Bayesian Nash equilibrium in all-pay auctions with interdependent types

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Abstract

We prove the existence of a behavioral-strategy Bayesian Nash equilibrium in all-pay auctions with statistically interdependent types (signals) under quite general assumptions on the values, costs and tie-breaking rules. Moreover, the set of equilibria is shown to be the same for any tie-breaking rule used in the auction.

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Key words: All-pay auctions, incomplete information, behavioral strategies, Bayesian Nash equilibrium, interdependent types.

1 Introduction

The path-breaking paper of Reny (1999) inspired numerous works on equilibrium existence in games with discontinuities, and, in particular, in discontinuous games with incomplete information.[1] For the latter category, conditions on the ex-post payoffs that guarantee "better-reply security" – the central hypothesis in Reny's result – were explicitly formulated in He and Yannelis (2016), Carbonell-Nicolau and McLean (2018) and Olszewski and Siegel (2023). Among other applications of their results on the existence of Bayesian Nash equilibrium in behavioral strategies, these authors

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[1] See, e.g., Reny (2020) for a survey.

considered all-pay auctions,[2] where all bidders bear the costs of their bids, because a particular feature of these auctions – raising one's bid infinitesimally never reduces the bidder's ex-post payoff – is very useful in establishing better-reply security.

The results obtained therein on equilibrium existence in all pay-auctions, although very comprehensive in terms of allowable cost and value functions and tie-breaking rules, leave out some cases of interest. He and Yannelis (2016) and Carbonell-Nicolau and McLean (2018) considered only common-value auctions because among their conditions for equilibrium existence is the requirement that the sum of all players' ex-post payoffs be upper semi-continuous in players' bids. With common value that requirement holds trivially since then all discontinuities in the probabilities of winning cancel each other out in the payoffs' sum, but if the value for winning differs across bidders then these discontinuities are typically inherited by the sum . Olszewski and Siegel (2023), on the other hand, allow general bidder-dependent valuations,[3] but unlike the two other works they restrict their result for all-pay auctions to the setting with independent types (or signals).

This paper aims to fill the gap by showing that a behavioral-strategy Bayesian Nash equilibrium exists in all-pay auctions under rather permissive conditions on (*bidder-dependent*) *values*, costs and tie-breaking, for general information structures (that can accommodate a vast range of statistical *type interdependency* because the only limitation that we place on the type space is the standard assumption of absolutely continuous information[4] due to Milgrom and Weber (1986)). We prove equilibrium existence in two stages, partially inspired by the method used in Fu et al. (2022) for multi-prize nested lottery contests.[5] We initially assume a specific tie-

[2] The concept of incomplete-information all-pay auctions is due to Amann and Leininger (1996) (and, somewhat more widely, Krishna and Morgan (1997)), who initiated their systematic study.

[3] Their condition is formulated using the upper semi-continuous *envelope* of the payoffs' sum, thereby obviating the need to have upper semi-continuity of the sum itself.

[4] Absolute continuity of information requires the joint distribution of the players' types to be absolutely continuous with respect to the product of its marginals. This condition is useful in applications because it holds, e.g., when the players' types are countable, or when they are uncountable but have joint density.

[5] The discontinuity of the probabilities to win in lottery contests is limited to the profile of zero bids, whereas in auctions the discontinuity occurs (for bidders tieing at the highest bid) at all bid profiles with such ties. But, although the method of Fu et al. (2022) is applied to games that are "almost continuous," the all-pay feature that is common to contests and the auctions considered here (alongside monotonicity properties of the probabilities to win) make that method useful in our

breaking rule that, at any type profile, deterministically awards the object to one of the top bidders at that profile that has the highest value.[6] It is then relatively straightforward to check that this auction satisfies the two conditions of the main equilibrium existence result in Carbonell-Nicolau and McLean (2018), uniform payoff security and aggregate-payoff upper semi-continuity. We then show that for any tie-breaking rule in an all-pay auction, any equilibrium puts zero ex-ante probability on bid profiles with ties, and that all unilateral strategic improvements can be done almost as profitably by avoiding ties. This implies that the particular tie-breaking rule bears no relevance on the expected payoff considerations in equilibrium, and thus does not affect the equilibrium set in the auction.[7] As a corollary, the claim of equilibrium existence applies to all-pay auctions under all tie-breaking rules.[8]

Our results, in effect, claim (and use) invariance of the equilibrium set in an all-pay auction under changes in a tie-breaking rule. This reflects some general results in Carmona and Podczeck (2018), who studied games with indeterminate outcomes (under complete and incomplete information) and were concerned with the existence of a common equilibrium for such games under all sharing rules. All-pay auctions fit their concept of a game with indeterminate outcomes: a tie at the highest bid means an indeterminate outcome, which is decided by a sharing (tie-breaking) rule. However, their incomplete information framework is more limited than ours, as it would require – for all-pay auctions – the utilities of winning to be continuous in types (and the type spaces themselves to be metric and compact). Since our type spaces are general (not necessarily topological) and the link of values and costs to types is only measurable (not necessarily continuous), we chose an independent approach.

Due to generality of our framework, equilibrium existence can only be established in behavioral Bayesian strategies, which allow type-dependent bid choices to be ran-

setting, when the games have major discontinuities.

[6]Such a rule is clearly artificial as it presupposes that the types (or, at least, the true values) of all bidders are fully known ex post, but it only serves us as a first step in a general existence result. Other rules that choose the highest-value bidders have been known in the literature; e.g., Maskin and Riley (2000) considered a rule that chooses a bidder based on the maximal expected payoff conditional on winning (given his realized type and the other bidders' bid functions).

[7]We do not have binding budget constraints in our framework. With such constraints, even in the complete information case equilibria may depend on a particular tie-breaking rule used in the all-pay auction (see Example 3 in Allison et al. (2022)).

