Nonlinear dynamics and pattern formation with applications to ecology

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Part II: Controling patterns by temporal and spatial periodic forcing

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Outine of Part II

Experimental studies of periodically forced pattern forming systems:

1. Temporally forced Belouzov-Zhabotinsky reaction
2. Spatially forced Rayleigh-Benard convection

Nonlinear analysis of temporally forced oscillating systems:

1. 2:1 resonance – frequency locking in uniform oscillations, phase front instabilities, Bloch-front turbulence, effects of pattern on frequency locking.
2. 4:1 resonance – phase front instabilities, multi-phase patterns

Nonlinear analysis of spatially forced systems:

Wavenumber locking can be dramatically different from frequency locking

Experimental studies of periodically forced systems

Temporal forcing of a spatially extended oscillatory system

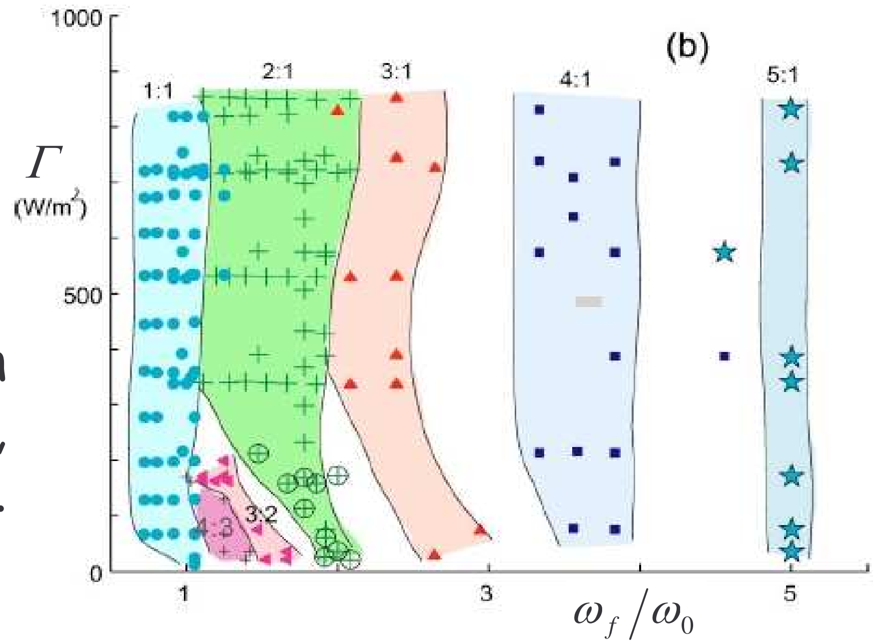
Experimental system: BZ reaction with a photo-sensitive catalyst, forced by periodic illumination in time, uniform in space.

The forcing is controlled by the light intensity Γ and the ratio of forcing frequency ω_f to the frequency ω_0 of the unforced reaction.

The experimental results reproduced main dynamical aspects of single forced oscillator, such as Arnold tongues of frequency locking within which

$$\omega_f / \omega = n/m \quad (\, n{:}m \text{ resonance})$$

where ω is the actual oscillation frequency of the forced reaction, and traces of Farey tree, e.g. (3:2) between (1:1) and (2:1).



Lin, Hagberg, Meron, Swinney, PRE 2004

Experimental studies of periodically forced systems

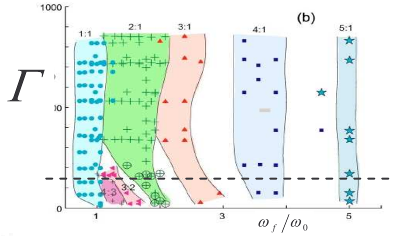
But the experimental results also revealed strong effects of the forcing on pattern formation:

Transitions from traveling spiral waves to standing labyrinthine patterns

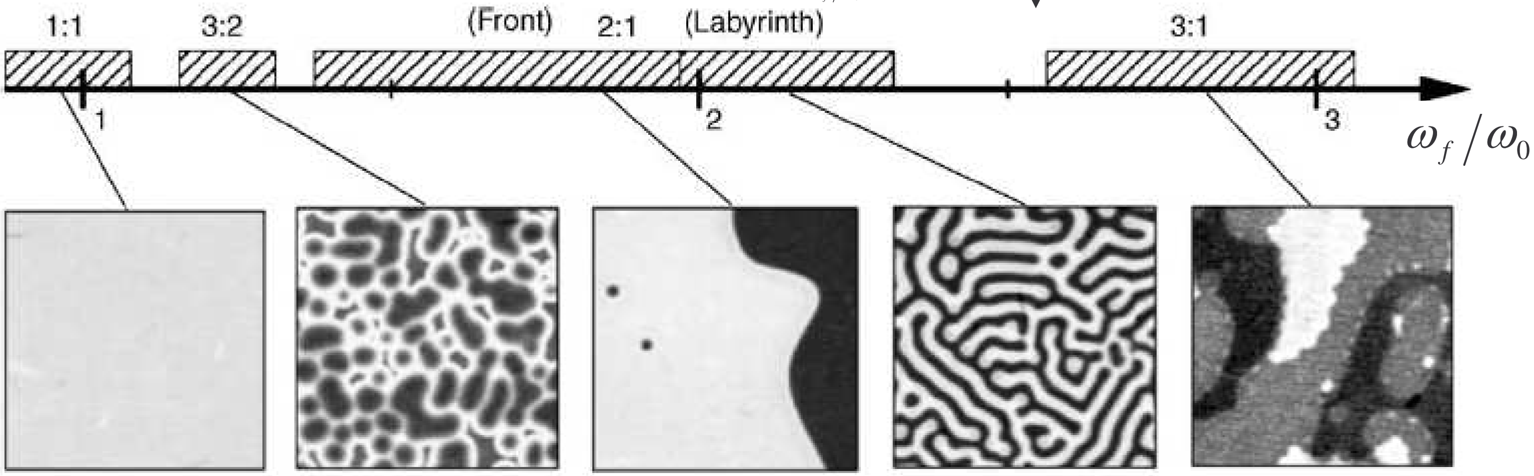
Unforced BZ

2:1 forced BZ

A wide variety of patterns:

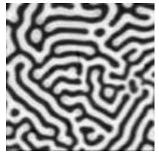


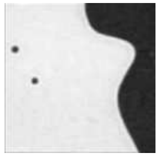
Petrov, Ouyang, Swinney
Nature (1997)



Experimental studies of periodically forced systems

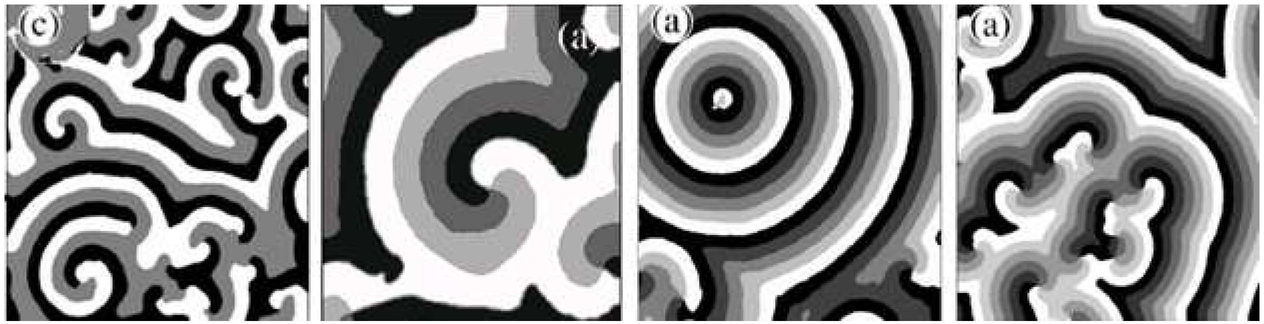
The 2:1 resonance include:

Patterns with a characteristic length-scale or wavenumber 

Amorphous domain patterns with fronts 

Are the former a result of a transverse front instability, or a result of a finite-wavenumber instability of a uniform state (i.e. class II or class I)?

Multiphase traveling waves:



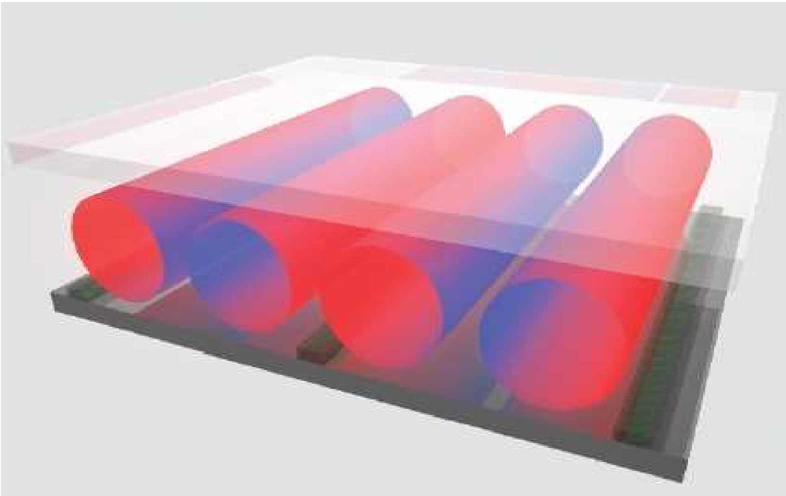
3-phase 4-phase 5-phase 6-phase

Lin, Hagberg, Meron, Swinney, PRE 2004

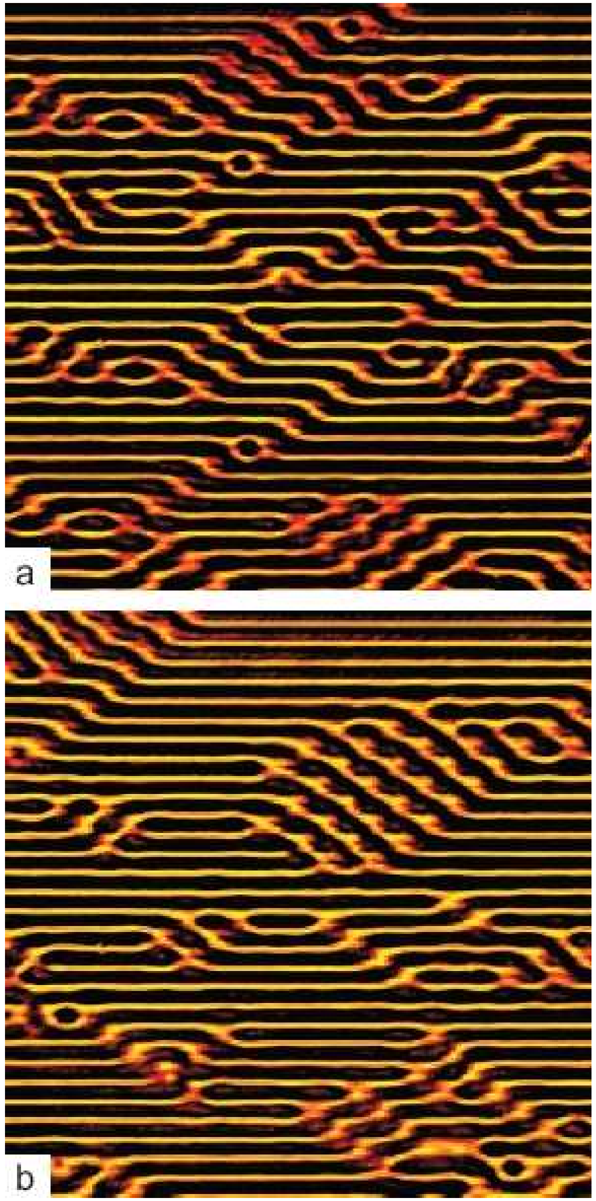
Experimental studies of periodically forced systems

