INEFFECTIVE PRIZES IN MULTI-DIMENSIONAL CONTESTS

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Abstract

We study k-dimensional contests wherein each of the k sub-contests the n agents compete against each other in a Tullock contest. The designer who wishes to maximize the total effort in the k sub-contests chooses the prize allocation which indicates the prize of every agent for any outcome. We show that in our simultaneous two-dimensional contest, if the number of agents is two, a prize for winning in one of the sub-contests is ineffective, namely, it does not (positively) affect the agents' efforts and therefore these prizes do not have to be awarded. In our sequential two-dimensional contest, if each agent wins in a different sub-contest, with a positive probability, the prizes should not be awarded. On the other hand, in simultaneous as well as in sequential two-dimensional contests, if the number of agents is larger than two, the prizes for winning in one sub-contest positively affect the agents' efforts. Then, we generalize the above results and find the ineffective prizes for any simultaneous k-dimensional contest with $k > 2$ symmetric sub-contests and with any number of agents that are either smaller than, equal to, or larger than the number of sub-contests.

Keywords: Multidimensional contests, Tullock contests, ineffective prizes

JEL classification: D44, O31, O32

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1 Introduction

There are several environments in which contests are multi-dimensional such that each agent has to choose an action in each sub-contest (dimension). Multi-dimensional contests could be either sequential or simultaneous. Examples of sequential multi-dimensional contests are dynamic models in which an agent allocates a resource over the stages. Each agent has a resource budget for which he decides how to allocate it along all the stages (see, among others, Ryvkin 2011, Sela and Erez 2013 and Klumpp et al. 2019). On the other hand, a well-known simultaneous multi-dimensional contest is the Colonel Blotto game in which two agents compete against each other in n different sub-contests. Each agent distributes a fixed amount of resource over the n sub-contests without knowing his opponent's distribution of the resource (see, among others, Snyder 1989, Roberson 2006, Hart 2008 and Kovenock and Roberson 2021).

In a different form of multi-dimensional contest, there is only one competition but each agent needs to choose several kinds of actions. The most well-known example of such contests is when each agent exerts an effort to increase his probability of winning but also exerts an effort to decrease the probability of his opponents to win, namely, to sabotage his opponents (see, among others, Konrad 2000, Chen 2003, Amegashie and Runkel 2007, Amegashie 2012, Guertler et al. 2013 and Dato and Nieken 2014, and Bernhardt and Ghosh 2020).

We study k-dimensional contests with $n \geq 2$ agents wherein each of the k sub-contests the winner is determined by the Tullock contest success function (see, among others, Tullock 1980, Skaperdas1996, Clark and Riis 1998, and Baye and Hoppe, H. 2003). We first consider the case when there are two asymmetric sub-contests. We assume that the agents act either simultaneously or sequentially. If an agent wins in both sub-contests he wins the prize which is normalized to be 1. If different agents win both sub-contests, the designer can decide whether the prize is awarded or not, and alternatively, he can make the contest asymmetric by choosing two different parameters of α and β, where the agent who wins in the first sub-contest only wins the entire prize sum with a probability of $1 \geq \alpha \geq 0$, and the agent who wins the second sub-contest wins the entire prize sum with a probability of $1 \geq \beta \geq 0$, where $\alpha + \beta \leq 1$. That is, if $\alpha + \beta < 1$, it means that there is a positive probability that the entire prize will not be allocated at all, or alternatively, only part of the entire prize will be allocated.

There is extensive literature on the optimal allocation of prizes in contests, most of which occur in one-stage contests (see, among others, Barut and kovenock 1998, Mokdovanu and Sela 2001, 2006, 2012, Schweinzer and Segev 2012, Akerlof and Holden, and Sela 2020) and others in multi-stage contests (see, among others, Rosen 1986, Fu and Lu 2012, Sela and Tsahi 2020, and Alshech and Sela 2021). In the present paper our goal is not to characterize the optimal design of prizes in multi-dimensional contests, but to indicate the ineffective prizes in such contests, namely, the prizes that do not positively affect the agents' efforts. In that case, the ineffective prizes should not be allocated in multi-dimensional contests where the rest of the prizes (the effective prizes) will be allocated.

In our model, we assume that the values of α and β, namely, the prizes for winning each of the two sub-contests, are endogenously determined by the contest designer. For example, suppose that two junior members in the same department at a university compete for one position. The two junior researchers are examined in two dimensions; the first is their research quality α and the second is their teaching quality β. If one of the researchers dominates the other in both dimensions then he wins the position for sure. However, if one of them is dominant in research and the other in teaching, then there might be some advantage to the researcher who is dominant in research, but there is a positive probability that none of these junior researchers will get the position since the department does not want to compromise on the quality. In that case, the parameters α and β are endogenous and their sum in our model is smaller than one.

Setting the values of the parameters α and β by the contest designer is actually the policy for a case of a draw when each of the winners has only one win. The literature offers several findings on the meaningfulness of the policy in the case of a tie or a draw. In the case of a tie in one-dimensional contests, a designer can decide to allocate the prize for each of the agents with the highest effort with the same probability, not to allocate a prize at all, or to allocate the prize with a positive probability which is smaller than 1. Cohen and Sela (2007), for example, show that in all-pay auctions with two agents under complete information, in the case of a tie it is optimal to allocate the prize with a probability of 1/3 or less.[1] Imhof and Krakel (2015) show that in the model of Lazear and Rosen (1981) introducing the possibility of a tie is beneficial for the principal because it makes the agent that benefits from the competitive advantage less likely to win. In multi-stage contests, however, Deng and Weng (2018) show that introducing draws into a Tullock contest can be optimal for the contest designer who aims to maximize the expected winner's effort when the agents are sufficiently heterogeneous.

In our simultaneous two-dimensional contest we show that with two agents, the agents' total effort does not depend on the values of the parameters α and β which make these prizes ineffective. However, for more than two agents, the agents' total effort is maximized for all α and β that satisfy $\alpha + \beta = 1$. The intuition behind these results is that when there are only two agents they can agree that each one exerts a low effort in a different sub-contest and in the other one he does not exert any effort. Then, every agent wins in one sub-contest by exerting low efforts. In order to avoid such a collusion, the prizes for one win should not be awarded. On the other hand, when the number of agents is larger than two, there is no possibility of a collusion, and then there are no ineffective prizes that do not positively affect the agents' efforts.

