EUROPHYSICS LETTERS 1 April 2006

Europhys. Lett., **74** (1), pp. 15–21 (2006)
DOI: 10.1209/epl/i2005-10501-8

Weak ergodicity breaking with deterministic dynamics

G. Bel(*) and E. Barkai
Physics Department, Bar-Ilan University - Ramat-Gan 52900, Israel

received 26 October 2005; accepted in final form 3 February 2006
published online 22 February 2006

PACS. 05.45.-a – Nonlinear dynamics and chaos.
PACS. 05.40.Fb – Random walks and Levy flights.
PACS. 74.40.+k – Fluctuations (noise, chaos, nonequilibrium superconductivity, localization, etc.).

Abstract. – The concept of weak ergodicity breaking is defined and studied in the context of deterministic dynamics. We show that weak ergodicity breaking describes a system whose dynamics is governed by a nonlinear map which generates subdiffusion deterministically. In the non-ergodic phase a non-trivial distribution of the fraction of occupation times is obtained. The visitation fraction remains uniform even in the non-ergodic phase. In this sense the non-ergodicity is quantified, leading to a statistical mechanical description of the system even though it is not ergodic.

Introduction. – The relation between deterministic non-linear dynamics and statistical mechanics is a topic of wide interest [1,2]. In particular, the domain of validity of the ergodic hypothesis is fundamental in our understanding of the foundations of statistical mechanics. Boltzmann's ergodic hypothesis states that a typical trajectory on the constant energy surface spends equal amounts of time in regions of equal measure, in the limit of infinite measurement time. Using Birkhoff's theorem one may show that a system is ergodic if time averages of physical observables are identical to ensemble averages in the limit of infinite measurement time. In laboratory, the infinite time limit is usually replaced by a time which is much longer than the characteristic microscopic time scale of the dynamics. The identification of this microscopic time scale may be difficult, still if this time scale diverges as some control parameters are changed, we may expect ergodicity breaking. More general theories of dynamical systems replace the constant energy surface concept with dynamics on an invariant measure and the problem of ergodicity is not limited to Hamiltonian systems [1]. Here we consider the problem of ergodicity for a class of one-dimensional maps.

Non-ergodic behaviour is usually related to the concept of a metrically decomposable system, where the phase space is divided into at least two regions such that a phase point starting in one region will always stay in that region. According to Dorfman [1], a system is ergodic if and only if it is metrically indecomposable. In particular if a system is non-ergodic it is metrically decomposable.

A very different approach introduced by Bouchaud [3] in the context of phenomenological models of glassy dynamics, is the fundamental concept of weak ergodicity breaking. Such systems are non-ergodic but their phase space is not *a priori* decomposable into inaccessible regions. We recently investigated occupation time statistics for the well-known continuous-time random walk [4], which is a stochastic model exhibiting weak ergodicity breaking (WEB).

(*) Present address: Chemistry Department, University of California - Santa Barbara, CA 93106-9510, USA.

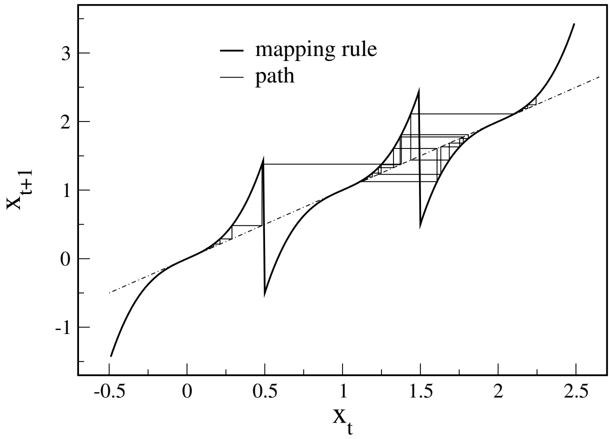

Fig. 1 – The mapping eqs. (1), (3) with $z = 3$ and $L = 3$; the dot-dashed line shows the curve $x_{t+1} = x_t$. Fixed points are at $x_t = 0, 1, 2$. A specific realization of the dynamics is also drawn (thin line).

We derived a generalized non-ergodic statistical law for the dynamics, which in the ergodic limit converges to Boltzmann statistics if detailed balance condition holds.

