Plug-in Measure-Transformed Quasi Likelihood Ratio Test for Random Signal Detection

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Outline

- Detection problem formulation
- Existing detectors
- Proposed detector
- Simulation studies
- Summary

Detection Problem Formulation

$$H_0 \quad : \quad \begin{cases} \mathbf{X}_n = \mathbf{W}_n, & n = 1, \ldots, N \\ \mathbf{Y}_m = \mathbf{W}_m^{(s)}, & m = 1, \ldots, M \end{cases}$$

$$H_1 \quad : \quad \begin{cases} \mathbf{X}_n = S_n \mathbf{a} + \mathbf{W}_n, & n = 1, \ldots, N \\ \mathbf{Y}_m = \mathbf{W}_m^{(s)}, & m = 1, \ldots, M, \end{cases}$$

- $\{\mathbf{X}_n \in \mathbb{C}^p\}$, $\{\mathbf{Y}_m \in \mathbb{C}^p\}$: Primary and secondary data
- $\{S_n \in \mathbb{C}\}$: i.i.d. zero-mean signal with *unknown* distribution
- $\mathbf{a} \in \mathbb{C}^p$: *Known* steering vector
- $\{\mathbf{W}_n\}$, $\{\mathbf{W}_m^{(s)}\}$: i.i.d. zero-mean symmetrically distributed homogeneous noise processes with *unknown* distribution
- $\{S_n\}$, $\{\mathbf{W}_n\}$ and $\{\mathbf{W}_m^{(s)}\}$ are mutually independent

Existing detectors

Gauss-Gauss detector *[Jin & Friedlander 2005]*

- GLRT-based detector
- Assumes normally distributed signal and noise
- Simple implementation and tractable performance analysis
- Sensitive to deviation from normality (e.g., in the case of heavy-tailed noise that produces outliers)

Existing detectors

CG-GLRT *[Gerlach 1999, Shuai et. al. 2010]*

- Assumes elliptical compound-Gaussian noise
- Resilient against heavy-tailed noise outliers
- Iterative ML-estimation of the noise scatter matrix
  - Converges under some regularity conditions
  - Each iteration involves matrix inversion
  - Does not reject large norm outliers
- Can be sensitive to non-elliptical noise

Proposed Detector

General operation principle

Selects a Gaussian model that best empirically fits a transformed probability distribution of the data

Advantages

- Resilient to outliers (rejects large-norm outliers)
- Involves higher-order statistical moments
- Significant mitigation of the model mismatch effect
- Computational and implementation simplicity

Probability Measure Transform

Definition

Given a non-negative function $u : \mathcal{X} \to \mathbb{R}_+$ satisfying

$$0 < \mathrm{E}\left[u\left(\mathbf{X}\right); P_{\mathbf{x};H}\right] < \infty.$$

A transform $\mathrm{T}_u : P_{\mathbf{x};H} \to Q^{(u)}_{\mathbf{x};H}$ is defined as:

$$\mathrm{T}_u\left[P_{\mathbf{x};H}\right]\left(A\right) = Q^{(u)}_{\mathbf{x};H}\left(A\right) \triangleq \int_A \varphi_u\left(\mathbf{x}; H\right) dP_{\mathbf{x};H}\left(\mathbf{x}\right),$$

where

$$\varphi_u\left(\mathbf{x}; H\right) \triangleq \frac{u\left(\mathbf{x}\right)}{\mathrm{E}\left[u\left(\mathbf{X}\right); P_{\mathbf{x};H}\right]}.$$

The function $u\left(\cdot\right)$ is called the *MT-function*.

