Strategic Gradual Learning and Information Transmission

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Strategic Gradual Learning and Information Transmission*

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Abstract

This paper addresses the problem of information transmission between a biased expert and a decision maker in an environment where the expert's learning is gradual and strategic. It is shown how the gradualness of the expert's learning can be exploited to enhance informativeness in communication. The result suggests that even in the absence of an "objective" reason to expedite information transmission, putting the expert under "strategic pressure" can increase the amount of transmitted information and be beneficial to both players.

1 Introduction

Consider the following scenario. A patient (decision maker) can be healthy (state 0) or ill, and in the latter case he can have either a mild disease (state 1) or a severe disorder (state 2). The three states are equally likely. The patient has to determine the level of his health-care expenditure a. His first-best such expenditure at state θ is $a = \theta$. The doctor (expert), who stands to benefit from the treatment, can advise the patient. If the patient is healthy, the doctor prefers not to induce any active treatment, and so the interests of the players are aligned at $\theta = 0$. If the patient is ill, the doctor is upwardly biased and her most-desired level of patient expenditure in state $\theta \in \{1, 2\}$ is $a = \theta + 1$. Both players minimize the quadratic loss (that is, the expected distance between the ideal point and the action taken).

The case in which the doctor submits her report when she is perfectly informed and the patient is uninformed corresponds to the model of cheap talk analyzed in Crawford and Sobel (1982). Under these assumptions on the players' information, in the "most informative" equilibrium, the patient can learn only whether he is healthy ($\theta = 0$) or ill ($\theta \in \{1, 2\}$).

To illustrate the key element of the model, I now enrich the standard cheap-talk environment with an explicit description of the process according to which information asymmetry between players arises. Assume that both players are initially uninformed but the doctor has the ability to explore the patient's

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condition as follows. Two medical examinations are available: ϵ_1 and ϵ_2. Examination ϵ_i is costless and it privately reveals to the doctor whether $\theta = i$ or not. The patient knows what examinations are available to the doctor but does not observe which, if any, are performed. It is known that each examination takes one unit of time. However, it is assumed that the time available to the doctor, before the patient has to make a decision, is enough to fully explore the patient's condition.

In this case, it turns out that the patient can accurately learn his health condition. The doctor's strategy described below is consistent with an equilibrium. First, the doctor performs examination ϵ_2. If the result is positive, she reports this fact *immediately*. Otherwise, she proceeds to examination ϵ_1. This takes another unit of time. When the result of ϵ_1 is obtained the doctor (truthfully) reports it.

To see why this is an equilibrium, it is important to note that when the doctor chooses her learning strategy, she takes into account that she has only one opportunity to induce the highest possible expenditure of the patient ($a = 2$). This can be done only by reporting that "$\theta = 2$" at the end of period 1. Since only one examination can be performed at each unit of time, the doctor has to choose which examination to perform first. It is most desirable from her perspective to begin with ϵ_2 and report that "$\theta = 2$" *only* if the result of ϵ_2 is positive. If ϵ_2 is negative, inducing $a = 2$ is not desirable because, at this point, the doctor does not know whether $\theta = 1$ or $\theta = 0$. She prefers to continue learning and report truthfully at the end of period 2.

The timing of the report provides the patient with some information regarding the doctor's information structure. The key element in the equilibrium presented above is that if a report is submitted at the end of period 1, the patient infers that the doctor could not perform both medical examinations before reporting. Although there is no objective reason to expedite information transmission, in the most informative equilibrium the doctor is put under *strategic pressure*, which induces a problem of *strategic scheduling* of the available examinations. In this case, the most informative equilibrium is fully revealing and, ex ante, this equilibrium is strictly preferred by both players to any other equilibrium.

This paper combines a dynamic cheap-talk environment with gradual learning on the part of an expert that is strategic, unobservable, and unverifiable. At the start, both the expert and the decision maker are uninformed about the state of the world, but the expert has the ability to explore it. The expert's learning is carried out under some technological limitations: only certain experiments can be performed. Nevertheless, as is often the case, some aspects of the exploration process are left to the expert's discretion, e.g., the selection of a particular collection of experiments to be performed and the ordering of these experiments. This part of the expert's strategy is assumed to be unobservable and unverifiable. To emphasize the particular effect of the gradualness in the expert's learning I assume that the learning is costless and that the time available for learning places no restrictions on the expert's ability to become fully informed. The purpose of this analysis is to show how the gradualness of an expert's learning can be exploited for the sake of acquiring more informative communication in this environment.

Several other works have considered certain aspects of strategic learning on the part of an expert, e.g., Austen-Smith (1994), Argenziano et al. (2011), and Di Pei (2013). The main difference is that while these models are essentially static, the key feature of our model is the dynamic nature of the expert's learning.

A more extensive discussion of these and other related papers is deferred to Section 8.

The rest of the paper is organized as follows. Section 2 presents the model. In Section 3, I analyze the model and discuss the interpretation of strategic pressure and the manner in which an upwardly biased expert explores the state of the world. Section 4 discusses the role of the special component of the state space assumed earlier and develops a related result in a more standard state space. In Section 5 the results given in previous sections are compared with the informed sender benchmark. Sections 6 and 7 illustrate how the ideas presented apply to other environments. Section 6 provides an example of a case where the expert is biased towards extreme actions and Section 7 discusses alternative learning technologies. Related literature is discussed in Section 8. Section 9 concludes.

2 Model

There are two players, sender and receiver, $N = \{S, R\}$.

State Space To present the forces behind the main result in the most transparent way, I begin the analysis with a convenient state space $\Theta = \{\phi, 1, 2, \ldots, n\}$, where ϕ represents a "categorically" different state (intuitively, ϕ corresponds to the state in which the patient is healthy in our doctor-patient example). Let P denote the common prior on Θ.

The inclusion of ϕ, along with the associated assumptions given below, has clear technical implications and this was my primary motivation. In Section 4, I discuss the role of ϕ and develop a related result in a state space that does not contain ϕ.

Sender's Learning For each $k \in \{1, \ldots, n\}$, there is a potential experiment $\epsilon_k = \mathbf{1}_{\theta=k}$ that reveals whether $\theta = k$ or not. All of the experiments entail no direct costs for any of the players; however, conducting an experiment takes one unit of time (parallel experimentation is not possible). Only the sender has access to the set of experiments $\{\epsilon_k | k \in \{1, \ldots, n\}\}$. The experiments she conducts and the order in which she conducts them cannot be observed or verified.

Notice that there does not exist an experiment ϵ_ϕ. In other words, the realization $\theta = \phi$ can be learned by the sender only by means of ruling out all other states (In our doctor-patient example, the doctor learns that the patient is healthy by ruling out all diseases.)

Sender's Reporting The sender is allowed to submit only one report, but she can choose when to submit it. Formally, let M be a set of available reports and let $\sigma \in M$ denote the "empty report" corresponding to the sender's choice to "remain silent." Assume that the set M is rich enough to transmit any of the information that was available to the sender. Let $\mathbf{T} = \{1, 2, \ldots, T\}$ be a discrete timeline and assume $T > n$. At each $t \in \mathbf{T}$, the sender may remain silent (i.e., submit an "empty report" $m_t = \sigma$) or submit an "active report" $m_t \in M - \{\sigma\}$. Any terminal history of reports contains at most one "active report." The reports are *cheap talk* in the sense that they have no direct effect on players' utility.

It is worthwhile to note that the main result of the paper does not rely on the selection of this particular reporting protocol. In other words, every equilibrium under the selected single-report protocol can be recovered in an environment that places no restrictions on the sender's reporting opportunities.

The assumed reporting protocol highlights the main aspect of the reporting strategy, namely, the timing of the report's submission.

Receiver's Action At period T, the receiver has to choose an action $a \in \{\phi\} \cup \mathbb{R}_+$. It is convenient to interpret the *action* ϕ as "the choice of not taking an active action." The assumption that $T > n$ implies that, in principle, the time available for learning places no restrictions on the expert's ability to become fully informed.

Utility Functions Player i's utility function is

$$U^i(a,\theta) = \begin{cases} u^i(a,\theta) & a,\theta \neq \phi \\ 0 & a = \theta = \phi \\ -\infty & a = \phi \oplus \theta = \phi, \end{cases}$$

where the symbol $\oplus$ denotes the "exclusive or" logical operator. The assumption that $U^i(a,\theta) = -\infty$ if $a = \phi \oplus \theta = \phi$ captures the intuition that any mismatch with respect to ϕ is very costly for the players (intuitively, operating on a healthy patient and not treating an ill patient are both strongly undesirable). The simplifying assumption of the infinite cost is stronger than required for the main result and it is discussed at the end of the analysis in Section 3.

