof of Lemma 6. We deduce from Lemma 26 and Lemma 27 that for $a + 2b - 2 > \gamma$ it must be that $\left|\left\{i \mid |q_i^n - h^n_{\mu^n(i)}| > n^{b-1}\right\}\right| < n^a$. Now assume to the contrary that there is a firm that is matched under the optimal assignment to a worker who has a human capital level far higher than the firm's quality (by "far higher" we mean $n^{\delta-1}$ for some $\delta \in (0,1)$), and show (using a somewhat involved counting argument) that there must be another firm that is matched to a worker with a human capital level far lower than the firm's own quality, and such that a switch between the workers employed by those two firms would yield an efficiency gain of $c_1 n^{2-2\delta}$ on the quality dimension for some $c_1 \in \mathbb{R}_+$. Then for each new match, try to find an alternating path (in the spirit of Theorem 1) to fix the efficiency on the noise dimension. This leads to an overall improvement in efficiency, which leads to a contradiction. □

C.2 Sketch of proof of Theorem 7

Sketch of proof. The proof follows immediately from Lemma 6 and similar arguments to those used in Theorem 1, applied within a band of qualities of width $\Theta\left(n^{-b}\right)$. □