For our sequential two-dimensional contest, we show that with two agents, the lowest total effort is obtained when $\beta = 0$ and $\alpha = 1$, and the highest is obtained when $\alpha + \beta < 1$. If $\alpha = \beta$ the highest total effort is obtained for $\alpha = \frac{1}{3}$, that is, in the case that each agent wins one contest, the entire prize is not awarded with a probability of 1/3. The intuition behind these results is that in the sequential contest, the designer can award prizes for one win, since after that one agent wins in the first stage, he has no incentive to satisfy any agreement with the other agent for the competition in the second stage. Thus, the prizes for winning only one sub-contest are not completely ineffective. For more than two agents, the highest total effort in the sequential contest when $\alpha = \beta$ is obtained for either $\alpha = \beta = 0$ or $\alpha = \beta = 0.5$. The intuition behind this result is as follows: 1) Since the number of agents is larger than the number of sub-contests, there is no reason not to award all the prizes. 2) there is an asymmetry in the second stage between the winner of the first stage and the other agents which vanishes when $\alpha = \beta = 0.5$. Then, when there is no

[1] Gelder et al. (2019) analyzed the all-pay auction under complete information with contnuous strategy sets.

asymmetry the total effort is higher. 3) The one-stage contest yields a higher total effort than any two-stage contest and therefore $\alpha = \beta = 0$. We can conclude that in our two-dimensional contest either simultaneous or sequential, if the number of agents is two, the prizes should not be awarded certainty. On the other hand, when there are more than two agents, awarding the entire prize for any outcome is optimal for a designer who wishes to maximize the total effort.

We also compare which kind of two-dimensional contest, sequential or simultaneous, is more beneficial for a designer who wishes to maximize the agents' total effort. With two agents, the highest total effort in the simultaneous contest is smaller than that of the sequential contest. For more than two agents, we show that the highest total efforts in the sequential and simultaneous contests are the same. Thus, in sum, the sequential two-dimensional contest dominates the simultaneous one with respect to the total effort. Arbatskaya and Mialon (2012) study a sequential two-dimensional contest with two agents who compete in a contest with a success function that is a combination of the Tullock contest success function and a Cobb-Douglas type (for an axiomatization of this contest success function for multi-activity contests, see Arbatskaya and Mialon 2010). They compare the total effort in this model and find that the total effort in the sequential contest is always smaller than that of the simultaneous one if the agents have asymmetric cost functions and both forms of contests yield the same total effort when the agents are symmetric.

Last, we examine if our result for the simultaneous two-dimensional contests can be generalized for simultaneous k-dimensional contests with n agents. We show that in that case, if the number of agents n is larger than the number of sub-contests k, like in the two-dimensional contests, every prize positively affects the agents' efforts. Furthermore, in that case when $n > k$, we show that the prize for one win might have the largest effect on the agents' efforts than all the other prizes. When the number of agents and sub-contests is the same, similar to the two-dimensional contest with two agents, the prize for one win does not affect the agents' efforts, while all the other prizes (for two or more wins) positively affect the agents' efforts. When the number of agents is smaller than the number of sub-contests, then the prizes for one win are ineffective but other prizes for winning more than one sub-contest might also be ineffective. Then, we find sufficient conditions that a prize negatively affects the agents' efforts in our k-dimensional contest.

A paper related to the present one is Clark and Konrad (2007) who study a similar simultaneous multi-dimensional contest with two symmetric agents. They find the optimal (maximization of the total effort) number of wins in order to win the contest as a function of the number of symmetric sub-contests, and for any number of wins, the minimum number of dimensions k such that each agent will have a positive expected payoff. They assume that there are two symmetric players and two symmetric sub-contests (dimensions). We, on the other hand, assume more than two agents, and when there are two agents, the sub-contests are not necessarily symmetric. The major difference, however, between their work and our model is that they assume that only one prize is awarded to the winner of the contest, while we assume that several prizes are awarded. Then, we study what are the ineffective prizes, namely, those prizes that do not positively affect the agents' efforts. The paper most related to our work is Feng and Lu (2018) who characterize the effort-maximizing prize allocation in multi-stage Tullock contests when the prize sum is fixed, but the agents' prizes are contingent on the number of wins. In other words, several prizes might be awarded. They study a sequential three-dimensional contest with two agents, while we also focus on simultaneous contests with any number of agents and dimensions. Moreover, they assume that the prize sum has to be awarded, while we show that it might be optimal not to award the entire prize sum with a positive probability. In addition, they assume that the prize monotonically increases in the number of wins, while we show that this is not always the optimal case. Last, we focus on ineffective prizes that do not positively affect the agents' efforts, and show that there is a difference in the consequences for sequential and simultaneous multi-dimensional contests wherein the simultaneous ones the phenomenon of ineffective prizes is more prominent.

The rest of the paper is organized as follows: In Section 2 we introduce our model. In Section 3 we analyze simultaneous two-dimensional contests, and in Section 4 we analyze sequential two-dimensional contests. In Section 5 we deal with simultaneous k-dimensional contests. Section 6 concludes. All the proofs appear in the Appendix.

2 The model

Consider n agents who compete in k sub-contests. Each agent $i, i = 1, ..., n$ exerts an effort of x_i^j in sub-contest $j, j = 1, ..., k$, and the cost of his efforts is $\sum_{j=1}^{k} x_i^j$. The probability that agent i wins in sub-contest j is $\frac{x_i^j}{\sum_{s=1}^{n} x_s^j}$ where x_s^j, is agent s's effort in contest j. This contest will be referred to as a k-dimensional contest. The designer has a prize sum that is normalized to be 1, and he chooses the prize allocation function, namely, for any possible outcome, there is a given allocation of prizes such that the sum of the winners' prizes is smaller than or equal to 1. Formally, let the vector $s_i = (s_{i1}, s_{i2}, ..., s_{ik})$ denotes the outcomes of agent i such that $s_{i,j} \in \{0, 1\}$ where $s_{i,j} = 1$ denotes that agent i won sub-contest j and $s_{i,j} = 0$ denotes that agent i did not win sub-contest j, $j = 1, ..., k$. Then, $w(s_i)$ is the prize sum awarded to agent i where $\sum_{i=1}^{n} w(s_i) \leq 1$. Note that $\sum_{i=1}^{n} w(s_i) < 1$ means that not all the entire prize sum is allocated. Our goal is to indicate for which outcomes the prizes are ineffective, namely, they do not positively affect the agents' efforts. In other words, when prizes are given, our analysis indicates which prizes should be excluded from the list of prizes.

3 Simultaneous two-dimensional contests

We begin with the analysis of two-dimensional contests ($k = 2$) in which $w(1, 1) = 1$, $w(1, 0) = \alpha$, $w(0, 1) = \beta$ and $w(0, 0) = 0$ where $\alpha + \beta \leq 1$. In other words, if an agent wins in both sub-contests he wins a prize of 1, if he wins only in the first sub-contest he wins a prize of α, and if he wins only in the second sub-contest he wins a prize of β. If he does not win any sub-contest, then he does not win anything. Thus, when $\alpha \neq \beta$ it means that the sub-contests are not symmetric, and when $\alpha + \beta < 1$ it means that if in each sub-contest there is a different winner, the entire prize sum is awarded with a probability that is smaller than 1. We assume now that the agents simultaneously compete in both sub-contests which will be denoted by A and B. Then, the maximization problem of agent 1 is

$$\max_{x_1^A, x_1^B} \frac{x_1^A}{\sum_{i=1}^{n} x_i^A} \frac{x_1^B}{\sum_{i=1}^{n} x_i^B} + \alpha \frac{x_1^A}{\sum_{i=1}^{n} x_i^A}(1 - \frac{x_1^B}{\sum_{i=1}^{n} x_i^B}) + \beta(1 - \frac{x_1^A}{\sum_{i=1}^{n} x_i^A}) \frac{x_1^B}{\sum_{i=1}^{n} x_i^B} - x_1^A - x_1^B. \quad (1)$$

where x_i^A and x_i^B are agent i's efforts in sub-contests A and B, respectively. The solution of (1) yields