The functions $u^i : \mathbb{R}_+ \times \{1,2,..,n\} \to \mathbb{R}$ for $i \in \{S,R\}$ satisfy the following assumptions. For each θ, the function $u^i(\cdot,\theta) \leq 0$ represents single-peaked preferences, $a^i(\theta) = argmax_a\, u^i(a,\theta)$ is (strictly) increasing in θ, and the sender is upwardly biased in the sense that for all $\theta < n$,

$$u^S(a^R(\theta),\theta) < u^S(a^R(\theta+1),\theta).$$

Strategies and Equilibrium The sender's strategy is twofold. She chooses both how to explore and how to report. Clearly, the flexibility of the reporting component of the strategy, $m = \{m_t\}_{t\in\mathbf{T}}$, depends on its learning component f because the report at each period can depend only on the information collected up to that period.

To make it precise, let $f : \mathbf{T} \to \{0,1,..,n\}$ be a given plan of learning, specifying which experiment $\epsilon_{f(t)}$ is to be conducted at time $t \in \mathbf{T}$, where ϵ_0 denotes "do nothing." Denote by $\mathcal{F}_t(f)$ the sender's information structure at period t if she follows the learning plan f. In particular, $\mathcal{F}_t(f) = \{\{f(s)\}, (\cup\{f(s)\})^c \,| s \leq t\}$. A report choice in period t, m_t, must be measurable with respect to $\mathcal{F}_t(f)$. The reporting component of the sender's strategy, $m = \{m_t\}_{t\in\mathbf{T}}$, is "*$f$-measurable*" if m_t is measurable with respect to $\mathcal{F}_t(f)$ for all t.

Note that the continuation of learning is relevant only if previous experiments have not revealed θ (and a "non-empty report" has not been submitted), Thus, it is without loss of generality to assume that the learning strategy f is selected at the beginning of the interaction and is not periodically revised. To summarize: a sender's strategy is $s \in \Delta\{(f,m)|f \in \{0,1,..,n\}^{\mathbf{T}}$ and m is *f-measurable*$\}$. Whenever f is not random, simply denote $s = (f, m^f)$, where m^f is *f-measurable* and possibly mixed.

The receiver chooses an action rule $\alpha(m) \in \Delta A$ that assigns to any terminal history of reports a distribution over the actions. The receiver's pure strategy is denoted by $a(m)$.

The applied solution concept is a sequential equilibrium.

3 Analysis

In the main part of the analysis I illustrate how strategic pressure can increase the amount of information transmitted by showing that, under some conditions, there exists an equilibrium in which the receiver reveals the state of the world. An equilibrium in which the sender acquires complete information and transmits it accurately is termed here a *fully informative equilibrium*. Proposition 1 characterizes the essentially unique sender's strategy in any fully informative equilibrium. Qualitatively, this has two implications. First, due to strategic pressure, the sender will perform her exploration as fast as possible. Strategic pressure implies that certain actions can be induced only during the early stages of the game. To decide whether to induce a particular action, the sender would like to proceed as much as she can with the most effective exploration program (from her perspective). Second, in a fully informative equilibrium, an upwardly biased sender will necessarily explore the states downwards (against the bias direction). Actions towards which she is "most biased" are the first to become no longer inducible.

Proposition 1 *In any fully informative equilibrium, the sender's strategy (f, m^f) satisfies, for every $t < n$,*

- $f(t) = n + 1 - t$, *and*
- $m_t^f \neq \sigma$ *if and only if* $\epsilon_{f(t)} = 1$.

Proof Let e be a fully informative equilibrium and denote by (f, m^f) the sender's strategy. For each $k \in \Theta - \{\phi\}$, denote by $t(k)$ the last period in which the information that $\theta = k$ is transmitted to the receiver in this equilibrium. A necessary condition for a fully informative equilibrium is that for every $k > 1$, $\{k-1\} \notin \mathcal{F}_{t(k)}(f)$. Otherwise, upon learning the fact that $\theta = k-1$, the sender is better off pretending that $\theta = k$. Thus, it follows that $t(n) < t(n-1) < ... < t(2) < t(1)$. Moreover, if $t(k) > n+1-k$ for some $k > 1$, it follows that $t(2) \geq n$. In this case, the sender has a profitable deviation to a strategy by which she performs all the experiments by $t = n$ and induces the action $argmax_a\, u^R(a, \theta = 2)$ in case $\theta = 1$.

I established that for any $k > 1$, $t(k) = n + 1 - k$. Thus, for $t < n$ it must hold that $f(t) = n + 1 - t$. Moreover, different realizations of ϵ_k (an experiment that is performed at $t(k)$) must induce distinct reports at $t(k)$. Submitting $m_{t(k)} \neq \sigma$ terminates the game. Since e is fully informative, $\epsilon_k = 0$ must be followed by $m_{t(k)} = \sigma$. QED

Next, a sufficient condition for a fully informative equilibrium is provided. This condition relates the sender's preferences to the prior distribution. It replaces the standard supermodularity condition that is often assumed in information economics when considering interactions with informed agents.

The sender's preferences satisfy *p-supermodularity* if, for every $\bar{a} > \underline{a}$ and $\bar{\theta} > \underline{\theta}$, the following inequality holds:

$$p(\bar{\theta})[u^S(\bar{a},\bar{\theta}) - u^S(\underline{a},\bar{\theta})] > p(\underline{\theta})[u^S(\bar{a},\underline{\theta}) - u^S(\underline{a},\underline{\theta})].$$

Proposition 2 *If the sender's preferences satisfy p-supermodularity, a fully informative equilibrium exists.*

Proof I begin the proof with the following observation.

Observation 1 *Since the learning is unobservable and unverifiable, and it consists of deterministic and costless experiments, it is without loss of generality to assume that the sender's (pure) learning plan is a permutation $f \in S_n$.*

Consider the sender's strategy $s^* = (f, m^f)$, where $f(t) = n + 1 - t$ and $m^f_\tau \neq \sigma$ if and only if $\epsilon_{f(\tau)} = 1$. The observable part of the sender's strategy (the reporting component) has full support (whenever $m_t \neq \sigma$ let the sender report a random message in M). Therefore, the whole system of the receiver's beliefs is pinned down by the Bayes' rule. Accordingly, the receiver's best response is to select $a^*(\{m_t\}_{t\in\mathbf{T}}) = a^R(n+1-\tau)$ if $m_\tau \neq \sigma$ for some $\tau \leq n$, and $a^*(\sigma) = \phi$ where σ denotes $m_t = \sigma$ for all $t \leq n$. It is left to show that (f, m^f) is the sender's best response to $a^*(\cdot)$.

First, I show that m^f is optimal given f. Since $\phi \in \Theta$, it cannot be optimal for the sender to submit a "false positive" report (to induce $a \neq \phi$ as long as the possibility that $\theta = \phi$ is not ruled out). On the other hand, once θ is revealed to the sender, she cannot benefit from delaying the report because $u^S(a,\theta)$ represents single-peaked preferences for each θ and $a^R(\theta) < a^S(\theta)$.

I now show that there does not exist an alternative learning plan g, and a "*g-measurable*" reporting strategy m^g such that (g, m^g) constitutes a profitable deviation.

Let $g \in S_n$ such that $g \neq f$. Let k be the minimal integer with the property that $g(k) > f(k)$. Define $g' = (g(k), g(k-1)) \circ g$ and let m^g and $m^{g'}$ denote some optimal reporting plans given g and g' respectively.[1] I now show that the sender prefers $(g', m^{g'})$ over (g, m^g). Since at least one of the pure learning plans in S_n is consistent with a best response, this concludes the proof.

Since $\phi \in \Theta$, the reporting strategies m^g and $m^{g'}$ are essentially the same (they induce identical receiver's actions) whenever $\theta \notin \{g(k), g(k-1)\}$: the sender never induces $a \neq \phi$ as long as $\theta = \phi$ is not ruled out, and $g'^{-1}(\theta) = g^{-1}(\theta)$ for every $\theta \notin \{g(k), g(k-1)\}$. That is, every such θ is learned by the

[1] For a permutation $f \in S_n$ and $i, j \in \{1, .., n\}$, $(i,j) \circ f$ denotes the permutation obtained by swapping the elements i and j in f.

sender at identical times under both learning plans g and g'. Thus, to compare g and g' it suffices to evaluate the change in the utilities at the states $\theta \in \{g(k), g(k-1)\}$.

From the definition of k, it follows that $g(k) > g(k-1)$. Otherwise, $g(k-1) \geq g(k)+1 > f(k)+1 = f(k-1)$.

