SUBSIDY AND TAXATION IN ALL-PAY AUCTIONS UNDER INCOMPLETE INFORMATION

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Subsidy and Taxation in All-Pay Auctions under Incomplete Information

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Abstract

We study all-pay auctions under incomplete information with n contestants who have non-linear cost functions. The designer may award two kinds of subsidy (taxation): one that decreases (increases) each contestant's marginal cost of effort and another that increases (decreases) each contestant's value of winning. The designer's expected payoff is the contestants' expected total effort minus the cost of subsidy or, alternatively, plus the tax payment. We show that when the resource of subsidy (the marginal taxation rate) is relatively small and the cost function is concave (convex), the designer's expected payoff in all-pay auctions with both kinds of a subsidy (taxation) is higher than in the same contest without any subsidy (taxation). We then compare both kinds of subsidy and demonstrate that if the resource of subsidy is relatively small and the cost functions are concave (convex), the cost subsidy is better than the prize subsidy for the designer who wishes to maximize his expected payoff.

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1 Introduction

Subsidy and taxation are well common economic policies (see, for example, Sav 2004, Zuniga Vicente et al. 2014, and Bisceglia 2020). Lichtenberg (1990), claims that the U.S. Department of Defense (DoD) encourages private military R&D investment not only by establishing prizes, but also by subsidizing expenditures (the costs of making efforts) dedicated toward winning the prize. In that case, he asks why does the government provide a subsidy for private military R&D, in addition to establishing prizes for innovation? Here we try to provide some answers to this question by focusing on the potential of using economic policies of subsidy and taxation in contests. Our purpose is to show that different forms of subsidy or taxation might be useful for a contest designer who wishes to maximize the contestants' efforts (outputs).

We are not the first to deal with the role of subsidy/taxation in contests, but in contrast to the current literature (see, for example, Glazer and Konrad 1999, Konrad 2000, Person and Sandmo 2005, Fu et al. 2012, Mealem and Nitzan 2014, Carpenter et al. 2016, and Thomas and Wang 2017) who study the role of subsidy/taxation in environments under complete information, we study these policies in all-pay auctions under incomplete information about the contestants' types. One of the differences of using subsidy/taxation in environments under complete and incomplete information is that while in an environment under complete information the designer can apply a different subsidy/tax for each contestant according to his type (see, for example, Nitzan and Mealem 2014), in an environment under incomplete information the contestants' types are ex-ante identical where each contestant knows his type (which is private information), and therefore the designer **who does not know the contestants' types,** has to apply a uniform policy of subsidy/taxation for all the contestants without the ability to discriminate among them.

We study the all-pay auction (contest) with $n \geq 2$ contestants under incomplete information and non-linear cost functions.[1] In other words, our contestants are not risk-neutral and, in particular, the revenue equivalence theorem (see Myerson 1981, and Riley and Samuelson 1981) does not hold in our environment. In such a case, the analysis of the optimal all-pay auction is complex and is generally unknown.

We first consider a cost subsidy where the designer has a monetary resource that can be used to subsidize all the contestants by decreasing their marginal costs of effort.[2] We can find several examples in the literature

[1] We study the same model of Moldovanu and Sela (2001, 2006), but with one prize only.

[2] This form of cost subsidy is studied by Glazer and Konrad (1999), and Thomas and Wang (2017) in a rent-seeking model

for such a kind of subsidy. For example, Thomas and Wang (2017) describe a class in which students exert efforts to achieve higher degree classifications, and their teacher offers marginal help to the weaker students in order to improve their performance and as such to maximize the average (total) performance of all the students. Lichtenberg (1990) claims that the U.S. Department of Defense frequently provides "implicit subsidies" to firms to help them win its design competitions. He then empirically shows that financial subsidies substantially improve the productivity of private military R&D. **Furthermore, Fu at al. (2012) show that in R&D contests, prizes and subsidies are complement and not substitute each other.** Miller (2009) argues that subsidy of teams in the four major sports leagues in the United States (football, baseball, hockey and basketball) generally comes from subsidizing the amenities in the teams' stadiums. This subsidy lowers the marginal cost of providing them to fans, driving ticket prices lower, and as such increases the teams' profits.

In our case of a cost subsidy, we assume that the contest designer may face a budget constraint on the amount that can be used as a subsidy. For instance, the firm may have a fixed amount of money available for providing subsidies, or the teacher may have a fixed amount of time for tutoring students. We also assume that the designer's expected payoff is equal to the contestants' expected total effort minus the cost of the subsidy. We show that in the all-pay auction if the resource of subsidy is relatively small, then, if the contestants' cost functions are concave (convex), the designer's expected payoff is larger (smaller) than in the same contest without any subsidy.

Similarly, we consider a cost taxation where the designer can tax all the contestants by increasing their marginal costs of effort. The literature provides several examples of such a kind of taxation. For example, Runkel (2006) addresses the prevalence of cost-raising policies (equivalent to taxation in our terms) by suggesting that "competitive balance" may be part of a contest designer's objective function together with maximization of total effort. He then claims that uniformly increasing effort costs becomes optimal. Ritz (2008) shows that in a contest model with participation fees, a policy which uniformly increases the contestants' effort costs can lead to an increase in total effort since it encourages weaker players (who otherwise would have stayed out) to participate in the contest. Paradoxically, a contest designer whose only

under complete information. Fu et al. (2012) call this form of subsidy an "efficiency-enhancing subsidy."

objective is to maximize total effort may thus wish to make rent-seeking "more difficult," namely, impose higher marginal cost functions.

We show that taxation by increasing the contestants' marginal costs can be useful for the contest designer who wishes to maximize the contestants' total effort even when contestants are ex-ante symmetric. In particular, we show that in the all-pay auction, if the taxation rate is relatively small and if the contestants' cost functions are convex (concave), the designer's expected payoff is larger (smaller) than in the same contest without any cost taxation. It is worth noting that if we combine the above results for cost subsidy and cost taxation, we obtain that if the contestants' cost functions are concave, the designer should apply a cost subsidy, while if they are convex, a cost taxation should be applied.

We then consider a different form of subsidy that will be referred to as a prize subsidy. One example of such a subsidy would be medals bonuses which are given in many countries to Olympic athletes who won medals. In the case of a prize subsidy, the designer can increase the winner's value of winning by awarding an extra prize. Then, the designer's expected payoff is the contestants' expected total effort (output) minus the cost of the prize subsidy. **Additionally, we can find this form of subsidy in procurement contracts in which a buyer who wishes to procure an exogenously given project** hires a contractor to perform the work. **The buyer** may provide incentives to the contractor such as an **additional compensation** (subsidy) which may increase the buyer's payoff by reducing ex post transaction costs due to costly renegotiation **(see Bajary and Tadelis, 2001, and Tadelis, 2012).** We show that in all-pay auctions, if the contestants' cost functions are strictly concave, there exists a prize subsidy such that the designer's expected payoff is larger for the same contest without any prize subsidy. Similarly, we consider a prize taxation such that the designer can decrease the winner's value of winning by imposing a tax only on the winner. For example, the U.S. tax system taxes prizes and awards, even Olympic athletes, if the recipient makes $ 1M a year or more. Then, the $37,500 monetary prize that accompanies a gold medal will also be taxed. In our model with a prize taxation, the designer's expected payoff is the contestants' expected total effort plus the tax of the winner. We show that if the cost functions are strictly convex, then there exists a positive tax such that the designer's expected payoff is larger for the same contest without any prize taxation.

