Radar Clutter Mitigation via Probability Measure Transform

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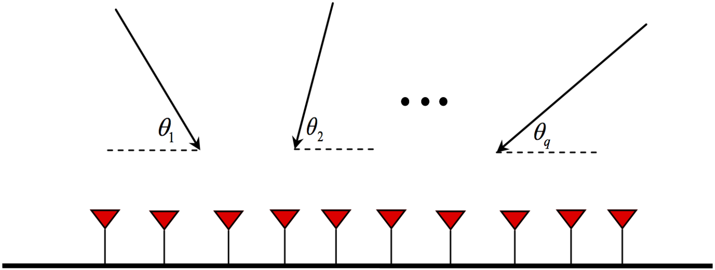
Outline

- Background and motivation
- Probability measure transform
- Measure transformed Gaussian QML estimator
- Application: robust direction finding in heavy-tailed clutter
- Summary

Background and motivation

Problem

In practical direction finding of radar targets accurate parametric models for the signals and clutter are unavailable



Background and motivation

Goal

Estimate $\boldsymbol{\theta}_0 \in \mathbb{R}^m$ given samples $\{\mathbf{X}_n\}_{n=1}^{N}$ from $P_{\mathbf{x};\boldsymbol{\theta}_0}$

Restriction

$P_{\mathbf{x};\boldsymbol{\theta}_0}$ belongs to *unknown* parametric family of probability measures

$$P_{\mathbf{x};\boldsymbol{\theta}_0} \in \{P_{\mathbf{x};\boldsymbol{\theta}} : \boldsymbol{\theta} \in \boldsymbol{\Theta}\}$$

- The maximum likelihood estimator cannot be implemented
- Resort to methods that require *partial statistical information*

Background and motivation

Gaussian QML estimator *[White 1982]*

- Minimize the *empirical KLD* between $P_{\mathbf{x};\boldsymbol{\theta}_0}$ and a Gaussian measure $\Phi_{\mathbf{x};\boldsymbol{\theta}}$ with mean $\boldsymbol{\mu}_{\mathbf{x}}(\boldsymbol{\theta})$ and covariance $\boldsymbol{\Sigma}_{\mathbf{x}}(\boldsymbol{\theta})$

- Amounts to maximization of

$$J(\boldsymbol{\theta}) \triangleq -D_{\mathrm{LD}}\left[\hat{\boldsymbol{\Sigma}}_{\mathbf{x}} || \boldsymbol{\Sigma}_{\mathbf{x}}(\boldsymbol{\theta})\right] - \|\hat{\boldsymbol{\mu}}_{\mathbf{x}} - \boldsymbol{\mu}_{\mathbf{x}}(\boldsymbol{\theta})\|^2_{(\boldsymbol{\Sigma}_{\mathbf{x}}(\boldsymbol{\theta}))^{-1}}$$

- The GQMLE:

$$\hat{\boldsymbol{\theta}} = \arg\max_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} J(\boldsymbol{\theta})$$

- Simple implementation, easy performance analysis

- Sensitive to model mismatch (e.g. in non-Gaussian clutter)

Background and motivation

Proposed approach

GQMLE under a transformed probability distribution of the data

Advantages

- Resilient to outliers
- Involves higher-order statistical moments
- Significant mitigation of the model mismatch effect
- Computational advantages of the first and second-order methods of moments

Probability Measure Transform

Definition

Given a non-negative function $u : \mathcal{X} \to \mathbb{R}_+$ satisfying

$$0 < \mathrm{E}\left[u\left(\mathbf{X}\right); P_{\mathbf{x};\boldsymbol{\theta}}\right] < \infty.$$

A transform $\mathrm{T}_u : P_{\mathbf{x};\boldsymbol{\theta}} \to Q^{(u)}_{\mathbf{x};\boldsymbol{\theta}}$ is defined as:

$$\mathrm{T}_u\left[P_{\mathbf{x};\boldsymbol{\theta}}\right](A) = Q^{(u)}_{\mathbf{x};\boldsymbol{\theta}}(A) \triangleq \int_A \varphi_u\left(\mathbf{x};\boldsymbol{\theta}\right) dP_{\mathbf{x};\boldsymbol{\theta}}\left(\mathbf{x}\right),$$

where

$$\varphi_u\left(\mathbf{x};\boldsymbol{\theta}\right) \triangleq \frac{u\left(\mathbf{x}\right)}{\mathrm{E}\left[u\left(\mathbf{X}\right); P_{\mathbf{x};\boldsymbol{\theta}}\right]}.$$

The function $u\left(\cdot\right)$ is called the *MT-function*.