Probability Measure Transform

The measure transformed mean and covariance

$$\boldsymbol{\mu}^{(u)}_{\mathbf{x};H} = \mathrm{E}\left[\mathbf{X}\varphi_u\left(\mathbf{X};H\right);P_{\mathbf{x};H}\right]$$

$$\boldsymbol{\Sigma}^{(u)}_{\mathbf{x};H} = \mathrm{E}\left[\mathbf{X}\mathbf{X}^H\varphi_u\left(\mathbf{X};H\right);P_{\mathbf{x};H}\right] - \boldsymbol{\mu}^{(u)}_{\mathbf{x};H}\boldsymbol{\mu}^{(u)H}_{\mathbf{x};H}$$

where

$$\varphi_u\left(\mathbf{x};H\right) \triangleq \frac{u\left(\mathbf{x}\right)}{\mathrm{E}\left[u\left(\mathbf{X}\right);P_{\mathbf{x};H}\right]} = \frac{dQ^{(u)}_{\mathbf{x};H}}{dP_{\mathbf{x};H}}$$

Conclusion

- *The mean and covariance under $Q^{(u)}_{\mathbf{x};H}$ can be estimated using only samples from $P_{\mathbf{x};H}$.*
- $u\left(\mathbf{x}\right)$ *non-constant & analytic* $\Rightarrow$ *the mean and covariance under $Q^{(u)}_{\mathbf{x};H}$ involve higher-order statistical moments of $P_{\mathbf{x};H}$.*

Probability Measure Transform

Proposition (Consistent empirical MT mean and covariance)

Let $\mathbf{X}_n$, $n = 1, \ldots, N$ denote a sequence of i.i.d. samples from $P_{\mathbf{X};H}$, and define the empirical mean and covariance estimates:

$$\hat{\boldsymbol{\mu}}_{\mathbf{X}}^{(u)} \triangleq \sum_{n=1}^{N} \mathbf{X}_n \hat{\varphi}_u\left(\mathbf{X}_n\right)$$

$$\hat{\boldsymbol{\Sigma}}_{\mathbf{X}}^{(u)} \triangleq \sum_{n=1}^{N} \mathbf{X}_n \mathbf{X}_n^H \hat{\varphi}_u\left(\mathbf{X}_n\right) - \hat{\boldsymbol{\mu}}_{\mathbf{X}}^{(u)} \hat{\boldsymbol{\mu}}_{\mathbf{X}}^{(u)H}$$

where $\hat{\varphi}_u\left(\mathbf{X}_n\right) \triangleq \frac{u(\mathbf{X}_n)}{\sum_{n=1}^{N} u(\mathbf{X}_n)}$. *If*

$$\mathrm{E}\left[\left\|\mathbf{X}\right\|^2 u\left(\mathbf{X}\right); P_{\mathbf{X};H}\right] < \infty,$$

then $\hat{\boldsymbol{\mu}}_{\mathbf{X}}^{(u)} \xrightarrow{w.p.1} \boldsymbol{\mu}_{\mathbf{X};H}^{(u)}$ *and* $\hat{\boldsymbol{\Sigma}}_{\mathbf{X}}^{(u)} \xrightarrow{w.p.1} \boldsymbol{\Sigma}_{\mathbf{X};H}^{(u)}$ *as* $N \to \infty$.

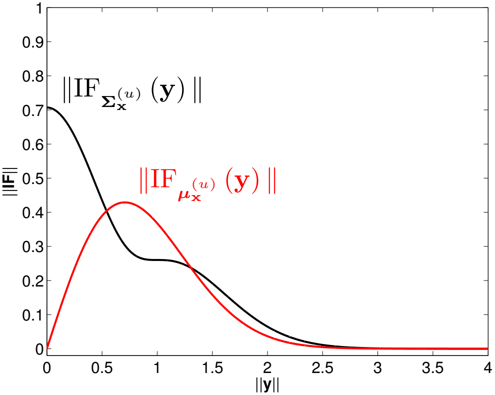
Probability Measure Transform

Proposition (Robustness to outliers)

If the MT-function $u(\mathbf{x})$ and $u(\mathbf{x})\|\mathbf{x}\|^2$ are bounded, then the influence functions [Hampel, 1974] of $\hat{\boldsymbol{\mu}}_{\mathbf{x}}^{(u)}$ and $\hat{\boldsymbol{\Sigma}}_{\mathbf{x}}^{(u)}$ are bounded.