[8]In contrast, it is known that in a standard (first-price) auction a change in the tie-breaking rule may affect equilibrium existence (see, e.g., Example 2 in Maskin and Riley (2000)).

dom. This is not surprising because even in complete-information all-pay auctions the equilibria are typically in strictly mixed strategies. Interestingly, however, pure Bayesian strategy equilibria may exist in a reasonably broad range of circumstances even with interdependent types: in a recent work, Prokopovych and Yannelis (2023) offer sufficient conditions for the existence of equilibrium in monotone pure strategies for all-pay auctions and contests when types are one-dimensional and affiliated.

Our paper is organized as follows. The framework is presented in sections 2. Section 3 states our claims, which are proved in Section 4. Section 5 discusses an extension that mixes auctions and contests.

2 All-pay auctions with incomplete information

2.1 The model

Members of $N = \{1, ..., n\}$, with $n \geq 2$, bid for a single object (or, in a common alternative interpretation, exert effort to win a prize). The information endowment of each bidder $i \in N$ is given by a measurable space $(T_i, \mathcal{T}_i)$ of signals, or *types*, and the bidders are assumed to have a common prior probability p on the product space $(T, \mathcal{T}) := (\times_{i \in N} T_i, \otimes_{i \in N} \mathcal{T}_i)$ of all type-profiles. In common with much of the literature, it will be postulated that p is absolutely continuous w.r.t. the product of its marginals, $\otimes_{i \in N} p_i$.

Upon privately observing their respective types, bidders simultaneously choose bids (that may alternatively be viewed as effort levels), and pay them irrespective of the final outcome. It is assumed that bids may not exceed some universal bound $M > 0$, and hence any bid profile $x = (x_1, ..., x_n)$ is an element of the cube $[0, M]^n$. The type-dependent cost of bid of each $i \in N$ is described by $c_i : T \times [0, M]^n \rightarrow \mathbb{R}$. The typical case is that $c_i(t, x)$ depends on just t and i's own bid x_i, but our formulation also allows dependence on the bids of others. This captures *inter alia* costs that are determined by min $\left\{x_i, \max_{j \in N \setminus \{i\}} x_j\right\}$ as in the war of attrition models.[9]

The type- (and possibly bid-)dependent value for the object of each bidder $i \in N$ is given by $V_i : T \times [0, M]^n \rightarrow \mathbb{R}_{++}$, i.e., if $t \in T$ and $x \in [0, M]^n$ are the realized type and bid profiles then i's value is $V_i(t, x) > 0$.

For any $x \in [0, M]^n$, $i \in N$ and $y_i \in [0, M]$, let $(y_i, x_{-i}) \in [0, M]^n$ be the profile

[9]See Krishna and Morgan (1997).

obtained from x by replacing x_i with y_i. The following assumptions will be made on the functions V_i and c_i :

(i) V_i and c_i are $\mathcal{T} \otimes \mathcal{B}([0, M]^n)$-measurable[10] and bounded;

(ii) $V_i(\cdot, (M, x_{-i})) - c_i(\cdot, (M, x_{-i})) < 0$ and $c_i(\cdot, (0, x_{-i})) \leq 0$ for any fixed $t \in T$ and $x_{-i} \in [0, M]^{n-1}$;[11]

(iii) the functions $\{V_i(t, \cdot)\}_{t \in T}$ are continuous and $\{c_i(t, \cdot)\}_{t \in T}$ are equicontinuous;

(**iv**) $V_i(t, (y_i, x_{-i}))$ is non-decreasing in y_i for any fixed $t \in T$ and $x_{-i} \in [0, M]^{n-1}$.

The object is awarded to one of the highest bidders, with ties broken probabilistically. This is fully described by a type-dependent *auction success function* $\rho : T \times [0, M]^n \to \Delta^n$, such that:

(**v**) ρ is $\mathcal{T} \otimes \mathcal{B}([0, M]^n)$-measurable;

(**vi**) for any $(t, x) \in T \times [0, M]^n$, $\rho(t, x)$ is a probability vector with a support on the set

$$N^{\max}(x) = \left\{ i \in N \mid x_i = \max_{j \in N} x_j \right\}$$

of the highest bidders at x.

Thus, $\rho_i(t, x) = 0$ if $i \notin N^{\max}(x)$ and, if $N^{\max}(x) = \{j\}$ – i.e., the highest bid is unique and belongs to bidder j – then $\rho_j(t, x) = 1$. Note that the discontinuity points of $\rho_i(t, \cdot)$ are confined to the set of bid profiles where i ties with at least one other bidder at the highest bid, namely,

$$X_i^{tie} = \left\{ x \in [0, M]^n \mid x_i = \max_{j \in N \setminus \{i\}} x_j \right\}, \tag{1}$$

and that outside X_i^{tie} the function $\rho_i(t, \cdot)$ can obtain two values only:

(**vii**) if $x \in X_i^{tie}$ then $\rho_i(t, (y_i, x_{-i})) = 1$ for every $y_i > x_i$ and $\rho_i(y_i, x_{-i}) = 0$ for every $y_i < x_i$. In particular, $\rho_i(t, (y_i, x_{-i}))$ is non-decreasing in $y_i \in [0, M]$ for any fixed $t \in T$ and $x_{-i} \in [0, M]^{n-1}$.

[10] Here and henceforth, $\mathcal{B}(A)$ denotes the σ-algebra of Borel sets in a closed subset A of some $\mathbb{R}^m$.

[11] Assumption (**ii**) implies that bidding 0 strictly dominates bidding M. In particular, our framework does not admit binding bid caps. With binding bid caps, it is known that tie-breaking rules may affect the equilibria even in a complete information case (see Example 3 in Allison e al. (2022)).