Proposition 1 *In the simultaneous two-dimensional contest with two asymmetric sub-contests and n symmetric agents, the symmetric equilibrium efforts in both sub-contests are*

$$\begin{aligned} x^A &= \frac{(n-1)}{n^3}((1-\beta)+(n-1)\alpha) \\ x^B &= \frac{(n-1)}{n^3}((1-\alpha)+(n-1)\beta). \end{aligned} \tag{2}$$

For $n = 2$, the agents' equilibrium efforts are

$$\begin{aligned} x^A &= \frac{1+\alpha-\beta}{8} \\ x^B &= \frac{1+\beta-\alpha}{8}. \end{aligned}$$

Inserting (2) into (1) gives the agents' symmetric expected payoff

$$U_{sim} = \frac{1}{n^3}\left(-n+2(n-1)\alpha+2(n-1)\beta+2\right), \tag{3}$$

and by (2), the agents' total effort is

$$TE_{sim} = n(x+y) = \frac{n-1}{n^2}(2+(n-2)\alpha+(n-2)\beta). \tag{4}$$

For $n = 2$ we obtain

$$TE_{sim}(n=2) = \frac{1}{2}. \tag{5}$$

Thus, by (4) and (5) we have

Proposition 2 *In the simultaneous two-dimensional contest with two symmetric agents, the agents' total effort does not depend on the values of the parameters α and β. If the number of agents is larger than two, then the agents' total effort is maximized for all α and β that satisfy $\alpha + \beta = 1$.*

One of the implications of Proposition 2 is that a designer who wishes to maximize the agents' total effort when there are two agents should not award a prize to an agent who does not win in both sub-contests. However, for all $n > 2$, the designer should award a prize for winning one contest where the distribution of the entire prize between the two winners of the two sub-contests does not affect the agents' total effort. The intuitive explanation for these results is that when there are only two agents, they can make an agreement that each one will win in a different sub-contest, and then the agents can exert relatively low efforts and win both prizes. On the other hand, when the number of agents is larger than two an agreement among the $n > 2$ agents is not possible, and therefore the designer does not have an incentive not to award prizes for winning in one sub-contest only. In other words, when $n > 2$ all the prizes positively affect the agents' efforts.

4 Sequential two-dimensional contests

Now, we assume that the n agents sequentially compete, first in sub-contest A and later in sub-contest B. In order to analyze the subgame-perfect equilibrium, we begin with the analysis of the second stage and go backward to the first one.

4.0.1 The second stage

Assume that agent 1 won in the first stage. Then, his maximization problem in the second stage is

$$\max_{x_1^B} \frac{x_1^B}{\sum_{i=1}^{n} x_i^B} + \alpha(1 - \frac{x_1^B}{\sum x_i^B}) - x_1^B. \tag{6}$$

The maximization problem of agent $j, j = 2, ..., n$ is

$$\max_{y_2} \beta \frac{x_j^B}{\sum_{i=1}^{n} x_i^B} - x_j^B. \tag{7}$$

The solution of (6) and (7) yields

Proposition 3 *In the sequential two-dimensional contest with n symmetric agents, the agents' equilibrium efforts in the second stage (sub-contest B) are*

$$\begin{aligned} x_1^B &= \beta(1-\alpha)\frac{n-1}{(n+\alpha+\beta-n\alpha-1)^2}(n+\alpha+2\beta-n\alpha-n\beta-1) \\ x^B &= x_j^B = \beta^2(1-\alpha)\frac{n-1}{(n+\alpha+\beta-n\alpha-1)^2}, \; j = 2, ..., n. \end{aligned}$$

where x_1^B *is the equilibrium effort of the winner in the first stage (sub-contest A), and* x^B *is the symmetric equilibrium effort of all the other* $n-1$ *agents. For* $n=2$ *we have*

$$\begin{aligned} x_1^B &= \beta(1-\alpha)^2\frac{1}{(1+\beta-\alpha)^2} \\ x_2^B &= \beta^2(1-\alpha)\frac{1}{(1+\beta-\alpha)^2}. \end{aligned} \tag{8}$$

We can see from (8) that in the sequential two-dimensional contest there exists

$$\frac{x_1^B}{x_2^B} = \frac{1-\alpha}{\beta}.$$

Thus, in the second stage, the equilibrium effort of the winner of sub-contest A is higher than the equilibrium effort of all the other agents iff $\alpha+\beta<1$. The utility of the winner of sub-contest A in the second stage is

$$\begin{aligned} U_{1\,\mathrm{seq}} &= (1-\alpha)\frac{(n-1)(1-\alpha)\beta((n-1)(1-\alpha)-(n-2)\beta)}{(1-\alpha)\beta^2(n-1)^2+(n-1)(1-\alpha)\beta((n-1)(1-\alpha)-(n-2)\beta)} \\ &\quad +\alpha-\frac{(n-1)(1-\alpha)\beta((n-1)(1-\alpha)-(n-2)\beta)}{(\beta+(n-1)(1-\alpha))^2} \\ &= \frac{1}{(n+\alpha+\beta-n\alpha-1)^2}\begin{pmatrix} -2n^2\alpha^2\beta+n^2\alpha^2-n^2\alpha\beta^2+4n^2\alpha\beta-2n^2\alpha+n^2\beta^2-2n^2\beta+n^2 \\ +4n\alpha^2\beta-2n\alpha^2+4n\alpha\beta^2-10n\alpha\beta+4n\alpha-4n\beta^2 \\ +6n\beta-2n-2\alpha^2\beta+\alpha^2-3\alpha\beta^2+6\alpha\beta-2\alpha+4\beta^2-4\beta+1 \end{pmatrix}, \end{aligned} \tag{9}$$

and the symmetric utility of all the other agents in the second stage is:

$$\begin{aligned} U_{\mathrm{seq}} &= U_{j\,\mathrm{seq}} = \frac{\beta(1-\alpha)\beta^2(n-1)}{(1-\alpha)\beta^2(n-1)^2+(n-1)(1-\alpha)\beta((n-1)(1-\alpha)-(n-2)\beta)} - \frac{(1-\alpha)\beta^2(n-1)}{(\beta+(n-1)(1-\alpha))^2} \\ &= \frac{\beta^3}{(n+\alpha+\beta-n\alpha-1)^2}\ ,\ j=2,3,...,n. \end{aligned} \tag{10}$$

For $n=2$, we have

$$\begin{aligned} U_{1\,\mathrm{seq}} &= \frac{1}{(\beta-\alpha+1)^2}\left(-2\alpha^2\beta+\alpha^2+\alpha\beta^2+2\alpha\beta-2\alpha+1\right) \\ U_{2\,\mathrm{seq}} &= \frac{\beta^3}{(\beta-\alpha+1)^2}\ , \end{aligned} \tag{11}$$

where agent 1 is the winner in the first stage.