The intuition behind the above results is that the efficiency of a subsidy or a taxation for the designer depends on whether the marginal increase (decrease) of the contestants' efforts is larger (smaller) than the marginal costs of the subsidy (taxation). Since the contestants' expected efforts and the cost of subsidy and taxation are forms of the inverse cost function, the results depend on its curvature. Furthermore, since the amount of either a subsidy or a taxation is relatively smaller than the expected contestant's effort, the fact that the curvature of the cost function is increasing or decreasing plays a key role and yields the same results whether or not the subsidy or the taxation are efficient for enhancing the designer's expected payoff.

Last, we compare which kind of subsidy is better for the designer. We demonstrate that if the resource of subsidy is sufficiently small then in an all-pay auction with a concave (convex) cost function, the designer's expected payoff is larger (smaller) with a cost subsidy than with a prize subsidy. Given our previous results according to which both forms of a subsidy are efficient for enhancing the total effort when the contestants' cost functions are concave, we can conclude that for sufficiently low levels of a subsidy, the cost subsidy is better than the prize subsidy. However, if the optimal value of the resource of subsidy is relatively large this comparison will not yield meaningful results.

As mentioned above, the optimal all-pay auction under incomplete information in which contestants have non-linear cost functions has not yet been conclusively analyzed. We do not claim that using a subsidy and/or a taxation are optimal economic policies to maximize the contestants' efforts. Indeed, the literature on contests mentions several other ways to enhance the contestants' efforts. Some examples include limiting the number of contestants by setting a minimum effort level (see Taylor 1995, Fullerton and McAfee 1999, Casson et al. 2010, Fu et al. 2015, and Kirkegaard 2022), imposing a maximum effort level (see Che and Gale 1998, Gavious et al. 2003, Megidish and Sela 2014, and Olszewski and Siegel 2019), allocating several prizes and punishments (see, Lazear and Rosen 1981, Green and Stokey 1983, Nalebuff and Stiglitz 1983, Moldovanu and Sela, 2001, 2006, Moldovanu et al. 2012, Olszewski and Siegel 2016, and Sela 2020), allocating head-starts or handicaps (see Kirkegaard 2012, Franke et al. 2013, Segev and Sela 2014, Drugov and Ryvkin 2017, and Fu and Wu 2020), or reimbursing some of the contestants' cost of efforts (see Cohen and Sela 2005, Matros 2012, Minchuk 2018 and Minchuk and Sela 2020). We do claim, however, that a subsidy or a taxation, with or without some of the above well-known methods could be a basic component in the optimal all-pay auction under incomplete information and non-linear cost functions.

The rest of the paper is organized as follows. In Section 2 we analyze the all-pay auction with a cost subsidy as well as with a cost taxation, and in Section 3 we analyze the all-pay auction with a prize subsidy as well as with a prize taxation. In Section 4 we compare between a cost subsidy (taxation) and a prize subsidy (taxation). Section 5 concludes. The proofs appear in the Appendix.

2 A cost subsidy and a cost taxation

Consider $n \geq 2$ contestants who compete in an all-pay auction for a single prize. Contestant i's value of winning is v_i, $i = 1, .., n$, and is private information. The contestants' values are drawn independently of each other from the interval $[0, 1]$ according to the distribution function F which is common knowledge. We assume that F is continuously differentiable and that $f(x) = F'(x) > 0$ for all $0 \leq x \leq 1$. The contestant with the highest effort wins and all the contestants pay the cost of their efforts where an effort of x has a cost of $\gamma(x), \gamma' > 0, \gamma(0) = 0$ in monetary units. In other words, γ transfers x units of effort to $\gamma(x)$ monetary units. We denote $g = \gamma^{-1}$.

The designer who has incomplete information about the contestants' types, has also a monetary resource of θ that can be used to subsidize the contestants by decreasing their marginal costs of effort. In such a case, contestant i's cost of effort will be $\beta\gamma(x_i)$ where $1 - \beta$, $0 < \beta \leq 1$, is referred to as the marginal subsidy rate. Since the allocated subsidy is equal to the designer's monetary resource θ we have

$$\theta = n(1 - \beta)E(\gamma(x)), \tag{1}$$

where the LHS of (1) is the designer's monetary resource for subsidy, and the RHS of (1) is the expected change of total effort as a result of the cost subsidy. The designer's expected payoff in effort units is

$$R_{cs} = TE - E\left(g\left(n(1 - \beta)\gamma(x)\right)\right), \tag{2}$$

where TE is the contestants' expected total effort, and $E\left(g\left(n(1 - \beta)\gamma(x)\right)\right)$ is the cost of the designer's subsidy in effort units.

Alternatively, the designer can also tax the contestants by increasing their marginal costs of effort. In such a case, contestant i's cost of effort will be $\beta\gamma(x_i)$ where $\beta - 1$, $\beta > 1$ is referred to as the marginal taxation rate. Then, the designer imposes a tax rate of $\beta - 1$ on each effort unit of a contestant in which case, the designer's expected payoff in effort units is

$$R_{ct} = TE + E\left(g(n(\beta - 1)\gamma(x))\right), \tag{3}$$

where TE is the contestants' expected total effort, and $E\left(g(n(\beta - 1)\gamma(x))\right)$ is the designer's expected profit from taxation in effort units.