In the literature, it is usually assumed that the recipient of the object in $N^{\max}(x)$ is chosen by a fair lottery (which is type-independent), i.e.,

$$\rho_i^*(x) = \begin{cases} \frac{1}{|N^{\max}(x)|}, & \text{if } i \in N^{\max}(x); \\ 0, & \text{otherwise} \end{cases}. \tag{2}$$

Our specification of ρ_i in (**vi**) coincides with (2) when $|N^{\max}(x)| = 1$ or $i \notin N^{\max}(x)$, but places no restriction on the lottery performed among the top bidders when $|N^{\max}(x)| \geq 2$. In particular, the lottery on $N^{\max}(x)$ may depend in the type profile t, reflecting potential biases that may exist in the case of ties at the highest bid in some (or all) states of nature.[12]

Definition 1. An (incomplete-information all-pay) *auction* is given by a collection $(N, (T_i, \mathcal{T}_i)_{i\in N}, p, \{V_i\}_{i\in N}, \{c_i\}_{i\in N}, \rho)$ of the above-described attributes. As all these attributes will be fixed throughout with the exception of the auction success function ρ (which will be allowed to vary for technical reasons), an auction will be denoted by $G(\rho)$.

For any realized type profile $t \in T$ and any bid profile $x \in [0, M]^n$, the payoff of each bidder $i \in N$ in an auction $G(\rho)$ is given by his expected share of the object's value net of his cost, namely,

$$u_i^\rho(t, x) = \rho_i(t, x) \cdot V_i(t, x) - c_i(t, x). \tag{3}$$

Below we list some obvious implications of (**i**) – (**vi**), which will be of use henceforth.

Fact 1. For any $i \in N$, the payoff function $u_i^\rho : T \times [0, M]^n \to \mathbb{R}$ has the following properties:

(**a**) u_i^ρ is $\mathcal{T} \otimes \mathcal{B}([0, M]^n)$-measurable and bounded (making the expected payoffs, introduced in the next section, well defined);

(**b**) $u_i^\rho(t, \cdot)$ is continuous on the open set $[0, M]^n \setminus X_i^{tie}$ for any $t \in T$.

2.2 Behavioral Bayesian strategies and equilibrium

The concept of behavioral Bayesian strategy allows randomness in a type-dependent choice of bids. Formally, as in Balder (1988), a *(behavioral Bayesian) strategy* of

[12] Even if a given success function is type-independent, such as ρ^* in (2), our proof of equilibrium existence uses an auxiliary success function that is type-dependent. Hence, we admit any type-dependent ρ from the start.

$i \in N$ in $G(\rho)$ is a mapping $\sigma_i : T_i \times \mathcal{B}([0, M]) \to [0, 1]$, such that $\sigma_i(t_i, \cdot)$ is a probability measure on $[0, M]$ for every $t_i \in T_i$ and $\sigma_i(\cdot, A)$ is $\mathcal{T}_i$-measurable for every $A \in \mathcal{B}([0, M])$. We denote by Σ_i the set of i's strategies, and by $\Sigma = \times_{i=1}^n \Sigma_i$ the set of strategy profiles.

For any strategy profile $\sigma = (\sigma_i)_{i \in N} \in \Sigma$, the expected payoff of bidder $i \in N$ is given by

$$U_i^\rho(\sigma) = \int_T \int_{[0,M]^n} u_i^\rho(t, x)\, \sigma_1(t_1, dx_1) ... \sigma_n(t_n, dx_n) p(dt). \tag{4}$$

Also, denote by $(\sigma_i', \sigma_{-i}) \in \Sigma$ the profile that is obtained from σ by replacing σ_i with some $\sigma_i' \in \Sigma_i$.

Definition 2. Strategy profile $\sigma^* = (\sigma_i^*)_{i \in N} \in \Sigma$ constitutes a *Bayesian Nash equilibrium* (or *BNE*, for short) of an auction $G(\rho)$ if

$$U_i^\rho(\sigma^*) \geq U_i^\rho(\sigma_i, \sigma_{-i}^*) \tag{5}$$

for every bidder $i \in N$ and every $\sigma_i \in \Sigma_i$.

3 BNE existence and independence of ρ

As a first step, BNE existence will be established in an auction in which all ties at the top bid are broken in favor of the bidder with the highest value for the object.[13] Let an auction success function $\widetilde{\rho}$ be defined, for any $(t, x) \in T \times [0, M]^n$, by

$$\widetilde{\rho}(t, x) = \left(1, \mathbf{0}_{-i(t,x)}\right), \tag{6}$$

where $(1, \mathbf{0}_{-i})$ stands for the i^{th} unit vector and $i(t, x)$ denotes the lowest-numbered bidder in the set $\left\{i \in N^{\max}(x) \mid V_i(t, x) = \max_{j \in N^{\max}(x)} V_j(t, x)\right\}$. By (**vii**), $\widetilde{\rho}(t, \cdot)$ may differ from another success function $\rho(t, \cdot)$ at a bid profile x only if there are ties at the highest bid in x (i.e., $x \in X_i^{tie}$ for some $i \in N$). The winner in $G(\widetilde{\rho})$ is then chosen deterministically as the (lowest-numbered) bidder with the highest value from the set $N^{\max}(x)$ of those who tie at the top bid in x.

Our initial BNE existence result, Proposition 1 below, is obtained as a fairly straightforward application of Theorem 1 in Carbonell-Nicolau and McLean (2018).

[13] Such a step is used in Fu et al. (2022), who show equilibrium existence for nested multi-prize lottery contests, and, in a much greater generality, in the proof of Theorem 5 in Carmona and Podczeck (2018).

That theorem identifies the conjunction of two conditions on ex-post payoffs, uniform payoff security and aggregate-payoff upper semi-continuity, as sufficient for equilibrium existence in Bayesian games.

Proposition 1. $G(\widetilde{\rho})$ possesses a BNE.

It turns out that any BNE of $G(\widetilde{\rho})$ remains a BNE when *any* auction success function is used instead of $\widetilde{\rho}$. In fact, the set of BNE in an auction does not depend on a particular auction success function:

Proposition 2. For any two auction success functions ρ and ρ', the auctions $G(\rho)$ and $G(\rho')$ have the same BNEs.

