Radar Clutter Mitigation via Probability Measure Transform

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Outline

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- Measure transformed Gaussian QML estimator
- Application: robust direction finding in heavy-tailed clutter
- Summary
Background and motivation

Problem
In practical direction finding of radar targets accurate parametric models for the signals and clutter are unavailable
Background and motivation

**Goal**
Estimate $\theta_0 \in \mathbb{R}^m$ given samples $\{X_n\}_{n=1}^N$ from $P_{x;\theta_0}$

**Restriction**
$P_{x;\theta_0}$ belongs to *unknown* parametric family of probability measures

\[ P_{x;\theta_0} \in \{ P_{x;\theta} : \theta \in \Theta \} \]

- The maximum likelihood estimator cannot be implemented
- Resort to methods that require *partial statistical information*
Background and motivation

Gaussian QML estimator [White 1982]

- Minimize the *empirical KLD* between $P_{X;\theta_0}$ and a Gaussian measure $\Phi_{X;\theta}$ with mean $\mu_{X}(\theta)$ and covariance $\Sigma_{X}(\theta)$

- Amounts to maximization of

$$ J(\theta) \triangleq -D_{LD} \left[ \hat{\Sigma}_{X} \| \Sigma_{X}(\theta) \right] - \| \hat{\mu}_{X} - \mu_{X}(\theta) \|^2_{(\Sigma_{X}(\theta))^{-1}} $$

- The GQMLE:

$$ \hat{\theta} = \arg \max_{\theta \in \Theta} J(\theta) $$

- Simple implementation, easy performance analysis

- Sensitive to model mismatch (e.g. in non-Gaussian clutter)
Background and motivation

Proposed approach
GQMLE under a transformed probability distribution of the data

Advantages

▶ Resilient to outliers
▶ Involves higher-order statistical moments
▶ Significant mitigation of the model mismatch effect
▶ Computational advantages of the first and second-order methods of moments
Probability Measure Transform

Definition
Given a non-negative function \( u : \mathcal{X} \rightarrow \mathbb{R}_+ \) satisfying

\[
0 < E [u (X) ; P_{X;\theta}] < \infty.
\]

A transform \( T_u : P_{X;\theta} \rightarrow Q_{X;\theta}^{(u)} \) is defined as:

\[
T_u [P_{X;\theta}] (A) = Q_{X;\theta}^{(u)} (A) \triangleq \int_A \varphi_u (x; \theta) dP_{X;\theta} (x),
\]

where

\[
\varphi_u (x; \theta) \triangleq \frac{u (x)}{E [u (X) ; P_{X;\theta}]}.\]

The function \( u (\cdot) \) is called the **MT-function**.
Probability Measure Transform

The measure transformed mean and covariance

\[
\mu_{X}^{(u)}(\theta) = \mathbb{E}[X \varphi_{u}(X; \theta); P_{X;\theta}]
\]

\[
\Sigma_{X}^{(u)}(\theta) = \mathbb{E}[XX^{H} \varphi_{u}(X; \theta); P_{X;\theta}] - \mu_{X}^{(u)}(\theta) \mu_{X}^{(u)H}(\theta)
\]

where

\[
\varphi_{u}(x; \theta) \triangleq \frac{u(x)}{\mathbb{E}[u(X); P_{X;\theta}]} = \frac{d\mathbb{Q}_{X;\theta}^{(u)}}{dP_{X;\theta}}
\]

Conclusion

- The mean and covariance under \(Q_{X;\theta}^{(u)}\) can be estimated using only samples from \(P_{X;\theta}\).

- \(u(x)\) non-constant & analytic \(\Rightarrow\) the mean and covariance under \(Q_{X;\theta}^{(u)}\) involve higher-order statistical moments of \(P_{X;\theta}\).
Probability Measure Transform

Proposition (Consistent empirical MT mean and covariance)

Let $X_n, n = 1, \ldots, N$ denote a sequence of i.i.d. samples from $P_{X;\theta}$, and define the empirical mean and covariance estimates:

$$\hat{\mu}_x (u) \triangleq \sum_{n=1}^{N} X_n \hat{\phi}_u (X_n)$$

$$\hat{\Sigma}_x (u) \triangleq \sum_{n=1}^{N} X_n X_n^H \hat{\phi}_u (X_n) - \hat{\mu}_x (u) \hat{\mu}_x (u)^H$$

where $\hat{\phi}_u (X_n) \triangleq \frac{u(X_n)}{\sum_{n=1}^{N} u(X_n)}$. If

$$\mathbb{E} \left[ \|X\|^2 u(X) ; P_X \right] < \infty,$$

then $\hat{\mu}_x (u) \xrightarrow{w.p.1} \mu_x (u) (\theta)$ and $\hat{\Sigma}_x (u) \xrightarrow{w.p.1} \Sigma_x (u) (\theta)$ as $N \to \infty$. 
Probability Measure Transform

Proposition (Robustness to outliers)

If the MT-function $u(x)$ and $u(x)\|x\|^2$ are bounded, then the influence functions [Hampel, 1974] of $\hat{\mu}_x^{(u)}$ and $\hat{\Sigma}_x^{(u)}$ are bounded.