Spatial forcing of a system supporting stripe (roll) patterns

Experimental system: Rayleigh-Benard convection with a periodically modulated bottom plate



One-dimensional forcing induces two-dimensional patterns – oblique domains in a roll background (entrained to the modulation). (a) and (b) – 5 hours apart.

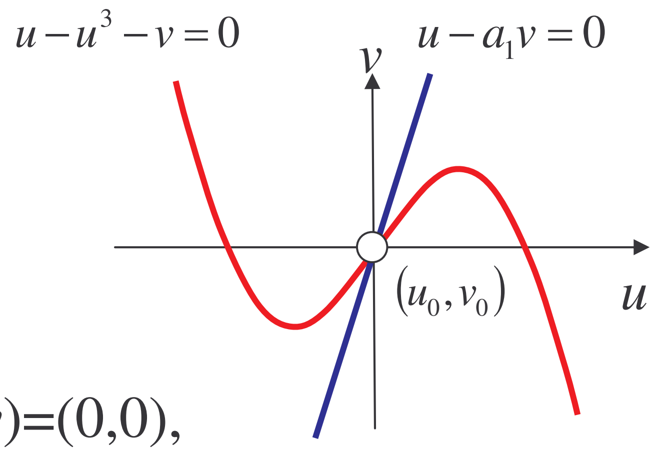


McCoy and Bodenschatz 2008

2:1 temporal forcing: amplitude equation

Consider the FHN model (Class I):

$$u_t = u - u^3 - v + \nabla^2 u$$
$$v_t = \varepsilon(u - a_1 v) + \delta\nabla^2 v \qquad a_1 < 1$$



The single stationary uniform state, $(u,v)=(0,0)$, loses stability to uniform oscillations as ε is decreased below

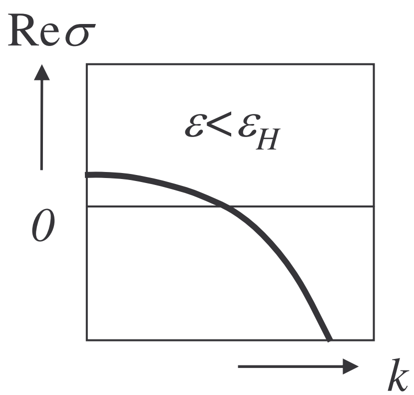
$$\varepsilon = \varepsilon_H = a_1^{-1}$$

The oscillation frequency at the instability point is :

$$\omega_H = \sqrt{a_1^{-1}(1-a_1)}$$

This is an example of a Hopf bifurcation where an oscillatory mode begins to grow:

$$u \sim e^{\sigma(k)t + i\omega_H t} + c.c.$$



2:1 temporal forcing: amplitude equation

Suppose we force the system at a frequency $\omega_f \approx 2\omega_H$ by adding to a_1 an oscillatory term:

$$a_1 \to a_1 + \Gamma \sin \omega_f t$$

Close to the Hopf bifurcation we can approximate solutions of the forced FHN model by

$$\begin{pmatrix} u \\ v \end{pmatrix} \cong \begin{pmatrix} 1 \\ c \end{pmatrix} A(x, y, t) e^{i(\omega_f/2)t} + c.c.$$

The Complex Ginzburg-Landau or CGL equation

where $A(x,y,t)$ satisfies the amplitude equation

$$A_t = (\lambda + i\nu)A + (1 + i\alpha)\nabla^2 A - (1 + i\beta)|A|^2 A + \gamma A^*$$

$\lambda = (\varepsilon_H - \varepsilon)/\varepsilon_H$ - distance from the Hopf bifurcation

$\nu = \omega_H - \omega_f/2$ - detuning (deviation from exact 2:1 resonance)

$\gamma \propto \Gamma$ - forcing intensity

Frequency locking of uniform oscillations

The temporally forced system has two frequencies, the Hopf frequency, ω_H , and the forcing frequency ω_f . In general such systems show quasi-periodic oscillations, but if the two frequencies are rationally related the dynamics is periodic in time.

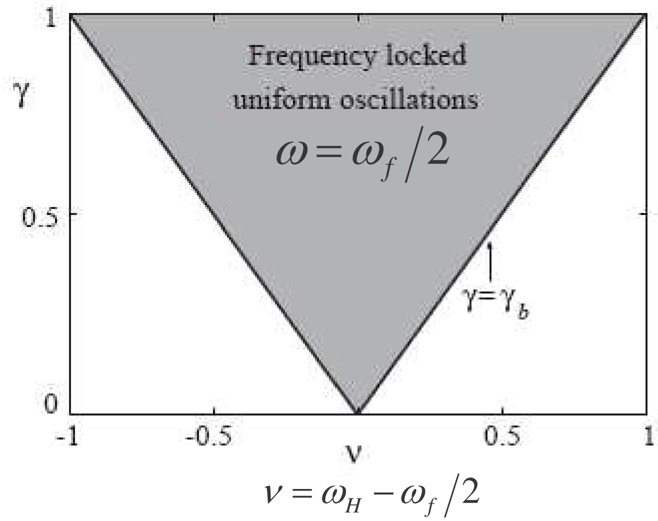
Time independent solutions of the amplitude equation represent periodic oscillations of the FHN system with a frequency $\omega_f/2$.

Let's consider first time independent solutions which are also uniform in space, i.e. solutions satisfying

$$(\lambda + i\nu)A - (1+i\beta)|A|^2 A + \gamma A^* = 0$$

Such solutions exist provided $\gamma > \gamma_b$ where

$$\gamma_b = \frac{\nu - \lambda\beta}{\sqrt{1+\beta^2}}$$



2:1 temporal forcing: frequency locking – patterns

The shaded area is the so called Arnold tongue of the 2:1 resonance. Within the tongue the system adjusts its oscillation frequency to exactly half the forcing frequency, even when $\omega_H \neq \omega_f / 2$. We say that the oscillation frequency is locked to the forcing.

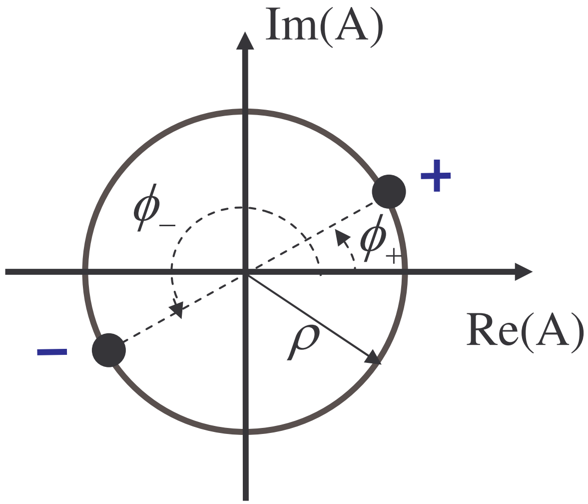
What types of spatial patterns the forcing can induce and how is frequency locking affected by pattern formation?

Recall the two classes of pattern-formation mechanisms: Class I (instabilities of uniform states) and Class II (multi-stability of states).

Class II pattern formation mechanisms induced by the forcing:

There are two stable constant solutions within the 2:1 Arnold tongue, whose phases (arguments) differ by π:

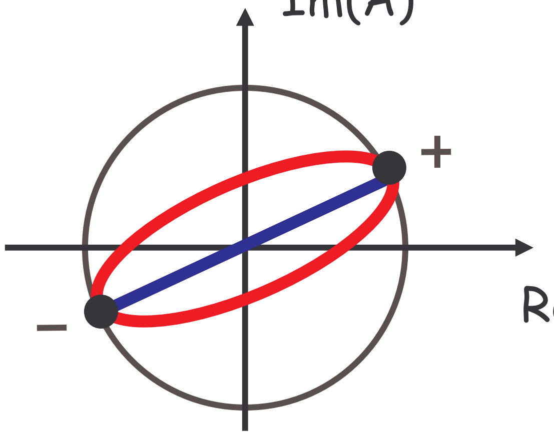
$$A_\pm = \rho e^{i\phi_\pm}, \qquad \phi_- = \phi_+ + \pi$$

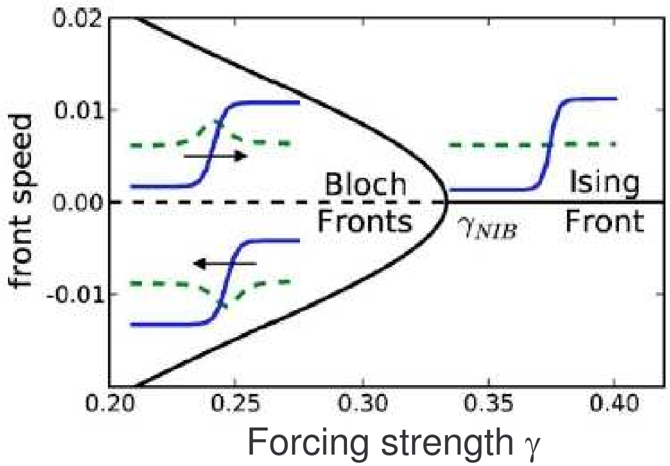


$\Rightarrow$ The forcing creates a bistable system

2:1 temporal forcing: frequency locking - patterns

and front solutions that can go through a NIB front bifurcation as the forcing strength is decreased (like in the unforced bistable FHN):