Let $\underline{a}$ and $\bar{a}$ denote the actions that are taken by the receiver after $m_t \neq \sigma$ at $t = k$ and $t = k-1$ respectively. Recall that $\underline{a} < \bar{a}$. As $g(k) > f(k)$, expediting $\epsilon_{g(k)}$ from $t = k$ to $t = k-1$ generates an (expected) improvement of

$$p(\theta = g(k)) \cdot [u^S(\bar{a}, g(k)) - u^S(\underline{a}, g(k))].$$

On the other hand, a delay of $\epsilon_{g(k-1)}$ from $t = k-1$ to $t = k$ can have one of two possible consequences. If $u^s(\bar{a}, g(k-1)) < u^s(\underline{a}, g(k-1))$, such a delay does not affect the sender's utility because the optimality of m^g implies that if $\epsilon_{g(k-1)} = 1$, then $m^g_\tau \neq \sigma$ for some $\tau \geq k$ (it induces an action lower than $\bar{a}$). This means that under g', the sender can induce the same action at $\theta = g(k-1)$ since, as in the case according to g, the experiment $\epsilon_{g(k-1)}$ is performed no later than τ.

Alternatively, if $u^s(\bar{a}, g(k-1)) \geq u^s(\underline{a}, g(k-1))$ a delay of $\epsilon_{g(k-1)}$ from $t = k-1$ to $t = k$ amounts to the expected loss of

$$p(\theta = g(k-1)) \cdot [u^S(\underline{a}, g(k-1)) - u^S(\bar{a}, g(k-1))].$$

In this case, if $\epsilon_{g(k-1)} = 1$ then $m^{g'}_k \neq \sigma$, because $u^s(\cdot, g(k-1))$ represents single-peaked preferences.

To summarize: the net value of the swap is

$$p(\theta = g(k)) \cdot [u^S(\bar{a}, g(k)) - u^S(\underline{a}, g(k))] + p(\theta = g(k-1)) \cdot max\{0, [u^S(\underline{a}, g(k-1)) - u^S(\bar{a}, g(k-1))]\} \geq$$

$$p(\theta = g(k)) \cdot [u^S(\bar{a}, g(k)) - u^S(\underline{a}, g(k))] + p(\theta = g(k-1)) \cdot [u^S(\underline{a}, g(k-1)) - u^S(\bar{a}, g(k-1))] > 0.$$

where the last inequality follows from the *p-supermodularity* of u^S.

Thus, $s^* = (f, m^f)$ is a best response to $a^*(\cdot)$ and the proposition follows. QED

I interpret this result as follows. Even though there is no "objective" reason to expedite information transmission, sometimes it is possible to put the sender under an effective form of "strategic" pressure. In particular, for the expert not to have an opportunity to inflate the report in the direction of her bias, it is important that the set of inducible actions shrink sufficiently fast over time in the direction opposite to the bias. Relatively high actions can be induced by the sender only during the early stages of the game. Of course, for the communication to be efficient, it is also important that the set of inducible actions shrink at a pace that is in keeping with the expert's physical ability to acquire information.

In Sections 6 and 7, I show that the intuitions of strategic pressure and exploration manner (in terms of direction) also apply to other situations in which the sender's information is acquired through a different learning technology and the bias is not of constant direction.

It is worthwhile to note that the assumption that $U^i(a, \theta) = -\infty$ if $a = \phi \oplus \theta = \phi$ is clearly stronger than required. As the example in the Introduction suggests, often a much weaker assumption will suffice to support this equilibrium. I include this assumption in order to clear the proof of additional elements that depend on the fine specifications of the game. It simplifies the exposition, highlights the general aspect, and helps in the later comparison with the informed sender benchmark in section 5.

4 The Role of ϕ

I now discuss the role of the special state ϕ and develop a related result in a state space not containing ϕ for the discrete uniform quadratic case (defined below). The absence of ϕ might severely complicate the derivation of most-desired equilibria from the ex-ante perspective. On the other hand, it is shown below that the effect of the existence of the state ϕ on the informativeness of such equilibria becomes insignificant if the state space is large relative to the divergence between the players' interests.

Consider again the sender's strategy in the fully informative equilibrium given in Proposition 2. As long as she has not learned θ, the mere possibility of $\theta = \phi$ prevents her from inducing an "active" action ($a \neq \phi$). Assume now that $\Theta = \{1, 2, .., n\}$ and that the sender follows the same learning plan as in Proposition 2. If the divergence in interests is not too great, during the early periods the sender will not have an incentive to submit a "false positive" report. By doing so she will induce a high action and benefit in the case where θ is sufficiently high (below the receiver's belief). On the other hand, if θ turns out to be very low, submitting a "false positive" report will induce an action which is too high even from the perspective of an upwardly biased sender. Thus, during the early stages of the game, the sender's behavior as in Proposition 2 seems plausible in this environment as well. However, as time goes by, the risk of "far low states" is reduced. For example, if $\epsilon_k = 0$ for all $k > 1$, then rather than wait another period, the sender is better off submitting a false positive report that corresponds to "$\epsilon_2 = 1$."

This suggests that if $\Theta = \{1, .., n\}$ one can proceed with the logic of the equilibrium as in Proposition 2, but not all the way down the state space. Instead, at some point, the players must engage in another reporting phase, which will not be fully revealing. Consider the following specification of the model.

A discrete uniform quadratic case: $\Theta = \{1, .., n\}$, $p(k) = \frac{1}{n}$ for each $k \in \Theta$, the players' preferences are represented by $u^R(a, \theta) = -(\theta - a)^2$, and $u^S(a, \theta, b) = -(\theta + b - a)^2$, where $b \in \mathbb{N}$. The set of available experiments is $\{\epsilon_k | k \in \Theta\}$. As before, the sender is allowed to submit only one report but she can choose the timing of the report's release.

In Proposition 3, I prove that for the discrete uniform quadratic case, there exists a threshold that depends on b (and does not depend on n), such that, in equilibrium, the receiver can discover every state above this threshold.

Proposition 3 *There exists an equilibrium that attains complete separation of every state $\theta \geq 10b$.*

Proof See Appendix. QED

The proof of this proposition is more involved than the proof of Proposition 2. An important feature in the proof of Proposition 2 is that, due to ϕ, we can limit attention to strategies in which a nonempty report is submitted *only* if the sender knows θ. In the proof of Proposition 3, however, strategies in which both the learning plan is changed and a false positive report is submitted should also be considered. Moreover, an important assumption in Proposition 2 is that ϵ_ϕ is not part of the sender's arsenal. In contrast, now the sender can directly explore any of the states.

The sender knows that she has a given amount of time before choosing whether to induce each action. She might find it optimal to invest this time in exploring some "low" states and upon ruling them out to induce a relatively high action. Dealing with these kinds of deviations is central in the proof of Proposition 3 (see step 3 in the proof). The threshold selected in this proposition ensures that it will take the sender too long (and hence be too costly) to rule out a sufficient amount of low states. For a threshold of the form $c \cdot b$, $c = 10$ is the minimal integer for which the proposition holds for all n and b. Selecting $c \leq 8$ will not suffice for any $b \in \mathbb{N}$, and, for sufficiently small values of b, $c = 9$ is also insufficient.

Proposition 3 does not present the optimal equilibrium. Sometimes the players can benefit if they choose a more complicated equilibrium below the threshold (for example, playing some equilibrium à la Crawford and Sobel (1982) in this region). The main purpose of this Proposition is to illustrate how the logic of the equilibrium in Proposition 2 can be applied in a more standard state space. Essentially, by giving up communication about states below a certain threshold, an artificial substitute for ϕ is created. In the suggested equilibrium, we again see a similar form of strategic pressure as high actions can be induced only during the early stages of the game. Importantly, the threshold in Proposition 3 is independent of n. In this sense, the effect of ϕ on the optimal equilibrium becomes negligible as the state space becomes large.

5 The Informed Sender Benchmark

At this point, it is natural to compare the results in Propositions 2 and 3 with equilibria à la Crawford and Sobel (CS equilibria), namely, the completely informed sender benchmark. I begin with a general observation regarding the structure of Pareto efficient equilibria when ϕ is an element of the state space and the sender is completely informed. Then I focus on the *optimal* CS equilibrium in the discrete uniform quadratic case (with and without ϕ).[2]

Observation 2 *In any (ex-ante) Pareto efficient equilibrium, $\{\phi\}$ is a completely isolated element in the receiver's information structure.*

[2]In the uniform quadratic case, both players rank different equilibria identically. Therefore, there exists an unambiguously ex-ante most-desirable equilibrium, which is termed the *optimal* equilibrium.