4.0.2 The first stage

In the first stage, the maximization problem of agent 1 is

$$\max_{x_1^A} U_{1\,\mathrm{seq}} \frac{x_1^A}{\sum_{i=1}^{n} x_i^A} + U_{\mathrm{seq}}\left(1 - \frac{x_1^A}{\sum_{i=1}^{n} x_i^A}\right) - x_1^A, \tag{12}$$

where $U_{1\,\mathrm{seq}}$ and U_{seq} are the expected utilities in the second stage and are given by (9) and (10). The solution of (12) yields

Proposition 4 *In the sequential two-dimensional contest with n symmetric agents, their symmetric equilibrium effort in the first stage (sub-contest A) is:*

$$x^A = (n-1)\frac{U_{1\,\mathrm{seq}} - U_{\mathrm{seq}}}{n^2} = \frac{1}{n^2}\frac{n-1}{(n+\alpha+\beta-n\alpha-1)^2}\cdot \tag{13}$$
$$\left(\begin{array}{c} -2n^2\alpha^2\beta + n^2\alpha^2 - n^2\alpha\beta^2 + 4n^2\alpha\beta - 2n^2\alpha + n^2\beta^2 - 2n^2\beta + \\ n^2 + 4n\alpha^2\beta - 2n\alpha^2 + 4n\alpha\beta^2 - 10n\alpha\beta + 4n\alpha - 4n\beta^2 \\ +6n\beta - 2n - 2\alpha^2\beta + \alpha^2 - 3\alpha\beta^2 + 6\alpha\beta - 2\alpha + 4\beta^2 - 4\beta + 1 - \beta^3 \end{array}\right)$$

For $n = 2$ we have

$$x^A = x_1^A = x_2^A = \frac{1}{4(\beta-\alpha+1)^2}(-2\alpha^2\beta + \alpha^2 + \alpha\beta^2 + 2\alpha\beta - 2\alpha + 1 - \beta^3). \tag{14}$$

The expected total effort in both stages together is

$$TE_{\mathrm{seq}} = nx^A + x_1^B + (n-1)x^B = \tag{15}$$
$$\frac{1}{n}\frac{n-1}{(n+\alpha+\beta-n\alpha-1)^2}\left(\begin{array}{c} -n^2\alpha^2\beta + n^2\alpha^2 - n^2\alpha\beta^2 + 2n^2\alpha\beta - 2n^2\alpha + n^2\beta^2 - n^2\beta + n^2 \\ +3n\alpha^2\beta - 2n\alpha^2 + 3n\alpha\beta^2 - 8n\alpha\beta + 4n\alpha - 3n\beta^2 + 5n\beta - 2n - 2\alpha^2\beta \\ +\alpha^2 - 3\alpha\beta^2 + 6\alpha\beta - 2\alpha - \beta^3 + 4\beta^2 - 4\beta + 1 \end{array}\right),$$

and for $n = 2$ we have

$$TE_{\mathrm{seq}-2} = 2x^A + x_1^B + x_1^B = \frac{1}{2(\beta-\alpha+1)^2}\left(\alpha^2 - \alpha\beta^2 - 2\alpha\beta - 2\alpha - \beta^3 + 2\beta^2 + 2\beta + 1\right). \tag{16}$$

For $n = 2$, the difference between the total effort in the simultaneous and sequential contests is

$$\Delta TE = TE_{sim-2} - TE_{\mathrm{seq}-2} = \frac{1}{2} - \frac{1}{2(\beta-\alpha+1)^2}\left(\alpha^2 - \alpha\beta^2 - 2\alpha\beta - 2\alpha - \beta^3 + 2\beta^2 + 2\beta + 1\right) = \frac{1}{2}\beta^2\frac{\alpha+\beta-1}{(\beta-\alpha+1)^2}.$$

It can be easily verified that $\Delta TE < 0$ iff $\alpha + \beta < 1$. Thus we can conclude as follows:

Proposition 5 *In the simultaneous two-dimensional contest with two agents the total effort is smaller than that of the sequential two-dimensional contest with two agents if $\alpha + \beta < 1$. The lowest total effort of the sequential two-dimensional contest with two agents is obtained when $\beta = 0$ and $\alpha = 1$ and then the total efforts in the simultaneous and sequential contests are the same. In particular, the highest total effort in the sequential two-dimensional contest with two agents is obtained when $\alpha + \beta < 1$. If $\alpha = \beta$ the highest total effort is obtained for $\alpha = \beta = \frac{1}{3}$.*

By Proposition 5, in the sequential two-dimensional contest with two agents when $\alpha = \beta$, it is optimal for the contest designer who wishes to maximize the agents' total effort not to award any prize with a probability of 1/3 if each agent won only once. The intuition behind this result is that the robustness of an agreement between both agents is weaker than in the simultaneous contest since after that one agent has one win in the first stage, he does not have an incentive to satisfy such an agreement. Thus, since the prizes for winning in sub-contest A or B positively affect the agents' efforts, it is optimal to allocate the prizes for one win only, but, to avoid any agreement between the agents such that it will not be profitable, the values of these prizes are reduced.

Consider now that $n > 2$ and $\alpha = \beta$. Then, by (15), we have

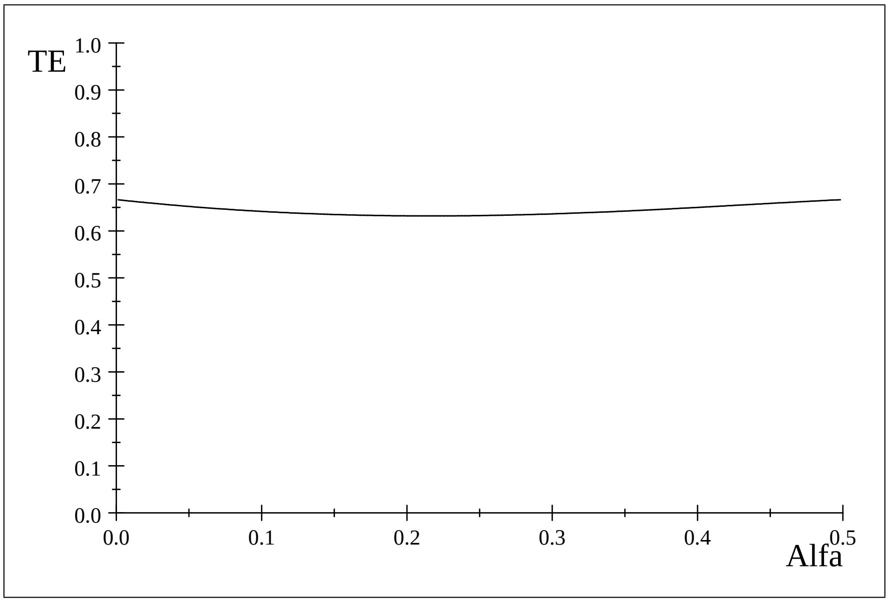
$$TE_{\text{seq}}(\alpha) = \frac{1}{n}(1-\alpha)\frac{n-1}{(n+2\alpha-n\alpha-1)^2}\left(2n^2\alpha^2 - 2n^2\alpha + n^2 - 6n\alpha^2 + 7n\alpha - 2n + 6\alpha^2 - 5\alpha + 1\right). \quad (17)$$

The following example illustrates the case of a sequential two-dimensional contest with three symmetric agents.