The proof of Proposition 2 is based on (a) ruling out the possibility that any BNE σ^* of $G(\rho')$ assigns a positive ex-ante probability to bid profiles with ties at the highest bid (where ρ' may disagree with ρ), thereby ensuring that the expected payoffs under σ^* are identical in both $G(\rho')$ and $G(\rho)$; and (b) showing that all profitable unilateral deviations from σ^* in $G(\rho)$ can be mimicked almost as profitably in $G(\rho')$, thus ensuring that σ^* is also a BNE of $G(\rho)$.

An immediate corollary of the two propositions is our main theorem:

Theorem 1. Any $G(\rho)$ possesses a BNE.

The proofs of propositions 1 and 2 are presented in the next section.

4 Proofs

4.1 A Lemma

We start with a lemma that is needed in the subsequent proofs, showing that the fall in i's payoff in $G(\rho)$ is minor if his bid is raised slightly (even if the other bids are also slightly varied).

Lemma 1. Consider an auction $G(\rho)$ and let $\varepsilon > 0$. Then, for all low enough $0 < \delta < 1$, the following holds: given any $(t, x) \in T \times [0, M]^n$ with $x_i < M$ for some $i \in N$, there exists a (relatively) open neighborhood $W_{x_{-i}} \subset [0, M]^{n-1}$ of x_{-i} such that

$$u_i^\rho(t, (x_i + \delta(M - x_i), z_{-i})) > u_i^\rho(t, x) - \varepsilon \tag{7}$$

for any $z_{-i} \in W_{x_{-i}}$.

Proof of Lemma 1. Since $\{c_i(t, \cdot)\}_{t \in T}$ are equicontinuous by assumption (**iii**) (and hence, uniformly so), for all low enough $0 < \delta < 1$ we have

$$|c_i(t, (y_i, x_{-i})) - c_i(t, x)| < \frac{\varepsilon}{2} \tag{8}$$

for any $i \in N$, $(t, x) \in T \times [0, M]^n$ and $y_i \in [0, M]$ with $|y_i - x_i| \leq \delta M$. We will show that (7) holds given any such δ.

First assume that $x \notin X_i^{tie}$. In this case, we have

$$u_i^\rho(t, (x_i, z_{-i})) > u_i^\rho(t, x) - \frac{\varepsilon}{2} \tag{9}$$

for any z_{-i} in some open neighborhood $W_{x_{-i}}$ of x_{-i} by Fact 1(**b**). Then, for any $z_{-i} \in W_{x_{-i}}$,

$$\begin{aligned}
& u_i^\rho(t, (x_i + \delta(M - x_i), z_{-i})) \\
= \;& \rho_i(t, (x_i + \delta(M - x_i), z_{-i})) \cdot V_i(t, (x_i + \delta(M - x_i), z_{-i})) \\
& - c_i(t, (x_i + \delta(M - x_i), z_{-i})) \\
\geq \;& \rho_i(t, (x_i, z_{-i})) \cdot V_i(t, (x_i, z_{-i})) - c_i(t, (x_i, z_{-i})) - \frac{\varepsilon}{2} && (10) \\
= \;& u_i^\rho(t, (x_i, z_{-i})) - \frac{\varepsilon}{2} > u_i^\rho(t, x) - \varepsilon. && (11)
\end{aligned}$$

Here, the inequality in (10) is implied by $\rho_i(t, x)$ and $V_i(t, x)$ being non-decreasing in x_i for a fixed x_{-i} (due to (**iv**) and (**vii**)), and by (8). The inequality in (11) is due to (9).

We now show that (7) also holds when $x \in X_i^{tie}$. In this case, since $0 < \delta < 1$ and $x_i < M$, clearly $(x_i + \delta(M - x_i), x_{-i}) \notin X_i^{tie}$. There exists an open neighborhood $W_{x_{-i}}$ of x_{-i} such that

$$u_i^\rho(t, (x_i + \delta(M - x_i), z_{-i})) > u_i^\rho(t, (x_i + \delta(M - x_i), x_{-i})) - \frac{\varepsilon}{2} \tag{12}$$

holds for any $z_{-i} \in W_{x_{-i}}$ by Fact 1(**b**). Then, for any $z_{-i} \in W_{x_{-i}}$,

$$\begin{aligned}
& u_i^\rho(t, (x_i + \delta(M - x_i), z_{-i})) \\
> \;& u_i^\rho(t, (x_i + \delta(M - x_i), x_{-i})) - \frac{\varepsilon}{2} && (13) \\
= \;& \rho_i(t, (x_i + \delta(M - x_i), x_{-i})) \cdot V_i(t, (x_i + \delta(M - x_i), x_{-i})) \\
& - c_i(t, (x_i + \delta(M - x_i), x_{-i})) - \frac{\varepsilon}{2} \\
\geq \;& \rho_i(t, x) \cdot V_i(t, x) - c_i(t, x_i) - \varepsilon = u_i^\rho(t, x) - \varepsilon. && (14)
\end{aligned}$$

Here, the inequality in (13) is due to (12), and the inequality in (14) follows from $\rho_i(t,x)$ and $V_i(t,x)$ being non-decreasing in x_i for a fixed x_{-i} and from (8). ■

4.2 Proof of Proposition 1

To establish BNE existence in $G(\widetilde{\rho})$, we first verify that $G(\widetilde{\rho})$ is *uniformly payoff-secure.* This property, formulated in Definition 9 in Carbonell-Nicolau and McLean (2018), involves pure (Bayesian) strategies. Let us denote by S_i the set of pure strategies of bidder $i \in N$, which are $\mathcal{T}_i$-measurable function $s_i : T_i \to [0, M]$. Uniform payoff security requires that, for any $i \in N$, $s_i \in S_i$ and $\varepsilon > 0$, there must exist $\overline{s}_i \in S_i$ with the following feature: for every $(t, x_{-i}) \in T \times [0, M]^{n-1}$ there is a (relatively) open neighborhood $W_{x_{-i}} \subset [0, M]^{n-1}$ of x_{-i} such that

$$u_i^{\widetilde{\rho}}(t, (\overline{s}_i(t_i), z_{-i})) > u_i^{\widetilde{\rho}}(t, (s_i(t_i), x_{-i})) - \varepsilon \tag{15}$$

whenever $z_{-i} \in W_{x_{-i}}$.