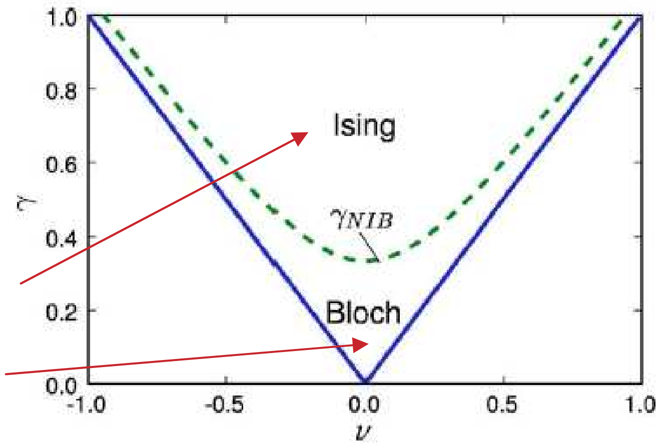


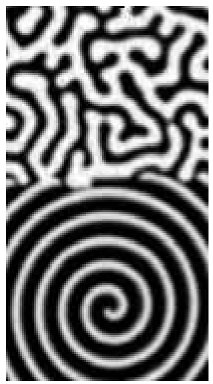


In the forcing plan the NIB threshold divides the Arnold tongue into two parts:

Ising: Stationary fronts and patterns

Bloch: Traveling fronts and patterns





Forced BZ
Petrov et al.

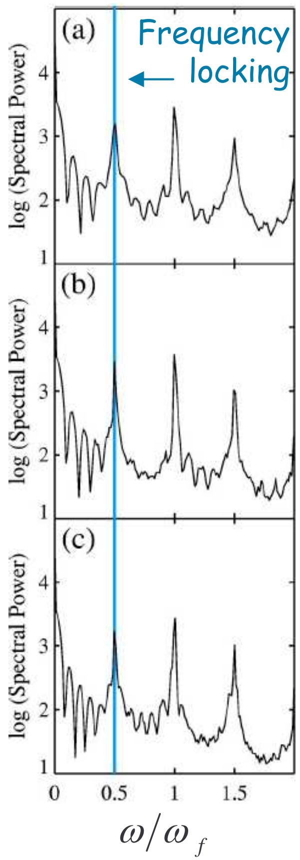
2:1 temporal forcing: frequency locking – patterns

Stationary solutions of the amplitude equation $\Rightarrow$ resonant oscillations at exactly $\omega_f/2$.

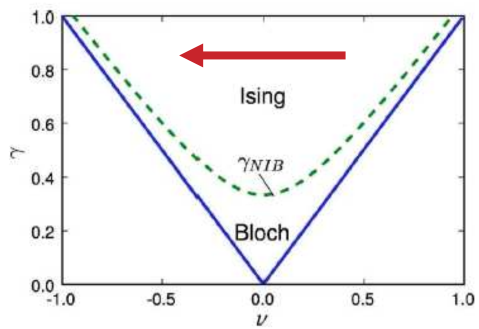
Ising regime remains resonant also when spatial patterns appear.

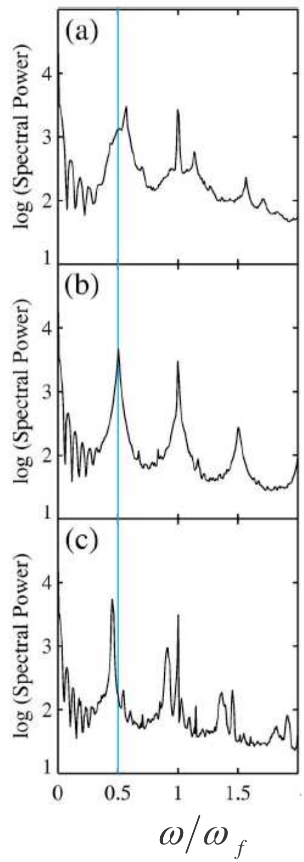
Bloch regime becomes non-resonant when patterns appear

Supported by experiments on the forced BZ reaction (Lin et al. PRE, 2004):

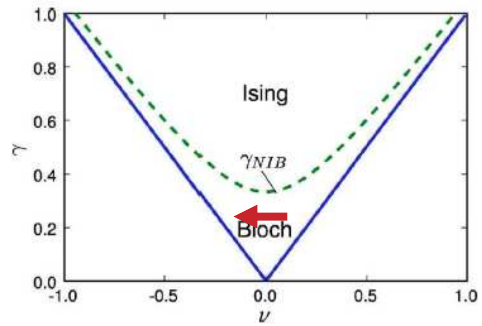


As the forcing frequency is varied across the tongue, the BZ frequency remains locked to $\omega_f/2$





As the forcing frequency is varied across the tongue, the BZ frequency is **not** locked to $\omega_f/2$

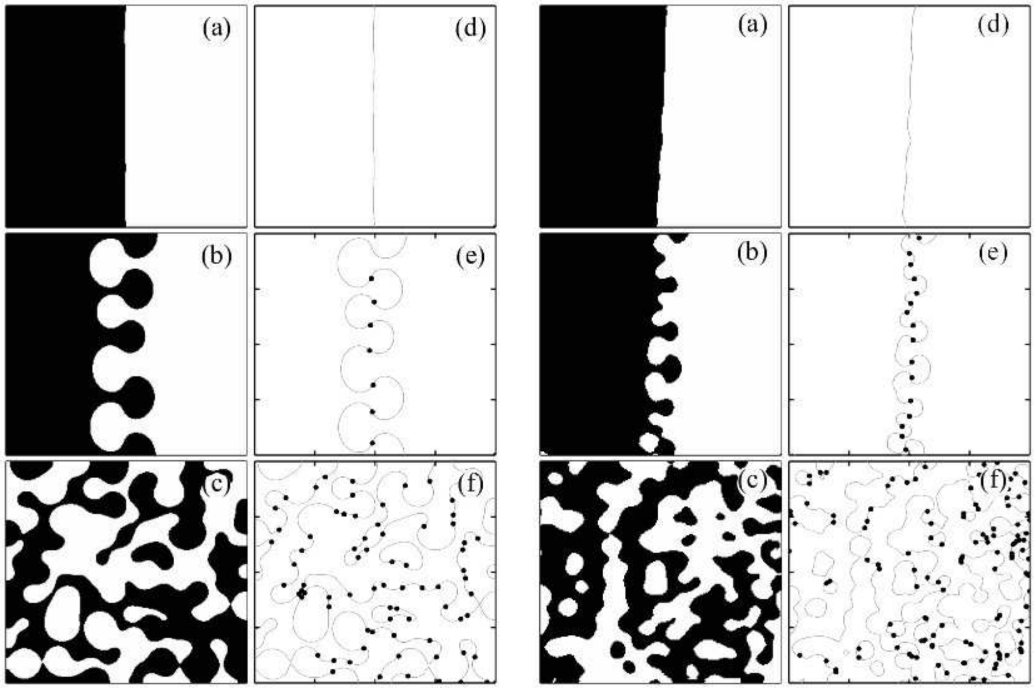


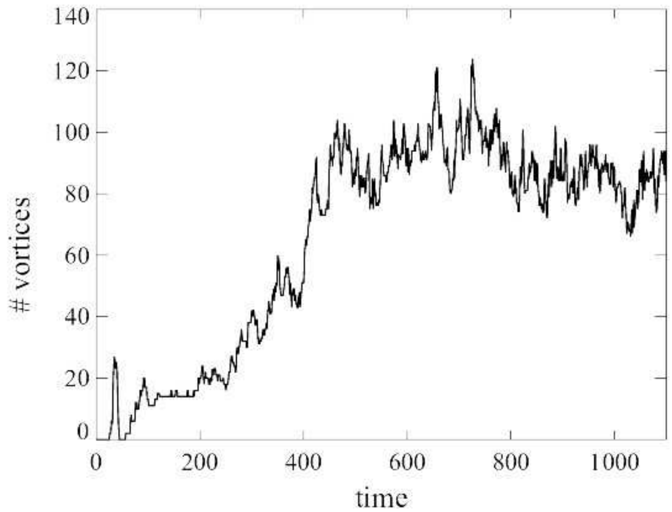
2:1 temporal forcing: frequency locking – patterns

Dynamics near the NIB bifurcation: can we find Bloch-front turbulence as in the (unforced) bistable FHN model?

Forced CGL simulations

Experimental observations





Marts, Hagberg, Meron and Lin et al. PRL, 2004

A non-resonant dynamics within Arnold tongue of resonant uniform oscillations.

2:1 temporal forcing: frequency locking - patterns

Class I pattern formation mechanisms induced by the forcing:

The forcing not only affects the oscillatory states, creating bistability of states, front structures, etc.; it can also induce a finite-wavenumber or Turing-like instability of the original zero state that underwent the Hopf bifurcation to uniform oscillations.