Remark

Condition is satisfied when $u(x) \in$ Gaussian family.
Measure Transformed Gaussian QML Estimator

The MT-GQMLE

- Minimize the empirical KLD between $Q_{X;\theta_0}^{(u)}$ and a Gaussian measure $\Phi_{X;\theta}^{(u)}$ with mean $\mu_{X}^{(u)}(\theta)$ and covariance $\Sigma_{X}^{(u)}(\theta)$.

- Amounts to maximization of

$$J_{u}(\theta) \triangleq -D_{LD} \left[ \hat{\Sigma}_{X}^{(u)} \parallel \Sigma_{X}^{(u)}(\theta) \right] - \left\| \hat{\mu}_{X}^{(u)} - \mu_{X}^{(u)}(\theta) \right\|^{2} \left( \Sigma_{X}^{(u)}(\theta) \right)^{-1}$$

- The MT-QMLE:

$$\hat{\theta}_{u} = \arg \max_{\theta \in \Theta} J_{u}(\theta)$$
Theorem (Strong consistency of $\hat{\theta}_u$)

Given a sequence of $N$ i.i.d. samples from $P_{X;\theta_0}$. Assume that the following conditions are satisfied:

1. The parameter space $\Theta$ is compact.
2. $\mu_X^{(u)}(\theta_0) \neq \mu_X^{(u)}(\theta)$ or $\Sigma_X^{(u)}(\theta_0) \neq \Sigma_X^{(u)}(\theta) \forall \theta \neq \theta_0$.
3. $\Sigma_X^{(u)}(\theta)$ is non-singular $\forall \theta \in \Theta$.
4. $\mu_X^{(u)}(\theta)$ and $\Sigma_X^{(u)}(\theta)$ are continuous in $\Theta$.
5. $\mathbb{E} \left[ \|X\|^2 u(X) ; P_{X;\theta_0} \right] < \infty$.

Then,

$$\hat{\theta}_u \xrightarrow{w.p.} \frac{1}{N} \rightarrow \theta_0 \quad \text{as} \quad N \rightarrow \infty$$
Measure Transformed Gaussian QML Estimator

Theorem (Asymptotic normality and unbiasedness of $\hat{\theta}_u$)

Given a sequence of $N$ i.i.d. samples from $P_{X;\theta_0}$. Assume that the following conditions are satisfied:

1. $\hat{\theta}_u \xrightarrow{P} \theta_0$ as $N \to \infty$.

2. $\theta_0$ lies in the interior of $\Theta$ which is assumed to be compact.

3. $\mu_X^{(u)}(\theta)$, $\Sigma_X^{(u)}(\theta)$ are twice continuously differentiable in $\Theta$.

4. $E\left[u^2(X) ; P_{X;\theta_0}\right] < \infty$ and $E\left[\|X\|^4 u^2(X) ; P_{X;\theta_0}\right] < \infty$.

Then,

$$\hat{\theta}_u \xrightarrow{a} \mathcal{N}(\theta_0, C_u(\theta_0))$$
Measure Transformed Gaussian QML Estimator

Asymptotic MSE

\[ C_u(\theta_0) = N^{-1} F_u^{-1}(\theta_0) G_u(\theta_0) F_u^{-1}(\theta_0) \]

where

\[ G_u(\theta) \triangleq \mathbb{E} \left[ u^2(X) \psi_u(X; \theta) \psi^T_u(X; \theta); P_{x;\theta_0} \right] \]

\[ \psi_u(X; \theta) \triangleq \nabla_{\theta_0} \log \phi^{(u)}(X; \theta) \]

\[ F_u(\theta) \triangleq -\mathbb{E} \left[ u(X) \Gamma_u(X; \theta); P_{x;\theta_0} \right] \]

\[ \Gamma_u(X; \theta) \triangleq \nabla^2_{\theta} \log \phi^{(u)}(X; \theta) \]
Measure Transformed Gaussian QML Estimator

Proposition (Relation to the CRLB)

\[ C_u(\theta_0) \succeq \text{CRLB}(\theta_0) \]

where equality holds if and only if

\[ \nabla_\theta \log f(X; \theta) \big|_{\theta = \theta_0} = \text{I}_{\text{FIM}}(\theta_0) F_u^{-1}(\theta_0) \psi_u(X; \theta_0) u(X) \quad \text{w.p. 1} \]

Remark

\( P_{X;\theta_0} \text{ Gaussian} \Rightarrow \text{Condition is satisfied only for } u(x) = \text{const} \)