The goal of this paper is to investigate WEB for a deterministic dynamical system which has no disorder built into it. It is shown for the first time that a non-linear deterministic system actually exhibits WEB. Non-trivial (in a way defined later) probability density function (PDF) of the fraction of occupation time is shown to describe equilibrium properties of the dynamical system. In this sense the non-ergodicity is quantified, enabling us to perform a statistical mechanical description of the system although it is not ergodic.

Geisel's maps. – Probably the simplest deterministic maps which lead to normal and anomalous diffusion are one-dimensional maps of the form

$$x_{t+1} = x_t + F(x_t), \tag{1}$$

with the following symmetry properties of $F(x)$: i) $F(x)$ is periodic with a period interval set to 1, *i.e.*, $F(x) = F(x + N)$, where N is an integer; ii) $F(x)$ is anti-symmetric, namely, $F(x) = -F(-x)$; note that t in eq. (1) is a discrete time. Geisel and Thomae [5] considered a wide family of such maps which behave like

$$F(x) = ax^z \quad \text{for} \quad x \to 0_+, \tag{2}$$

where $z > 1$. Equation (2) defines the property of the map close to its unstable fixed point. In numerical experiments introduced later we will use the following map:

$$F(x) = (2x)^z, \qquad 0 \le x \le 1/2 \tag{3}$$

which together with the symmetry properties of the map define the mapping for all x. The functional form of the map on a unit interval was introduced by Pomeau and Manneville to describe intermittency [6]. Variations of these maps have been investigated by taking into account time-dependent noise [7], quenched disorder [8], and additional uniform bias which breaks the symmetry of the map [9]. We consider the map with L unit cells and periodic boundary conditions. In fig. 1 we show the map, and one realization of a path along three unit cells.

In an ongoing process a walker following the iteration rules may get stuck close to the vicinity of one of the unstable fixed points of the map. It has been shown both analytically and numerically, that the PDF of escape times from the vicinity of the fixed points decays as a power law [5]. To see this, one considers the dynamics in a half unit cell, say $0 < x < 1/2$. Assume that at time $t = 0$ the particle is on x^* residing in the vicinity of the fixed point $x = 0$. Close to the fixed point we may approximate the map eq. (1) with the differential equation $\mathrm{d}x/\mathrm{d}t = F(x) = ax^z$. This equation is reminiscent of the equation defining the q generalized Lyapunov exponent [10–13]. The solution is written in terms of the q *exponential* function, $\exp_q[y] \equiv [1 + (1-q)\,y]^{1/(1-q)}$, where $q = z$ and

$$x(t) = x^* \exp_z\left[a x^{*(z-1)} t\right]. \tag{4}$$

We invert eq. (4) to obtain the escape time from x^* to a boundary on $b(x^* < b < 1/2)$ which is $t \simeq \int_{x^*}^{b} [F(x)]^{-1}\,\mathrm{d}x$, using eq. (2) we obtain $t \simeq (1/a)\left[(x^{*1-z})/(z-1) - (b^{1-z})/(z-1)\right]$. The PDF of escape times $\psi(t)$ is related to the unknown PDF of injection points $\eta(x^*)$, through the chain rule $\psi(t) = \eta(x^*)\,|\mathrm{d}x^*/\mathrm{d}t|$. Assuming that $\eta(x^*)$ is smooth and non-zero around the unstable fixed point $x^* = 0$, one finds that for large escape times

$$\psi(t) \sim \frac{A}{|\Gamma(-\alpha)|} t^{-1-\alpha}, \quad \alpha = (z-1)^{-1}, \tag{5}$$

where A depends on the PDF of injection points, namely on how trajectories are injected from one cell to the other. The parameter A is sensitive to the detailed shape of the map while the parameter z depends only on the behaviour of the map close to the unstable fixed points. When $z > 2$ the average escape time diverges [5,14,15], which in turn yields aging [16], non-stationarity [17], anomalous diffusion [18,19] and as we show here WEB.

