Plug-in Measure-Transformed Quasi Likelihood Ratio Test for Random Signal Detection

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Outline

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- Existing detectors
- Proposed detector
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Detection Problem Formulation

\[ H_0 : \begin{cases} 
X_n = W_n, & n = 1, \ldots, N \\
Y_m = W_m^{(s)}, & m = 1, \ldots, M 
\end{cases} \]

\[ H_1 : \begin{cases} 
X_n = S_n a + W_n, & n = 1, \ldots, N \\
Y_m = W_m^{(s)}, & m = 1, \ldots, M, 
\end{cases} \]

- \{X_n \in \mathbb{C}^p\}, \{Y_m \in \mathbb{C}^p\}: \text{Primary and secondary data}
- \{S_n \in \mathbb{C}\}: \text{i.i.d. zero-mean signal with unknown distribution}
- a \in \mathbb{C}^p: \text{Known steering vector}
- \{W_n\}, \{W_m^{(s)}\}: \text{i.i.d. zero-mean symmetrically distributed homogeneous noise processes with unknown distribution}
- \{S_n\}, \{W_n\} \text{ and } \{W_m^{(s)}\} \text{ are mutually independent}
Existing detectors

Gauss-Gauss detector \cite{Jin & Friedlander 2005}

- GLRT-based detector
- Assumes normally distributed signal and noise
- Simple implementation and tractable performance analysis
- Sensitive to deviation from normality (e.g., in the case of heavy-tailed noise that produces outliers)
Existing detectors

CG-GLRT [Gerlach 1999, Shuai et. al. 2010]

- Assumes elliptical compound-Gaussian noise
- Resilient against heavy-tailed noise outliers
- Iterative ML-estimation of the noise scatter matrix
  - Converges under some regularity conditions
  - Each iteration involves matrix inversion
  - Does not reject large norm outliers
- Can be sensitive to non-elliptical noise
Proposed Detector

General operation principle
Selects a Gaussian model that best empirically fits a transformed probability distribution of the data

Advantages
- Resilient to outliers (rejects large-norm outliers)
- Involves higher-order statistical moments
- Significant mitigation of the model mismatch effect
- Computational and implementation simplicity
Probability Measure Transform

Definition
Given a non-negative function \( u: \mathcal{X} \to \mathbb{R}_+ \) satisfying

\[
0 < \mathbb{E} [u (X); P_{x;H}] < \infty.
\]

A transform \( T_u: P_{x;H} \to Q_{x;H}^{(u)} \) is defined as:

\[
T_u [P_{x;H}] (A) = Q_{x;H}^{(u)} (A) \triangleq \int_A \varphi_u (x; H) dP_{x;H} (x),
\]

where

\[
\varphi_u (x; H) \triangleq \frac{u (x)}{\mathbb{E} [u (X); P_{x;H}]}.
\]

The function \( u (\cdot) \) is called the MT-function.
Probability Measure Transform

The measure transformed mean and covariance

\[ \mu_{x;H}^{(u)} = \mathbb{E} [X \varphi_u (X; H); P_{x;H}] \]

\[ \Sigma_{x;H}^{(u)} = \mathbb{E} [XX^H \varphi_u (X; H); P_{x;H}] - \mu_{x;H}^{(u)} \mu_{x;H}^{(u)H} \]

where

\[ \varphi_u (x; H) \triangleq \frac{u (x)}{\mathbb{E} [u (X); P_{x;H}]} = \frac{dQ_{x;H}^{(u)}}{dP_{x;H}} \]

Conclusion

- The mean and covariance under \( Q_{x;H}^{(u)} \) can be estimated using only samples from \( P_{x;H} \).