Recall the amplitude (CGL) equation

$$A_t = (\lambda + i\nu)A + (1 + i\alpha)\nabla^2 A - (1 + i\beta)|A|^2 A + \gamma A^*$$

Linear stability analysis of the zero state $A=0$:

$$\delta A(x,t) = a e^{\sigma(k)t + ikx} + c.c. \qquad |a| \to 0$$

gives

$$\sigma(k) = \lambda - k^2 + \sqrt{\gamma^2 - (\nu - \alpha k^2)^2}$$

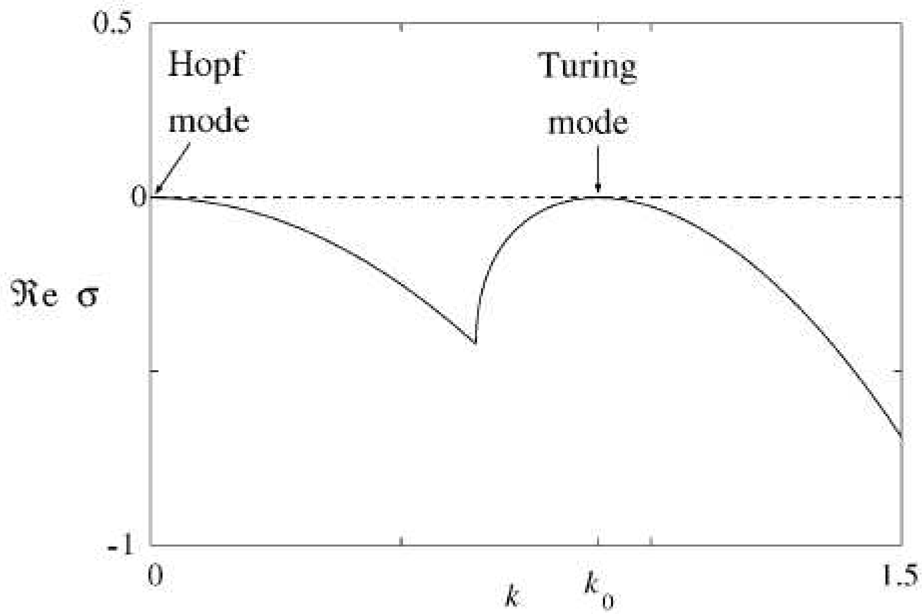
By tuning the forcing strength γ and distance λ from the Hopf bifurction we can identify a surface in parameter space, or a codimension-2 point, $\lambda = 0,$ $\quad \gamma = \nu / \sqrt{1 + \alpha^2}$ at which both a

2:1 temporal forcing: frequency locking – patterns

Hopf mode (k=0, $\omega=\omega_0$) and a Turing (finite-wavenumebr) mode ($k=k_0$, ω=0) begin to grow simultaneously:

$$k_0^2 = \frac{\nu\alpha}{1+\alpha^2},$$

$$\omega_0 = \frac{\nu\alpha}{\sqrt{1+\alpha^2}}$$



In the vicinity of this codimension-2 point we can approximate a solution of the CGL equation as

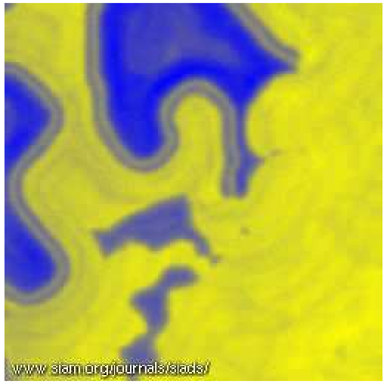
$$A(x,y,t) \cong B_0(x,y,t)e^{i\omega_0 t} + B_k(x,y,t)e^{ikx} + c.c.$$

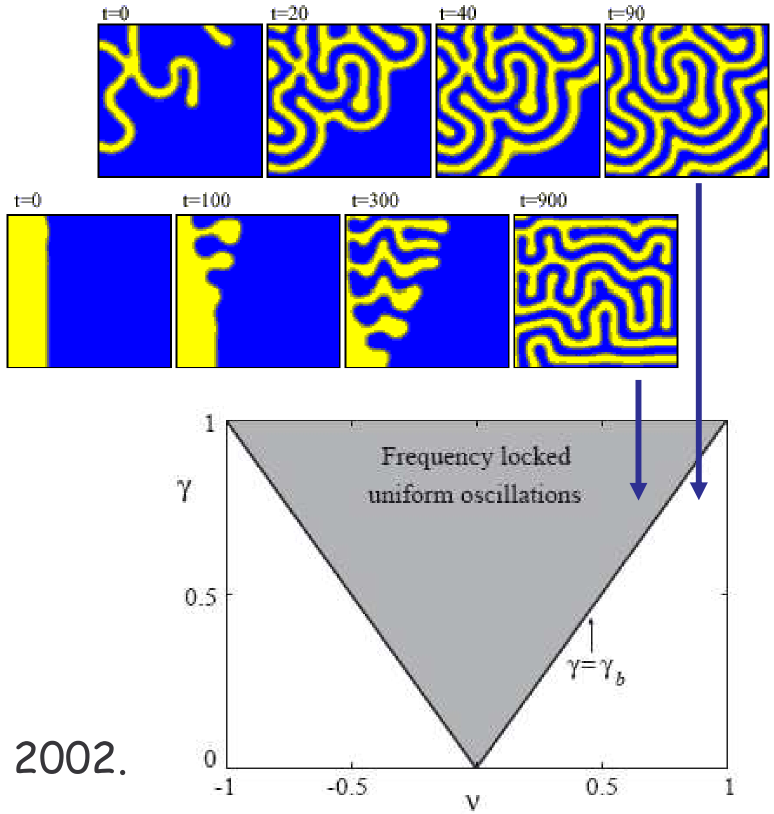
and derive coupled equations for the amplitudes B_0 and B_k , the so called Hopf-Turing amplitude equations.

2:1 temporal forcing: frequency locking – patterns

A significant outcome of these equations is the possibility of having stationary labyrinthine patterns outside the Arnold tongue boundaries, implying resonant standing waves in a parameter range where uniform oscillations are not resonant.

These patterns are to be contrasted with stationary labyrinthine patterns inside the Arnold tongue resulting from a transverse instability of an Ising front.





Experimental results, obtained in the forced BZ reaction, that show resonant standing waves invading into quasi-periodic oscillations outside the Arnold tongue.

Yochelis, Hagberg, Meron, Lin, Swinney, SIADS 2002.

4:1 temporal forcing: amplitude equation

To study periodic forcing we may consider again the FHN model except that now we force the cubic term:

$$u_t = u - \left(1 + \Gamma \sin \omega_f t\right) u^3 - v + \nabla^2 u \qquad \omega_f \approx 4\omega_H$$

$$v_t = \varepsilon(u - a_1 v) + \delta \nabla^2 v \qquad a_1 < 1$$

Close to the Hopf bifurcation where $\lambda = \left(\varepsilon_H - \varepsilon\right)/\varepsilon_H$ is small, we approximate a solution to the forced FHN equations as

$$\begin{pmatrix} u \\ v \end{pmatrix} \cong \begin{pmatrix} 1 \\ c \end{pmatrix} A(x, y, t) e^{i(\omega_f/4)t} + c.c.$$

The amplitude equation now reads

$$A_t = (\lambda + i\nu)A + (1 + i\alpha)\nabla^2 A - (1 + i\beta)|A|^2 A + \gamma A^{*3}$$

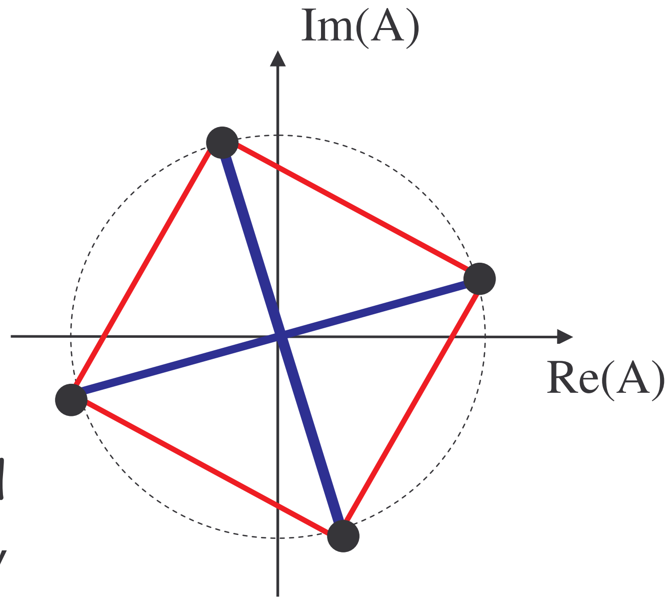
where $\nu = \omega_H - \omega_f/4$ is the deviation from exact 4:1 resonance, and γ is proportional to the forcing strength Γ.

4:1 temporal forcing: phase front instability

The forcing creates four stable constant solutions within the 4:1 Arnold tongue, whose phases (arguments) differ by $\pi/2$. It also creates two types of front solutions:

π-fronts: shift the phase by π

$\pi/2$ -fronts: shift the phase by $\pi/2$



Analysis of the amplitude equation reveals the following:

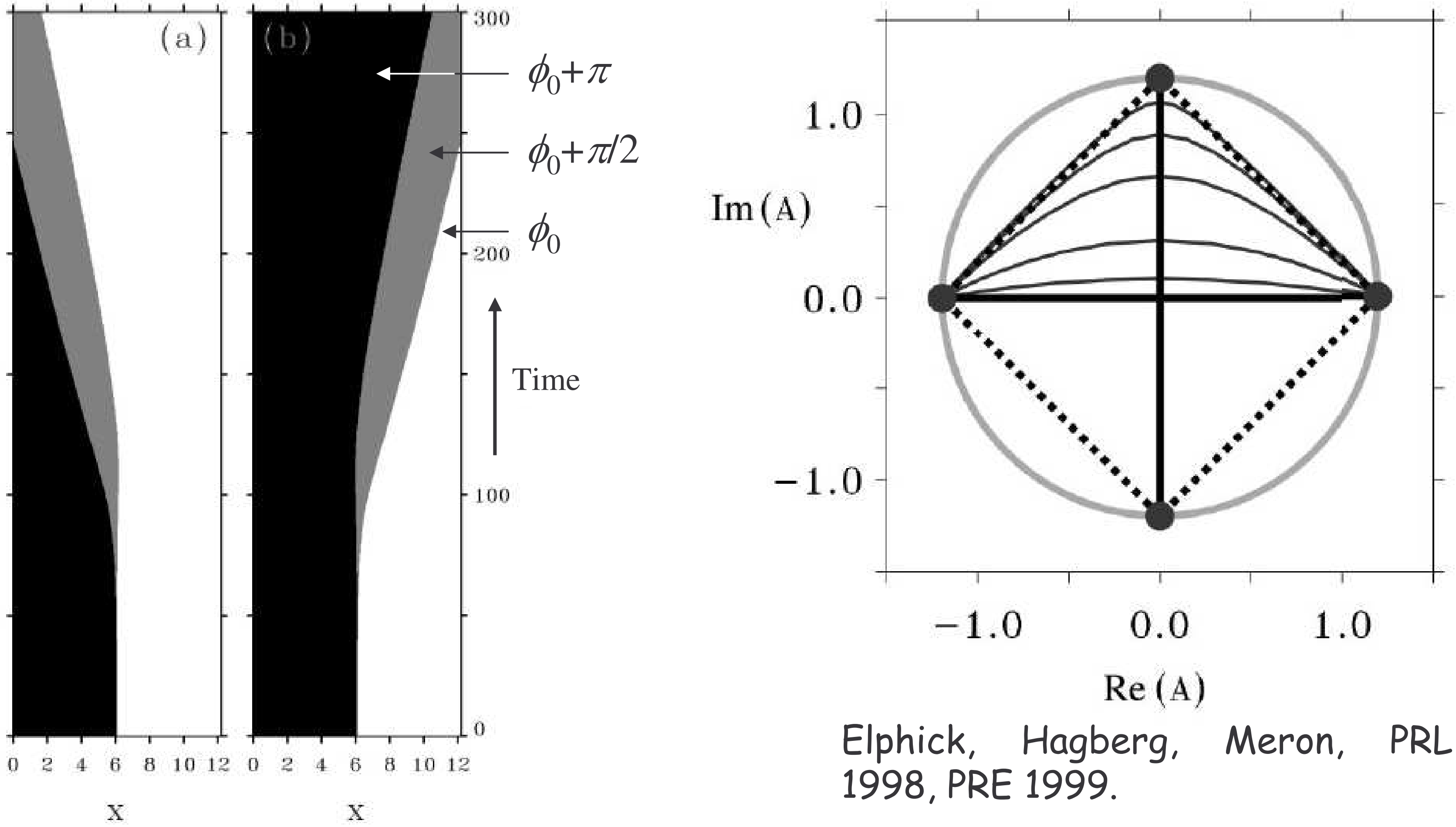
1. π -fronts are always stationary and are stable for forcing strengths γ that exceed a threshold value γ_c .
2. Decreasing the forcing strength below γ_c, π-fronts lose stability to pairs of propagating $\pi/2$-fronts that repel one another. Conversely, increasing γ beyond γ_c, propagating $\pi/2$-fronts attract one another and merge to form stationary π-fronts.