Weak ergodicity breaking. – The map is said to be ergodic if for almost any initial condition (excluding paths starting at fixed points) the path spends equal amounts of time in each cell in the limit of long measurement time. Namely $t^1/t = 1/L$, where t^1 is the time spent in one of the lattice cells and t is the total measurement time. Another method to quantify the dynamics is to consider the fraction of number of visits per cell n^1/n, where n^1 is the number of visits in one of the lattice cells and n is the total number of visits (or total number of jumps between cells). For maps with finite average escape time $\langle\tau\rangle = \int_0^\infty \tau\psi(\tau)\,\mathrm{d}\tau$, *i.e.*, $z < 2$, the total measurement time $t \sim n\langle\tau\rangle$ and the occupation time of one cell $t^1 \sim n^1\langle\tau\rangle$ hence $n^1/n = 1/L$. The equalities above should be understood in statistical sense, namely the fluctuations vanish at large n. We call n^1/n the visitation fraction, which plays an important role in WEB.

We denote the fraction of occupation time in $m \leq L$ cells as t^m/t, and the visitation fraction of these m cells as n^m/n. For ergodic systems the PDF of the fraction of occupation time is a delta function,

$$f\left(\frac{t^m}{t}\right) = \delta\left(\frac{t^m}{t} - \frac{m}{L}\right), \tag{6}$$

and the visitation fraction is

$$\frac{n^m}{n} = \frac{m}{L} \tag{7}$$

in statistical sense, both equalities are valid only in the limit of long measurement time ($t \to \infty$) (at long times the average number of jumps behaves as $\langle n\rangle \sim t^{\frac{1}{z-1}}$ in the non-ergodic case). Note that the specific choice of the m cells is not important. Maps exhibiting

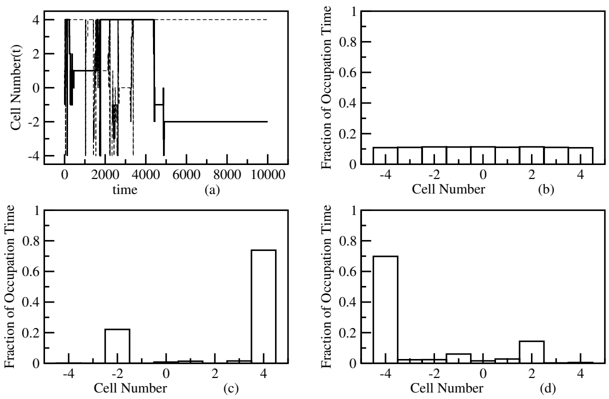

Fig. 2 – (a) Two paths (dashed and solid curves) generated according to the mapping rule with $z = 3$; for each path the particle gets stuck in one of the cells for a time which is of the order of the measurement time. (b) Histogram of the fraction of occupation for a path with $z = 1.5$. For this ergodic case the fraction of occupation times in the cells is uniform. (c), (d) The same as in (b), but this time for the non-ergodic phase with $z = 3$.

strong non-ergodicity do not obey eqs. (6), (7). Dynamics obeying eq. (7) but not eq. (6) are said to be weakly non-ergodic. As we will show, when $z > 2$ WEB is found.

We start our analysis by discussing numerical simulations of the map with a system of size $L = 9$. In fig. 2(a) we show two paths generated according to the mapping rule eqs. (2), (3) with $z = 3$. As one can see, each path gets stuck at one of the cells for a time which is of the order of the measurement time. During the measurement the path visits all the L cells many times. In fig. 2(b) we consider the case $z = 1.5$; we show that the fraction of occupation time obeys eq. (6) with $m = 1$, namely $t^1/t = 1/L$, with $L = 9$. On the other hand, in figs. 2(c,d) we show the fraction of occupation time histogram for two initial conditions obtained using $z = 3$. Since each path is localized in one of the cells for a time which is of the order of the total measurement time, the histogram for this case looks very different from that of the ergodic case and clearly eq. (6) is not valid. Histograms of the visitation fraction in each cell are presented in fig. 3, for both ergodic ($z = 1.2$) (a) and non-ergodic ($z = 3$) (b) maps. The figure demonstrates that for both cases eq. (7) holds, *i.e.*, the visitation fraction in each cell is given by $1/L$. From fig. 2 we see that the fraction of occupation time is not uniform, hence we investigate its distribution. We will find universal PDFs of occupation time, which generalizes, the ergodic rule eq. (6).