Probability Measure Transform

The measure transformed mean and covariance

$$\boldsymbol{\mu}_{\mathbf{X}}^{(u)}(\boldsymbol{\theta}) = \mathrm{E}\left[\mathbf{X}\varphi_u(\mathbf{X};\boldsymbol{\theta}); P_{\mathbf{x};\boldsymbol{\theta}}\right]$$

$$\boldsymbol{\Sigma}_{\mathbf{X}}^{(u)}(\boldsymbol{\theta}) = \mathrm{E}\left[\mathbf{X}\mathbf{X}^H\varphi_u(\mathbf{X};\boldsymbol{\theta}); P_{\mathbf{x};\boldsymbol{\theta}}\right] - \boldsymbol{\mu}_{\mathbf{X}}^{(u)}(\boldsymbol{\theta})\,\boldsymbol{\mu}_{\mathbf{X}}^{(u)H}(\boldsymbol{\theta})$$

where

$$\varphi_u(\mathbf{x};\boldsymbol{\theta}) \triangleq \frac{u(\mathbf{x})}{\mathrm{E}\left[u(\mathbf{X});P_{\mathbf{x};\boldsymbol{\theta}}\right]} = \frac{dQ_{\mathbf{x};\boldsymbol{\theta}}^{(u)}}{dP_{\mathbf{x};\boldsymbol{\theta}}}$$

Conclusion

- *The mean and covariance under $Q_{\mathbf{x};\boldsymbol{\theta}}^{(u)}$ can be estimated using only samples from $P_{\mathbf{x};\boldsymbol{\theta}}$.*
- $u(\mathbf{x})$ *non-constant & analytic* $\Rightarrow$ *the mean and covariance under $Q_{\mathbf{x};\boldsymbol{\theta}}^{(u)}$ involve higher-order statistical moments of $P_{\mathbf{x};\boldsymbol{\theta}}$.*

Probability Measure Transform

Proposition (Consistent empirical MT mean and covariance)

Let $\mathbf{X}_n$, $n = 1, \ldots, N$ denote a sequence of i.i.d. samples from $P_{\mathbf{X};\boldsymbol{\theta}}$, and define the empirical mean and covariance estimates:

$$\hat{\boldsymbol{\mu}}_{\mathbf{X}}^{(u)} \triangleq \sum_{n=1}^{N} \mathbf{X}_n \hat{\varphi}_u\left(\mathbf{X}_n\right)$$

$$\hat{\boldsymbol{\Sigma}}_{\mathbf{X}}^{(u)} \triangleq \sum_{n=1}^{N} \mathbf{X}_n \mathbf{X}_n^H \hat{\varphi}_u\left(\mathbf{X}_n\right) - \hat{\boldsymbol{\mu}}_{\mathbf{X}}^{(u)} \hat{\boldsymbol{\mu}}_{\mathbf{X}}^{(u)H}$$

where $\hat{\varphi}_u\left(\mathbf{X}_n\right) \triangleq \frac{u(\mathbf{X}_n)}{\sum_{n=1}^{N} u(\mathbf{X}_n)}$. *If*

$$\mathrm{E}\left[\left\|\mathbf{X}\right\|^2 u\left(\mathbf{X}\right); P_{\mathbf{X}}\right] < \infty,$$

then $\hat{\boldsymbol{\mu}}_{\mathbf{X}}^{(u)} \xrightarrow{w.p.1} \boldsymbol{\mu}_{\mathbf{X}}^{(u)}\left(\boldsymbol{\theta}\right)$ *and* $\hat{\boldsymbol{\Sigma}}_{\mathbf{X}}^{(u)} \xrightarrow{w.p.1} \boldsymbol{\Sigma}_{\mathbf{X}}^{(u)}\left(\boldsymbol{\theta}\right)$ *as* $N \to \infty$.