4:1 temporal forcing: phase front instability

Numerical demonstration of an instability of a stationary π-front to pairs of propagating $\pi/2$-fronts ("decomposition instability"):

Space-time plot

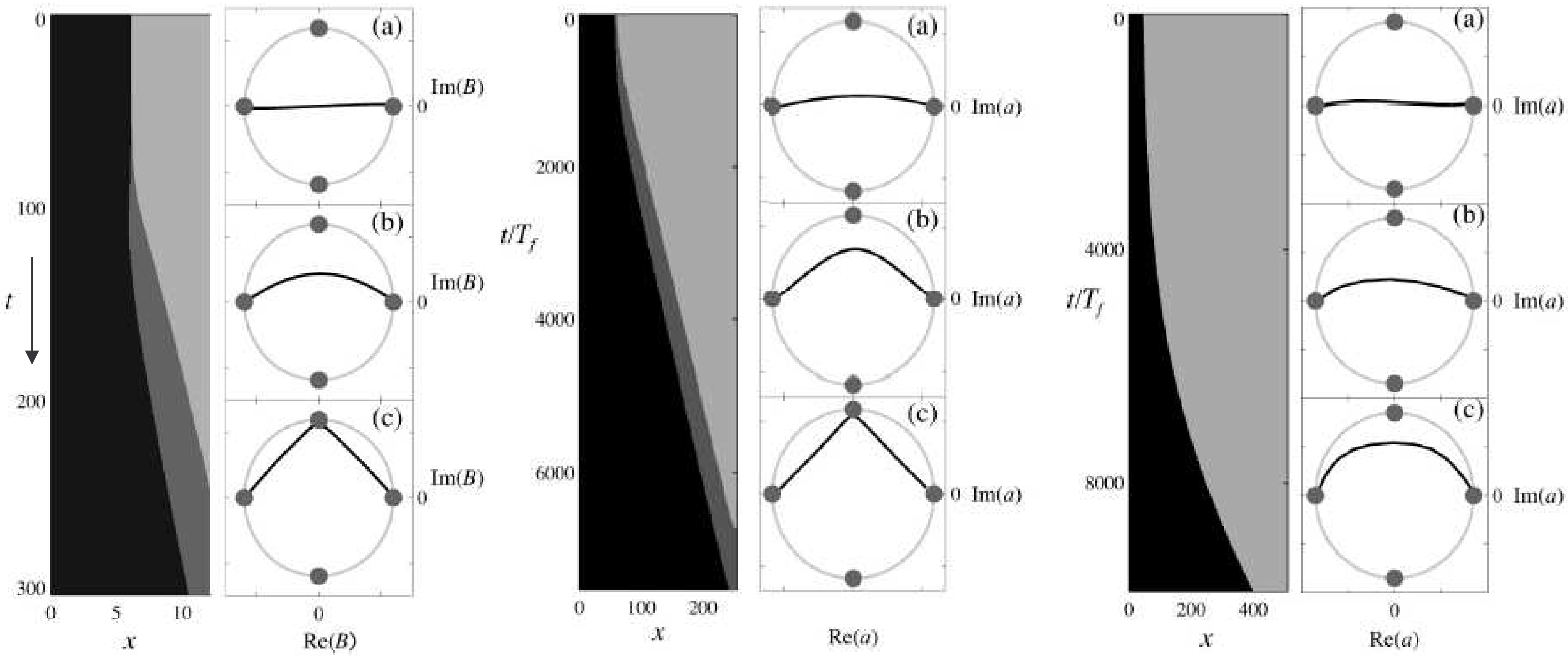
Phase-space plot



Elphick, Hagberg, Meron, PRL 1998, PRE 1999.

4:1 temporal forcing: phase front instability

Testing the decomposition instability using the FHN model:



Amplitude equation

FHN model close to Hopf

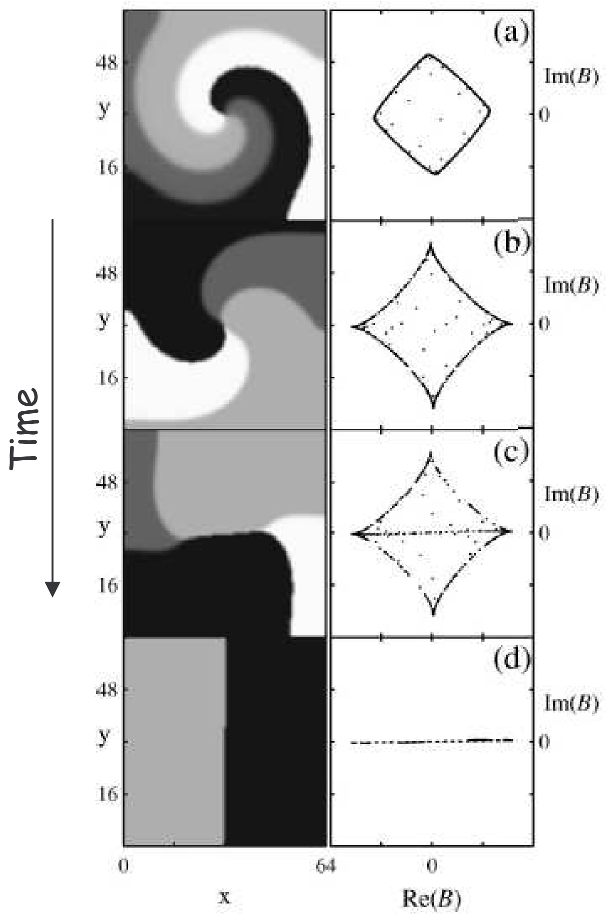
FHN model far from Hopf

The amplitude equation does not provide a good approximation far from onset.

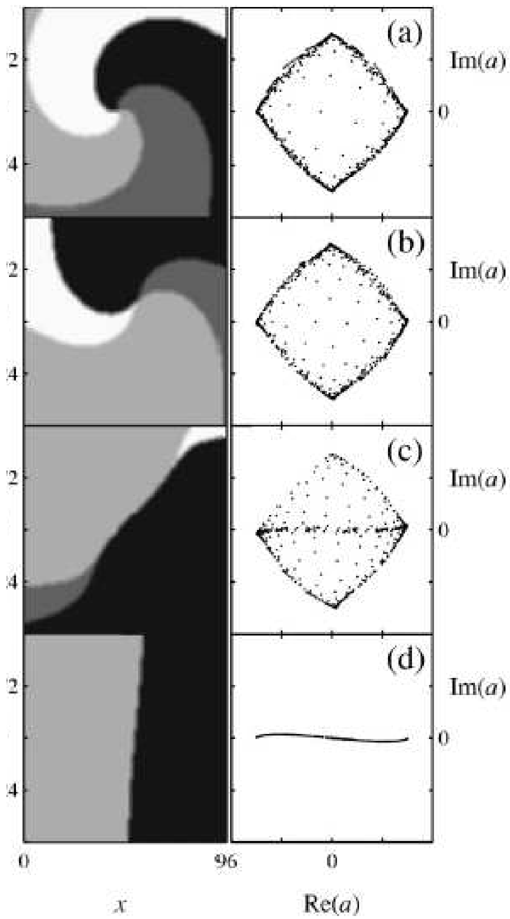
Lin, Hagberg, Ardelea, Bertram, Swinney, Meron, PRE 2000

4:1 temporal forcing: implications for pattern formation

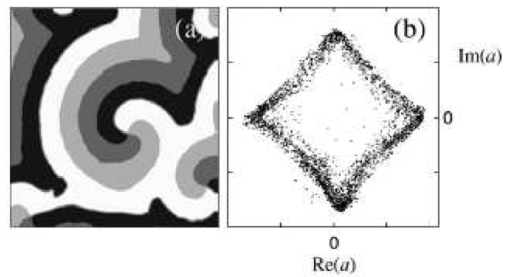
The decomposition instability designates a transition from four-phase traveling waves at low forcing strength to two-phase stationary patterns at high forcing strengths.