The PDF for t^m (the occupation time of m cells) in the case of diverging mean escape time is now derived. In particular, we show that the assumption of eq. (7) with power law waiting times, leads to non-trivial PDFs of the fraction of occupation times of the map. We denote by $f_{n,t}^{out}(t^m)$ the PDF of t^m, in the case that the path is not within the m cells at time t and that during the measurement time t there were n jumps. Then

$$f_{n,t}^{out}(t^m) = \left\langle \delta\left(t^m - \sum_{k=1}^{n^m} \tau_k^m\right) I\left(T_n < t < T_{n+1}\right)\right\rangle, \qquad (8)$$

where the summation in the delta function is over all sojourn times within the m cells, namely τ_k^m is the k-th sojourn time within the m cells, T_n is the time after n jumps and $I(x)$ is equal to 1 if x is true and 0 otherwise, the brackets denote average over all sojourn times τ. Double

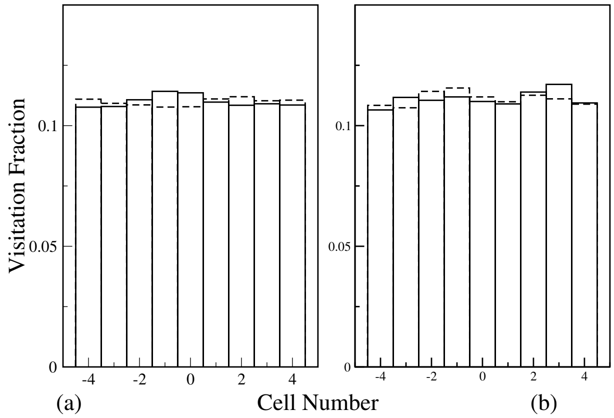

Fig. 3 – Histogram of the visitation fraction in each of the 9 lattice cells. (a) Ergodic phase, $z = 1.2$; dashed and solid lines correspond to two different initial conditions. (b) Non-ergodic phase with $z = 3$. In both ergodic and non-ergodic cases the visitation fractions in each cell are equal, with small fluctuations.

Laplace transform (LT) of eq. (8) yields

$$\widetilde{f}^{out}_{n,s}(u) = \left\langle \int_0^\infty \int_0^\infty e^{-ut^m} e^{-st} \delta\left(t_i - \sum_{k=1}^{n^m} \tau_k^m\right) I\left(T_n < t < T_{n+1}\right) \mathrm{d}t_i \mathrm{d}t \right\rangle$$
$$= \frac{\widetilde{\psi}^{n^m}(u+s)\, \widetilde{\psi}^{n-n^m}(s)\left[1-\widetilde{\psi}(s)\right]}{s}, \qquad (9)$$

where $\widetilde{\psi}(s)$ is the LT of $\psi(t)$. We assumed that waiting times in cells are not correlated. This and other assumptions are checked later using numerical simulations. In a way similar to eq. (9) we denote by $f^{in}_{n,t}(t^m)$ the PDF of t^m in the case that the particle is within the m cells at the end of the measurement, then in double Laplace space we find

$$\widetilde{f}^{in}_{n,s}(u) = \widetilde{\psi}^{n^m}(u+s)\, \widetilde{\psi}^{n-n^m}(s)\, \frac{1-\widetilde{\psi}^{n^m+1}(u+s)}{u+s}. \qquad (10)$$

Considering an ensemble of initial conditions uniformly distributed in a cell, the probability for the path to be within the m cells at the end of the measurement is m/L, thus the double-Laplace-transformed PDF of the occupation time of the cell, given that there were n visits during the measurement is

$$\widetilde{f}_{n,s}(u) = \frac{m}{L}\widetilde{f}^{in}_{n,s}(u) + \left(1-\frac{m}{L}\right)\widetilde{f}^{out}_{n,s}(u). \qquad (11)$$

Using eqs. (9), (10) and the assumption $n^m/n = m/L$, eq. (7), we rewrite $\widetilde{f}_{n,s}(u)$ as

$$\widetilde{f}_{n,s}(u) = \left[\frac{m}{L}\frac{1-\widetilde{\psi}(u+s)}{s+u} + \frac{L-m}{L}\frac{1-\widetilde{\psi}(s)}{s}\right]\left(\widetilde{\psi}^{m/L}(u+s)\,\widetilde{\psi}^{\frac{L-m}{L}}(s)\right)^n. \qquad (12)$$