Proof Note that the receiver's information partition $\Theta_\phi = \{\{\phi\}, \Theta - \{\phi\}\}$ is consistent with equilibrium. Denote this equilibrium by e_ϕ. Recall that whenever $a = \phi \oplus \theta = \phi$, $U^i(a,\theta) = -\infty$. Thus, the equilibrium e_ϕ is mutually preferred by both players over any equilibrium in which ϕ is not isolated. QED

It is easy to see that any equilibrium on the state space $\{1, 2, .., n\}$ corresponds to an equilibrium on $\{\phi, 1, 2, .., n\}$ in which ϕ is completely isolated (and vice versa). Thus, it is sufficient to determine the receiver's information structure that is obtained in the optimal CS equilibrium on $\Theta = \{1, 2, .., n\}$.

Essentially, CS equilibria in a discrete state space are similar to the continuous case. However, now the sender's types have a strictly positive measure. Thus, in general, limiting attention to partitional equilibria is with loss of generality. Yet, in a separate technical report it is shown that in the discrete uniform quadratic case, the optimal equilibrium is always partitional and it can be derived according to the following procedure:[3]

Procedure 1 Define $x_1 = 1$ and $x_{j+1} = x_j + 4b - 2$ for $j \in \mathbb{N}$. Next, find the index i^* that

$$\sum_{i=1}^{i^*} x_i \leq n < \sum_{i=1}^{i^*+1} x_i$$

and denote the residual by

$$y = n - \sum_{i=1}^{i^*} x_i.$$

Let $\{y_i\}_{i=1}^{i^*}$ be the maximal (in a lexicographic sense) weakly increasing sequence where $y_i \in \{0\} \cup \mathbb{N}$ and $\sum_{i=1}^{i^*} y_i = y$.

The optimal CS equilibrium partitions the states $\{1, 2, ..., n\}$ into "intervals" of magnitude $x_i + y_i$.

For example, let $\Theta = \{\phi, 1, 2, .., 50\}$ and $b = 2$. Then, in the optimal equilibrium ϕ is isolated. The first elements of the x-sequence are $x_1 = 1$, $x_2 = 7$, $x_3 = 13$ $x_4 = 19$, $x_5 = 25$, accordingly $i^* = 4$ and $y = 10$. The lexicographically maximal weakly increasing sequence $\{y_i\}_{i=1}^{4}$ is then given by $\{2, 2, 3, 3\}$.

ϕ	1,..,3	4,..,12	13,..,28	29,..,50

A standard feature of this equilibrium is that higher "intervals" contain more elements. Namely, when b is fixed, as n increases the receiver obtains less precise information regarding the high states of the world. This is, of course, not the case in Propositions 2 and 3. Recall that under the assumptions of Proposition 2 (which are clearly satisfied here) a fully informative equilibrium exists. If ϕ is not part of the state space, then, according to Proposition 3, a complete separation of states $\theta \geq 20$ is attainable in equilibrium.

[3]Details are given in Frug (2014b).

6 Bias towards Extreme Actions

This section illustrates by way of example how the insights presented in Section 3 (exploration direction and strategic pressure) can be applied to a situation in which the sender's bias is not of constant direction.

Let $\Theta = \{-2, -1, 0, 1, 2\}$ and assume that $p(\theta) = \frac{1}{5}$ for each $\theta \in \Theta$. Let $u^R(a, \theta) = -(\theta - a)^2$ and $u^S(a, \theta) = -(\theta + sign(\theta) - a)^2$ be the players' utility functions.[4] These functions capture the idea that, relative to the receiver, the sender is biased towards the extreme actions. In particular, within each region (positive/negative), the sender has a constant bias of 1.

It is easy to see that if the sender is completely informed, under the optimal equilibrium the receiver learns whether the state is negative, positive, or 0 (the unique point of agreement).

Now I incorporate gradual learning. Assume that for each $k \in \Theta$, there is an experiment ϵ_k which reveals to the sender whether $\theta = k$. As before, each experiment takes one unit of time, the sender can choose when to submit a report, and at most one report is allowed.

Observe that, similar to the logic in Proposition 1, if a fully informative equilibrium exists, it must be the case that in such equilibrium ϵ_2 is performed before ϵ_1 and that ϵ_{-2} is performed before ϵ_{-1}. I now show that such an equilibrium exists.

Claim 1 *A fully informative equilibrium exists.*

Proof Consider the sender's strategy $s^* = (f, m^f)$, where $f(t) = (-1)^t \lfloor 3 - \frac{t}{2} \rfloor$, that is,

$$\begin{pmatrix} t: & 1 & 2 & 3 & 4 & 5 \\ f(t): & -2 & 2 & -1 & 1 & 0 \end{pmatrix},$$

and $m_t^f \neq \sigma$ if and only if $\epsilon_{f(t)} = 1$. The observable part of this strategy has full support. Thus, the receiver's beliefs are derived from the Bayes' rule and the receiver's best reply is

$$a^*(m) = \begin{cases} (-1)^t \lfloor 3 - \frac{t}{2} \rfloor & m_t \neq \sigma,\ t \leq 4 \\ 0 & m_t = \sigma,\ \forall t \leq 4. \end{cases}$$

I now show that s^* is the sender's best response to $a^*(m)$. Denote by $d_s(\theta)$ the distance between the sender's most-desirable action and the action induced by the strategy s at state θ. The sender's expected utility if she plays according to s is simply $E[u^S|s] = -\frac{1}{5} \sum_{\theta \in \Theta} (d_s(\theta))^2$. Under s^*, whenever $\theta \neq 0$ the sender experiences a loss of 1, and thus $E[u^S|s^*] = -\frac{4}{5}$.

[4] Recall that $sign(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases}$ for $x \in \mathbb{R}$.

Clearly, m^f is optimal given f. Note that if $s' = (g, m^g)$ is a profitable deviation, it must be the case that the sender is better off at $\theta = 1$ or at $\theta = -1$.

First, assume that the sender is better off at $\theta = -1$. In this case, under s', the sender induces the action $a = -2$ at the state $\theta = -1$. If $g(1) = -1$, and $m_1 \neq \sigma$ if and only if $\epsilon_{-1} = 1$, the action $a = -2$ is not induced at the state $\theta = -2$. In this case $d_{s'}(-2) \geq 2$, and $E[u^S|s'] \leq -\frac{1}{5}((d_{s'}(-2))^2 + (d_{s'}(2))^2) \leq -1 < E[u^S|s^*]$. Otherwise, it must be the case that according to the sender's strategy, $\epsilon_{g(1)} = 0 \Rightarrow m_1 \neq \sigma$. In this case, there are at least four states in which the action $a = -2$ is induced. Thus, $d_{s'}(\eta) > 2$ for some $\eta > 0$ and $E[u^S|s'] \leq -\frac{1}{5}(d_{s'}(\eta))^2 < -1 < E[u^S|s^*]$.

Now, assume that the sender is better off at $\theta = 1$. In this case, under s', the sender induces the action $a = 2$ at the state $\theta = 1$. If $1 \in \{g(1), g(2)\}$, and $m_2 \neq \sigma$ if and only if $\epsilon_1 = 1$, then either $2 \notin \{g(1), g(2)\}$ or $-2 \notin \{g(1), g(2)\}$. Consequently, $d_{s'}(\eta) \geq 2$ for some $\eta \in \{-2, 2\}$, and so $E[u^S|s'] \leq -\frac{1}{5}((d_{s'}(-2))^2 + (d_{s'}(2))^2) \leq -1 < E[u^S|s^*]$. Otherwise, it must be the case that according to the sender's strategy, $\epsilon_{g(1)} = 0 \wedge \epsilon_{g(2)} = 0 \Rightarrow m_2 \neq \sigma$. In this case, the action $a = 2$ is induced in at least three different states, and so there is $\eta \leq 0$ with $d_{s'}(\eta) \geq 2$, and therefore $E[u^S|s'] \leq -\frac{1}{5}((d_{s'}(\eta))^2 + (d_{s'}(2))^2) \leq -1 < E[u^S|s^*]$.
QED

Again, compared to the completely informed sender benchmark, the players manage to attain a mutually preferred equilibrium in which the sender is put under strategic pressure. Certain actions can be induced only during the early stages of the game.

In this section, the logic of Proposition 1 is applied to positive and negative regions separately. It may seem tempting to consider another equilibrium candidate in which, roughly speaking, the whole problem is separated into two sub-problems where the sender explores, say, the positive region first and afterwards turns to the negative region. It turns out that such an intuition is inconsistent with equilibrium. This is because, contrary to the example in the Introduction, now the sender can directly explore any of the states (ϵ_0 *is* part of the sender's arsenal). To achieve complete separation in equilibrium, it is important that states in a given region (negative/positive) not be completely ruled out as long as there is still more than one state to be examined within the opposite region.