- \( u (x) \) non-constant & analytic \( \Rightarrow \) the mean and covariance under \( Q_{x;H}^{(u)} \) involve higher-order statistical moments of \( P_{x;H} \).
Probability Measure Transform

Proposition (Consistent empirical MT mean and covariance)

Let $X_n$, $n = 1, \ldots, N$ denote a sequence of i.i.d. samples from $P_{X;H}$, and define the empirical mean and covariance estimates:

$$\hat{\mu}_X^{(u)} \triangleq \sum_{n=1}^{N} X_n \hat{\phi}_u (X_n)$$

$$\hat{\Sigma}_X^{(u)} \triangleq \sum_{n=1}^{N} X_n X_n^H \hat{\phi}_u (X_n) - \hat{\mu}_X^{(u)} \hat{\mu}_X^{(u)H}$$

where $\hat{\phi}_u (X_n) \triangleq \frac{u(X_n)}{\sum_{n=1}^{N} u(X_n)}$. If

$$E \left[ \|X\|^2 u(X) ; P_{X;H} \right] < \infty,$$

then $\hat{\mu}_X^{(u)} \xrightarrow{w.p.1} \mu_{X;H}^{(u)}$ and $\hat{\Sigma}_X^{(u)} \xrightarrow{w.p.1} \Sigma_{X;H}^{(u)}$ as $N \to \infty$. 
Probability Measure Transform

Proposition (Robustness to outliers)

If the MT-function $u(x)$ and $u(x)\|x\|^2$ are bounded, then the influence functions [Hampel, 1974] of $\hat{\mu}_x^{(u)}$ and $\hat{\Sigma}_x^{(u)}$ are bounded.

Remark

Condition is satisfied when $u(x) \in$ Gaussian family.

![Graph showing the influence functions for $\mu_x^{(u)}$ and $\Sigma_x^{(u)}$.](image-url)
Measure Transformed Gaussian Quasi LRT

MT-GQLRT [Todros & Hero, 2016]

Compares the empirical KLDs between $Q^{(u)}_{X;H}$ and two Gaussian measures $\Phi(\mu^{(u)}_{X;H_0}, \Sigma^{(u)}_{X;H_0})$ and $\Phi(\mu^{(u)}_{X;H_1}, \Sigma^{(u)}_{X;H_1})$

$$T_u = D_{LD} \left[ \hat{\Sigma}^{(u)}_X \| \Sigma^{(u)}_{X;H_0} \right] + \| \hat{\mu}^{(u)}_X - \mu^{(u)}_{X;H_0} \|^2 \left( \Sigma^{(u)}_{X;H_0} \right)^{-1}$$

$$- D_{LD} \left[ \hat{\Sigma}^{(u)}_X \| \Sigma^{(u)}_{X;H_1} \right] - \| \hat{\mu}^{(u)}_X - \mu^{(u)}_{X;H_1} \|^2 \left( \Sigma^{(u)}_{X;H_1} \right)^{-1} \overset{H_1}{\underset{H_0}{\gtrless}} \tau,$$
Measure Transformed Gaussian Quasi LRT

MT-GQLRT for the considered detection problem

- Class of MT-functions:
  \[
  \{ u(x) = v(P_a x), \quad v : \mathbb{C}^p \rightarrow \mathbb{R}_+, \quad v(x) = v(-x) \}
  \]

- MT-mean and MT-covariance under $H_0$ and $H_1$:
  \[
  \mu_{x;H_k}^{(u)} = 0, \quad k = 0, 1
  \]
  \[
  \Sigma_{x;H_0}^{(u)} = \Sigma_w^{(u)} \quad \text{and} \quad \Sigma_{x;H_1}^{(u)} = \sigma^2 s a a^H + \Sigma_w^{(u)}
  \]

- Equivalent test statistic:
  \[
  T_u' = a^H \left( \Sigma_w^{(u)} \right)^{-1} \hat{C}_x^{(u)} \left( \Sigma_w^{(u)} \right)^{-1} a
  \]
Plug-in Measure Transformed Gaussian Quasi LRT

Plug-in MT-GQLRT

Plug-in the empirical MT-covariance of the noise obtained from noise-only secondary data \( \{Y_m\} \)

\[
T''_u \triangleq a^H \left( \hat{\Sigma}_Y^{(u)} \right)^{-1} \hat{C}_X^{(u)} \left( \hat{\Sigma}_Y^{(u)} \right)^{-1} a \xrightarrow{H_1 \gg H_0} t,
\]

Theorem (Asymptotic normality)

Assume that

1. \( \sigma^2_S > 0, \Sigma^{(u)}_W \) is non-singular
2. \( \mathbb{E} \left[ u^2(X) ; P_{X;H} \right], \mathbb{E} \left[ \|X\|^4 u^2(X) ; P_{X;H} \right] \) are finite