Remark

Condition is satisfied when $u(\mathbf{x}) \in$ Gaussian family.



Measure Transformed Gaussian Quasi LRT

MT-GQLRT [*Todros & Hero, 2016*]

Compares the *empirical KLDs* between $Q^{(u)}_{\mathbf{x};H}$ and two Gaussian measures $\Phi(\boldsymbol{\mu}^{(u)}_{\mathbf{x};H_0}, \boldsymbol{\Sigma}^{(u)}_{\mathbf{x};H_0})$ and $\Phi(\boldsymbol{\mu}^{(u)}_{\mathbf{x};H_1}, \boldsymbol{\Sigma}^{(u)}_{\mathbf{x};H_1})$

$$\begin{aligned} T_u &= D_{\mathrm{LD}}\left[\hat{\boldsymbol{\Sigma}}^{(u)}_{\mathbf{x}} || \boldsymbol{\Sigma}^{(u)}_{\mathbf{x};H_0}\right] + \left\|\hat{\boldsymbol{\mu}}^{(u)}_{\mathbf{x}} - \boldsymbol{\mu}^{(u)}_{\mathbf{x};H_0}\right\|^2_{\left(\boldsymbol{\Sigma}^{(u)}_{\mathbf{x};H_0}\right)^{-1}} \\ &- D_{\mathrm{LD}}\left[\hat{\boldsymbol{\Sigma}}^{(u)}_{\mathbf{x}} || \boldsymbol{\Sigma}^{(u)}_{\mathbf{x};H_1}\right] - \left\|\hat{\boldsymbol{\mu}}^{(u)}_{\mathbf{x}} - \boldsymbol{\mu}^{(u)}_{\mathbf{x};H_1}\right\|^2_{\left(\boldsymbol{\Sigma}^{(u)}_{\mathbf{x};H_1}\right)^{-1}} \underset{H_0}{\overset{H_1}{\gtrless}} \tau, \end{aligned}$$

Measure Transformed Gaussian Quasi LRT

MT-GQLRT for the considered detection problem

- Class of MT-functions:

$$\left\{u\left(\mathbf{x}\right)=v\left(\mathbf{P}_{\mathbf{a}}^{\perp}\mathbf{x}\right),\ v:\mathbb{C}^{p}\rightarrow\mathbb{R}_{+},v(\mathbf{x})=v(-\mathbf{x})\right\}$$

- MT-mean and MT-covariance under H_0 and H_1:

$$\boldsymbol{\mu}_{\mathbf{x};H_k}^{(u)}=\mathbf{0},\qquad k=0,\,1$$

$$\boldsymbol{\Sigma}_{\mathbf{x};H_0}^{(u)}=\boldsymbol{\Sigma}_{\mathbf{w}}^{(u)}\ \text{ and }\ \boldsymbol{\Sigma}_{\mathbf{x};H_1}^{(u)}=\sigma_S^2\mathbf{a}\mathbf{a}^H+\boldsymbol{\Sigma}_{\mathbf{w}}^{(u)}$$

- Equivalent test statistic:

$$T_u'=\mathbf{a}^H\left(\boldsymbol{\Sigma}_{\mathbf{w}}^{(u)}\right)^{-1}\hat{\mathbf{C}}_{\mathbf{x}}^{(u)}\left(\boldsymbol{\Sigma}_{\mathbf{w}}^{(u)}\right)^{-1}\mathbf{a}$$

Plug-in Measure Transformed Gaussian Quasi LRT

Plug-in MT-GQLRT

Plug-in the empirical MT-covariance of the noise obtained from noise-only secondary data $\{\mathbf{Y}_m\}$

$$T''_u \triangleq \mathbf{a}^H \left(\hat{\boldsymbol{\Sigma}}_{\mathbf{Y}}^{(u)}\right)^{-1} \hat{\mathbf{C}}_{\mathbf{X}}^{(u)} \left(\hat{\boldsymbol{\Sigma}}_{\mathbf{Y}}^{(u)}\right)^{-1} \mathbf{a} \underset{H_0}{\overset{H_1}{\gtrless}} t,$$