7 Alternative Learning Technologies

To demonstrate how the gradualness of the sender's learning can enhance informativeness in communication I assumed a particular learning technology. Refer to the sender's learning in the previous sections as *learning by identification* because at each stage only one element can be inspected. Clearly, the result of having a fully revealing equilibrium in Proposition 2 is an extreme case, which relies on this particular learning technology. In this section I address other learning technologies and elaborate on the intuition of strategic pressure.

I begin with introducing a slightly more general model which accounts for a more general sender's experimentation. This model will serve as a unified framework for what has been presented above and all the following examples.

An *experiment* $\mathbf{Q} = \{Q(\theta) : \theta \in \Theta\}$ is a partition of Θ with the standard interpretation that upon performing $\mathbf{Q}$, the sender learns that $\theta \in Q(\theta)$. A *learning technology* is characterized by the set of available experiments. This set, denoted by $EX = \{\mathbf{Q}_i | i \in I\}$, where I is a (finite) set of indices, is commonly known. Each experiment takes one unit of time and it is impossible to conduct more than one experiment at a time. The experiments performed and their outcomes are unobservable and nonverifiable.

At the end of each period, the sender submits a cheap-talk report regarding her progress in exploration. (In previous sections, it was convenient and without loss of generality to restrict the sender's reporting opportunities. With more general learning technologies, allowing for only one report is with loss of generality.) Action $a \in \mathbb{R}$ must be selected by the receiver at time $T > |\Theta|$. This captures the idea that, in principle, the time available for exploration places no restrictions on the expert's ability to explore the state.

Clearly, the completely uninformative equilibrium always exists. Also, if the set EX is *rich* in the sense that upon performing all of the experiments in EX the sender becomes completely informed, it is immediate that any equilibrium of the informed sender benchmark can be reproduced. I now turn to analyze two examples of alternative learning technologies that may be natural under some circumstances, and then conclude this section with several general observations.

7.1 Criteria Learning

The total value may be determined by several different criteria of varying importance. Exploring these criteria separately one after another is a natural way of learning in this environment. Regardless of the order in which the criteria are explored, the information is acquired gradually. Consider the following example in which state θ is determined by the values of two different variables.

Let $\zeta, \eta \sim U\{0, .., k\}$ and let $\theta = (k+1)\zeta + \eta$. To verify that $\theta \sim U\{0, k^2 + 2k\}$, write every $x \in \{0, .., k^2 + 2k\}$ in terms of its unique representation $x = (k+1)x_1 + x_2$, where $x_i \in \{0, .., k\}$. Then, $Pr(\theta \leq x) = Pr(\zeta < x_1) + Pr(\zeta = x_1) \cdot Pr(\eta \leq x_2) = \frac{x_1}{k+1} + \frac{1}{k+1} \cdot \frac{x_2+1}{k+1} = \frac{1}{(k+1)^2} \cdot (x+1)$. Again, assume that the players minimize the (expected) quadratic loss and denote the sender's bias by $b \in \mathbb{N}$.

Claim 2 *Let $b = 1$. The optimal equilibrium under the completely informed sender benchmark splits Θ into $k+1$ "intervals" of different lengths.*

Proof To see this, observe that for each I, we have $\sum_{i=1}^{I}(1 + 2(i-1)) = I^2$. According to procedure 1, the length of the i-th interval is exactly $1 + 2(i-1)$. The claim follows as $|\Theta| = (k+1)^2$. QED

Comment Clearly, for every $b > 1$, every equilibrium splits Θ into a smaller number of intervals.

I now introduce strategic gradual learning. Assume that two experiments are available,

$$EX = \{check_{\zeta}, check_{\eta}\},$$

such that each experiment returns the realized value of the corresponding random variable.

If a decision maker has to explore the state himself and he has only limited time to explore, it is natural that he would focus on the most important criterion. On the other hand, if a decision maker has sufficient time to check all the attributes before taking an action, the ordering of the experiments would not matter. In our setup, the decision maker gathers information via reports that are submitted by a biased expert. In this case, even in the absence of direct time constraints, optimal equilibrium again involves indirect pressure that forces the expert to submit an early report. Contrary to *learning by identification*, if the bias is sufficiently small, additional reporting can be useful.

Claim 3 *Let* $b \leq \lceil \frac{k}{2} \rceil$*. There exists an equilibrium that is strictly preferred to the optimal equilibrium under the completely informed sender benchmark.*

Proof I construct an equilibrium. Consider the following strategy profile: the sender performs $check_{\zeta}$ (explores the most important criterion) and submits a "separating" report at $t = 1$, denoted by $m^*(\zeta)$. Regardless of the report at $t = 1$, no valuable information is transmitted at $t = 2$ (the sender "babbles"). The receiver plays his best response (denoted by $a^*(m)$). The choice of $b \leq \lceil \frac{k}{2} \rceil$ ensures that, provided that the sender's information structure corresponds to knowing only the value of ζ, she is willing to reveal the truth. To show that this is indeed an equilibrium I now show that the sender cannot benefit from performing $check_{\eta}$ instead of $check_{\zeta}$. Let $a^S(\eta)$ denote the most-desirable action, from the sender's perspective, given a realization of η (while ζ is unknown). Notice that

$$a^S(\eta) = argmax_a \, E[u^S(a, \theta) | \theta \in \{(k+1)\zeta + \eta\}] = \frac{k(k+1)}{2} + \eta + b.$$

Substitute and rearrange terms to get

$$E[u^S(a^*(m(\eta)), \theta)] \leq -\frac{1}{(k+1)^2} \sum_{\eta=0}^{k} \sum_{\zeta=0}^{k} (\zeta \cdot (k+1) + \eta + b - a^S(\eta))^2 = -(k+1) \sum_{\zeta=0}^{k} (\zeta - \frac{k}{2})^2 =$$

$$= -\frac{k(k+2)(k+1)^2}{12} \underset{(*)}{<} -(\frac{(k+1)^2}{12} + 1) < -\frac{k^2 + 2k + 12}{12} = E[u^S(a^*(m^*(\zeta)), \theta)].$$

The RHS is the sender's expected utility if she follows the equilibrium strategy. The LHS is her utility if she deviates and privately explores η instead of ζ. The inequality denoted by $(*)$ holds for every integer $k \geq 2$. It is straightforward to verify that for $k = 1$, the suggested profile of strategies also constitutes an equilibrium. In the latter case, however, it is a weak equilibrium.

I established a possible equilibrium in the environment of strategic gradual learning. To complete the proof, note that the suggested equilibrium splits Θ into $k+1$ intervals of the same length. Thus, the variance of this partition is necessarily lower than the variance of the unbalanced partition that is obtained under the optimal equilibrium in the completely informed sender benchmark (see claim 2). QED

Note that claim 3 does not characterize the optimal equilibrium under strategic learning. Contrary to previous sections, here the players can benefit from allowing for more than one round of meaningful communication. If $4b \leq k+1$, after revealing ζ at $t=1$ the players can have another report regarding the residual state space in which they play the standard CS equilibrium. For example, if $k=3$ and $b=1$, the resulting receiver's information partition under the optimal equilibrium is $\{\underbrace{\{0\},\{1,2,3\}}_{\zeta=0},\underbrace{\{4\},\{5,6,7\}}_{\zeta=1},\underbrace{\{8\},\{9,10,11\}}_{\zeta=2},\underbrace{\{12\},\{13,14,15\}}_{\zeta=3}\}$.

Comment In effect, I assumed two criteria such that ζ is lexicographically more important than η. It was convenient in the proof that the resulting distribution of θ be uniform so that I could directly apply procedure 1. This is obtained by setting the coefficient of ζ to be equal to $k+1$. It is easy to see that for any coefficient greater than $k+1$, the suggested play remains an equilibrium and whenever the coefficient is not too large, gradualness of learning makes both players (strictly) better off.

7.2 Cutoff Learning

The current learning technology, contrary to *learning by identification,* admits the extreme opposite result. It is shown below that if the sender can at each period select an element $x \in \Theta$ and privately learn whether $\theta \geq x$, gradual learning does not increase the amount of information that can be transmitted in equilibrium, relative to the informed sender benchmark.

A "cutoff k" experiment, $c_k = \mathbf{1}_{\theta \geq k}$, reveals to the sender whether $\theta \geq k$. The set of available experiments in the cutoff learning regime is given by the collection

$$EX = \{c_k : k \in \Theta\}.$$

In each period the sender privately chooses a cutoff k_t, which may depend on the results of previous experiments. Thus, a sender's strategic learning is a plan $(k_t(h_t))_{t \in T}$, where $h_t = (c_{k_s})_{s<t}$. Given any history, the experiment $c_{min\,\Theta}$ is uninformative from the sender's perspective and so it can be interpreted as "doing nothing." In this sense, our definition of a learning plan allows the sender to avoid acquiring information.