Amplitude equation



FHN model



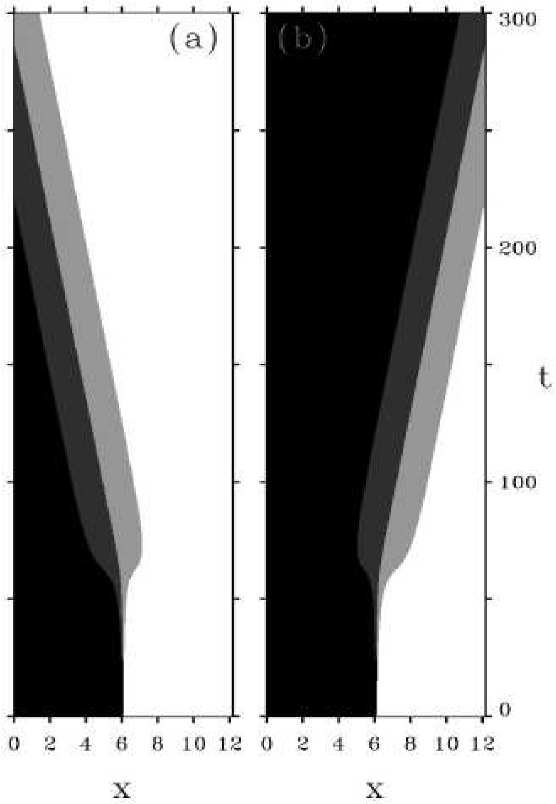
BZ experiment

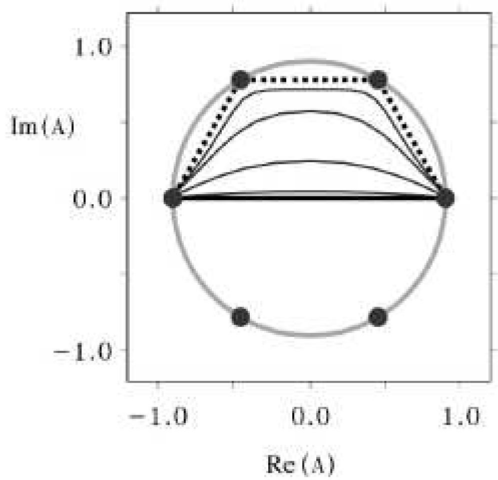
A decomposition instability has been found so far in the experiments.

Lin, Hagberg, Ardelea, Bertram, Swinney, Meron, PRE 2000

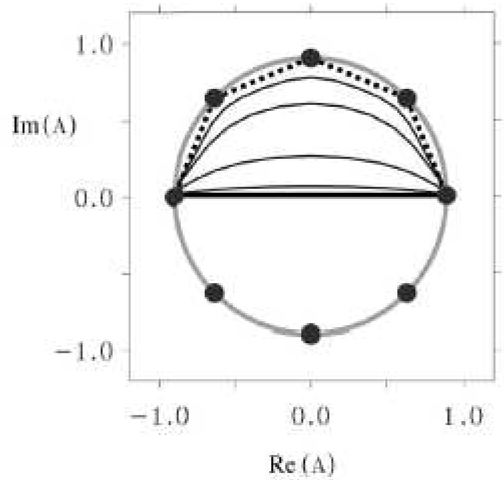
2n:1 temporal forcing (n=2,3,4, …): frequency locking

Similar decomposition instabilities are found in 6:1 and higher even forcing (using the appropriate amplitude equations).





Instability to 3 $\pi/3$-fronts



Instability to 4 $\pi/4$-fronts

The implication for frequency locking is that within any $2n$:1 resonance tongue (n =2,3,…) the oscillations are frequency locked only above the decomposition instability where the (two-phase) patterns are stationary. Unlike the 2:1 resonance the forcing in the higher resonances cannot induce a secondary Turing instability of the zero state, which can affect frequency locking.

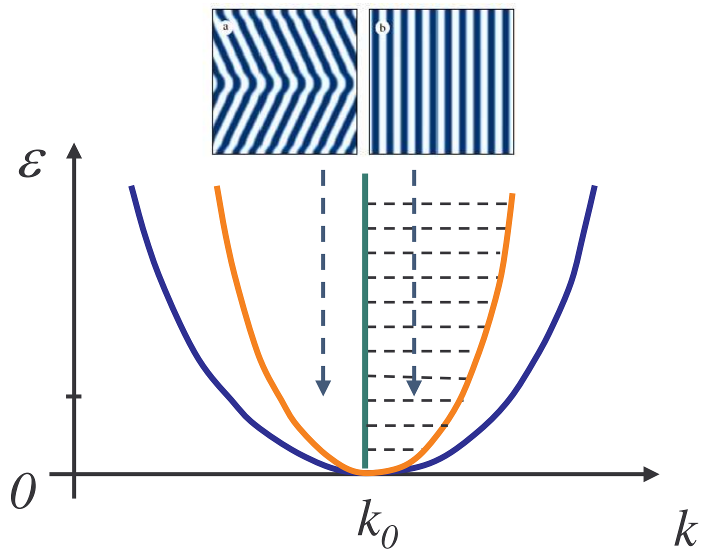
Spatial forcing and wavenumber locking

Consider a system that supports stripe patterns and is subjected to one-dimensional spatially periodic forcing. This is the spatial counterpart of forced oscillations. Despite the apparent similarity between the two problems, there are significant differences.

For the sake of concreteness consider the SH model,

$$u_t = \varepsilon u - u^3 - \left(\nabla^2 + k_0^2\right)^2 u$$

The patterns that develop above but close to the instability point $\varepsilon=0$ of the zero state have wavenumbers k_0 or close to it. Stripe patterns with $k > k_0$ are stable but those with $k < k_0$ are unstable to zigzag perturbations.



Spatial forcing and wavenumber locking

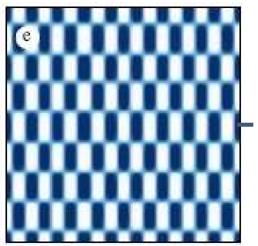
We force the SH equation by making the replacement:

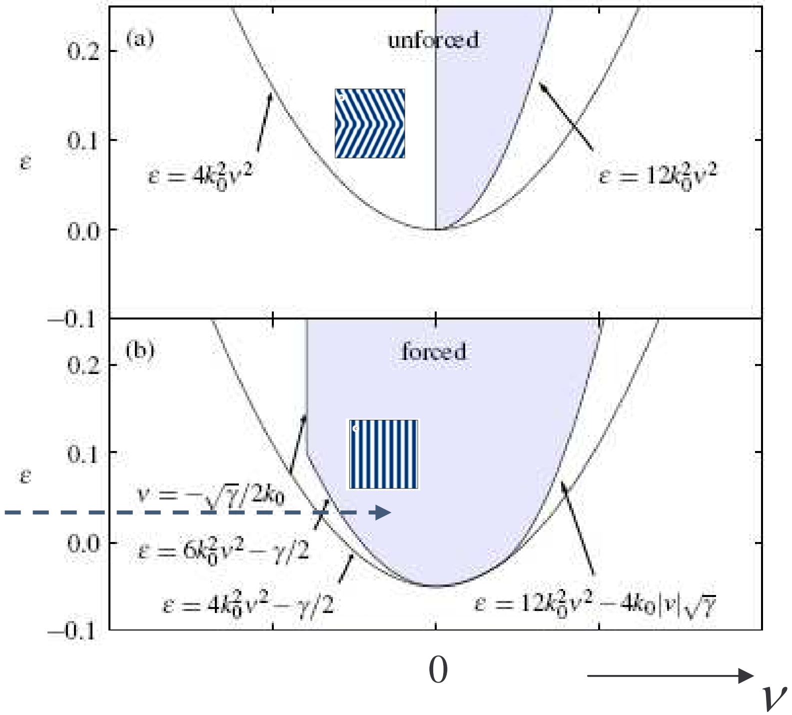
$$\varepsilon \rightarrow \varepsilon + \gamma \cos(k_f x)$$

where k_f is the forcing wavenumber. We will consider k_f values about $2k_0$ and smaller, and introduce a detuning $\nu = k_f/2 - k_0$.

The forcing act to stabilize the zigzag instability if we start with a stripe solution.

But if we start with random initial conditions about the zero state $u=0$ we get instead rectangular patterns



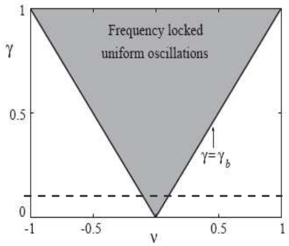


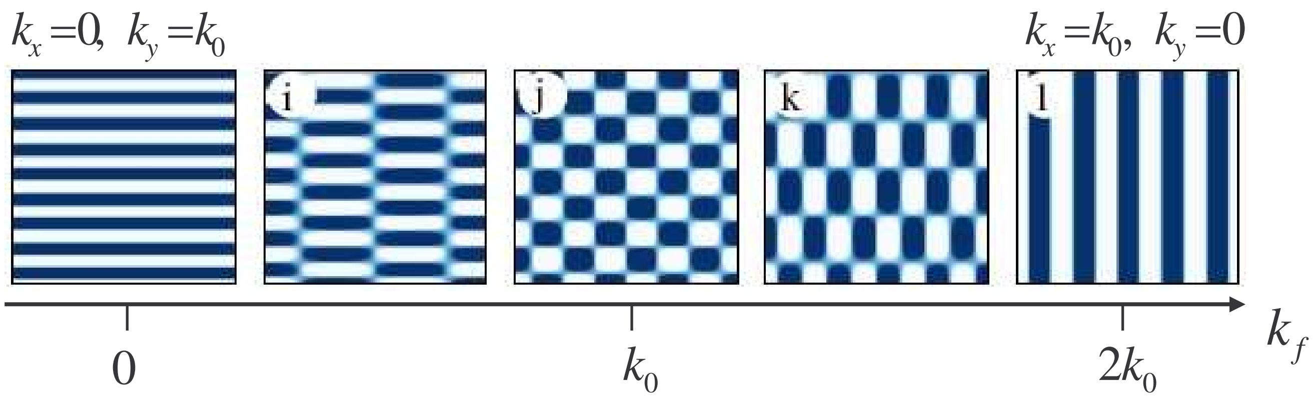
Spatial forcing and wavenumber locking

Why do we get rectangular patterns when the unforced system has only stripe solutions and we apply a stripe forcing?

The answer: unlike forced oscillations where the time axis is intrinsically one-dimensional, here the system can respond in the second space dimension. It does so by first locking the x component of its wavevector to the forcing and then compensating for the unfavorable wavenumber by developing a y component.

Unlike normal locking phenomena where the locking range is very small for weak forcing (the Arnold-tongue width), in this case the locking range can be very wide even for small forcing:





$$k_x = k_f/2$$

$$k_y = \sqrt{k_0^2 - k_x^2}$$

All patterns are wavenumber locked: k_x is exactly $k_f/2$.