Let $i \in N$, $s_i \in S_i$ and $\varepsilon > 0$, and define

$$\overline{s}_i(t_i) := \begin{cases} s_i(t_i) + \delta(M - s_i(t_i)), & \text{if } s_i(t_i) < M; \\ 0, & \text{if } s_i(t_i) = M \end{cases} \tag{16}$$

for any $t_i \in T_i$, where $0 < \delta < 1$ is chosen to be such that the assertion of Lemma 1 holds for the auction $G(\widetilde{\rho})$ and ε. Thus, the requisite $W_{x_{-i}}$ exists and (15) holds whenever $s_i(t_i) < M$. Thus, only the case of $s_i(t_i) = M$ needs to be addressed. But then bidding $\overline{s}_i(t_i) = 0$ constitutes a strict improvement for bidder i regardless of the possibly changing bids by $N \backslash \{i\}$. Formally, for any $z_{-i} \in [0, M]^{n-1}$,

$$\begin{aligned}
& u_i^{\widetilde{\rho}}(t, (\overline{s}_i(t_i), z_{-i})) \\
= \; & u_i^{\widetilde{\rho}}(t, (0, z_{-i})) = \widetilde{\rho}_i(t, (0, z_{-i})) \cdot V_i(t, (0, z_{-i})) - c_i(t, (0, z_{-i})) \\
\geq \; & -c_i(t, (0, z_{-i})) \geq 0 > V_i(t, (M, x_{-i})) - c_i(t, (M, x_{-i})) \\
\geq \; & \widetilde{\rho}_i(t, (M, x_{-i})) \cdot V_i(t, (M, x_{-i})) - c_i(t, (M, x_{-i})) \\
= \; & \widetilde{\rho}_i(t, (s_i(t_i), x_{-i})) \cdot V_i(t, (s_i(t_i), x_{-i})) - c_i(t, (s_i(t_i), x_{-i})) \\
= \; & u_i^{\widetilde{\rho}}(t, (s_i(t_i), x_{-i})).
\end{aligned} \tag{17}$$

Here, the last two inequalities in (17) are due to assumption (**ii**). We conclude that $G(\widetilde{\rho})$ is uniformly payoff-secure.

We now show that $G(\widetilde{\rho})$ is, furthermore, *aggregate-payoff upper semi-continuous*, i.e., that for any $(t,x) \in T \times [0,M]^n$,

$$\limsup_{y \to x} \sum_{i \in N} u_i^{\widetilde{\rho}}(t,y) \leq \sum_{i \in N} u_i^{\widetilde{\rho}}(t,x). \tag{18}$$

Indeed, for any y in some open neighborhood of x we have $N^{\max}(y) \subset N^{\max}(x)$, and so

$$\begin{aligned}
& \limsup_{y \to x} \sum_{i \in N} u_i^{\widetilde{\rho}}(t,y) \\
= \;& \limsup_{y \to x} \sum_{i \in N^{\max}(y)} \widetilde{\rho}_i(t,y) \cdot V_i(t,y) - \sum_{i \in N} c_i(t,x) \qquad (19) \\
\leq \;& \limsup_{y \to x} \max_{i \in N^{\max}(x)} V_i(t,y) - \sum_{i \in N} c_i(t,x) \\
= \;& \max_{i \in N^{\max}(x)} V_i(t,x) - \sum_{i \in N} c_i(t,x) \qquad (20) \\
= \;& \sum_{i \in N} \widetilde{\rho}_i(t,x) \cdot V_i(t,x) - \sum_{i \in N} c_i(t,x) = \sum_{i \in N} u_i^{\widetilde{\rho}}(t,x), \qquad (21)
\end{aligned}$$

where the equalities in (19) and (20) are due to assumption (**iii**), and the first equality in (21) is due to (6). Thus, (18) holds for any $(t,x) \in T \times [0,M]^n$.

Since $G(\widetilde{\rho})$ is uniformly payoff-secure and aggregate-payoff upper semi-continuous, the hypotheses of Theorem 1 in Carbonell-Nicolau and McLean (2018) are satisfied,[14] and that theorem guarantees existence of a BNE in behavioral strategies for $G(\widetilde{\rho})$. ■

4.3 Proof of Proposition 2

It clearly suffices to check that any fixed BNE $\sigma^* = (\sigma_1^*, ..., \sigma_n^*) \in \Sigma$ of $G(\rho')$ is also a BNE of $G(\rho)$. The proof proceeds in two steps.

Step 1. We will show that the ex-ante probability under σ^* that the realized bid profile has ties at the highest bid is zero, i.e.,

$$\int_T \int_{[0,M]^n} \chi_{\bigcup_{i \in N} X_i^{tie}}(x)\, \sigma_1^*(t_1, dx_1)...\sigma_n^*(t_n, dx_n) p(dt) = 0, \tag{22}$$

where χ_A denotes the characteristic function of a set $A \in \mathcal{B}([0,M]^n)$.

[14]The general conditions of the model in Carbonell-Nicolau and McLean (2018) are also satisfied by our assumptions on values, costs and the common prior.