Then

\[
\sqrt{N} \left( T''_u - \eta_H^{(u)} \right) \xrightarrow{D \, N \to \infty} \mathcal{N} \left( 0, \lambda_H^{(u)} \right) \quad \text{for} \quad H = H_0, H_1
\]
Plug-in Measure Transformed Gaussian Quasi LRT

Threshold determination

\[ t = \hat{\eta}_{H_0}^{(u)} + Q^{-1}(\alpha) \sqrt{\hat{\lambda}_{H_0}^{(u)}}, \]

- Guarantees asymptotic CFAR = $\alpha$
- \[ \hat{\eta}_{H_0}^{(u)} \triangleq \mathbf{a}^H \left( \hat{\Sigma}_{Y}^{(u)} \right)^{-1} \mathbf{a} \]
- \[ \hat{\lambda}_{H_0}^{(u)} \triangleq \frac{M}{N} \sum_{m=1}^{M} \varphi_u^{2}(\mathbf{Y}_m) \left( \left| \mathbf{a}^H \left( \hat{\Sigma}_{Y}^{(u)} \right)^{-1} \mathbf{Y}_m \right|^2 - \hat{\eta}_{H_0}^{(u)} \right)^2 \]
Plug-in Measure Transformed Gaussian Quasi LRT

Selection of the MT-function to induce outlier resilience

- **Gaussian MT-function:**
  \[ u_G(x; \omega) = \exp \left( -\| \mathbf{P}_a x \|^2 / \omega^2 \right) \]

- Define the **Asymptotic Relative Local Power Sensitivity** to change in \( \sigma_S^2 \) (under nominal Gaussian distribution):
  \[ R(\omega, \Sigma_W) \triangleq \left. \frac{\partial \beta_{uG}}{\partial \sigma_S^2} / \frac{\partial \beta_{LRT}}{\partial \sigma_S^2} \right|_{\sigma_S^2=0} = \frac{\sqrt{NG}(\omega, \Sigma_W) + Q^{-1}(\alpha)}{\sqrt{N} + Q^{-1}(\alpha)} \]

- Select the lowest \( \omega \) such that \( R(\omega, \Sigma_W) \geq r, \ 0 << r < 1 \)

- In practice \( \Sigma_W \) is replaced by:
  \[ \hat{\Sigma}_W(\omega) = \left( \hat{\Sigma}_Y^{(uG)}(\omega) - \mathbf{P}_a / \omega^2 \right)^{-1} \]
Simulation Studies

Setup

- BPSK signal with variance $\sigma^2_S$
- Steering vector: $a \triangleq \frac{1}{\sqrt{p}} [1, e^{-i\pi \sin(\theta)}, \ldots, e^{-i\pi(p-1) \sin(\theta)}]^T$, $p = 8$, $\theta = 60^\circ$
- Sample size in primary and secondary data: $N = 1000$, $M = 5000$
- False alarm rate $\alpha = 0.05$
- Asymptotic Relative Local Power Sensitivity: $r = 0.9$
Simulation Studies

Detection in Gaussian noise
Zero-mean Gaussian noise with Toeplitz structured covariance $\Sigma$:

$$\begin{align*}
[\Sigma]_{i,j} & \triangleq \begin{cases} 
\sigma_w^2 b^{j-i}, & i \leq j \\
\sigma_w^2 (b^{i-j})^*, & i > j,
\end{cases} \quad |b| < 1,
\end{align*}$$

SNR [dB]

Power

0
0.2
0.4
0.6
0.8
1

Omniscient LRT
Gauss-Gauss detector
CG-GLRT
Plug-in MT-GQLRT (empirical)
Plug-in MT-GQLRT (asymptotic)

SNR [dB]

Power

0
0.2
0.4
0.6
0.8
1

-20 -18 -16 -14 -12 -10 -8
Simulation Studies

Detection in elliptical CG noise

Elliptical $t$-distributed noise with $\lambda = 0.2$ degrees of freedom and dispersion matrix $\Sigma$
Simulation Studies

Detection in non-elliptical noise

Elliptical $t$-distributed noise $+$ BPSK interference ($SIR = -5$ [dB])
Conclusions

- Plug-in MT-GQLRT for detection of a random signal that lies on a known rank-one subspace
- Simple implementation and tractable performance analysis
- Less sensitive to model mismatch as compared to other model based detectors
- Extension to non-homogeneous noise environment