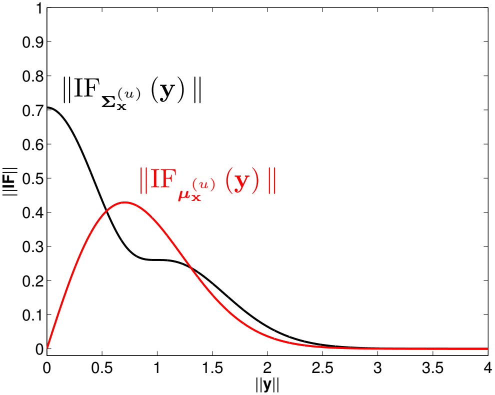
Probability Measure Transform

Proposition (Robustness to outliers)

If the MT-function $u(\mathbf{x})$ and $u(\mathbf{x})\|\mathbf{x}\|^2$ are bounded, then the influence functions [Hampel, 1974] of $\hat{\boldsymbol{\mu}}_{\mathbf{x}}^{(u)}$ and $\hat{\boldsymbol{\Sigma}}_{\mathbf{x}}^{(u)}$ are bounded.

Remark

Condition is satisfied when $u(\mathbf{x}) \in$ Gaussian family.



Measure Transformed Gaussian QML Estimator

The MT-GQMLE

- Minimize the *empirical KLD* between $Q^{(u)}_{\mathbf{x};\boldsymbol{\theta}_0}$ and a Gaussian measure $\Phi^{(u)}_{\mathbf{x};\boldsymbol{\theta}}$ with mean $\boldsymbol{\mu}^{(u)}_{\mathbf{x}}(\boldsymbol{\theta})$ and covariance $\boldsymbol{\Sigma}^{(u)}_{\mathbf{x}}(\boldsymbol{\theta})$.
- Amounts to maximization of

$$J_u(\boldsymbol{\theta}) \triangleq -D_{\mathrm{LD}}\left[\hat{\boldsymbol{\Sigma}}^{(u)}_{\mathbf{x}} || \boldsymbol{\Sigma}^{(u)}_{\mathbf{x}}(\boldsymbol{\theta})\right] - \left\|\hat{\boldsymbol{\mu}}^{(u)}_{\mathbf{x}} - \boldsymbol{\mu}^{(u)}_{\mathbf{x}}(\boldsymbol{\theta})\right\|^2_{\left(\boldsymbol{\Sigma}^{(u)}_{\mathbf{x}}(\boldsymbol{\theta})\right)^{-1}}$$

- The MT-QMLE:

$$\hat{\boldsymbol{\theta}}_u = \arg\max_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} J_u(\boldsymbol{\theta})$$

Measure Transformed Gaussian QML Estimator

Theorem (Strong consistency of $\hat{\boldsymbol{\theta}}_u$)

Given a sequence of N i.i.d. samples from $P_{\mathbf{X};\boldsymbol{\theta}_0}$. Assume that the following conditions are satisfied:

1. *The parameter space $\boldsymbol{\Theta}$ is compact.*
2. $\boldsymbol{\mu}_{\mathbf{X}}^{(u)}(\boldsymbol{\theta}_0) \neq \boldsymbol{\mu}_{\mathbf{X}}^{(u)}(\boldsymbol{\theta})$ *or* $\boldsymbol{\Sigma}_{\mathbf{X}}^{(u)}(\boldsymbol{\theta}_0) \neq \boldsymbol{\Sigma}_{\mathbf{X}}^{(u)}(\boldsymbol{\theta})$ $\forall \boldsymbol{\theta} \neq \boldsymbol{\theta}_0$.
3. $\boldsymbol{\Sigma}_{\mathbf{X}}^{(u)}(\boldsymbol{\theta})$ *is non-singular* $\forall \boldsymbol{\theta} \in \boldsymbol{\Theta}$.
4. $\boldsymbol{\mu}_{\mathbf{X}}^{(u)}(\boldsymbol{\theta})$ *and* $\boldsymbol{\Sigma}_{\mathbf{X}}^{(u)}(\boldsymbol{\theta})$ *are continuous in* $\boldsymbol{\Theta}$.
5. $\mathrm{E}\left[\|\mathbf{X}\|^2 u(\mathbf{X}) ; P_{\mathbf{X};\boldsymbol{\theta}_0}\right] < \infty$.