Let $f_t(t^m)$ be the PDF of the occupation time of the m cells, its double LT is given by

$$\widetilde{f}_s(u) = \sum_{n=0}^{\infty}\widetilde{f}_{n,s}(u) = \left[\frac{m}{L}\frac{1-\widetilde{\psi}(u+s)}{s+u} + \frac{L-m}{L}\frac{1-\widetilde{\psi}(s)}{s}\right]\frac{1}{1-\widetilde{\psi}^{m/L}(u+s)\,\widetilde{\psi}^{\frac{L-m}{L}}(s)}. \qquad (13)$$

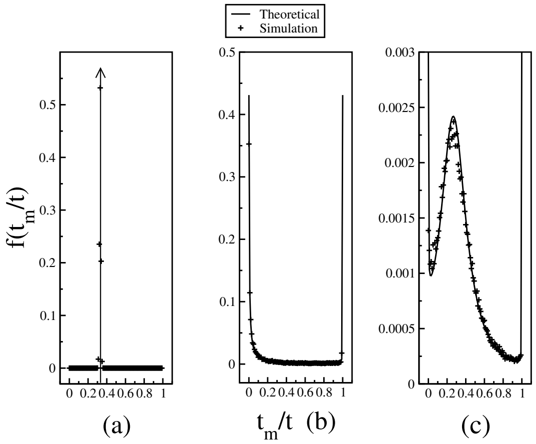

Fig. 4 – (a) The PDF of the fraction of occupation time of $m = 3$ cells for a system with $L = 9$ cells and $z = 1.5$. In this ergodic phase the PDF is very narrow and centered around m/L. (b) Same as (a), however here $z = 3$. The PDF has a U shape, with maxima at 0 and 1. (c) Same as (a) with $z = 2.25$. In this case a W-shaped PDF is obtained, with a peak at the vicinity of m/L, but yet the maxima are at 0 and 1. The curve is eq. (14) without fitting. Only the parameter z determines the shape of the PDFs, while a detailed form of the map is unimportant.

Taking the limit $u, s \to 0$ (using the asymptotic behaviour of the escape time PDF $\widetilde{\psi}(s) \sim 1 - As^{\frac{1}{z-1}}$), and inverting the double LT, one obtains Lamperti's PDF [20] of the fraction of the occupation time,

$$f\left(\frac{t^m}{t}\right) = \frac{\sin\left(\frac{\pi}{z-1}\right)}{\pi} \frac{R\left(\frac{t^m}{t}\right)^{\frac{2-z}{z-1}}\left(1-\frac{t^m}{t}\right)^{\frac{2-z}{z-1}}}{R^2\left(1-\frac{t^m}{t}\right)^{\frac{2}{z-1}} + \left(\frac{t^m}{t}\right)^{\frac{2}{z-1}} + 2R\left(1-\frac{t^m}{t}\right)^{\frac{1}{z-1}}\left(\frac{t^m}{t}\right)^{\frac{1}{z-1}}\cos\left(\frac{\pi}{z-1}\right)}, \quad (14)$$

where $R = \frac{m}{L-m}$, which is valid only when t is large. Equation (14) for $m = 1$ is in agreement with the formalism developed in [4]. Here we showed that the WEB condition $n^m/n = m/L$ and the power law behaviour of the sojourn time PDF leads to the statistical law eq. (14).

The assumptions made in the derivation of eq. (14) are tested using numerical simulations of the map. Plots of the PDF of the fraction of occupation time for three different values of z are shown in fig. 4. We used 10^6 trajectories and measured the occupation time of $m = 3$ cells in a system of size $L = 9$. The simulations time is $t = 10^6$, and it was verified that the PDF of occupation times already converges for such a measurement time. In the ergodic phase, $z < 2$, the PDF is just a delta function centered at m/L (see fig. 4 (a)). In the non-ergodic cases the maxima of $f(t^m/t)$ are at 0 and 1. These events $t^m/t \approx 1 (t^m/t \approx 0)$ correspond to trajectories where the particle occupies (does not occupy) one of the observed cells for the whole duration of the measurement, respectively. In the non-ergodic phase, $z > 2$, two types of behaviour are found. For $z = 2.25$, the PDF of the fraction of occupation time has a W shape, with a peak in the vicinity of m/L. While for $z = 3$, we find a U-shaped PDF and the probability of obtaining a fraction of occupation time equal to m/L is almost zero. Comparison of theory, eq. (14), with the numerical simulations of the map shows excellent agreement without any fitting, thus justifying our assumptions. Note that if we limit our model to integer z (*i.e.*, assuming an analytical behaviour) then we either observe WEB ($z > 2$) or the special border point between WEB and ergodicity $z = 2$. The latter point exhibits very slow convergence and logarithmic corrections which should be the subject of future work.