Spatial forcing and wavenumber locking

To analyze two-dimensional response to one-dimensional forcing we approximate a solution of the forced SH equation as

$$u \cong a e^{i(k_x x + k_y y)} + b e^{i(k_x x - k_y y)} + c.c.$$

where the amplitudes a and b are small and vary weakly in time and space. Multiple time-scale analysis gives the amplitude equations:

$$a_t = \varepsilon a + 4(k_x\partial_x + k_y\partial_y)^2 a - 3(|a|^2 + 2|b|^2)a + \frac{\gamma}{2}b^*$$

$$b_t = \varepsilon b + 4(k_x\partial_x - k_y\partial_y)^2 b - 3(|b|^2 + 2|a|^2)b + \frac{\gamma}{2}a^*$$

These equations have the following family of constant solutions

$$a_0 = \rho_0 e^{i\alpha}, \qquad b_0 = \rho_0 e^{-i\alpha}, \qquad \rho_0 = \frac{1}{3}\sqrt{\varepsilon + \frac{\gamma}{2}}$$

where α is an arbitrary constant describing translations in the y direction (the forcing breaks the symmetry only in the x direction) .

Spatial forcing and wavenumber locking

These solutions describe rectangular patterns. Linear stability analysis shows that they are stable in the range

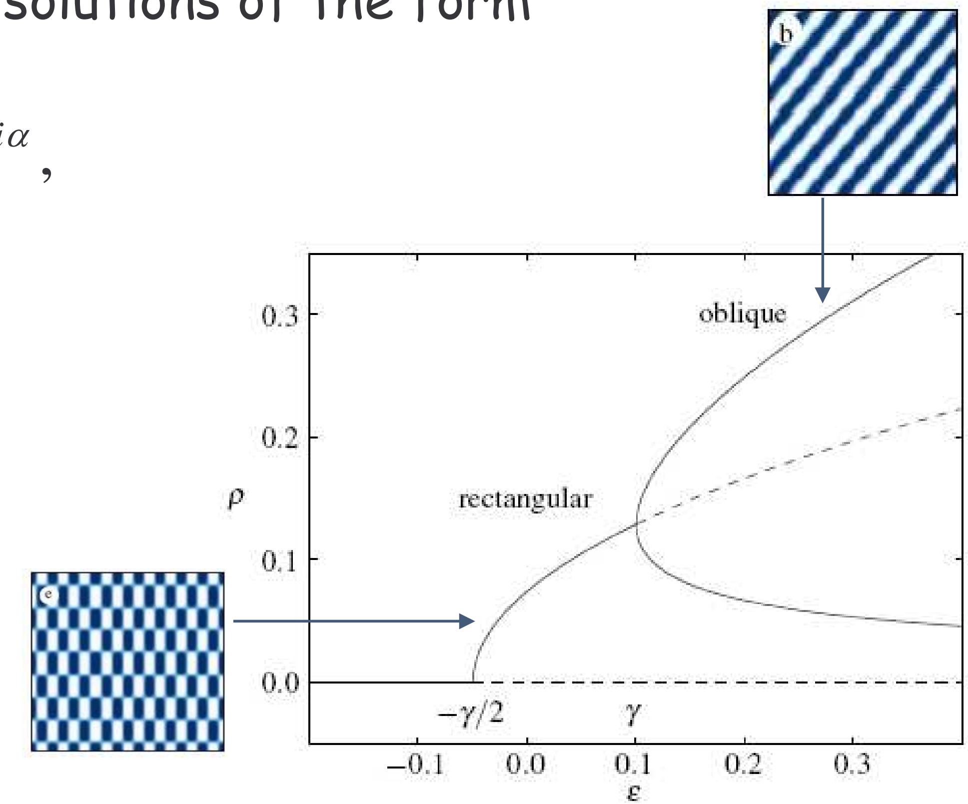
$$-\frac{\gamma}{2} < \varepsilon < \gamma$$

At $\varepsilon=\gamma$ they lose stability to solutions of the form

$$a_{\pm} = \rho_{\pm} e^{i\alpha}, \qquad b_{\mp} = \rho_{\mp} e^{-i\alpha},$$

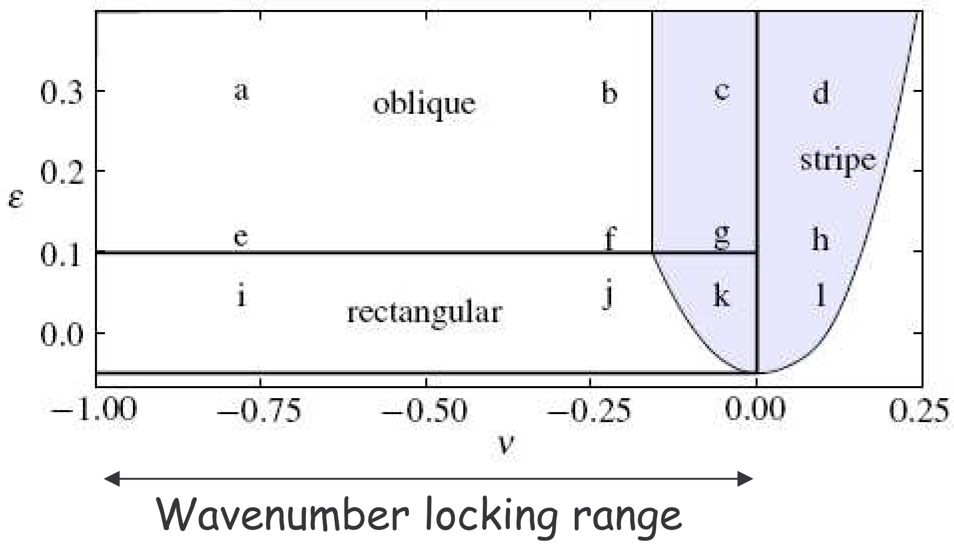
$$\rho_{\pm} = \sqrt{\frac{\varepsilon \pm \sqrt{\varepsilon^2 - \gamma^2}}{6}}$$

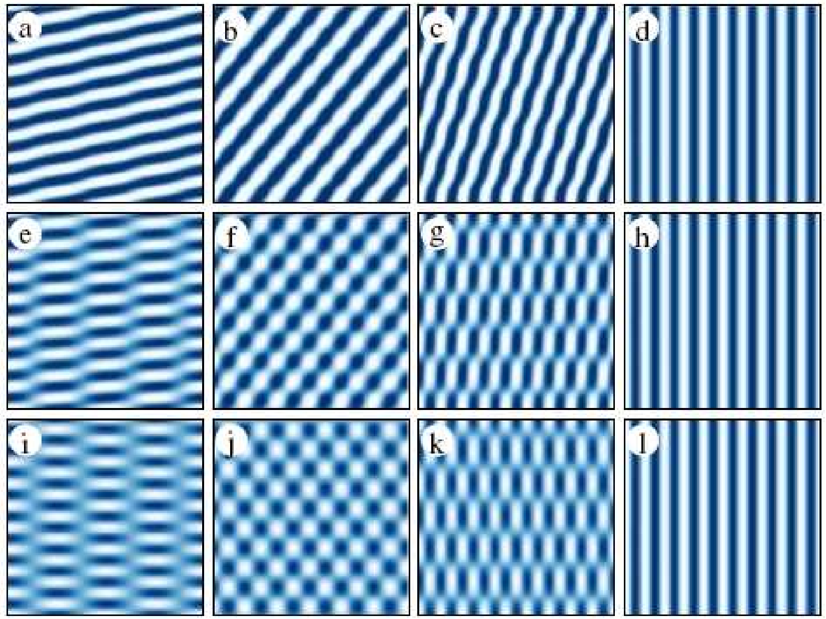
These solutions break the symmetry between the two modes and describe oblique patterns.



Spatial forcing and wavenumber locking

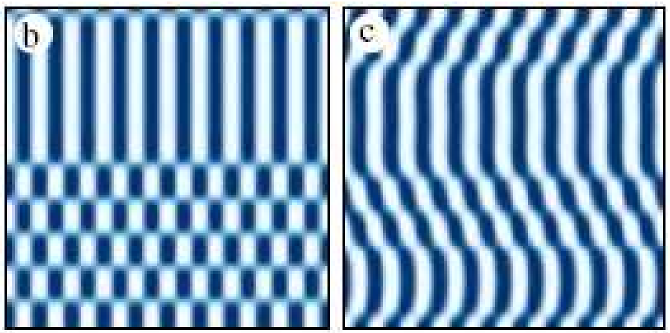
Altogether:





Note the multistability ranges of stripes and rectangles and of stripes and oblique patterns.

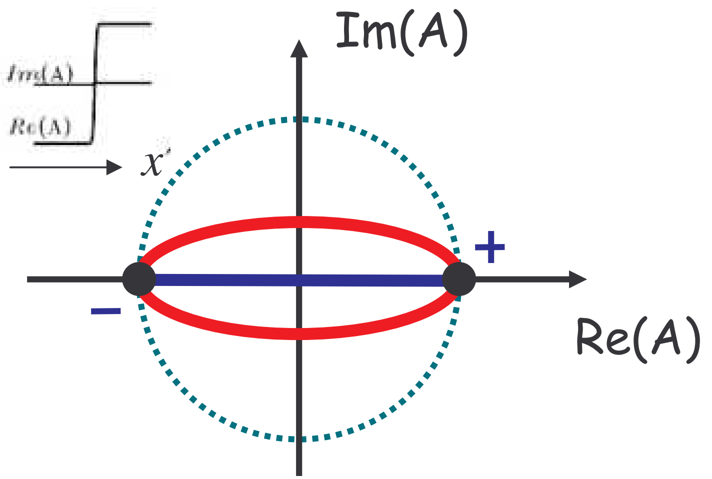
In these ranges stable spatially mixed patterns exist:



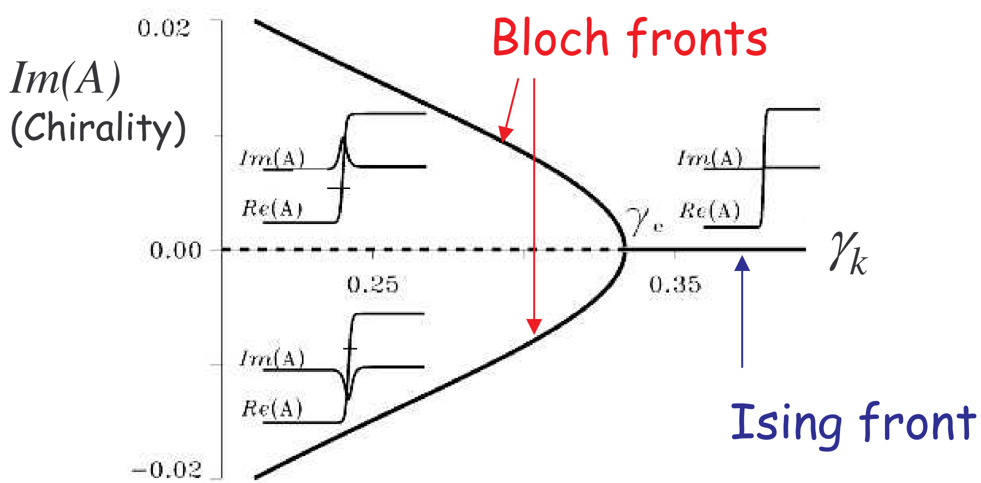
Spatial forcing and wavenumber locking

The 2D patterns induced by the forcing, appear from the zero state $u=0$, even below the instability to stripes (i.e. for $\varepsilon<0$), and therefore represent a class I pattern formation process. It is reminiscent of the Turing instability that temporal periodic forcing of oscillatory systems induces even below the Hopf bifurcation.