The following result is given under fairly general assumptions. Let Θ be a linearly ordered discrete state space (for a state space like that presented in Section 2, let $\phi < \theta$ for every $\theta \neq \phi$). Assume that for each θ, $u^R(\cdot,\theta)$ is concave and that for each θ, $u_1^R(\cdot,\theta) = 0$ for some a. This ensures that, given any information, there is a unique optimal action from the receiver's perspective. Also assume that u^i is supermodular.

Proposition 4 *Under cutoff learning regime, every equilibrium is equivalent (in terms of the transmitted information) to some CS equilibrium.*

Proof See Appendix. QED

I conclude this section with several general observations about the relation between the learning technology and equilibrium informativeness. Observation 3 shows that even in the enriched environment assumed in Section 2, fully revealing equilibria are indeed rare. It is shown that *learning by identification* is "almost" the unique set of experiments that allows for complete separation. One particular implication of this observation is that adding an experiment to a rich collection of experiments can make the players *worse off*. On the other hand, in Observation 4 it is shown that adding experiments to rich collections of experiments can also make the players *better off*. Observation 5 shows that giving up experiments that are unused in the optimal equilibrium can also make the players *better off*.

Observation 3 *Consider the environment of Section 2 ($\Theta = \{\phi, 1, .., n\}$ and $EX = \{\epsilon_k | k \in \Theta - \{\phi\}\}$) and an experiment $\mathbf{Q} \notin EX$. If there is $A \subset \mathbf{Q}$ such that $\phi \notin \cup A$ and $1 + min \cup A \in \cup A$, then adding the experiment $\mathbf{Q}$ to the set EX makes full revelation inconsistent with equilibrium.*

Proof Assume by way of contradiction that a fully informative equilibrium exists and denote it by e. Similarly to Proposition 1, for each $k \in \Theta$ denote by $t(k)$ the last period in which the information that $\theta = k$ is transmitted under e. It must be the case that, according to the sender's equilibrium learning plan, $\{k-1\}$ is not an element of the sender's information structure of the period $t(k)$, for any $k > 1$. Thus, it follows that $t(n) < t(n-1) < ... < t(2) < t(1)$. The sender is better off performing $\mathbf{Q}$ at $t(1 + min \cup A)$ and inducing the action $a^R(1 + min \cup A)$ at $\theta = min \cup A$. QED

This implies that adding an experiment $\mathbf{Q}$ is quite likely to make the players worse off. For example, if $\mathbf{Q}$ contains a nontrivial convex element, enriching EX with $\mathbf{Q}$ makes the players worse off in equilibrium. An example of an experiment $\mathbf{Q}$ such that the set of experiments $\{\mathbf{Q}, \epsilon_k | k \in \Theta - \{\phi\}\}$ allows for a fully informative equilibrium under the uniform quadratic case with $b = 1$ is $\mathbf{Q} = \{\{\phi, 2\}, \{\phi, 2\}^c\}$.

Observation 4 *Adding experiments to a rich collection of experiments can make the players better off.*

Example Consider the uniform quadratic case with $\Theta = \{1, .., 10\}$ and $b = 1$. By Proposition 4 and procedure 1, the optimal equilibrium under the *cutoff learning* regime induces the partition $\{\{1\}, \{2, .., 4\}, \{5, .., 10\}\}$ for the receiver. Let $\mathbf{Q} = \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}, \{9, 10\}\}$. If $\mathbf{Q}$ is added to the set of available experiments the following constitutes an equilibrium. The sender performs $\mathbf{Q}$ at $t = 1$

and *immediately* reports truthfully the outcome of this experiment (all subsequent reports are completely uninformative). Provided that the sender's information structure is given by $\mathbf{Q}$, it is immediate that truth-telling is incentive compatible and a direct calculation shows that the sender's expected payoff in this equilibrium is $-\frac{5}{4}$. To see that the sender does not have an incentive to deviate and perform another experiment, note that *every* cutoff experiment partitions Θ in two. This implies that performing such an experiment at $t = 1$ will induce at most two receiver's actions. Thus, there are at least four states with $d_{s'}(\theta) \geq 2$ where s' denotes a sender's strategy by which she performs any experiment other than $\mathbf{Q}$ at $t = 1$.[5] Therefore, $E[u^S(a^*, \theta)|c_j] < -\frac{1}{10} \cdot 4 \cdot 2^2 < -\frac{5}{4}$.

Observation 5 *Dispensing with experiments that are unused in the optimal equilibrium can make the players better off.*

Example Consider the uniform quadratic case with $\Theta = \{1, .., 5\}$ and $b = 1$. By Proposition 4 and procedure 1, the optimal equilibrium under the *cutoff learning* regime induces the partition $\{\{1\}, \{2, .., 5\}\}$ for the receiver. Moreover, this can be attained by a report at $t = 1$ after the sender performs only c_2. The set of experiments $EX' = \{c_2, c_5\}$ is obtained by dispensing with some unused experiments. The following sender's strategy is consistent with equilibrium. Perform c_5 at $t = 1$ and report truthfully. If $\theta < 5$, perform c_2 at $t = 2$ and report truthfully. The partition obtained in this equilibrium refines the receiver's information structure under the optimal equilibrium with $EX = \{c_k : k \in \Theta\}$. Thus, both players are now better off.

8 Related Literature

The literature on costless communication (cheap talk) between informed experts and uninformed decision makers began with the contributions of Crawford and Sobel (1982) and Green and Stokey (1980).[6] Since then, many authors have studied a variety of cheap-talk environments under different specifications. Sobel (2010) provides a comprehensive literature review of the communication literature. Several previous works consider environments where the sender is imperfectly informed. Fisher and Stocken (2001) showed that, in equilibrium, the accuracy of the receiver's information is not monotonic in the quality of the sender's information. This was extended by Ivanov (2010) who characterized the optimal static information structure from the receiver's perspective for the leading uniform-quadratic case. In a recent contribution by Ivanov (2012) an optimal two-stage learning protocol is suggested. The author assumes that the receiver can perfectly control the dynamic structure of the sender's information. By coupling distant separable elements[7] at the early stage and by conditioning future experiments on early truthful communication, the

[5] Recall that $d_{s'}(\theta)$ denotes the distance between the sender's most-desirable action and the action induced by the strategy s' at state θ.

[6] Earlier literature on costly signaling can be traced back to Spence (1973).

[7] A similar idea appears in Golosov et al. (2013).

receiver can successfully elicit full information. The sender is initially uninformed; however, her learning is not strategic but designed by the receiver in these models.

Several works have also considered certain aspects of the sender's strategic learning. Austen-Smith (1994) proposed a model in which the sender chooses whether to learn (perfectly) the state and has the ability to prove information acquisition. The choice whether to become informed depends on the realization of the cost of learning that is privately observed. The author shows that, relative to Crawford and Sobel (1982), more information can be transmitted in equilibrium. As in our model, the choice whether to become informed is left to the sender's discretion. Unlike in our model, the learning is costly, verifiable, and non-gradual. In Argenziano et al. (2011) the sender can affect the quality of her information by selecting how many Bernoulli trials to perform. The trials are costly for the sender. The authors compare "covert" and "overt" selection of the number of trials to be performed and show that under the overt regime it is possible to force the sender to over invest in learning. In Kamenica and Gentzkow (2011) the sender selects a "signal," i.e., an information structure used to persuade a decision maker. While the selection of the information structure is costless and left to the sender's discretion, the receiver observes both the information structure and its realization. In Gentzkow and Kamenica (2012) the sender also publicly selects the information structure but now it is no longer costless. A more informative information structure is associated with a higher cost. Since the information is costly for the sender, endogenous information will always be disclosed in equilibrium and so disclosure requirements have no effect on the set of equilibrium outcomes. A related result appears in Di Pei (2013). In that paper the sender gathers costly information before advising the receiver. The sender communicates all her information and all equilibria are less informative than the most informative one in Crawford and Sobel (1982).

All of the above models that contain strategic learning are essentially static. However, the key feature of the model studied in this paper is the dynamic nature of the sender's learning. The timing of the sender's reports provides a signal to the receiver about the quality of the sender's information. Frug (2014a) considers a model of gradual exogenous learning and communication. That paper focuses on the design of efficient reporting protocols.