Conclusion. – The weak ergodicity breaking we find is related to the dynamics of trajectories close to the unstable fixed points. On the coarse-grained level of cells the system seems to be in equilibrium. However there is a subtle point to the dynamics. The particle is ejected in the vicinity of fixed points, the closer the particle is to the fixed point the longer it takes it to escape. Since in the non-ergodic case the average escape time diverges, there is no microscopic time scale and ergodicity is broken. A detailed characterization of weak ergodicity breaking was established as follows: i) The lattice is not divided into mutually inaccessible regions, thus a trajectory visits all cells for almost all (all but a set of measure zero) initial conditions. ii) The visitation fraction fluctuates very slightly in the limit $t \to \infty$ (note that n and n^m exhibit strong fluctuations). iii) The visitation fraction is determined by the probability of finding a member of an ensemble of particles within the given cell. iv) The total time the system spends in each cell fluctuates strongly, and is described by eq. (14), implying that statistics of occupation times depend only on the parameter z describing the non-linearity and not on the shape of the map far from the fixed points (*e.g.*, a in eq. (2)). A relation between deterministic anomalous diffusion and weak ergodicity breaking was found. Since the former behaviour is wide spread in dynamical systems, it is possible that the non-ergodic dynamical scenario might emerge in other fractional dynamical systems, and then the present concepts naturally lead to a new type of weakly non-ergodic statistical mechanics.

* * *

EB thanks the Center for Complexity Science Jerusalem for support.

REFERENCES

[1] Dorfman J. R., *An Introduction to Chaos in Nonequilibrium Statistical Mechanics, Cambridge Lect. Notes Phys.* (Cambridge University Press, Cambridge) 1999.

[2] Zaslavsky G. M., *Phys. Today*, **8** (1999) 39.

[3] Bouchaud J. P., *J. Phys. I*, **2** (1992) 1705.

[4] Bel G. and Barkai E., *Phys. Rev. Lett.*, **94** (2005) 240602.

[5] Geisel T. and Thomae S., *Phys. Rev. Lett.*, **52** (1984) 1936.

[6] Pomeau Y. and Manneville P., *Commun. Math. Phys.*, **74** (1980) 189; Manneville P., *J. Phys.*, **41** (1980) 1235.

[7] Bettin R., Mannella R., West B. J. and Grigolini P., *Phys. Rev. E*, **51** (1995) 212.

[8] Radons G., *Phys. Rev. Lett.*, **77** (1996) 4748.

[9] Barkai E. and Klafter J., *Phys. Rev. Lett.*, **79** (1997) 2245.

[10] Baldovin F. and Robledo A., *Europhys. Lett.*, **60** (2002) 518.

[11] Robledo A., *Physics A*, **342** (2004) 104.

[12] Ananos G. F. J. and Tsallis C., *Phys. Rev. Lett.*, **93** (2004) 020601.

[13] Grassberger P., *Phys. Rev. Lett.*, **95** (2005) 140601.

[14] Zaslavski G. M., *Phys. Rep.*, **371** (2002) 461.

[15] Geisel T., Nierwetberg J. and Zacherl A., *Phys. Rev. Lett.*, **54** (1985) 616.

[16] Barkai E., *Phys. Rev. Lett.*, **90** (2003) 104101.

[17] Ignaccolo M., Grigolini P. and Rosa A., *Phys. Rev. E*, **64** (2001) 026210.

[18] Zumofen G. and Klafter J., *Phys. Rev. E*, **47** (1993) 851.

[19] Korabel N., Chechkin A. V., Klages R., Sokolov I. M. and Gonchar V. Yu., *Europhys. Lett.*, **70** (2005) 63.

[20] Lamperti J., *Trans. Am. Math. Soc.*, **88** (1958) 380.