Then,

$$\hat{\boldsymbol{\theta}}_u \xrightarrow{w.p.\ 1} \boldsymbol{\theta}_0 \text{ as } N \rightarrow \infty$$

Measure Transformed Gaussian QML Estimator

Theorem (Asymptotic normality and unbiasedness of $\hat{\boldsymbol{\theta}}_u$)

Given a sequence of N i.i.d. samples from $P_{\mathbf{X};\boldsymbol{\theta}_0}$. Assume that the following conditions are satisfied:

1. $\hat{\boldsymbol{\theta}}_u \xrightarrow{P} \boldsymbol{\theta}_0$ *as* $N \to \infty$.
2. $\boldsymbol{\theta}_0$ *lies in the interior of* $\boldsymbol{\Theta}$ *which is assumed to be compact.*
3. $\boldsymbol{\mu}_{\mathbf{X}}^{(u)}(\boldsymbol{\theta})$, $\boldsymbol{\Sigma}_{\mathbf{X}}^{(u)}(\boldsymbol{\theta})$ *are twice continuously differentiable in* $\boldsymbol{\Theta}$.
4. $\mathrm{E}\left[u^2(\mathbf{X}); P_{\mathbf{X};\boldsymbol{\theta}_0}\right] < \infty$ *and* $\mathrm{E}\left[\|\mathbf{X}\|^4 u^2(\mathbf{X}); P_{\mathbf{X};\boldsymbol{\theta}_0}\right] < \infty$.

Then,

$$\hat{\boldsymbol{\theta}}_u \overset{a}{\sim} \mathcal{N}\left(\boldsymbol{\theta}_0, \mathbf{C}_u(\boldsymbol{\theta}_0)\right)$$

Measure Transformed Gaussian QML Estimator

Asymptotic MSE

$$\mathbf{C}_u(\boldsymbol{\theta}_0) = N^{-1}\mathbf{F}_u^{-1}(\boldsymbol{\theta}_0)\,\mathbf{G}_u(\boldsymbol{\theta}_0)\,\mathbf{F}_u^{-1}(\boldsymbol{\theta}_0)$$

where

$$\mathbf{G}_u(\boldsymbol{\theta}) \triangleq \mathrm{E}\left[u^2(\mathbf{X})\,\boldsymbol{\psi}_u(\mathbf{X};\boldsymbol{\theta})\,\boldsymbol{\psi}_u^T(\mathbf{X};\boldsymbol{\theta}); P_{\mathbf{x};\boldsymbol{\theta}_0}\right]$$

$$\boldsymbol{\psi}_u(\mathbf{X};\boldsymbol{\theta}) \triangleq \nabla_{\boldsymbol{\theta}_0}\log\phi^{(u)}(\mathbf{X};\boldsymbol{\theta})$$

$$\mathbf{F}_u(\boldsymbol{\theta}) \triangleq -\mathrm{E}\left[u(\mathbf{X})\,\boldsymbol{\Gamma}_u(\mathbf{X};\boldsymbol{\theta}); P_{\mathbf{x};\boldsymbol{\theta}_0}\right]$$

$$\boldsymbol{\Gamma}_u(\mathbf{X};\boldsymbol{\theta}) \triangleq \nabla^2_{\boldsymbol{\theta}}\log\phi^{(u)}(\mathbf{X};\boldsymbol{\theta})$$