Like in the temporal case, the spatial forcing also induces class II processes, for it fixes the phase of a stationary stripe pattern at 0 or π, creating bistability of states, Ising and Bloch fronts and vortices.



(Coullet et al. PRL 1990)



More about that in Part III.

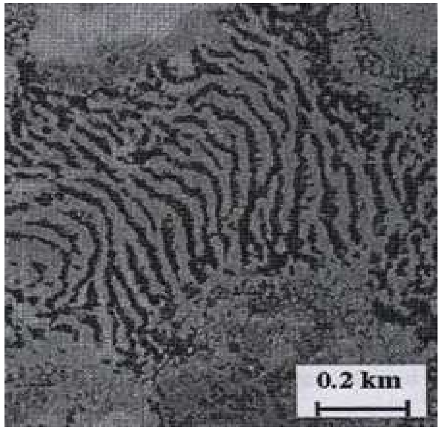
Spatial forcing: application to restoration of vegetation

Vegetation on hill slopes in arid and semi-arid regions tend to self-organize in the form of stripes perpendicular to the slope direction.

Arid regions are vulnerable to desertification which can result in the loss of the vegetation.

Efforts to restore the vegetation in such regions employ water harvesting techniques, such as dikes that capture runoff and along which the vegetation is planted.

These practices are empirical with very little theory behind. Very little is known about the wavenumber that a given plant species will form in a given environment.

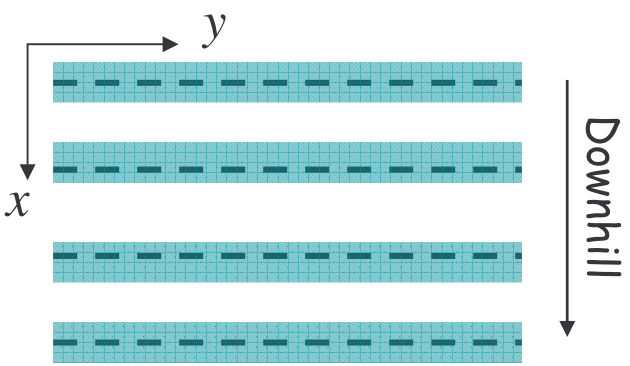


Clos-Arceduc, 1956

Spatial forcing: application to restoration of vegetation

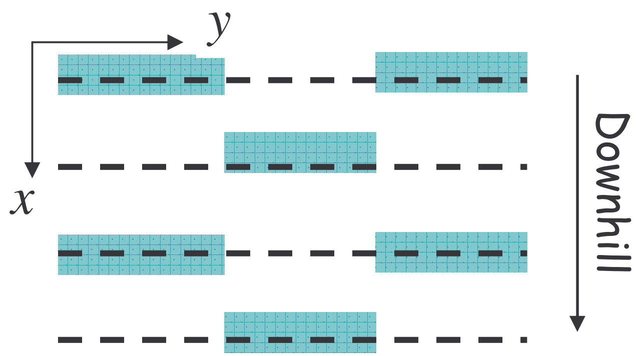
The general results of our analysis are particularly relevant to such situations where there is a big uncertainty in k_0 .

Traditional restoration



In both cases total wavenumber is k_0

Alternative restoration



Advantage: total biomass is higher

Disadvantage: narrow locking range under conditions where k_0 is not known or fluctuates with environmental changes.

$\Rightarrow$ risk of massive biomass decay

Advantage: wide locking range - k_x remains fixed, k_y adjusts itself to changes in k_0

$\Rightarrow$ Small risk of vegetation collapse

Disdvantage: total biomass is lower

More research is needed using vegetation pattern formation models

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Summary

Spatially uniform, temporal periodic forcing of spatially extended oscillatory systems reproduce the typical behavior of a single forced oscillator (frequency locking, Arnold resonance tongues), but induce, in addition, class I and class II pattern formation phenomena:

Class I: A Turing instability of the zero state that leads to a Hopf-Turing codimension-2 bifurcation.

Class II: Fronts that go through a NIB bifurcation (2:1 resonance) and decomposition instabilities (4:1, 6:1, … resonances), which designate transitions from stationary patterns to traveling waves. And a transverse front instability in the 2:1 resonance that couples with the NIB bifurcation to yield Bloch-front turbulence.

The patterns that the forcing induces affect frequency locking: They reduce the range of resonant oscillations within $2n{:}1$ Arnold tongues, but also extend the resonance range outside the tongue (2:1 resonance only).

Summary

Spatial forcing of a system that goes through a finite-k instability to stationary stripe patterns, induces an additional instability of the zero state to two-dimensional patterns (rectangular or oblique).

Wavenumber locking in spatially forced systems differ significantly from frequency locking in temporally forced systems in that the locking range is very wide even for weak forcing. This is due to the freedom of the system to respond in a direction orthogonal to the forcing and form 2D patterns.

Conclusions are general because they are based on universal normal form or amplitude equations. We used specific models to test their predictions but the conclusions apply more generally.

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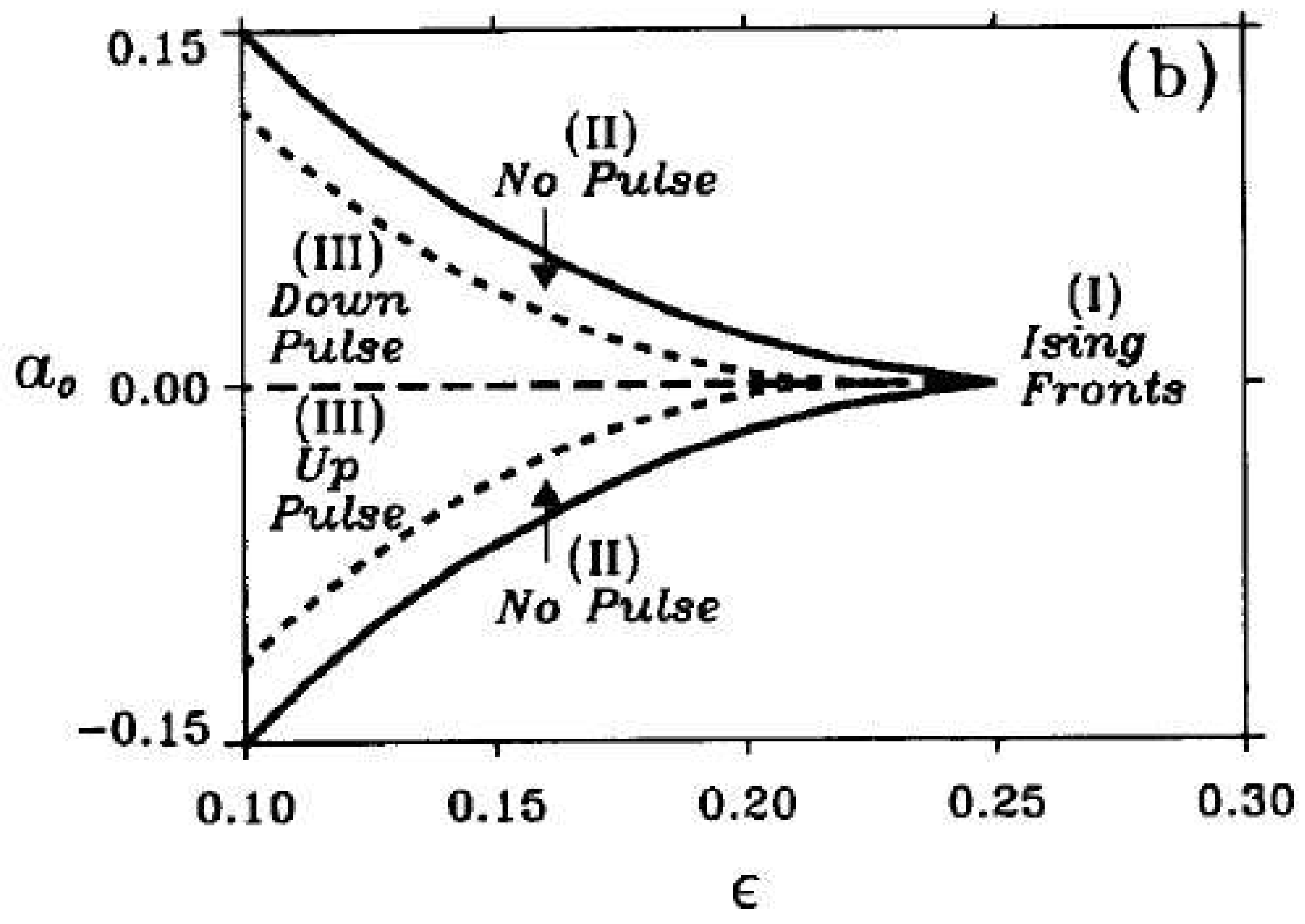

(b)
(II)
No Pulse
(III)
Down
Pulse
(III)
Up
Pulse
(II)
No Pulse
(I)
Ising
Fronts
0.15
0.00
−0.15
α_o
0.10
0.15
0.20
0.25
0.30
ϵ