2.1 A cost subsidy

We first study the all-pay auction with a cost subsidy. If there is a symmetric monotonically increasing equilibrium effort function $x(v_i)$, the utility function of contestant $i, i = 1, .., n$, is

$$U(v_i) = v_i G(v_i) - \beta\gamma(x(v_i)), \tag{4}$$

where $G(v_i) = F^{n-1}(v_i)$ is the probability that the value v_i is the highest among all the n contestants, and the marginal subsidy rate satisfies $0 < \beta \leq 1$. The first order condition (FOC) of the maximization problem of contestant i's expected payoff given by (4) is

$$G'(v_i)v_i - \beta\left(\gamma(x(v_i))\right)' = 0.$$

Rearranging yields

$$\beta\gamma(x(v_i)) = \int_0^{v_i} sG'(s)ds + k.$$

Since $\gamma(x(0) = 0$, we have

$$\gamma(x(v_i)) = \frac{1}{\beta}\int_0^{v_i} sG'(s)sds.$$

Integrating by parts and rearranging yields the equilibrium effort of contestant i, $i = 1, 2, ..., n$ as follows:

$$x_{cs}(v_i) = g\left(\frac{1}{\beta}\left(v_i G(v_i) - \int_0^{v_i} G(s)ds\right)\right). \tag{5}$$

It can be easily verified that the above equilibrium effort is monotonically increasing.[3] Then, contestant i with value v_i has the the following expected payoff:

$$\begin{aligned} U(v_i) &= v_i G(v_i) - \beta\gamma(x) \\ &= \int_0^{v_i} G(s)ds, \end{aligned}$$

[3] Similarly as in Moldovanu et al. (2001, 2006), it is not difficult to show that this is the unique symmetric equilibrium.

which is exactly the contestant's expected payoff in the standard all-pay auction with linear cost functions (see Krishna 2010). Thus, we can conclude that the contestants are indifferent between having or not having a cost subsidy. We show below that the designer might have an incentive to apply a cost subsidy in all-pay auctions.

Since the allocated subsidy should be equal to the designer's resource θ, we have

$$\begin{aligned}\theta &= n(1-\beta)E(\gamma(x)) \\ &= n(1-\beta)\int_0^1 \frac{1}{\beta}\left(vG(v) - \int_0^v G(s)ds\right) f(v)dv = \frac{(1-\beta)}{\beta}R_{lin},\end{aligned}$$

This yields that that the marginal taxation rate is

$$\beta = \frac{R_{lin}}{\theta + R_{lin}}, \tag{6}$$

where $R_{lin} = n\int_0^1 \left(vG(v) - \int_0^v G(s)ds\right) f(v)dv$ is the designer's expected payoff in the standard all-pay auction with linear cost functions and without subsidy.

The designer's expected payoff in an all-pay auction with a cost subsidy will be denoted by R_{cs} and is equal to the contestants' expected total effort minus the cost of a subsidy in effort units.

$$\begin{aligned}R_{cs} &= n\int_0^1 g\left(\frac{1}{\beta}\left(vG(v) - \int_0^v G(s)ds\right)\right) f(v)dv - E\left(g\left(n(1-\beta)\gamma(x)\right)\right) \qquad (7)\\ &= n\int_0^1 g\left(\frac{1}{\beta}\left(vG(v) - \int_0^v G(s)ds\right)\right) f(v)dv - \int_0^1 g\left(\frac{n(1-\beta)}{\beta}\left(vG(v) - \int_0^v G(s)ds\right)\right) f(v)dv.\end{aligned}$$

Note that the first part of (7) is the contestants' expected total effort, while the second part is the subsidy paid by the designer in effort units. The following result demonstrates the conditions under which a cost subsidy is either profitable or not for the contest designer.

Proposition 1 *In the all-pay auction with a monetary resource of $\theta \leq \frac{R_{lin}}{n-1}$, if the cost function γ is concave (convex) then the designer's expected payoff is larger (smaller) than in the same contest without any subsidy.*

Proof. See Appendix. ■

We point out that in the case of linear cost functions, by the revenue equivalence theorem (RET), the designer's expected payoff is the same with and without a subsidy of any θ. However, when the cost functions are non-linear, the RET no longer holds. Then, by Proposition 1, if the cost function γ is concave, a relatively small subsidy increases the designer's expected payoff. The intuition behind this result is that for a subsidy to be profitable for the designer depends on whether the marginal increase in the contestant' expected total effort is larger than the marginal cost of a subsidy. The marginal increase of efforts and the marginal cost of a subsidy are both depend on $g'(x)$ which is the derivative of the inverse cost function g. Since g is convex such that $g'(x)$ is an increasing function, for low-type contestants whose expected efforts are smaller than the cost of the subsidy, the marginal increase of the expected total effort is smaller than the marginal increase of the cost of a subsidy. On the other hand, for high-type contestants whose expected efforts are larger than the cost of subsidy, the marginal increase of the expected total effort is larger than the marginal increase of the cost of a subsidy. Thus, we actually show that, independent of the distribution of the contestants' types, the effect of the high-type contestants is larger than that of the low-type contestants on the designer's expected payoff. Therefore, we can conclude that it is profitable for the designer to allocate a subsidy if the cost function γ is concave, and vice versa when it is convex.

By (7), the optimal subsidy is as follows:

Proposition 2 *In the all-pay auction, if the cost function γ is concave, the optimal marginal subsidy rate is $1 - \beta^* = \frac{1}{n}$ and the monetary resource is $\theta^* = \frac{R_{lin}}{n-1}$. Then, the optimal amount of subsidy per-contestant decreases.*

Proof. See Appendix. ■

By Propositions (1) and (2) we can see that the optimal monetary resource θ^* is equal to its upper bound.

2.2 A cost taxation

Now we consider the all-pay auction with a cost taxation. As in the previous case, if there is a symmetric monotonically increasing equilibrium effort function $x(v_i)$, the utility function of contestant $i, i = 1, .., n$, is

$$U(v_i) = v_i G(v_i) - \beta\gamma(x(v_i)) \tag{8}$$

where the cost taxation rate satisfies $\beta > 1$. We can see that the contestants' equilibrium efforts have the same form as with a cost subsidy except that β has different values. However, the designer's expected payoff in the all-pay auction with a cost taxation (denoted by R_{ct}) has a different form than in the all-pay auction with a cost subsidy and is given by

$$\begin{aligned} R_{ct} &= n\int_0^1 g\left(\frac{1}{\beta}\left(vG(v)-\int_0^v G(s)ds\right)\right)f(v)dv + E\left(g(n(\beta-1)\gamma(x))\right) \qquad (9) \\ &= n\int_0^1 g\left(\frac{1}{\beta}\left(vG(v)-\int_0^v G(s)ds\right)\right)f(v)dv + \int_0^1 g\left(\frac{n(\beta-1)}{\beta}\left(vG(v)-\int_0^v G(s)ds\right)\right)f(v)dv \end{aligned}$$

Note that the first part of (9) is the contestants' expected total effort, while the second part is the tax payment in effort units. The following result demonstrates the conditions under which taxation is either profitable or not for the contest designer.

Proposition 3 *In the all-pay auction with a cost taxation rate of* $0 < \beta \leq 1 + \frac{1}{n}$, *if the cost function* γ *is convex (concave), the designer's expected payoff is larger (smaller) than in the same contest without any cost taxation.*

Proof. See Appendix. ■

According to Proposition 3, if the cost taxation rate is sufficiently low and if the contestants' cost functions are convex, taxation will be profitable for the contest designer. The intuition behind this result is that the marginal decrease in the contestants' efforts is smaller than the marginal profit from taxation for high-type contestants, and the opposite for low-type contestants. Since the tax payment and the expected level of the equilibrium effort are functions of g' (which is a decreasing function) we obtain that by applying the taxation, for high-type contestants the marginal decrease of the expected total effort is smaller than the marginal increase of the tax payment, and the opposite for low-type contestants. Our result shows that, independent of the distribution of the contestants' types, the marginal decrease of the expected total effort is smaller than the marginal increase of the tax payment. Therefore, it is profitable for the designer to set a tax if the cost function γ is convex, but it is not profitable when the cost function is concave.