Example 1 *By (17), the total effort in the sequential two-dimensional contest with three symmetric agents is*

$$TE_{\text{seq}} = -\frac{2}{3(\alpha-2)^2}\left(6\alpha^3 - 8\alpha^2 + 6\alpha - 4\right).$$

In the following figure we can see the total effort as a function of $\alpha \leq 0.5$.



Note that the total effort obtains its maximum values at the end points $\alpha = 0$ and $\alpha = 0.5$. Then,

$$TE(\alpha = 0) = -\frac{2}{3\left(-2\right)^2}\left(-4\right) = \frac{2}{3},$$

and

$$TE(\alpha = 0.5) = -\frac{2}{3\left(0.5-2\right)^2}\left(6(0.5)^3 - 8(0.5)^2 + 6(0.5) - 4\right) = \frac{2}{3}.$$

Furthermore, by (4), the total effort is

$$TE_{sim} = \frac{4}{9}(\alpha + 1).$$

We can see that the highest total effort in the simultaneous two-dimensional contest with three symmetric agents is obtained for $\alpha = 0.5$ and is equal to $\frac{2}{3}$. Therefore, the highest total efforts in the simultaneous and sequential contests with three symmetric agents are the same.

The next result generalizes the findings of the above example for two-dimensional contests with any number ($n > 2$) of agents.

Proposition 6 *The highest total effort in the sequential two-dimensional contests with $n > 2$ symmetric agents where $\alpha = \beta$ is obtained for either $\alpha = \beta = 0$ or $\alpha = \beta = 0.5$. Furthermore, the highest total efforts in the sequential and simultaneous two-dimensional contests are the same and are equal to $\frac{n-1}{n}$.*

The intuition behind this result is that when there are no prizes for winning in one sub-contest, the agents do not compete in the second stage since they do not have a chance to win. Then we actually have a one-stage contest. When the prizes for winning one sub-contest are relatively large, the agents compete also in the second stage since there is a significant prize there such that the larger the value of this prize is the larger the agents' efforts in the second stage. By Propositions 5 and 6, we can see that in the sequential two-dimensional contest, for any number of agents, the designer who wishes to maximize the agents' total effort, does not have to award a prize when two different agents win in both stages. In other words, it is not necessary to choose a winner in the sequential two-dimensional contest.

5 Simultaneous k-dimensional contests

We now consider simultaneous k-dimensional contests with n symmetric agents. In this section we also assume that the k sub-contests are symmetric such that $w(j)$ is the prize for an agent who wins in j sub-contests, $j = 1, ..., k$. The maximization problem of agent 1 is

$$\max_{x_1^1,\dots,x_1^k} \sum_{j=1}^{k} w(j) \begin{pmatrix} k \\ j \end{pmatrix} \prod_{l=1}^{j} \frac{x_1^l}{\sum_{i=1}^n x_1^i} \prod_{m=j+1}^{k} (1 - \frac{x_1^m}{\sum_{i=1}^n x_i^m}) - \sum_{j=1}^{k} x_1^j, \tag{18}$$

where $w(k) = 1$, and if $j + m \leq k$ then $w(j) + w(m) \leq 1$. The solution of (18) yields

Proposition 7 *In the simultaneous k-dimensional contest with n symmetric agents, the symmetric equilibrium effort is*

$$x = \frac{n-1}{n^{k+1}}(w(k) + \sum_{i=1}^{k-1} w(k-i)(n-1)^{i-1} \left((n-1) \begin{pmatrix} k-1 \\ i \end{pmatrix} - \begin{pmatrix} k-1 \\ i-1 \end{pmatrix} \right) . \tag{19}$$

Then, by the symmetric equilibrium effort we can calculate the ineffective prizes, namely, the prizes that do not positively affect the agents' total effort.

Proposition 8 *In the simultaneous k-dimensional contest with n agents,*

1) If $n > k$, every prize positively affects the agents' symmetric effort.

2) If $n = k$, the prize for one win does not affect the agents' symmetric effort, while all the other prizes (for two or more wins) positively affect the agents' symmetric effort.

3) If $n < k$, let $\widetilde{i}$ satisfies $\widetilde{i} \geq \frac{(n-1)k}{n}$ and $\widetilde{i} - 1 < \frac{(n-1)k}{n}$. Then all the prizes for $k - i$ wins, $k > i \geq \widetilde{i}$, do not positively affect the agents' symmetric effort.

By Proposition 8, if the number of agents n is smaller than the number of sub-contests k, there might be several ineffective prizes. For example,when $n = 2$ and $k = 4$, let $\widetilde{i} = 2$. Then, we have $2 \geq \frac{(2-1)4}{2}$ and $2 - 1 < \frac{(2-1)4}{2}$. Thus, by Proposition 8, the prizes for one win and for two wins are both ineffective, namely, they do not positively affect the agents' symmetric effort. However, when $n > k$, all the prizes positively affect the agents' efforts. The intuition behind these results is that for $n < k$ the designer should avoid any combination of prizes that may yield an outcome of n different winners, each of which wins a positive prize, since then the group of agents may sign an agreement such that each agent is the only one to exert a positive effort in a different sub-contest. As we can see in the following example, when $n > k$, the prize for one win may have the highest marginal effect on the agents' symmetric effort.

Example 2 *Assume a simultaneous three-dimensional contest with five symmetric agents. By (18), the maximization problem of agent 1 is*

$$\begin{aligned}
&\max_{x_1^1, x_1^2, x_1^3, w_1} w(3)\frac{x_1^1}{\sum_{i=1}^5 x_i^1}\frac{x_1^2}{\sum_{i=1}^5 x_i^2}\frac{x_1^3}{\sum_{i=1}^5 x_i^3} \\
&+w(2)\frac{x_1^1}{\sum_{i=1}^5 x_i^1}\frac{x_1^2}{\sum_{i=1}^5 x_i^2}(1-\frac{x_1^3}{\sum_{i=1}^5 x_i^3}) + w(2)\frac{x_1^1}{\sum_{i=1}^5 x_i^1}\frac{x_1^3}{\sum_{i=1}^5 x_i^2}(1-\frac{x_1^2}{\sum_{i=1}^5 x_i^2}) \\
&+w(2)\frac{x_1^2}{\sum_{i=1}^5 x_i^2}\frac{x_1^3}{\sum_{i=1}^5 x_i^3}(1-\frac{x_1^1}{\sum_{i=1}^5 x_i^1}) \\
&+w(1)\frac{x_1^1}{\sum_{i=1}^5 x_i^1}(1-\frac{x_1^2}{\sum_{i=1}^5 x_i^2})(1-\frac{x_1^3}{\sum_{i=1}^5 x_i^3}) \\
&+w(1)\frac{x_1^2}{\sum_{i=1}^5 x_i^2}(1-\frac{x_1^1}{\sum_{i=1}^5 x_i^1})(1-\frac{x_1^3}{\sum_{i=1}^5 x_i^3}) \\
&+w(1)\frac{x_1^3}{\sum_{i=1}^5 x_i^3}(1-\frac{x_1^1}{\sum_{i=1}^5 x_i^1})(1-\frac{x_1^2}{\sum_{i=1}^5 x_i^2}) \\
&-x_1^1 - x_1^2 - x_1^3
\end{aligned}$$