Indeed, suppose by way of contradiction that there is $i \in N$ with

$$\int_T \int_{[0,M]^n} \chi_{X_i^{tie}}(x)\, \sigma_1^*(t_1, dx_1) ... \sigma_n^*(t_n, dx_n) p(dt) > 0.$$

Since i ties in X_i^{tie} with someone else, there is $j \in N \backslash \{i\}$ such that, in fact,

$$\int_T \int_{[0,M]^n} \chi_{X_i^{tie} \cap X_j^{tie}}(x)\, \sigma_1^*(t_1, dx_1) ... \sigma_n^*(t_n, dx_n) p(dt) > 0. \tag{23}$$

For any $t \in T$ and $x \in X_i^{tie} \cap X_j^{tie}$, denote by $k(t,x)$ the lowest-numbered bidder $k \in \{i,j\}$ for whom $\rho_k'(t,x) \leq \frac{1}{2}$. Since, clearly,

$$\chi_{X_i^{tie} \cap X_j^{tie}} = \chi_{\left\{x' \in X_i^{tie} \cap X_j^{tie} | k(t,x') = i\right\}} + \chi_{\left\{x' \in X_i^{tie} \cap X_j^{tie} | k(t,x') = j\right\}},$$

we may assume w.l.o.g. – based on (23) – that

$$\int_T \int_{[0,M]^n} \chi_{\left\{x' \in X_i^{tie} \cap X_j^{tie} | k(t,x') = i\right\}}(x)\, \sigma_1^*(t_1, dx_1) ... \sigma_n^*(t_n, dx_n) p(dt) > 0.$$

Since σ_i^* is a BNE strategy and bidding 0 strictly dominates bidding M (due to (**ii**)), for p-almost every $t \in T$ we have $\sigma_i^*(t_i, \{M\}) = 0$, and hence

$$\int_T \int_{[0,M]^n} \chi_{F_i(t)}(x)\, \sigma_1^*(t_1, dx_1) ... \sigma_n^*(t_n, dx_n) p(dt) > 0 \tag{24}$$

for $F_i(t) := \left\{x \in X_i^{tie} \cap X_j^{tie} \mid k(t,x) = i \text{ and } x_i < M\right\}$.

Now, given $\varepsilon > 0$, choose and fix $0 < \delta < 1$ such that

$$u_i^{\rho'}(t, (x_i + \delta(M - x_i), x_{-i})) > u_i^{\rho'}(t,x) - \varepsilon \tag{25}$$

and

$$|c_i(t, (x_i + \delta(M - x_i), x_{-i})) - c_i(t,x)| < \varepsilon. \tag{26}$$

for every $(t,x) \in T \times [0,M]^n$ (the existence of such a δ is guaranteed by Lemma 1 and equicontinuity of $\{c_i(t,\cdot)\}_{t \in T}$). Let $\sigma_i^\delta \in \Sigma_i$ be a strategy determined by the equality

$$\sigma_i^\delta(t, [a, M]) = \sigma_i^*\left(t, [\frac{a - \delta M}{1 - \delta}, M]\right). \tag{27}$$

for any $t \in T$ and $a \in [\delta M, M]$. (That is, if X_i is a $\sigma_i^*(t,\cdot)$-distributed random variable on $[0,M]$, then $Y_i := X_i + \delta(M - X_i)$ is $\sigma_i^\delta(t,\cdot)$-distributed.)

When (t,x) is such that $x \in F_i(t)$, then

$$\begin{aligned} & u_i^{\rho'}(t, (x_i + \delta(M - x_i), x_{-i})) - u_i^{\rho'}(t,x) && (28) \\ \geq\ & \frac{1}{2} V_i(t,x) + c_i(t, (x_i + \delta(M - x_i), x_{-i})) - c_i(t,x) && (29) \\ \geq\ & \frac{1}{2} V_i(t,x) - \varepsilon. && (30) \end{aligned}$$

The inequality in (29) follows from the definition of $F_i(t)$, the fact that $x_i+\delta(M-x_i)$ is a bid that yields a certain winning in $G(\rho')$[15] (unlike x_i, which leads to i's winning with probability at most $\frac{1}{2}$, by the assumption that $i = k(t,x)$), and $V_i(t,x)$ being non-decreasing in x_i for a fixed x_{-i} by (**iv**). The inequality in (30) is due to (26). Thus, the following holds:

$$
\begin{aligned}
& U_i^{\rho'}(\sigma_i^{\delta},\sigma_{-i}^{*}) - U_i^{\rho'}(\sigma^{*}) && (31)\\
= \quad & \int_T \int_{[0,M]^n} \begin{bmatrix} u_i^{\rho'}(t,(x_i+\delta(M-x_i),x_{-i})) \\ -u_i^{\rho'}(t,x) \end{bmatrix} \sigma_1^{*}(t_1,dx_1)...\sigma_n^{*}(t_n,dx_n)p(dt) \\
\geq \quad & -\varepsilon + \int_T \int_{[0,M]^n} \frac{1}{2} V_i(t,x)\chi_{F_i(t)}(x)\,\sigma_1^{*}(t_1,dx_1)...\sigma_n^{*}(t_n,dx_n)p(dt). && (32)
\end{aligned}
$$

Here, the inequality in (32) follows from (25) and (28)–(30).

The second term in the expression in (32) is positive, as implied by (24) and the strict positivity of the function V_i, and it does not depend on ε. By taking ε to be sufficiently small, it therefore follows from (31)–(32) that $U_i^{\rho'}(\sigma_i^{\delta},\sigma_{-i}^{*}) - U_i^{\rho'}(\sigma^{*}) > 0$ for an appropriately chosen δ, which contradicts the assumption that σ^* is a BNE of $G(\rho')$. We conclude that (22) holds, after all.