Proposition 3 indicates that a cost taxation might be profitable for the designer who wishes to maximize his expected payoff, but the value of the optimal cost taxation rate β^* could be relatively large. The optimal cost taxation is as follows

Proposition 4 *In the all-pay auction, if the cost function γ is convex, the optimal marginal taxation rate is $\beta^* = 1 + \frac{1}{n}$. Then, the marginal taxation rate decreases in the number of contestants.*

Proof. See Appendix. ■

By Propositions (3) and (4) we can see that the optimal cost taxation rate β^* is equal to its upper bound.

3 A prize subsidy and a prize taxation

We consider the same model of the all-pay auction as in the previous section, but now the designer can increase the winner's value of winning by awarding a prize subsidy (bonus) of $\theta > 0$. Then, his expected payoff is

$$R_{ps} = TE - g(\theta) \tag{10}$$

where TE is the contestants' expected total effort, and $g(\theta)$ is the cost of subsidy in effort units.

Alternatively, the designer can decrease the winner's value of winning by imposing a tax of $\theta > 0$. Then, his expected payoff is

$$R_{pt} = TE + g(\theta) \tag{11}$$

where TE is the contestants' expected total effort, and $g(\theta)$ is the tax payment in effort units. Notice that in contrast to the model with a cost subsidy (taxation) in which all the contestants are subsidized (taxed), in this model with prize subsidy (taxation) only the winner is subsidized (taxed).

3.1 A prize subsidy

We first study the all-pay auction with a prize subsidy in which the designer awards a positive subsidy of $\theta > 0$ for the winner of the contest. If there is a symmetric monotonically increasing equilibrium effort function $x(v_i) : [0, 1] \to [0, 1]$, the utility function of contestant $i, i = 1, .., n$, is

$$U_i(v_i) = (v_i + \theta) G(v_i) - \gamma(x_i). \tag{12}$$

By a similar analysis of the all-pay auction with a cost subsidy, we obtain that the equilibrium effort function is as follows

Proposition 5 *The equilibrium effort function in the all-pay auction with a prize subsidy of θ is*

$$x_{ps}(v) = g\left((v+\theta)G(v) - \int_0^v G(s)ds\right). \tag{13}$$

The designer's expected payoff (denoted by R_{ps}) is the contestants' expected total effort minus the cost of the prize subsidy $g(\theta)$ in effort units as follows:

$$R_{ps} = n\int_0^1 g\left((v+\theta)G(v) - \int_0^v G(s)ds\right)f(v)dv - g(\theta). \tag{14}$$

The following result demonstrates the conditions under which a prize subsidy increases the designer's expected payoff.

Proposition 6 *In the all-pay auction, if the cost function γ is strictly concave on $(0,1]$, then there exists a sufficiently small subsidy of $\theta > 0$ such that the designer's expected payoff is larger than in the same contest without any prize subsidy.*

Proof. See Appendix. ■

The intuition behind the result of Proposition 6 according to which prize subsidy is profitable for the contest designer when the cost function is concave is quite similar to the intuition for the result of Proposition 1 according to which the cost subsidy is profitable to the contest designer for every concave cost function.

By the proof of Proposition 6, if the prize subsidy is relatively small then it is profitable to the contest designer. In the following example, we show that the optimal prize subsidy is not necessarily small and its effect on the designer's expected payoff is significant.

Example 1 *Consider an all-pay auction with two contestants where each contestant's value is distributed according to $F(v) = v$. The cost function is concave and is given by $\gamma(x) = x^{0.5}$. Then, by (14), the designer's expected payoff is*

$$\begin{aligned} R_{ps} &= 2\int_0^1 \left((v+\theta)v - \int_0^v sds\right)^2 dv - (\theta)^2 \\ &= -\frac{1}{3}\theta^2 + \frac{1}{2}\theta + \frac{1}{10} \end{aligned}$$

Figure 1 describes the designer's expected payoff as a function oft the prize subsidy θ:

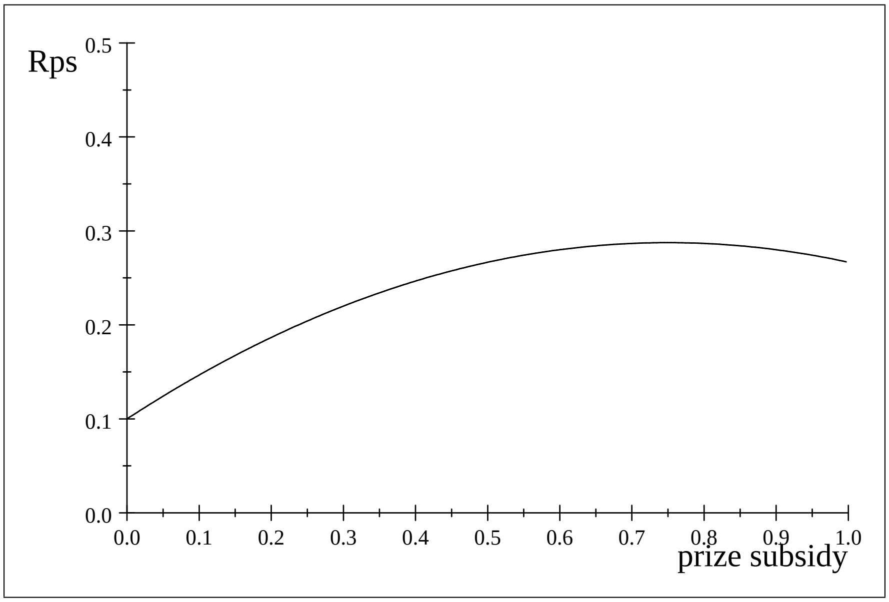

Figure 1: the designer's expected payoff as a function of the prize subsidy

In order to find the optimal prize subsidy, we obtain

$$\frac{dR_{ps}}{d\theta} = -\frac{2}{3}\theta + \frac{1}{2}$$

This implies (as we can see in Figure 1) that the optimal prize subsidy is

$$\theta^* = \frac{3}{4}$$

We can see that the prize subsidy makes the following change in the designer's expected payoff

$$R_{ps}(\theta = \frac{3}{4}) - R(\theta = 0) = 0.1875$$

Note that this difference in the designer's expected payoff is larger than his expected payoff when there is no subsidy ($\theta = 0$).