Measure Transformed Gaussian QML Estimator

Proposition (Relation to the CRLB)

$$\mathbf{C}_u(\boldsymbol{\theta}_0) \succeq \textbf{CRLB}(\boldsymbol{\theta}_0)$$

where equality holds if and only if

$$\nabla_{\boldsymbol{\theta}} \log f(\mathbf{X};\boldsymbol{\theta})|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} = \mathbf{I}_{\text{FIM}}(\boldsymbol{\theta}_0)\,\mathbf{F}_u^{-1}(\boldsymbol{\theta}_0)\,\boldsymbol{\psi}_u(\mathbf{X};\boldsymbol{\theta}_0)\,u(\mathbf{X}) \quad w.p.\ 1$$

Remark

$P_{\mathbf{x};\boldsymbol{\theta}_0}$ *Gaussian* $\Rightarrow$ *Condition is satisfied only for* $u(\mathbf{x}) = const$

Conclusion

$P_{\mathbf{x};\boldsymbol{\theta}_0}$ *Gaussian &* $u(\mathbf{x}) \neq const \Rightarrow \mathbf{C}_u(\boldsymbol{\theta}_0) \succ \textbf{CRLB}(\boldsymbol{\theta}_0)$

Measure Transformed Gaussian QML Estimator

Theorem (Empirical asymptotic MSE)

Define the empirical asymptotic MSE:

$$\hat{\mathbf{C}}_u(\hat{\boldsymbol{\theta}}_u) \triangleq N^{-1}\hat{\mathbf{F}}_u^{-1}(\hat{\boldsymbol{\theta}}_u)\hat{\mathbf{G}}_u(\hat{\boldsymbol{\theta}}_u)\hat{\mathbf{F}}_u^{-1}(\hat{\boldsymbol{\theta}}_u)$$

where

$$\hat{\mathbf{G}}_u(\boldsymbol{\theta}) \triangleq N^{-1}\sum_{n=1}^{N} u^2(\mathbf{X}_n)\,\boldsymbol{\psi}_u(\mathbf{X}_n;\boldsymbol{\theta})\,\boldsymbol{\psi}_u^T(\mathbf{X}_n;\boldsymbol{\theta})$$

$$\hat{\mathbf{F}}_u(\boldsymbol{\theta}) \triangleq -N^{-1}\sum_{n=1}^{N} u(\mathbf{X}_n)\,\boldsymbol{\Gamma}_u(\mathbf{X}_n;\boldsymbol{\theta})$$

Furthermore, assume that the following conditions are satisfied:

1. $\hat{\boldsymbol{\theta}}_u \xrightarrow{P} \boldsymbol{\theta}_0$ *as* $N \to \infty$.
2. $\boldsymbol{\mu}_{\mathbf{X}}^{(u)}(\boldsymbol{\theta})$, $\boldsymbol{\Sigma}_{\mathbf{X}}^{(u)}(\boldsymbol{\theta})$ *are twice continuously differentiable in* $\boldsymbol{\Theta}$.
3. $\mathrm{E}\left[u^2(\mathbf{X});P_{\mathbf{X};\boldsymbol{\theta}_0}\right] < \infty$ *and* $\mathrm{E}\left[\|\mathbf{X}\|^4 u^2(\mathbf{X});P_{\mathbf{X};\boldsymbol{\theta}_0}\right] < \infty$.

Then,

$$N\|\hat{\mathbf{C}}_u(\hat{\boldsymbol{\theta}}_u) - \mathbf{C}_u(\boldsymbol{\theta}_0)\| \xrightarrow{P} 0 \text{ as } N \to \infty.$$

Measure Transformed Gaussian QML Estimator

Optimal choice of the MT-function

- Specify the MT-function within some parametric family

$$\{u(\mathbf{X};\boldsymbol{\omega}), \boldsymbol{\omega} \in \boldsymbol{\Omega} \subseteq \mathbb{C}^r\}$$

- In order to gain *robustness against outliers* the *Gaussian family* would be a good choice