Theorem (Asymptotic normality)

Assume that

1. $\sigma_S^2 > 0$, $\boldsymbol{\Sigma}_{\mathbf{W}}^{(u)}$ *is non-singular*
2. $\mathrm{E}\left[u^2(\mathbf{X}); P_{\mathbf{X};H}\right]$, $\mathrm{E}\left[\|\mathbf{X}\|^4 u^2(\mathbf{X}); P_{\mathbf{X};H}\right]$ *are finite*

Then

$$\sqrt{N}\left(T''_u - \eta_H^{(u)}\right) \xrightarrow[N\to\infty]{D} \mathcal{N}\left(0, \lambda_H^{(u)}\right) \quad \textit{for } H = H_0, H_1$$

Plug-in Measure Transformed Gaussian Quasi LRT

Threshold determination

$$t = \hat{\eta}_{H_0}^{(u)} + Q^{-1}\left(\alpha\right)\sqrt{\hat{\lambda}_{H_0}^{(u)}},$$

- Guarantees asymptotic CFAR$= \alpha$
- $\hat{\eta}_{H_0}^{(u)} \triangleq \mathbf{a}^H \left(\hat{\mathbf{\Sigma}}_{\mathbf{Y}}^{(u)}\right)^{-1} \mathbf{a}$
- $\hat{\lambda}_{H_0}^{(u)} \triangleq \frac{M}{N} \sum\limits_{m=1}^{M} \hat{\varphi}_u^2\left(\mathbf{Y}_m\right) \left(\left|\mathbf{a}^H \left(\hat{\mathbf{\Sigma}}_{\mathbf{Y}}^{(u)}\right)^{-1} \mathbf{Y}_m\right|^2 - \hat{\eta}_{H_0}^{(u)}\right)^2$

Plug-in Measure Transformed Gaussian Quasi LRT

Selection of the MT-function to induce outlier resilience

- Gaussian MT-function:

$$u_G\left(\mathbf{x};\omega\right) = \exp\left(-\|\mathbf{P}_{\mathbf{a}}^{\perp}\mathbf{x}\|^2/\omega^2\right)$$

- Define the Asymptotic Relative Local Power Sensitivity to change in σ_S^2 (under nominal Gaussian distribution):

$$R\left(\omega,\mathbf{\Sigma_W}\right) \triangleq \frac{\partial\beta_{u_G}}{\partial\sigma_S^2}\Big/\frac{\partial\beta_{LRT}}{\partial\sigma_S^2}\bigg|_{\sigma_S^2=0} = \frac{\sqrt{N}G\left(\omega,\mathbf{\Sigma_W}\right)+Q^{-1}\left(\alpha\right)}{\sqrt{N}+Q^{-1}\left(\alpha\right)}$$

- Select the lowest ω such that $R\left(\omega,\mathbf{\Sigma_W}\right)\geq r$, $0 << r < 1$

- In practice $\mathbf{\Sigma_W}$ is replaced by:

$$\hat{\mathbf{\Sigma}}_{\mathbf{W}}\left(\omega\right) = \left(\hat{\mathbf{\Sigma}}_{\mathbf{Y}}^{(u_G)}\left(\omega\right) - \mathbf{P}_{\mathbf{a}}^{\perp}/\omega^2\right)^{-1}$$