3.2 A prize taxation

Now we consider the all-pay auction with a prize taxation in which the winner has to pay a tax of $\theta > 0$. In that case, if the tax is larger than contestant i's type, $\theta > v_i$, he will stay out of the contest. Then, If there is a symmetric monotonically increasing equilibrium effort function $x(v_i) : [0, 1] \to [0, 1]$, the utility function of contestant $i, i = 1, .., n$, is

$$U_i(v_i) = (v_i - \theta)\, G(v_i) - \gamma(x_i). \tag{15}$$

Similar to the previous sections, the symmetric equilibrium effort function is

$$x_{pt}(v) = \begin{cases} 0 & 0 \leq v < \theta, \\ g\left((v-\theta)\,G(v) - \int_\theta^v G(s)ds\right) & \theta \leq v \leq 1 \ . \end{cases} \tag{16}$$

Then, the designer's expected payoff is

$$\begin{aligned} R_{pt} &= n\int_\theta^1 x_{pt}(v)f(v)dv + g(\theta)\Pr(\text{there is a winner}) = \\ &= n\int_\theta^1 g\left((v-\theta)\,G(v) - \int_\theta^v G(s)ds\right) f(v)dv + g(\theta)\left(1 - F^n(\theta)\right). \end{aligned} \tag{17}$$

The following result demonstrates the condition under which a prize taxation increases the designer's expected payoff.

Proposition 7 *In the all-pay auction, if the cost function γ is either linear or convex on $(0,1]$, then there exists a sufficiently small $\theta > 0$ such that the designer's expected payoff is larger than in the same contest without any prize taxation.*

Proof. See Appendix. ■

The intuition behind the result of Proposition 7 according to which a prize taxation is profitable for the designer when the cost function is convex is similar to the intuition for the result of Proposition 3 according to which the cost taxation is profitable for every convex cost function. Note that a prize taxation "serves" also as a reserve price and it is well known that a reserve price is a profitable tool to enhance the contestants' expected total effort.

In the following example, we show that even for a linear cost function the optimal prize subsidy is non-negligible and its effect on the designer's expected payoff is significant.

Example 2 *Consider an all-pay auction with two contestants where each contestant's value is distributed according to $F(v) = v$. The cost function is linear and is given by $\gamma(x) = x$. Then, by (17), the designer's expected payoff is*

$$\begin{aligned} R_{st} &= 2\int_\theta^1 \left((v-\theta)(v) - \int_\theta^v sds\right) dv + (\theta)\left(1-(\theta)^2\right) \\ &= -\frac{4}{3}\theta^3 + \theta^2 + \frac{1}{3}, \end{aligned}$$

Figure 2 describes the designer's expected payoff as a function of the taxation θ :

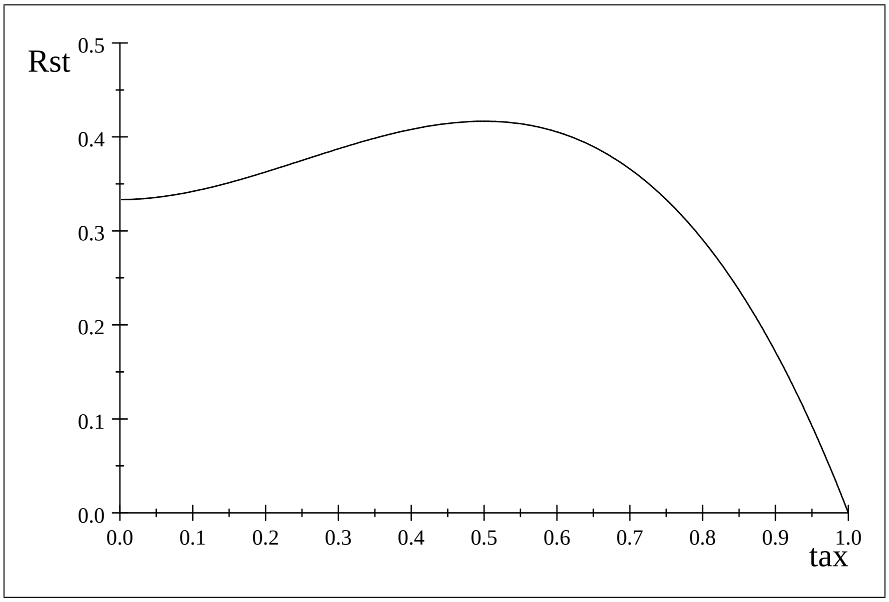

Figure 2: the designer's expected payoff as a function of the prize taxation

In order to find the optimal prize taxation, we obtain

$$\frac{dR_{ts}}{d\theta} = -2\theta\,(2\theta - 1)\,.$$

Thus (as we can see in Figure 2), the optimal prize taxation is

$$\theta^{*} = 0.5$$

We can see that the prize taxation makes the following change in the designer's expected payoff

$$R_{ts}(\theta = 0.5) - R(\theta = 0) = 0.083$$

Note that $R(\theta = 0) = \frac{1}{3}$ *such that the increase in the designer's expected payoff by the prize taxation is about 25%. For a convex cost function, such an increase will be even larger than for a linear cost function.*

4 A cost subsidy/taxation vs. a prize subsidy/taxation

So far we have shown that the designer who wishes to maximize his expected payoff can apply either cost (taxation) subsidy or prize subsidy (taxation). In the following example, we compare between a cost subsidy and a prize subsidy.

Example 3 *Consider an all-pay auction with two contestants where each contestant's value is distributed according to $F(v) = v$. The cost function is concave and is given by $\gamma(x) = x^{0.5}$, and $\beta = \frac{R_{lin}}{R_{lin}+\theta} = \frac{1}{3\theta+1}$. Then, by (14) and (17) we have*

$$R_{ps} - R_{cs} = -\frac{1}{3}\theta^2 + \frac{1}{2}\theta + \frac{1}{10} - \frac{(3\theta+1)^2}{10} + \frac{9\theta^2}{5}. \tag{18}$$

When the cost function is $\gamma(x) = x^{\frac{1}{3}}$, by (14) and (17) we have

$$R_{ps} - R_{cs} = \frac{3}{5}\theta^2 + \frac{1}{4}\theta + \frac{1}{28} - \frac{(3\theta+1)^2}{28} + \frac{47\theta^3}{14}. \tag{19}$$

Figure 3 describes for both concave cost functions, the difference $R_{ps} - R_{cs}$ as a function of the monetary resource θ:

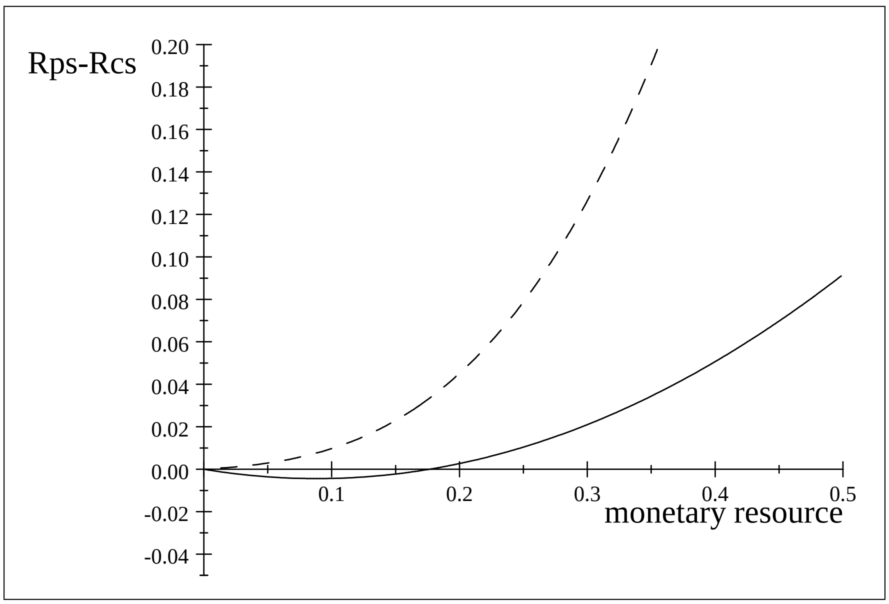

Figure 3: the difference of the designer's payoff with prize and cost subsidies as a function of the monetary resource

We can see for small (large) values of the monetary resource of θ, the designer's expected payoff with a cost (prize) subsidy is larger than in the same contest with a prize (cost) subsidy. When the cost function is $\gamma(x) = x^{0.5}$ (the solid curve) this results looks clear in Figure 3. When the cost function is $\gamma(x) = x^{\frac{1}{3}}$ (the DotDash curve) it is hard to see in Figure 3 that for small values of the monetary resource of θ, the designer's expected payoff with a cost subsidy is larger than in the same contest with a prize subsidy/. But, equation (19) confirms this result.

The next result establishes the findings in the last example.

Proposition 8 *In the all-pay auction, if the monetary resource of* θ *is sufficiently small, then if the cost function* γ *is concave (convex) the designer's expected payoff with a cost (prize) subsidy is larger than in the same contest with a prize (cost) subsidy.*

Proof. See Appendix ■

Since by Propositions 1 and 6, a subsidy increases the designer's expected payoff when the contestants' cost function is concave, Proposition 8 implies that a cost subsidy is better than a prize subsidy if the monetary resource is sufficiently small. The comparison between a cost taxation and a prize taxation is not clear since in an all-pay auction with a cost taxation there is no tax resource as with a prize subsidy. However, when the cost function is linear this comparison becomes clear since by Proposition 7, for linear cost functions, the designer's expected payoff with a prize taxation is higher than in the same contest without any taxation. On the other hand, for a cost taxation, the revenue equivalence theorem holds, such that the designer's expected payoff is the same as in that contest without any taxation. Thus we have

Corollary 4 *In the all-pay auction with a linear cost function, the designer's expected payoff is larger with a prize taxation than in the same contest with a cost taxation.*

The intuition behind this result is that in contrast to a cost taxation, a prize taxation acts also as an entry fee, and this implies a larger expected total effort in the case of a prize taxation as long as the tax is sufficiently small.

5 Conclusion

Lichtenberg (1988) and other researchers raised the question "why does the government provide a subsidy for private military R&D, in addition to establishing prizes for innovation." In order to answer this question we focused on all-pay auctions (contests) with n contestants who have private information about their values of winning and have non-linear cost functions. The optimal structure of such a contest is unknown to a designer who wishes to maximize the contestants' expected total effort. We suggest two forms of a subsidy and a taxation and show that they both make the contest more profitable. The first is a cost subsidy (taxation) that increases (decreases) all the contestants' marginal costs, and the second is a prize subsidy (taxation) that increases (decreases) the winner's value of winning. We showed that in the case of convex cost functions, a sufficiently small taxation of both forms is profitable to the designer, while in the case of concave cost functions, a subsidy of both forms will be profitable. The majority of the considered cost functions in the economics literature are convex, so according to our findings the designer should apply a taxation. On the other hand, for concave cost functions, the designer should apply a subsidy. In addition, we showed that even in the case of a linear cost function, taxation could be a good substitute to other well-known methods for enhancing the designer's expected payoff, and, in particular, the contestants' expected total effort. We also compared both forms of a subsidy and showed that if the monetary resource is sufficiently small and the contestants' cost functions are concave, then a cost subsidy is superior to a prize subsidy from the designer's point of view. However, since we have shown that the subsidy is optimal when the level of the resource of subsidy is relatively large, this comparison has limited significance.

It is worth noting that we showed that in a case of cost subsidy for low-type contestants, the marginal increase of the expected total effort is smaller than the marginal increase of the cost of a subsidy, but for high-type contestants, the marginal increase of the expected total effort is larger than the marginal increase of the cost of a subsidy. In other words, the cost subsidy significantly increases the high-type contestants, and therefore if the designer's goal is to maximize the contestants' highest effort instead of their total effort, a cost subidy is an efficient tool for him. For similar reasons, if the designer wishes to maximize the contestants' highest effort a prize subsidy is efficient as well.

As we mentioned, we show that depending on the form of the contestants' cost function, either a subsidy or taxation are efficient tools for increasing the contestants' total efforts. However, in order to maximize the contestants' total effort, the designer should use in addition other well-known tools in contest theory such as minimal effort constraints or entry fees that limit the number of contestants that participate in the contest.

6 Appendix

6.1 Proof of Proposition 1

If γ is concave and strictly increasing, its inverse function $g = \gamma^{-1}$ is convex. If $0 < \beta \leq 1$, there exists

$$g\left(vG(v) - \int_0^v G(s)ds\right) = g\left(\left(vG(v) - \int_0^v G(s)ds\right)\frac{1}{\beta}\beta\right) \leq g\left(\left(vG(v) - \int_0^v G(s)ds\right)\frac{1}{\beta}\right)\beta. \tag{20}$$

Hence, by (2),

$$\begin{aligned} R_{cs} &= n\int_0^1 g\left(\frac{1}{\beta}\left(vG(v) - \int_0^v G(s)ds\right)\right)f(v)dv - E\left(g\left(n(1-\beta)\gamma(x)\right)\right) \\ &= n\int_0^1 g\left(\frac{1}{\beta}\left(vG(v) - \int_0^v G(s)ds\right)\right)f(v)dv - \int_0^1 g\left(\frac{n(1-\beta)}{\beta}\left(vG(v) - \int_0^v G(s)ds\right)\right)f(v)dv. \end{aligned}$$