Step 2. Here we check that σ^* is also a BNE of $G(\rho)$. We begin by observing that, since the ex-ante probability under σ^* of ties at the highest bid (i.e., of the realized bid profile being in $\bigcup\limits_{i\in N} X_i^{tie}$) is 0 by (22), and ρ_i' may differ from ρ_i only for bid profiles in X_i^{tie} by (**vii**), the expected payoffs of the bidders under σ^* are identical in $G(\rho)$ and $G(\rho')$. That is, for every $i\in N$,

$$U_i^{\rho}(\sigma^{*}) = U_i^{\rho'}(\sigma^{*}). \tag{33}$$

Now assume, by way of contradiction, that σ^* fails to be a BNE of $G(\rho)$. It follows that there exists $i\in N$ (w.l.o.g., $i=1$) and $\sigma_1\in\Sigma_1$ such that

$$U_1^{\rho}(\sigma^{*}) < U_1^{\rho}(\sigma_1,\sigma_{-1}^{*}); \tag{34}$$

fix one such σ_1. Since bidding 0 strictly dominates bidding M, it can be assumed w.l.o.g. that $\sigma_1(\cdot,\{M\}) = 0$.

[15] That is because $x_i+\delta(M-x_i) > x_i = \max_{j\in N\setminus\{i\}} x_j$ (the equality is due to $F_i(t)\subset X_i^{tie}$).

Consider a collection of strategies $\{\sigma_1^\delta\}_{0<\delta<1} \subset \Sigma_1$, defined as in (27) for *all* $0 < \delta < 1$, and denote

$$
\begin{aligned}
I_\delta &:= \int_T \int_{[0,M]^n} \chi_{X_1^{tie}}(x)\, \sigma_1^\delta(t_1, dx_1)\sigma_2^*(t_2, dx_2)...\sigma_n^*(t_n, dx_n)p(dt) &(35)\\
&= \int_T \int_{[0,M]^n} \chi_{X_1^{tie}}(x_1 + \delta(M - x_1), x_{-1})\, \sigma_1(t_1, dx_1)\sigma_2^*(t_2, dx_2)...\sigma_n^*(t_n, dx_n)p(dt) &(36)
\end{aligned}
$$

Notice that the integrands in (36) – namely, the functions $\chi_{X_1^{tie}}(x_1 + \delta(M - x_1), x_{-1})$ – have disjoint supports for distinct values of δ when they are restricted to $[0, M)\times [0, M]^{n-1}$. (Indeed, if $\chi_{X_1^{tie}}(x_1 + \delta(M - x_1), x_{-1}) = 1$ then $x_1+\delta(M-x_1) = \max_{j\in N\setminus\{1\}} x_j$, and so $x_1+\delta'(M-x_1) \neq \max_{j\in N\setminus\{1\}} x_j$ for any $\delta' \neq \delta$, implying that $\chi_{X_1^{tie}}(x_1 + \delta'(M - x_1), x_{-1})$ $= 0$.) Thus, given any sequence of distinct numbers $\{\delta_k\}_{k\geq 1} \subset (0, 1)$,

$$
\sum_{k\geq 1} \chi_{X_1^{tie}}(x_1 + \delta_k(M - x_1), x_{-1}) \leq 1 \tag{37}
$$

for all $x \in [0, M) \times [0, M]^{n-1}$. Because σ_1 places zero mass on M, (35)–(36) and (37) imply that the sum $\sum_{k\geq 1} I_{\delta_k}$ cannot exceed 1. It follows that $I_\delta > 0$ for at most countably many values of δ. Consequently, there exists a sequence $\{\delta_k^0\}_{k=1}^\infty \subset (0, 1)$ such that $\lim_{k\to\infty\delta} \delta_k^0 = 0$ and $I_{\delta_k^0} = 0$ for every k, that is,

$$
\int_T \int_{[0,M]^n} \chi_{X_1^{tie}}(x)\, \sigma_1^{\delta_k^0}(t_1, dx_1)\sigma_2^*(t_2, dx_2)...\sigma_n^*(t_n, dx_n)p(dt) = 0. \tag{38}
$$

Let

$$
\varepsilon := \frac{1}{2}\left[U_1(\sigma_1, \sigma_{-1}^*) - U_1(\sigma^*)\right] > 0 \tag{39}
$$

(ε is positive by (34)). By Lemma 1, for all sufficiently large k and for every $(t, x) \in T \times [0, M]^n$,

$$
u_1^\rho(t, (x_1 + \delta_k^0(M - x_1), x_{-1})) > u_1^\rho(t, x) - \varepsilon. \tag{40}
$$

It follows from (40) that

$$
\begin{aligned}
&U_1^\rho(\sigma_1^{\delta_k^0}, \sigma_{-1}^*) &(41)\\
=\;& \int_T \int_{[0,M]^n} u_1^\rho(t, x)\, \sigma_1^{\delta_k^0}(t_1, dx_1)\sigma_2^*(t_2, dx_2)...\sigma_n^*(t_n, dx_n)p(dt)\\
=\;& \int_T \int_{[0,M]^n} u_1^\rho\left(x_1 + \delta_k^0(M - x_1), x_{-1}\right) \sigma_1(t_1, dx_1)\sigma_2^*(t_2, dx_2)...\sigma_n^*(t_n, dx_n)p(dt)\\
\geq\;& \int_T \int_{[0,M]^n} u_1^\rho(t, x)\, \sigma_1(t_1, dx_1)\sigma_2^*(t_2, dx_2)...\sigma_n^*(t_n, dx_n)p(dt) - \varepsilon\\
=\;& U_1^\rho(\sigma_1, \sigma_{-1}^*) - \varepsilon. &(42)
\end{aligned}
$$

Since, by (38), the ex-ante probability under $(\sigma_1^{\delta_k^0}, \sigma_{-1}^*)$ of the realized bid profile being in X_1^{tie} is 0, and ρ_1' may differ from ρ_1 only on X_1^{tie} by (**vii**), the expected payoffs of bidder 1 under $(\sigma_1^{\delta_k^0}, \sigma_{-1}^*)$ are identical in $G(\rho')$ and $G(\rho)$: for every $i \in N$,

$$U_1^{\rho'}(\sigma_1^{\delta_k^0}, \sigma_{-1}^*) = U_1^{\rho}(\sigma_1^{\delta_k^0}, \sigma_{-1}^*). \tag{43}$$