9 Conclusion

In this paper, I showed how the players can benefit from the gradualness of an expert's learning when it is strategic and unobservable. The main intuition is that of "strategic pressure." A common feature of desirable equilibria is that certain actions can be induced only during the early stages of the game. Since learning takes time, the expert's information structure in the early stages is still not refined, regardless of her particular learning strategy. Knowing that, the expert strategically chooses which experiments to perform and in what order. The set of inducible actions shrinks sufficiently fast with time, in synch with the expert's physical ability to acquire information. This provides the decision maker with an idea about the expert's information structure and how it develops over time.

The main ingredients of the environment in this paper are (1) gradual endogenous learning and (2) the time available for learning places no direct restrictions on the overall quality of the expert's information (before the decision maker is called to play the expert can become completely informed at no cost). If the expert is expected to submit a report at the end of her learning (as she becomes completely informed), the problem of experiment scheduling is completely vacuous. On the other hand, the artificial pressure, presented in most of the results in this paper, raised an important issue of *strategic scheduling* of experiments for the expert. The only motivation for the scheduling problem in our environment is strategic. This may suggest a broader question of strategic scheduling both in the environment of observable experiments (unobservable outcomes) and unobservable experimentation.

More generally, most of the literature on extensive games with asymmetric information assumes that all players' private information is known to them from the onset. Additional strategic considerations might be at play when private information is learned gradually. Thus, gradual attainment of private information can be interesting from a broader perspective.

10 Appendix

Proposition 3 *There exists an equilibrium that attains complete separation of every state $\theta \geq 10b$.*

Proof Consider the following profile of strategies: sender's strategy $s^* = (f, m^f)$: $f(t) = n+1-t$ for all $t \in \{1,..,n\}$, $m^f_\tau \neq \phi$ iff $\epsilon_{f(\tau)} = 1$ and $\tau \in \{1,..,n+1-10b\}$; receiver's strategy: $a^*(\{m_t\}) = n+1-\tau$ if $m_\tau \neq \sigma$ for some $\tau \in \{1,..,n+1-10b\}$ and $a(\{m_t\}) = 5b$ if $m_t = \sigma$ for all $t \in \{1,..,n+1-10b\}$. First, it is clear that the receiver's strategy is a best response to the sender's strategy. I now prove, in three steps, that s^* is a sender's best response to the receiver's strategy, and thus the suggested profile constitutes an equilibrium.

Step 1: Optimality of m^f given the learning plan f. The sender cannot benefit from postponing a report once θ is revealed because this will induce a lower receiver's action, which is worse for the sender. Also, the sender cannot benefit from a "false positive" report $m_{t(\bar{a})} \neq \sigma$ given that $\epsilon_{f(\tau)} = 0$ for all $\tau \leq t(\bar{a})$, for some $\bar{a} \geq 10b$. It is shown below that the sender's expected utility from a "false positive" report at $t(\bar{a})$ (LHS) is lower than her expected utility from continuing to follow the strategy s^* (RHS):

$$-\frac{1}{\bar{a}-1}\sum_{\theta=1}^{\bar{a}-1}(\theta+b-\bar{a})^2 < -\frac{1}{\bar{a}-1}\left(\sum_{\theta=1}^{\bar{a}-10b}(b)^2 + \sum_{\theta=\bar{a}-10b+1}^{\bar{a}-1}(\theta+b-\bar{a})^2\right) =$$

$$= -\frac{1}{\bar{a}-1}\left(\sum_{\theta=1}^{10b-1}(\theta+b-10b)^2 + (\bar{a}-10b)b^2\right) < -\frac{1}{\bar{a}-1}\left(\sum_{\theta=1}^{10b-1}(\theta+b-5b)^2 + (\bar{a}-10b)b^2\right).$$

It is left to show that there does not exist an alternative learning plan $g : \{1,..n\} \to \Theta$ and a g-*measurable* reporting policy m^g such that $s = (g, m^g)$ is a profitable deviation for the sender.

Let $K = \{(g, m^g) : \forall a \geq 10b, m_{t(a)} \neq \sigma \implies \exists t \leq t(a), \epsilon_{g(t)} = 1\}$. Also, let us denote by $a(\theta|s)$ the action induced at state θ if the sender plays according to strategy s.[8]

Step 2: The strategy $s^* = (f, m^f)$ is optimal in K. Let $s = (g, m^g) \in K$ such that $g \neq f$. I show that s is suboptimal. Let k be the minimal integer with $g(k) \neq f(k)$. Therefore, $g(k) < f(k)$. Let $g' = (g(k), f(k)) \circ g$ be the learning plan in which the experiments $\epsilon_{f(k)}$ and $\epsilon_{g(k)}$ are swapped. If $a(g(k)|s) \leq a(f(k)|s)$, g' is clearly preferred to g, as the sender is better off in state $f(k)$ (because a higher action can be induced) and her payoffs remain the same everywhere else. If $a(g(k)|s) > a(f(k)|s)$, g' allows the sender to induce the *higher* action $a(g(k)|s)$ in the *higher* state $f(k)$ and the *lower* action $a(f(k)|s)$ in the *lower* state $g(k)$ without changing the state-to-action mapping at other states. Supermodularity of the sender's preferences ensures that g' is preferred in this case as well. There exists an optimal strategy in K with a pure learning plan, and thus f is optimal in K.

Step 3. To conclude I show that f is better than every strategy outside K. These strategies have the property that there is $a \geq 10b$, such that after performing $t(a)$ "unsuccessful" experiments, the sender induces the action a ($m_{t(a)} \neq \sigma$). I now add some formal notation.

Let $N_{\bar{t}} = \{(g, m^g) : \forall t < \bar{t} \leq t(10b), \epsilon_{g(t)} = 0 \implies m_{\bar{t}} \neq \sigma\}$ and $N = \bigcup_{\bar{t} \leq t(10b)} N_{\bar{t}}$. Also, in the following steps, for a strategy $s = (g, m^g)$ to be considered, I denote $\zeta = max\, \Theta - \{g(t)|t \leq \bar{t}\}$ and $\eta = min\, \Theta - \{g(t)|t \leq \bar{t}\}$.

Step 3.1: An optimal strategy in $N_{\bar{t}}$ satisfies $\eta \leq \theta \leq \zeta \implies \theta \in \Theta - \{g(t) : t \leq \bar{t}\}$. Let $s = (g, m^g) \in N_{\bar{t}}$. Assume that there exists $\theta \in \{g(t)|t \leq \bar{t}\}$ such that $\eta < \theta < \zeta$. Recall that $\omega \in \Theta - \{g(t)|t \leq \bar{t}\} \Rightarrow a(\omega|s) = a(\bar{t}) = n - \bar{t} + 1$.

If $a(\theta|s) > n - \bar{t} + 1$, by supermodularity, the sender is better off performing ϵ_ζ instead of ϵ_θ and inducing the action $a(\theta|s)$ in state ζ (and action $n - \bar{t} + 1$ at θ).

If $a(\theta|s) < n - \bar{t} + 1$, by supermodularity, the sender is better off performing ϵ_η instead of ϵ_θ and inducing the action $a(\theta|s)$ in state η (and action $n - \bar{t} + 1$ at θ).

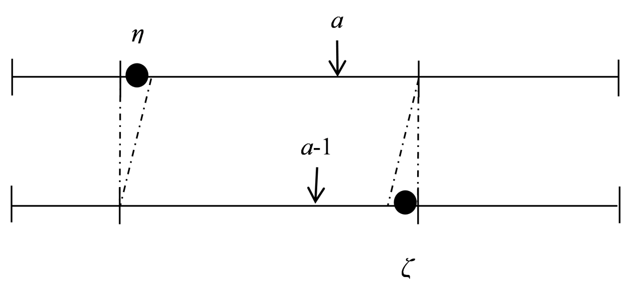
If $a(\theta|s) = n - \bar{t} + 1$, the sender gains nothing by performing ϵ_θ. If, at η, the sender prefers one of the inducible actions that are below $n - \bar{t} + 1$, she can be better off if instead of ϵ_θ she performs ϵ_η. Alternatively, $u^S(n - \bar{t} + 1, \zeta) < u^S(n - \bar{t} + 2, \zeta)$ and $\eta > 1$. Moreover, it must be the case that either $g^{-1}(\eta - 1) < \bar{t}$ or $g^{-1}(\theta) < \bar{t}$. Then, the sender can be better off if ϵ_ζ is performed instead of $\epsilon_{\eta-1}$ or ϵ_θ.

Step 3.2: Let $s = (g, m^g)$ be an optimal strategy in $N_{\bar{t}}$; then, $g^{-1}(\theta_0) > g^{-1}(\theta_1) > g^{-1}(\theta_2)$ for $\theta_0 < \eta$ and $\zeta < \theta_1 < \theta_2$. Again, this is a direct consequence of the supermodularity of the sender's preferences. Let $\hat{\theta}$ be the highest state that is examined after some $\theta < \hat{\theta}$. The sender is better off changing the order of these two experiments.