Since $\theta \leq \frac{R_{lin}}{n-1}$ where $R_{lin} = \int_0^1 n\left(vG(v) - \int_0^v G(s)ds\right)f(v)dv$ is the expected payoff in the all-pay auction with linear cost functions, by (6) we get $n(1-\beta) \leq 1$. Thus, by (20) we have

$$\begin{aligned} R_{cs} &\geq n\int_0^1 g\left(\frac{1}{\beta}\left(vG(v) - \int_0^v G(s)ds\right)\right)f(v)dv - n(1-\beta)\int_0^1 g\left(\frac{1}{\beta}\left(vG(v) - \int_0^v G(s)ds\right)\right)f(v)dv \\ &= n\int_0^1 \beta g\left(\frac{1}{\beta}\left(vG(v) - \int_0^v G(s)ds\right)\right)f(v)dv, \end{aligned}$$

which implies that

$$R_{cs} \geq n\int_0^1 \beta g\left(\frac{1}{\beta}\left(vG(v) - \int_0^v G(s)ds\right)\right)f(v)dv \geq n\int_0^1 g\left(vG(v) - \int_0^v G(s)ds\right)f(v)dv = R,$$

where R is the contestants 'expected total effort without a subsidy.

Similarly, if γ is convex and strictly increasing, its inverse function $g = \gamma^{-1}$ is concave, and then

$$g\left(vG(v) - \int_0^v G(s)ds\right) = g\left(\left(vG(v) - \int_0^v G(s)ds\right)\frac{1}{\beta}\beta\right) \geq g\left(\left(vG(v) - \int_0^v G(s)ds\right)\frac{1}{\beta}\right)\beta.$$

Likewise, by the same analysis for concave cost functions we obtain the opposite inequality

$$R_{cs} \leq n\int_0^1 g\left(vG(v) - \int_0^v G(s)ds\right)f(v)dv = R.$$

□

6.2 Proof of Proposition 2

The optimal resource of subsidy is determined by

$$\begin{aligned}
\frac{\partial R_{cs}}{\partial \theta} &= -n\frac{1}{\beta^2}\frac{\partial \beta}{\partial \theta}\int_0^1 g'\left(\frac{1}{\beta}\left(vG(v)-\int_0^v G(s)ds\right)\right)\left(vG(v)-\int_0^v G(s)ds\right)f(v)dv \\
&\quad +n\frac{1}{\beta^2}\frac{\partial \beta}{\partial \theta}\int_0^1 g'\left(\frac{n(1-\beta)}{\beta}\left(vG(v)-\int_0^v G(s)ds\right)\right)\left(vG(v)-\int_0^v G(s)ds\right)f(v)dv \\
&= \frac{n}{R_{lin}}\int_0^1\left[g'\left(\frac{1}{\beta}\left(vG(v)-\int_0^v G(s)ds\right)\right)-g'\left(\frac{n(1-\beta)}{\beta}\left(vG(v)-\int_0^v G(s)ds\right)\right)\right] \\
&\quad \cdot\left(vG(v)-\int_0^v G(s)ds\right)f(v)dv \\
&= 0
\end{aligned} \tag{21}$$

and

$$\begin{aligned}
\frac{\partial^2 R_{cs}}{\partial \theta^2} &= \frac{n}{R_{lin}^2}\int_0^1\left[g''\left(\frac{1}{\beta}\left(vG(v)-\int_0^v G(s)ds\right)\right)-ng''\left(\frac{n(1-\beta)}{\beta}\left(vG(v)-\int_0^v G(s)ds\right)\right)\right] \\
&\quad \cdot\left(vG(v)-\int_0^v G(s)ds\right)^2 f(v)dv.
\end{aligned} \tag{22}$$

By (2) we can see that $\frac{\partial R_{cs}}{\partial \theta}=0$ only when $n(1-\beta)=1$. When $n(1-\beta)=1$, by (22), if $g''>0$ (that is, γ is a concave function), we obtain that $\frac{\partial R_{cs}^2}{\partial \theta^2}|_{n(1-\beta)=1}<0$, and then the optimal values are $\beta^*=\frac{n-1}{n}$ and $\theta^*=\frac{R_{lin}}{n-1}$. Since $\frac{\partial(1-\beta^*)}{\partial n}<0$, when the number of contestants increases, the amount of subsidy per-contestant decreases. □

6.3 Proof of Proposition 3

If γ is concave and strictly increasing, its inverse function $g=\gamma^{-1}$ is convex. If $\beta>1$, then $\frac{1}{\beta}<1$ and there exists

$$g\left(\left(vG(v)-\int_0^v G(s)ds\right)\frac{1}{\beta}\right)\leq g\left(vG(v)-\int_0^v G(s)ds\right)\frac{1}{\beta}. \tag{23}$$

Hence, by (3), and Jensen's inequality, since g is convex we have

$$R_{ct}=n\int_0^1 g\left(\frac{1}{\beta}\left(vG(v)-\int_0^v G(s)ds\right)\right)f(v)dv+\int_0^1 g\left(n\frac{\beta-1}{\beta}\left(vG(v)-\int_0^v G(s)ds\right)\right)f(v)dv.$$

Since $n(\beta-1)\leq 1$, by (23) we get

$$\begin{aligned}
R_{ct} &\leq n\int_0^1 g\left(\frac{1}{\beta}\left(vG(v)-\int_0^v G(s)ds\right)\right)f(v)dv+n(\beta-1)\int_0^1 g\left(\frac{1}{\beta}\left(vG(v)-\int_0^v G(s)ds\right)\right)f(v)dv \\
&= n\int_0^1 \beta g\left(\frac{1}{\beta}\left(vG(v)-\int_0^v G(s)ds\right)\right)f(v)dv \leq n\int_0^1 \beta\frac{1}{\beta}g\left(\left(vG(v)-\int_0^v G(s)ds\right)\right)f(v)dv \\
&= n\int_0^1 g\left(vG(v)-\int_0^v G(s)ds\right)f(v)dv = R.
\end{aligned}$$

where R is the contestants' expected total effort in the all-pay auction without any taxation.