We thus obtain

$$U_1^{\rho'}(\sigma_1^{\delta_k^0}, \sigma_{-1}^*) \geq U_1^{\rho}(\sigma_1, \sigma_{-1}^*) - \varepsilon > U_1^{\rho}(\sigma^*) = U_i^{\rho'}(\sigma^*),$$

where the first inequality follows from combining (43) with the inequality established in (41)–(42), the second inequality uses the definition of ε in (39), and the third inequality is just (33). But the implication is that $U_1^{\rho'}(\sigma_1^{\delta_k^0}, \sigma_{-1}^*) > U_i^{\rho'}(\sigma^*)$, which cannot be the case because σ^* is a BNE of $G(\rho')$. This contradiction shows the impossibility of a profitable unilateral deviation (34) in $G(\rho)$, and proves that σ^* is also a BNE of $G(\rho)$. ■

5 An extension: mixing auctions and contests

In addition to the extensively discontinuous success functions in all-pay auctions we may, in principle, consider functions that – at least for some type profiles – are discontinuous only on a strict a subset of $\bigcup_{i \in N} X_i^{tie}$. The corresponding game may sometimes be amenable to a treatment similar to that offered in Section 4. The well known and much used model of contests represents one interesting case of limited discontinuity.

Contests can be easily incorporated into our framework. Let us assume that, for a subset $T^{cont} \in \mathcal{T}$ of type profiles, instead of an all-pay auction the bidders engage in an "imperfectly discriminating" *contest*, where below-top bids also have a chance to win. Formally, for any $t \in T^{cont}$, $\rho(t, \cdot) : [0, M]^n \to \Delta^n$ is assumed to be *continuous* on $[0, M]^n \setminus \{\mathbf{0}\}$, and the discontinuity at the zero-bid profile $\mathbf{0}$ is brought about by an additional assumption that $\rho_i(t, (x_i, \mathbf{0}_{-i})) = 1$ for any $x_i > 0$, i.e., that any bid ensures winning if it is the only positive one.[16] The category of such success functions – taken from the description of general "pre-Tullock" contests in Haimanko (2021) –

[16] In line with the description of pre-Tullock contests in Haimanko (2021), we further postulate that, for any $i \in N$, $\rho_i(t, x)$ is non-decreasing in x_i for any fixed $x_{-i} \in [0, M]^{n-1}$ (just as is trivially the case in all-pay auctions, see (**vii**)).

captures many contest models, from lottery contests to more general Tullock contests (Tullock (1980)) to their extensions in Szidarovszky and Okuguchi (1997). For type profiles in $T \backslash T^{cont}$, a regular all-pay auction is performed. Thus, a bidder i may be uncertain (given his type t_i) whether the winner will be determined with perfect or imperfect discrimination, or the uncertainty may be only ex ante if his type t_i fully reveals whether $t \in T^{cont}$ or $t \in T \backslash T^{cont}$.[17]

It is easy to see what needs to be changed in our approach in order to accommodate a given hybrid auction $G^{hyb}(\rho)$ described above. Let us modify the notion of X_i^{tie} by making it type-dependent: $X_i^{tie}(t)$ remains as in (1) for $t \in T \backslash T^{cont}$, but turns into $X_i^{tie}(t) = \{\mathbf{0}\}$ for $t \in T^{cont}$; thus, $X_i^{tie}(t)$ now represents the set of bid profiles where $\rho_i(t, \cdot)$ is discontinuous. Also, $\widetilde{\rho}(t, \cdot)$ will now depend on the given $\rho(t, \cdot)$: let $\widetilde{\rho}(t, \cdot)$ be defined by (6) for $t \in T \backslash T^{cont}$, and by

$$\widetilde{\rho}(t, x) = \begin{array}{ll} \rho(t, x), & \text{if } x \neq \mathbf{0}, \\ \left(1, \mathbf{0}_{-i(t,\mathbf{0})}\right) & \text{if } x = \mathbf{0} \end{array}$$

for $t \in T^{cont}$. Thus, as in an all-pay auction, here $\widetilde{\rho}(t, \cdot)$ may differ from $\rho(t, \cdot)$ only on bid profiles in $\bigcup_{i \in N} X_i^{tie}(t)$. When $t \in T^{cont}$, $\widetilde{\rho}(t, \mathbf{0})$ prescribes declaring as a winner the bidder with the highest value in N at the zero-bid profile.

Our Proposition 1 on BNE existence (with the success function being $\widetilde{\rho}$) and its proof apply fully to the hybrid auction $G^{hyb}(\widetilde{\rho})$. In the premise of Proposition 2 on the invariance of BNE when ρ' is replaced by ρ, one now needs to specifically require that, for any $t \in T$, $\rho(t, \cdot)$ and $\rho'(t, \cdot)$ may only differ on bid profiles in $\bigcup_{i \in N} X_i^{tie}(t)$ (this condition holds by definition in the case of an all-pay auction). The proof of Proposition 2 then also goes through. As a consequence, the existence of BNE is established for any $G^{hyb}(\rho)$. Finally, note that by taking $T^{cont} = T$, BNE existence is implied also for (pure) contests (it was established in Haimanko (2021) using a different method[18]).

[17] Our narrative assumes for simplicity that knowing $t \in T$ fully determines whether an auction or a contest are used in determining the winner. If a realized decision mechanism (including its specific tie-breaking rule) depends on extraneous factors, one may simply add to N a dummy player ("nature"), whose type set T_0 accounts for the choice of the decision mechanism. Our proofs and the changes described next will still achieve their goal, with some straightforward adjustments.

[18] Haimanko (2021) showed that a BNE in a contest can be obtained by taking an appropriate limit of a sequence of BNEs in contests with positive but diminishing floor on bids (efforts), where ex-post payoffs are continuous and BNE existence is guaranteed by Balder (1988).

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