Conclusion

\( P_{X;\theta_0} \text{ Gaussian } \& \ u(x) \neq \text{const} \Rightarrow C_u(\theta_0) \succ \text{CRLB}(\theta_0) \)
Measure Transformed Gaussian QML Estimator

Theorem (Empirical asymptotic MSE)

Define the empirical asymptotic MSE:

\[ \hat{C}_u(\hat{\theta}_u) \triangleq N^{-1} \hat{F}_u^{-1}(\hat{\theta}_u) \hat{G}_u(\hat{\theta}_u) \hat{F}_u^{-1}(\hat{\theta}_u) \]

where

\[ \hat{G}_u(\theta) \triangleq N^{-1} \sum_{n=1}^{N} u^2(X_n) \psi_u(X_n; \theta) \psi^T_u(X_n; \theta) \]

\[ \hat{F}_u(\theta) \triangleq -N^{-1} \sum_{n=1}^{N} u(X_n) \Gamma_u(X_n; \theta) \]

Furthermore, assume that the following conditions are satisfied:

1. \( \hat{\theta}_u \xrightarrow{P} \theta_0 \) as \( N \to \infty \).
2. \( \mu^{(u)}(\theta), \Sigma^{(u)}(\theta) \) are twice continuously differentiable in \( \Theta \).
3. \( E \left[ u^2(X) ; P_{X;\theta_0} \right] < \infty \) and \( E \left[ \| X \|^4 u^2(X) ; P_{X;\theta_0} \right] < \infty \).

Then,

\( N \| \hat{C}_u(\hat{\theta}_u) - C_u(\theta_0) \| \xrightarrow{P} 0 \) as \( N \to \infty \).
Optimal choice of the MT-function

- Specify the MT-function within some parametric family
  \[ \{ u(X; \omega), \omega \in \Omega \subseteq \mathbb{C}^r \} \]

- In order to gain *robustness against outliers* the *Gaussian family* would be a good choice

- An optimal choice of the MT-function parameter \( \omega \) would be this that minimizes the empirical asymptotic MSE
  \[ \omega_{opt} \triangleq \arg \min_{\omega \in \Omega} \hat{C}_u \left( \hat{\theta}_u(\omega); \omega \right) \]
Application

Robust direction finding in heavy-tailed clutter

\[ X_n = S_n a(\theta_0) + W_n, \quad n = 1, \ldots, N \]

- \( S_n \): emitted signal with \textit{unknown} symmetric distribution
- \( W_n \): clutter with \textit{unknown} spherically symmetric distribution
- \( S_n \) & \( W_n \) statistically independent and first-order stationary
- \textbf{Gaussian MT-function}: \( u(x; \omega) \triangleq \exp(-\|x\|^2/\omega^2) \)
  - MT-Mean: \( \mu^{(u)}_X(\theta; \omega) = 0 \)
  - MT-Covariance: \( \Sigma^{(u)}_X(\theta; \omega) = r_S(\omega) a(\theta) a^H(\theta) + r_W(\omega) I \)
- \textbf{MT-GQMLE}: \( \hat{\theta}_u(\omega) = \arg \max_{\theta \in \Theta} a^H(\theta) \hat{C}^{(u)}_X(\omega) a(\theta) \)
Application

Robust direction finding in heavy-tailed clutter

- BPSK signal, 4-element ULA, $\theta = 30^\circ$
- Impulsive K-distributed clutter
- $N = 3000$ snapshots, GSNR = $-15$ [dB]
Application

Robust direction finding in heavy-tailed clutter

- BPSK signal, 4-element ULA, $\theta = 30^\circ$
- Impulsive *K*-distributed clutter
- $N = 3000$ snapshots
A new estimator, called MT-GQMLE, was derived by applying a transform to the probability distribution of the data.

By specifying the MT-function in the Gaussian family, the MT-GQMLE was applied to robust direction finding.

Exploration of other MT-functions may result in additional estimators in this class that have different useful properties.
Application

Robust direction finding in heavy-tailed clutter

- BPSK signal, 4-element ULA, $\theta = 30^\circ$
- Impulsive Cauchy clutter
- $N = 3000$ snapshots, GSNR = $-5$ [dB]
Application

Robust direction finding in heavy-tailed clutter

- BPSK signal, 4-element ULA, $\theta = 30^\circ$
- Impulsive Cauchy clutter
- $N = 3000$ snapshots
Application

Robust direction finding in Gaussian clutter

- BPSK signal, 4-element ULA, $\theta = 30^\circ$
- Gaussian clutter
- $N = 3000$ snapshots, GSNR $= -5$ [dB]
Application

Robust direction finding in Gaussian clutter

- BPSK signal, 4-element ULA, $\theta = 30^\circ$
- Gaussian clutter
- $N = 3000$ snapshots