Simulation Studies

Setup

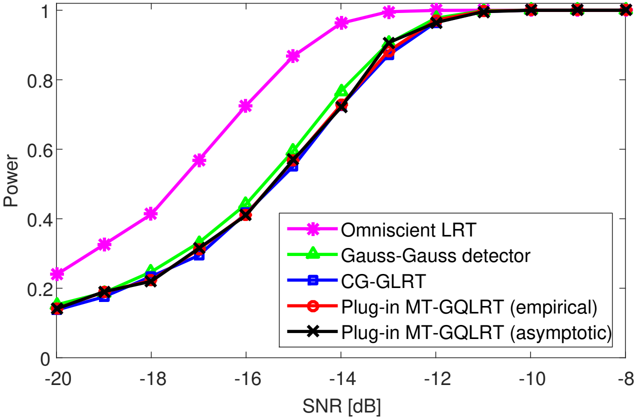
- BPSK signal with variance σ_S^2
- Steering vector: $\mathbf{a} \triangleq \frac{1}{\sqrt{p}} \left[1, e^{-i\pi \sin(\theta)}, \ldots, e^{-i\pi (p-1) \sin(\theta)}\right]^T$, $p = 8$, $\theta = 60^\circ$
- Sample size in primary and secondary data: $N = 1000$, $M = 5000$
- False alarm rate $\alpha = 0.05$
- Asymptotic Relative Local Power Sensitivity: $r = 0.9$

Simulation Studies

Detection in Gaussian noise

Zero-mean Gaussian noise with Toeplitz structured covariance $\boldsymbol{\Sigma}$:

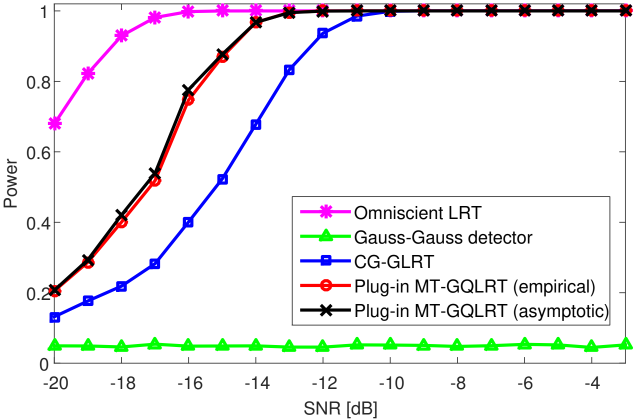
$$[\boldsymbol{\Sigma}]_{i,j} \triangleq \begin{cases} \sigma_{\mathbf{w}}^2 b^{j-i}, & i \leq j \\ \sigma_{\mathbf{w}}^2 \left(b^{i-j}\right)^*, & i > j \end{cases}, \qquad |b| < 1,$$



Simulation Studies

Detection in elliptical CG noise

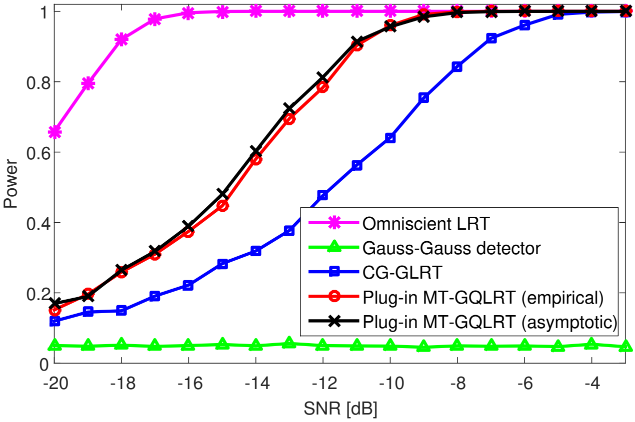
Elliptical t-distributed noise with $\lambda = 0.2$ degrees of freedom and dispersion matrix $\boldsymbol{\Sigma}$



Simulation Studies

Detection in non-elliptical noise

Elliptical t-distributed noise + BPSK interference ($\mathrm{SIR} = -5$ [dB])



Conclusions

- Plug-in MT-GQLRT for detection of a random signal that lies on a known rank-one subspace
- Simple implementation and tractable performance analysis
- Less sensitive to model mismatch as compared to other model based detectors
- Extension to non-homogeneous noise environment