Step 3.3: Let $s = (g, m^g)$ be an optimal strategy in $N_{t(\hat{a})}$; then $\eta \leq \hat{a} - 3b$. Assume by contradiction that $\hat{a} - 3b \in \{g(t) : t \leq t(\hat{a})\}$. From 3.1 and the fact that $|\Theta - \{g(t) : t \leq t(\hat{a})\}| \geq \hat{a} - 1$, it follows that $\zeta > \hat{a} + 6b$. Hence, $d_s(\zeta) \geq 7b$. If the sender performs ϵ_ζ before any experiment ϵ_ω s.t. $\omega < \eta$, she will be able to induce the action ζ at state ζ. Thus, at this state a loss of $(d_s(\zeta))^2$ is replaced with a loss of b^2. On the other hand, giving up $\epsilon_{\hat{a}-3b}$ increases the loss in this state by at most $(2b)^2$.

[8] $a(\theta|s)$ is well defined whenever the randomness in the sender's strategy is confined to the selection among the nonempty messages.

Step 3.4: The optimal sender's strategy in N is an element of $N_{t(10b)}$. Let $\bar{t} < t(10b)$. Consider an optimal strategy in $N_{\bar{t}}$, and let $a > 10b$ be such that $t(a) = \bar{t}$. I now show that there is a better strategy (from the sender's perspective) in $N_{\bar{t}+1}$. From 3.2, first the sender explores all the states above ζ (from top to bottom) and then turns to the exploration of states below η. Action a is induced at each of the states $\eta, \eta+1, .., \zeta-1, \zeta$. Consider the strategy $s' = (g', m^{g'}) \in N_{\bar{t}+1}$, where s and s' differ only in that before exploring states below η, the sender performs ϵ_ζ. $m_{t(\zeta)} \neq \sigma$ iff $\epsilon_\zeta = 1$, and the action $a-1$ is induced at period $\bar{t}+1$. To see that the sender prefers s' to s, it is sufficient to note that the sender's payoff at ζ under s' is higher than her payoff at η under s. This is illustrated in the figure below.



It is immediate that the sender obtains identical payoffs at all states above ζ and below η under both strategies s and s'. Also, it is easy to see that the sender's payoff at $\theta \in \{\eta+1, .., \zeta\}$ under s is equal to her payoff at $\theta - 1$ under s'. To conclude, note that the sender's payoff at ζ, under s', is $-b^2$. From 3.3, $\eta \leq a - 3b$, which means that the sender's payoff at η, under s, is at most $-(2b)^2$.

Let $s = (g, m^g)$ denote the optimal strategy in $N_{t(10b)}$. I now conclude the proof by showing that the sender prefers s^* to s. Under the strategy s, at least at $10b-1$ states, $\theta_1 < \theta_2 < ... < \theta_{10b-1}$, the action $a = 10b$ is induced. First, assume that $\theta_1 \geq 3b$. The vector of smallest possible distances $(d_s(\theta_j))_{j=1}^{10b-1}$ is given by

$$(5b-1, 5b-2, ..., 1, 0, 1, ..., 5b-2, 5b-1).$$

On the other hand, the distances $(d_{s^*}(\theta_j))_{j=1}^{10b-1}$ are at most

$$(b+1, \ldots, 6b-1, \underbrace{b, \ldots, b}_{5b}).$$

Notice that for any $\theta \in \{g(t) : t < t(10b)\}$ we have $a(\theta|s) = a(\theta|s^*)$. Since the uniform quadratic case is assumed, s and s^*can be compared by summing up the square distances $d_s(\theta)$ and $d_{s^*}(\theta)$ on $\{\theta_1, ..., \theta_{10b-1}\}$ and comparing the costs (negative utility). The following inequality holds[9] for any $b \in \mathbb{N}$:

$$\sum_{j=1}^{10b-1} d_s(\theta_j) \geq 2 \sum_{k=1}^{5b-1} k^2 > (5b)b^2 + \sum_{k=1+b}^{6b-1} k^2 \geq \sum_{j=1}^{10b-1} d_{s^*}(\theta_j);$$

[9]By using the formula for the sum of squares of the first K naturals: $\sum_{k=1}^{K} k^2 = \frac{K^3}{3} + \frac{K^2}{2} + \frac{K}{6}$

therefore, s^* is better than s. By replacing one of the states in $\{\theta_1, ..., \theta_{10b-1}\}$ with a state $\theta < 3b$ the LHS of the strict inequality increases by, at least, $(\theta + b - 10b)^2 - (5b-1)^2$ and the RHS increases by, at most, $(\theta+b-5b)^2-(6b-1)^2$. For any $b \in \mathbb{N}$ and $\theta < 3b$, we have $(\theta+b-10b)^2-(5b-1)^2 > (\theta+b-5b)^2-(6b-1)^2$. Thus, the sender is better off under s^* rather than under s even if we dispense with the assumption that $\theta_1 \geq 3b$. Therefore, s^* is better than s, and the proposition follows. QED

Proposition 4 *Under the cutoff learning regime, every (nonredundant) equilibrium induces a receiver's information partition that is consistent with some CS equilibrium.*

Proof First, observe that the sender's information structure at each time $t \in \mathbf{T}$ is monotonic: for $J_t \subset \Theta$, which is an element of the sender's information partition at time t, if $k_1, k_2 \in J_t$, then every $k_1 \leq k \leq k_2$ satisfies $k \in J_t$. Otherwise there exist $i < j$ such that $c_i = 0$ and $c_j = 1$, a contradiction.

I now show that every equilibrium is monotonic. Assume by contradiction that e is a nonmonotonic equilibrium. By the properties of u^R and as e is nonredundant, every element $\mu \in \tilde{\mu}_e$ induces a distinct, unique action. There exists a maximal element $\hat{\theta} \in \Theta$ such that, under e, $\bar{a}$ is induced with positive probability at $\hat{\theta}$, $\underline{a}$ is induced with positive probability at $\hat{\theta}+1$, and $\underline{a} < \bar{a}$. In particular, it follows that $c_{\hat{\theta}+1}$ is part of the sender's exploration program and that it is performed (weakly) before the reports that induce the actions $\underline{a}$ and $\bar{a}$ are expected at e.

By supermodularity of u^S, if $u^s(\underline{a}, \hat{\theta}+1) \geq u^s(\bar{a}, \hat{\theta}+1)$ then $u^s(\underline{a}, \hat{\theta}) > u^s(\bar{a}, \hat{\theta})$. Thus, the sender is better off replacing $c_{\hat{\theta}+1}$ with $c_{\hat{\theta}}$ because this allows her to join the state $\hat{\theta}$ to the reporting program that induces $\underline{a}$ without affecting the state-to-action mapping anywhere else. On the other hand, assume that $u^s(\underline{a}, \hat{\theta}+1) < u^s(\bar{a}, \hat{\theta}+1)$. If $\hat{\theta}+1 = n$, the sender is better off simply changing the reporting strategy as the element $\hat{\theta}+1$ is isolated by the time $\bar{a}$ is induced. In the case $\hat{\theta}+1 < n$, the sender is better off replacing $c_{\hat{\theta}+1}$ with $c_{\hat{\theta}+2}$. Again, after each of these changes she can join the state $\hat{\theta}+1$ to the reporting program that induces $\bar{a}$ without affecting the state-to-action mapping anywhere else. Thus, in the cutoff learning regime nonmonotonic equilibria are impossible.

Let e be an equilibrium. For $\mu \in \tilde{\mu}_e$ denote by $a(\mu)$ the action selected by the receiver if he holds the belief μ. Assume by contradiction that the induced belief structure is inconsistent with a CS equilibrium. Then, there are two adjacent elements $\mu_1, \mu_2 \in \tilde{\mu}_e$ such that either $u^S(a(\mu_1), min\, supp(\mu_2)) > u^S(a(\mu_2), min\, supp(\mu_2))$ or $u^S(a(\mu_2), max\, supp(\mu_1)) > u^S(a(\mu_1), max\, supp(\mu_1))$. Both actions $a(\mu_1)$ and $a(\mu_2)$ are still inducible at the time in which $c_{min\, supp(\mu_2)}$ is performed. The inequalities above imply that the sender can benefit from replacing $c_{min\, supp(\mu_2)}$ with either $c_{min\, supp(\mu_2)-1}$ or $c_{min\, supp(\mu_2)+1}$, contradicting the assumption that e is an equilibrium. QED

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