- An optimal choice of the MT-function parameter $\boldsymbol{\omega}$ would be this that minimizes the empirical asymptotic MSE

$$\boldsymbol{\omega}_{\rm opt} \triangleq \arg\min_{\boldsymbol{\omega}\in\boldsymbol{\Omega}} \hat{\mathbf{C}}_u\left(\hat{\boldsymbol{\theta}}_u(\boldsymbol{\omega});\boldsymbol{\omega}\right)$$

Application

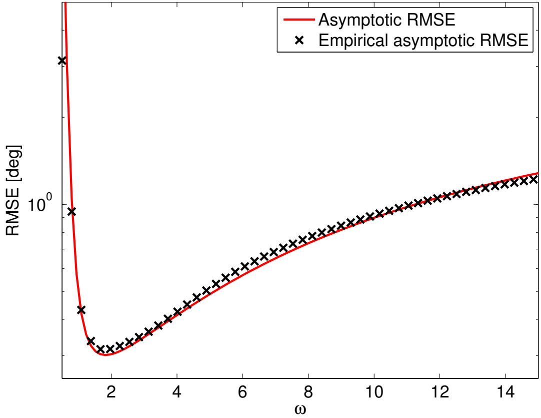
Robust direction finding in heavy-tailed clutter

$$\mathbf{X}_n = S_n \mathbf{a}(\theta_0) + \mathbf{W}_n, \;\; n = 1, \ldots, N$$

- S_n: emitted signal with *unknown* symmetric distribution
- $\mathbf{W}_n$: clutter with *unknown* spherically symmetric distribution
- S_n & $\mathbf{W}_n$ statistically independent and first-order stationary
- Gaussian MT-function: $u(\mathbf{x}; \omega) \triangleq \exp(-\|\mathbf{x}\|^2/\omega^2)$
  - MT-Mean: $\boldsymbol{\mu}_{\mathbf{X}}^{(u)}(\theta; \omega) = \mathbf{0}$
  - MT-Covariance: $\boldsymbol{\Sigma}_{\mathbf{X}}^{(u)}(\theta; \omega) = r_S(\omega)\, \mathbf{a}(\theta)\, \mathbf{a}^H(\theta) + r_W(\omega)\, \mathbf{I}$
- MT-GQMLE: $\hat{\theta}_u(\omega) = \arg\max_{\theta \in \Theta} \mathbf{a}^H(\theta)\, \hat{\mathbf{C}}_{\mathbf{X}}^{(u)}(\omega)\, \mathbf{a}(\theta)$

Application

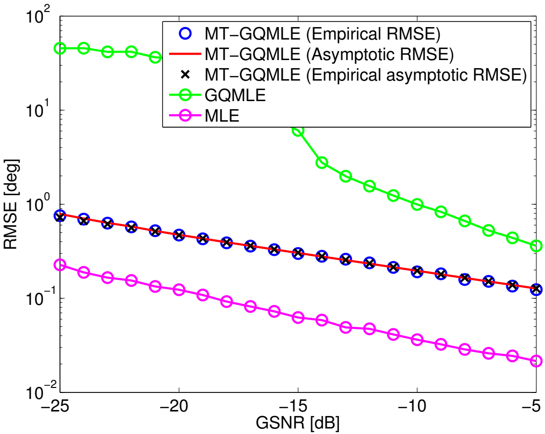
Robust direction finding in heavy-tailed clutter



- BPSK signal, 4-element ULA, $\theta = 30^\circ$
- Impulsive K-distributed clutter
- $N = 3000$ snapshots, $\mathrm{GSNR} = -15$ [dB]