The FOC (the derivative according to x_1^1) is

$$\begin{aligned}
& w(3)\frac{\sum_{i=2}^5 x_i^1}{(\sum_{i=1}^5 x_i^1)^2}\frac{x_1^2}{\sum_{i=1}^5 x_i^2}\frac{x_1^3}{\sum_{i=1}^5 x_i^3} \\
& +p(2)\frac{\sum_{i=2}^5 x_i^1}{(\sum_{i=1}^5 x_i^1)^2}\frac{x_1^2}{\sum_{i=1}^5 x_i^2}(1-\frac{x_1^3}{\sum_{i=1}^5 x_i^3})+p(2)\frac{\sum_{i=2}^5 x_i^1}{(\sum_{i=1}^5 x_i^1)^2}\frac{x_1^3}{\sum_{i=1}^5 x_i^2}(1-\frac{x_1^2}{\sum_{i=1}^5 x_i^2}) \\
& -p(2)\frac{x_1^2}{\sum_{i=1}^5 x_i^2}\frac{x_1^3}{\sum_{i=1}^5 x_i^3}\frac{\sum_{i=2}^5 x_i^1}{(\sum_{i=1}^5 x_i^1)} \\
& +p(1)\frac{\sum_{i=2}^5 x_i^1}{(\sum_{i=1}^5 x_i^1)^2}(1-\frac{x_1^2}{\sum_{i=1}^5 x_i^2})(1-\frac{x_1^3}{\sum_{i=1}^5 x_i^3})-p(1)\frac{x_1^2}{\sum_{i=1}^5 x_i^2}\frac{\sum_{i=2}^5 x_i^1}{(\sum_{i=1}^5 x_i^1)^2}(1-\frac{x_1^2}{\sum_{i=1}^5 x_i^2}) \\
& -p(1)\frac{x_1^3}{\sum_{i=1}^5 x_i^3}\frac{\sum_{i=2}^5 x_i^1}{(\sum_{i=1}^5 x_i^1)^2}(1-\frac{x_1^2}{\sum_{i=1}^5 x_i^2}) \\
= {} & 1
\end{aligned}$$

By symmetry of the sub-contests $x_i^j = x_i, j = 1, 2, 3$. Likewise, by symmetry of the agents $x = x_i$, $i = 1, ..., 5$. Thus, we obtain that

$$\begin{aligned}
& w(3)\frac{4x}{25x^2}\frac{1}{25}+w(2)(\frac{4x}{25x^2}(2\frac{4}{25}-\frac{1}{25}) \\
& +w(1)(\frac{4x}{25x^2}(\frac{16}{25}-2\frac{4}{25}) \\
= {} & 1
\end{aligned}$$

This implies that the symmetric equilibrium effort is

$$x = x^1 = x^2 = x^3 = w(3)\frac{4}{(25)^2}+w(2)\frac{28}{(25)^2}+w(1)\frac{32}{(25)^2}$$

This example shows that the marginal effect of the prize for winning a single sub-contest on the agents' symmetric equilibrium effort might be larger than the marginal effect of the other prizes, while the marginal effect of the prize for winning all the sub-contests on the symmetric equilibrium effort is the smallest one.

6 Conclusion

We demonstrated that in multi-dimensional contests, for a designer who wishes to increase the agents' efforts, the prize sum is not necessarily awarded for any outcome. In other words, we show that for some outcomes, only part of the entire prize should be awarded. We also show that in simultaneous multi-dimensional contests if the number of agents is smaller than or equal to the number of sub-contests, prizes for winning a small number of wins might be ineffective, namely, these prizes will not positively affect the agents' efforts. On the other hand, when the number of agents is larger than the number of sub-contests, there are no ineffective prizes, that is, prizes that do not positively affect the agents' efforts. We also show that in sequential multi-dimensional contests, even if the number of agents is larger than the number of sub-contests there are outcomes for which not all the entire prize sum has to be allocated. The reason is that with one prize for winning, the agents compete in the first stage only, while by awarding prizes for winning in other stages, the competition becomes intensive also in these stages. However, if competition in one stage yields a higher total effort than in multi-stage contests, the designer should allocate only one prize for the winner of all the sub-contests. In sum, we can conclude that not every prize, even if it is costless for the designer, should be awarded in multi-dimensional contests, since there are prizes that do not positively affect the agents' efforts.

7 Appendix

7.1 Proof of Proposition 1

The first-order conditions (FOC) of agent 1's maximization problem (1) are

$$\begin{aligned} & \frac{\sum_{i=1}^n x_i^A - x_1^A}{(\sum_{i=1}^n x_i^A)^2}\frac{x_1^B}{\sum_{i=1}^n x_i^B} + \alpha\frac{\sum_{i=1}^n x_i^A - x_1^A}{(\sum_{i=1}^n x_i^A)^2}(1-\frac{x_1^B}{\sum_{i=1}^n x_i^B}) - \beta\frac{\sum_{i=1}^n x_i^A - x_1^A}{(\sum_{i=1}^n x_i^A)^2}\frac{x_1^B}{\sum_{i=1}^n x_i^B} \\ = & \frac{\sum_{i=1}^n x_i^A - x_1^A}{(\sum_{i=1}^n x_i^A)^2\sum_{i=1}^n x_i^B}(x_1^B(1-\beta)+\alpha(\sum_{i=1}^n x_i^B - x_1^B) = 1, \end{aligned} \tag{20}$$

and

$$\begin{aligned} & \frac{x_1^A}{\sum_{i=1}^n x_i^A}\frac{\sum_{i=1}^n x_i^B - x_1^B}{(\sum_{i=1}^n x_i^B)^2} - \alpha\frac{x_1^A}{\sum_{i=1}^n x_i^A}\frac{\sum_{i=1}^n x_i^B - x_1^B}{(\sum_{i=1}^n x_i^B)^2} + \beta(1-\frac{x_1^A}{\sum_{i=1}^n x_i^A})\frac{\sum_{i=1}^n x_i^B - x_1^B}{(\sum_{i=1}^n x_i^B)^2} \\ = & \frac{\sum_{i=1}^n x_i^B - x_1^B}{\sum_{i=1}^n x_i^A(\sum_{i=1}^n x_i^B)^2}(x_1^A(1-\alpha)+\beta(\sum_{i=1}^n x_i^A - x_1^A)) = 1. \end{aligned} \tag{21}$$

By symmetry of the agents $x^A = x_i^A, i = 1, ..., n$ and $x^B = x_i^B, i = 1, ..., n$. Then, by (20) and (21), the agents' symmetric equilibrium efforts are

$$\begin{aligned} x^A &= \frac{(n-1)}{n^3}((1-\beta) + (n-1)\alpha) \\ x^B &= \frac{(n-1)}{n^3}((1-\alpha) + (n-1)\beta). \end{aligned}$$

Q.E.D.

7.2 Proof of Proposition 3

The FOC of the agents' maximization problems in sub-contest B (6) and (7) are

$$\begin{aligned} \frac{(\sum_{i=1}^n x_i^B - x_1^B)(1-\alpha)}{(\sum_{i=1}^n x_i^B)^2} &= 1 \\ \frac{\beta(\sum_{i=1}^n x_i^B - x_j^B)}{(\sum_{i=1}^n x_i)^2} &= 1\ . \end{aligned}$$

By symmetry, $x^B = x_i^B, i = 2, ..., n$. Then,

$$\begin{aligned} \frac{(\sum_{i=1}^n x_i^B - x_1^B)(1-\alpha)}{(\sum_{i=1}^n x_i^B)^2} &= \frac{(n-1)x^B(1-\alpha)}{((n-1)x^B + x_1^B)^2} = 1 \\ \frac{\beta(\sum_{i=1}^n x_i^B - x_j^B)}{(\sum_{i=1}^n x_i)^2} &= \frac{\beta((n-2)x^B + x_1^B)}{((n-1)x^B + x_1^B)^2} = 1\ . \end{aligned} \tag{22}$$

If we divide both FOC by each other we obtain

$$(n-1)x^B(1-\alpha) = \beta((n-2)x^B + x_1^B),$$

or, alternatively,

$$x_1^B = x^B \frac{(n-1)(1-\alpha) - (n-2)\beta}{\beta}. \tag{23}$$

Then, insert (23) into (22) and obtain the equilibrium efforts of the agents in the second stage (sub-contest B),

$$\begin{aligned} x_1^B &= \beta(1-\alpha)\frac{n-1}{(n+\alpha+\beta-n\alpha-1)^2}(n+\alpha+2\beta-n\alpha-n\beta-1) \\ x^B &= x_j^B = \beta^2(1-\alpha)\frac{n-1}{(n+\alpha+\beta-n\alpha-1)^2}\ , j = 2, ..., n. \end{aligned}$$

For $n = 2$, we have

$$x_1^B = \beta(1-\alpha)^2 \frac{1}{(1+\beta-\alpha)^2}$$
$$x^B = \beta^2 (1-\alpha) \frac{1}{(1+\beta-\alpha)^2}.$$

Q.E.D.

7.3 Proof of Proposition 4

The FOC of the agents' maximization problem (12) in the first stage (sub-contest A) is

$$(U_{1\,\text{seq}} - U_{\text{seq}}) \frac{\sum_{i=1}^{n} x_i^A - x_1^A}{(\sum_{i=1}^{n} x_i^A)^2} = 1.$$

By symmetry, $x^A = x_1^A = x_2^A =, ..., x_n^A$, and then the symmetric equilibrium effort in the first stage is

$$x^A = (n-1)\frac{U_{1\,\text{seq}} - U_{\text{seq}}}{n^2} =$$
$$\frac{1}{n^2}\frac{n-1}{(n+\alpha+\beta-n\alpha-1)^2}\left(\begin{array}{c} -2n^2\alpha^2\beta + n^2\alpha^2 - n^2\alpha\beta^2 + 4n^2\alpha\beta - 2n^2\alpha + n^2\beta^2 - 2n^2\beta + \\ n^2 + 4n\alpha^2\beta - 2n\alpha^2 + 4n\alpha\beta^2 - 10n\alpha\beta + 4n\alpha - 4n\beta^2 \\ +6n\beta - 2n - 2\alpha^2\beta + \alpha^2 - 3\alpha\beta^2 + 6\alpha\beta - 2\alpha + 4\beta^2 - 4\beta + 1 - \beta^3 \end{array}\right).$$

For $n = 2$, we have

$$x^A = x_1^A = x_2^A = \frac{1}{4(\beta-\alpha+1)^2}(-2\alpha^2\beta + \alpha^2 + \alpha\beta^2 + 2\alpha\beta - 2\alpha + 1 - \beta^3).$$

Q.E.D.

7.4 Proof of Proposition 5

By (16), the extreme points of TE_{seq} are obtained by

$$\frac{dTE_{\text{seq}}}{d\alpha} = \frac{1}{2}\frac{\beta^2}{(\beta-\alpha+1)^3}(1-\alpha-3\beta) = 0$$
$$\frac{dTE_{\text{seq}}}{d\beta} = \frac{1}{2}\frac{\beta}{(\beta-\alpha+1)^3}\left(2 + 2\alpha^2 + 3\alpha\beta - 4\alpha - \beta^2 - 3\beta\right) = 0.$$

A solution of these equations is

$$\alpha^* = 1, \beta^* = 0$$

Then, the total effort $TE_{\text{seq}}(\alpha^* = 1, \beta^* = 0)$ is not defined. By L'hopital rule we obtain that

$$\lim_{\substack{\alpha \to 1 \\ \beta = 0}} TE_{\text{seq}} = \frac{2a-2}{-4(-a+1)} = 0.5.$$

Note that when $\alpha = 1$ and $\beta = 0$ there exists

$$\begin{aligned}
\frac{d^2 TE_{\text{seq}}}{d\alpha^2} &= -\frac{\beta^2}{(\beta-\alpha+1)^4}(\alpha+5\beta-1) > 0 \\
\frac{d^2 TE_{\text{seq}}}{d\beta^2} &= (\frac{1}{2}\frac{\beta}{(\beta-\alpha+1)^3}\left(2+2\alpha^2+3\alpha\beta-4\alpha-\beta^2-3\beta\right)) = -\frac{(\alpha-1)^2}{(\beta-\alpha+1)^4}(\alpha+5\beta-1) > 0
\end{aligned}$$

and

$$\frac{d^2 TE_{\text{seq}}}{d\alpha d\beta} = 0.$$

Now, suppose that $\alpha = \beta$. Then, by (16), we have

$$TE_{\text{seq}} = -\alpha^3 + \frac{1}{2}\alpha^2 + \frac{1}{2}.$$

The optimal α is obtained by

$$\frac{dTE_{\text{seq}}}{d\alpha} = -3\alpha^2 + \alpha = 0.$$

Thus, $\alpha = \frac{1}{3}$ yields the highest total effort which is

$$TE_{\text{seq}} = -(\frac{1}{3})^3 + \frac{1}{2}(\frac{1}{3})^2 + \frac{1}{2} = 0.518.$$

In other words, if each agent has one win only, then prizes are not awarded with a probability of $\frac{1}{3}$. $Q.E.D.$