Application

Robust direction finding in heavy-tailed clutter



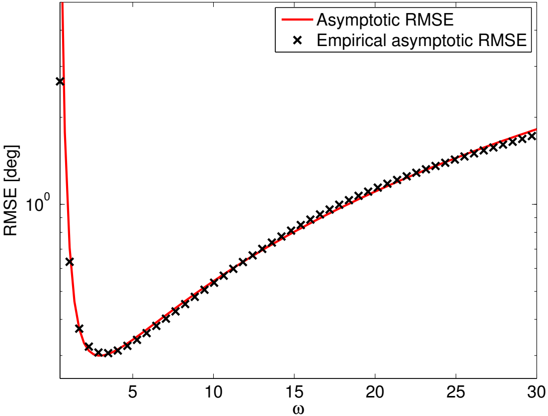
- BPSK signal, 4-element ULA, $\theta = 30^\circ$
- Impulsive K-distributed clutter
- $N = 3000$ snapshots

Summary

- A new estimator, called MT-GQMLE, was derived by applying a transform to the probability distribution of the data.
- By specifying the MT-function in the Gaussian family, the MT-GQMLE was applied to robust direction finding.
- Exploration of other MT-functions may result in additional estimators in this class that have different useful properties.

Application

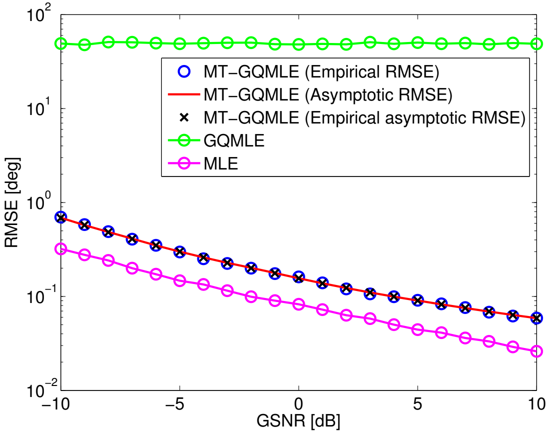
Robust direction finding in heavy-tailed clutter



- BPSK signal, 4-element ULA, $\theta = 30^\circ$
- Impulsive Cauchy clutter
- $N = 3000$ snapshots, $\mathrm{GSNR} = -5$ [dB]

Application

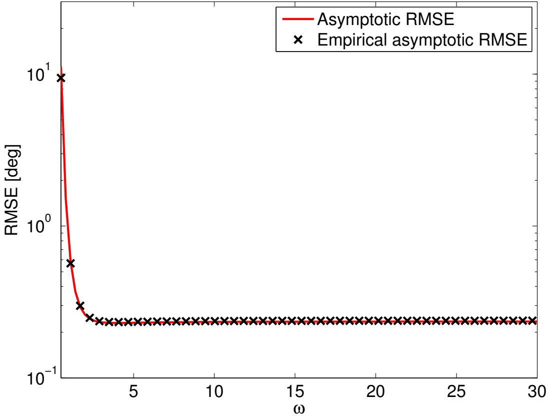
Robust direction finding in heavy-tailed clutter



- BPSK signal, 4-element ULA, $\theta = 30^\circ$
- Impulsive Cauchy clutter
- $N = 3000$ snapshots

Application

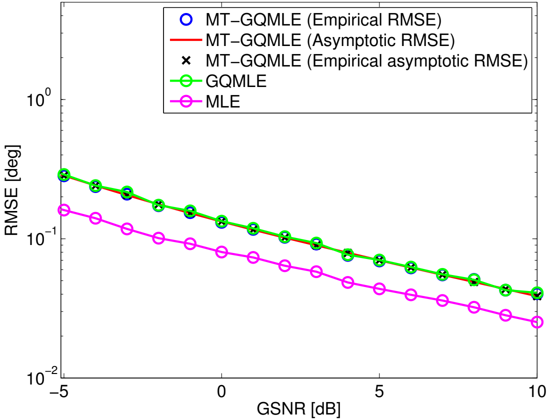
Robust direction finding in Gaussian clutter



- BPSK signal, 4-element ULA, $\theta = 30^\circ$
- Gaussian clutter
- $N = 3000$ snapshots, $\mathrm{GSNR} = -5$ [dB]

Application

Robust direction finding in Gaussian clutter



- BPSK signal, 4-element ULA, $\theta = 30^\circ$
- Gaussian clutter
- $N = 3000$ snapshots