7.5 Proof of Proposition 6

By (4) and (15) we obtain that

$$TE_{\text{seq}}(\alpha = \beta = 0) = TE_{\text{seq}}(\alpha = \beta = 0.5) = TE_{sim}(\alpha = \beta = 0.5) = \frac{n-1}{n}.$$

By (4), we have

$$\frac{dTE_{sim}}{d\alpha}(\alpha=\beta)=\frac{2(n-1)(n-2)}{n^2}>0.$$

Thus,

$$\max_{\alpha} TE_{sim}(\alpha)=TE_{sim}(0.5)=\frac{n-1}{n}.$$

To end this proof we need to show that

$$\max_{\alpha} TE_{\text{seq}}(\alpha)=TE_{\text{seq}}(0)=TE_{\text{seq}}(0.5)=\frac{n-1}{n}.$$

By (15), we have

$$\frac{dTE_{\text{seq}}}{d\alpha}=-\frac{1}{n}\frac{n-1}{(n+2\alpha-n\alpha-1)^3}\begin{pmatrix} -2n^3\alpha^3+6n^3\alpha^2-5n^3\alpha+n^3+10n^2\alpha^3-24n^2\alpha^2+19n^2\alpha-4n^2 \\ -18n\alpha^3+36n\alpha^2-24n\alpha+5n+12\alpha^3-18\alpha^2+10\alpha-2 \end{pmatrix}.$$

In order to show that TE_{seq} obtains its maximum in one of the end points, either 0 or 0.5, it is sufficient to show that $\frac{dTE_{\text{seq}}}{d\alpha}$ monotonically increases. Note that

$$\frac{d^2TE_{\text{seq}}}{d\alpha^2}=\frac{2}{n}\frac{(n-1)^2}{(n+2\alpha-n\alpha-1)^4}(n^3(1-\alpha)-2n^2+1-2\alpha).$$

Then, it can be verified that $\frac{d^2TE_{\text{seq}}}{d\alpha^2}>0$ for all $0\le\alpha\le 0.5$ and $n\ge 3$. *Q.E.D.*

7.6 Proof of Proposition 7

By symmetry of the sub-contests $x=x_i^j$ for all $j=1,2,...,k$, the FOC of the maximization problem (18) is

$$\begin{aligned}
& w(k)\frac{(n-1)x}{n^2x^2}(\frac{1}{n})^{k-1} \\
& +w(k-1)\frac{(n-1)x}{n^2x^2}(k-1)((\frac{1}{n})^{k-2}(1-\frac{1}{n})-(\frac{1}{n})^{k-1})) \\
& +w(k-2)\frac{(n-1)x}{n^2x^2}(\frac{(k-1)!}{(k-3)!2!})((\frac{1}{n})^{k-3}(1-\frac{1}{n})^2-\frac{(k-1)!}{(k-2)!1!}(\frac{1}{n})^{k-2}(1-\frac{1}{n})) \\
& +w(k-3)\frac{(n-1)x}{n^2x^2}(\frac{(k-1)!}{(k-4)!3!})(\frac{1}{n})^{k-4}(1-\frac{1}{n})^3-\frac{(k-1)!}{(k-3)!2!}(\frac{1}{n})^{k-3}(1-\frac{1}{n})^2) \\
& +....... \\
& +w(2)\frac{(n-1)x}{n^2x^2}(\frac{(k-1)!}{1!(k-2)!}(\frac{1}{n})(\frac{n-1}{n})^{k-2}-\frac{(k-1)!}{2!(k-3)!}(\frac{1}{n})^2(1-\frac{1}{n})^{k-3}) \\
& +w(1)(\frac{(n-1)x}{n^2x^2})((\frac{n-1}{n})^{k-1}-(k-1)\frac{1}{n}(1-\frac{1}{n})^{k-2}) \\
= \; & 1.
\end{aligned}$$

This is equivalent to

$$\frac{(n-1)x}{n^2x^2}(w(k)(\frac{1}{n})^{k-1}+\sum_{i=1}^{k-1}w(k-i)(\begin{pmatrix} k-1 \\ i \end{pmatrix})(\frac{1}{n})^{k-i-1}(1-\frac{1}{n})^{i}-(\begin{pmatrix} k-1 \\ i-1 \end{pmatrix}(\frac{1}{n})^{k-i}(1-\frac{1}{n})^{i-1})=1.$$

Thus, the symmetric equilibrium effort is

$$x=\frac{n-1}{n^{k+1}}(w(k)+\sum_{i=1}^{k-1}w(k-i)(n-1)^{i-1}\left((n-1)\begin{pmatrix} k-1 \\ i \end{pmatrix}-\begin{pmatrix} k-1 \\ i-1 \end{pmatrix}\right)).$$

Q.E.D.

7.7 Proof of Proposition 8

By (19), we can write the symmetric equilibrium effort as follows:

$$x=\frac{n-1}{n^{k+1}}(w(k)+\sum_{i=1}^{k-1}w(k-i)(n-1)^{i-1}\Delta_i(k,n),$$

where

$$\Delta_i(k,n)=(n-1)\begin{pmatrix} k-1 \\ i \end{pmatrix}-\begin{pmatrix} k-1 \\ i-1 \end{pmatrix}$$

Thus, $\frac{dx}{dp(k)}>0$. And, $\frac{dx}{dp(k-i)}>0$, $i=1,...,k-1$ iff $\Delta_i(k,n)>0$. This is equivalent to

$$\begin{aligned}\Delta_i(k,n) &= \frac{(n-1)(k-1)!}{(k-1-i)!i!}-\frac{(k-1)!}{(k-i)!(i-1)!} \\ &= \frac{(k-1)!}{(k-1-i)!(i-1)!}(\frac{n-1}{i}-\frac{1}{(k-i)})>0\end{aligned}$$

Suppose that $n=k$, then we have $\Delta_i(k,k)>0$ iff $(k-1)(k-i)-i>0$. The last inequality satisfies iff $i<k-1$.Thus, $\Delta_i(k,k)>0$ for all $i<k-1$ and is equal to zero for $i=k-1$. In other words, a prize for winning in a single sub-contest does not affect the agents' equilibrium effort.

Note that for $n\neq k$ we have $\Delta_i(k,n)>0$ iff $(n-1)(k-i)-i>0$. The last inequality holds iff $i<\frac{(n-1)k}{n}$. Since for $n>k$ we have $i<\frac{(n-1)k}{n}$ for all $i\leq k-1$. Thus, we obtain that all the prizes, even for winning a single sub-contest, positively affect the agents 'equilibrium effort.

On the other hand, for $n < k$, since $k - 1 > \frac{(n-1)k}{k}$, we have $\Delta_{k-1}(k, n) < 0$ which implies that the prize for winning a single sub-contest negatively affects the agents' equilibrium strategy. In that case, if $\widetilde{i} > \frac{(n-1)k}{n}$, then $\Delta_{\widetilde{i}}(k, n) < 0$ which implies that all the prizes for $k - \widetilde{i}$ wins or less, negatively affect the agents' equilibrium efforts, and therefore should not be awarded if the goal is to increase the agents' efforts. *Q.E.D.*

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