Similarly, if γ is convex and strictly increasing, its inverse function $g=\gamma^{-1}$ is concave, and then

$$g\left(\left(vG(v)-\int_0^v G(s)ds\right)\frac{1}{\beta}\right)\geq g\left(vG(v)-\int_0^v G(s)ds\right)\frac{1}{\beta}.$$

By the same analysis for concave cost functions we have the opposite inequality

$$R_{ct}\geq n\int_0^1 g\left(vG(v)-\int_0^v G(s)ds\right)f(v)dv=R.$$

□

6.4 Proof of Proposition 4

The optimal value of β is determined by

$$\begin{aligned}
\frac{\partial R_{ct}}{\partial\beta} &= -n\frac{1}{\beta^2}\int_0^1 g'\left(\frac{1}{\beta}\left(vG(v)-\int_0^v G(s)ds\right)\right)\left(vG(v)-\int_0^v G(s)ds\right)f(v)dv \qquad (24)\\
&\quad +n\frac{1}{\beta^2}\int_0^1 g'\left(\frac{n(\beta-1)}{\beta}\left(vG(v)-\int_0^v G(s)ds\right)\right)\left(vG(v)-\int_0^v G(s)ds\right)f(v)dv=0
\end{aligned}$$

and

$$\frac{\partial R_{ct}^2}{\partial\beta^2}=\frac{n}{\beta^4}\int_0^1\left[g''\left(\frac{1}{\beta}\left(vG(v)-\int_0^v G(s)ds\right)\right)+ng''\left(\frac{n(\beta-1)}{\beta}\left(vG(v)-\int_0^v G(s)ds\right)\right)\right]\left(vG(v)-\int_0^v G(s)ds\right)^2 f(v)dv. \qquad (25)$$

Thus, when $g''<0$ (that is, γ is convex), we obtain that $\frac{\partial R_{ct}^2}{\partial\beta^2}<0$, and by (24) the optimal β satisfies $n(\beta^*-1)=1$ or $\beta^*=\frac{n+1}{n}$. Since $\frac{\partial(\beta^*-1)}{\partial n}<0$, the marginal taxation rate decreases when the number of contestants increases. □

6.5 Proof of Proposition 6

Differentiating the designer's expected payoff (10) with respect to the prize subsidy θ yields

$$\frac{\partial R_{ps}}{\partial \theta} = n\int_0^1 g'\left((v+\theta)G(v) - \int_0^v G(s)ds\right) G(v)f(v)dv - g'(\theta).$$

When θ approaches zero we get

$$\lim_{\theta\to 0} \frac{\partial R_{ps}}{\partial \theta} = n\int_0^1 g'\left(vG(v) - \int_0^v G(s)ds\right) G(v)f(v)dv - g'(0).$$

Note that since $G(v) = F^{n-1}(v)$ we have

$$\begin{aligned} n\int_0^1 g'\left(vG(v) - \int_0^v G(s)ds\right) G(v)f(v)dv &= \int_0^1 g'\left(vG(v) - \int_0^v G(s)ds\right) nF^{n-1}(v)f(v)dv = \\ \int_0^1 g'\left(vG(v) - \int_0^v G(s)ds\right) dF^n(v) &= E_{\max}\left(g'\left(vG(v) - \int_0^v G(s)ds\right)\right). \end{aligned}$$

Since the inverse cost function function $g = \gamma^{-1}$ is convex (which implies that g' is increasing), and since for all $v > 0$, $vG(v) - \int_0^v G(s)ds > 0$, we obtain that for all $v > 0$, $g'\left(vG(v) - \int_0^v G(s)ds\right) > g'(0)$. Thus, we have

$$g'(0) \leq E_{\max}\left(g'\left(vG(v) - \int_0^v G(s)ds\right)\right),$$

which yields that $\lim_{\theta\to 0} \frac{\partial R_{ps}}{\partial \theta} > 0$. Thus, a relatively small total prize taxation will increase the designer's expected payoff. When the cost function γ is convex we have the opposite result. □

6.6 Proof of Proposition 7

Differentiating (11) with respect to the prize taxation θ we get

$$\frac{\partial R_{pt}}{\partial \theta} = n\int_\theta^1 g'\left((v-\theta)G(v) - \int_\theta^v G(s)ds\right)(G(\theta) - G(v))f(v)dv + g'(\theta)(1 - F^n(\theta)) - g(\theta)nG(\theta)f(\theta).$$

When θ approaches zero we have

$$\lim_{\theta\to 0} \frac{\partial R_{pt}}{\partial \theta} = g'(0) - n\int_0^1 g'\left(vG(v) - \int_0^v G(s)ds\right) G(v)f(v)dv.$$

Note that $n\int_0^1 g'\left(vG(v) - \int_0^v G(s)ds\right) G(v)f(v)dv$ is actually the derivative of the effort function of the contestant with the highest value of winning. Since the highest equilibrium effort is larger than zero, and the inverse cost function function $g = \gamma^{-1}$ is convex and in particular g' is increasing, we obtain that $\lim_{\theta \to 0} \frac{\partial R_{ps}}{\partial \theta} < 0$. Thus, a relatively small total prize taxation will decrease the designer's expected payoff when the cost function γ is concave. When the cost function γ is convex, we have the opposite result.

Now suppose that γ is linear. Differentiating (11) with respect to θ we get

$$\frac{\partial R_{pt}}{\partial \theta} = nG(\theta)\left(1 - F(\theta) - \theta f(\theta)\right).$$

Note that for a sufficiently small θ we get $\frac{\partial R_{pt}}{\partial \theta} > 0$. Thus, we obtain that even when the cost functions are linear a relatively small total prize taxation will increase the designer's expected payoff. $\square$

6.7 Proof of Proposition 8

By (2) and (10), the difference between the designer's expected payoff in the all-pay auction with a cost subsidy and a prize subsidy is

$$\begin{aligned} R_{cs} - R_{ps} &= n\int_0^1 \left[g\left(\frac{1}{\beta}\left(vG(v) - \int_0^v G(s)ds\right)\right) - g\left((v+\theta)G(v) - \int_0^v G(s)ds\right)\right] f(v)dv \\ &\quad - \int_0^1 g\left(\frac{n(1-\beta)}{\beta}\left(vG(v) - \int_0^v G(s)ds\right)\right) f(v)dv + g(\theta). \end{aligned}$$

Since $\beta = \frac{R_{lin}}{\theta + R_{lin}}$, differentiating the designer's expected payoff in the case of a cost subsidy is

$$\begin{aligned} \frac{\partial R_{cs}}{\partial \theta} &= \frac{\partial R_{cs}}{\partial \beta}\frac{\partial \beta}{\partial \theta} = n\int_0^1 \left[g'\left(\frac{1}{\beta}\left(vG(v) - \int_0^v G(s)ds\right)\right)\frac{1}{\beta^2}\left(vG(v) - \int_0^v G(s)ds\right)\frac{\beta}{\theta + R_{lin}}\right] f(v)dv \\ &\quad - n\int_0^1 \left[g'\left(n\frac{1-\beta}{\beta}\left(vG(v) - \int_0^v G(s)ds\right)\right)\frac{1}{\beta^2}\left(vG(v) - \int_0^v G(s)ds\right)\frac{\beta}{\theta + R_{lin}}\right] f(v)dv \\ &= \int_0^1 \left[ng'\left(\frac{1}{\beta}\left(vG(v) - \int_0^v G(s)ds\right)\right)\frac{vG(v) - \int_0^v G(s)ds}{R_{lin}}\right] f(v)dv \\ &\quad - n\int_0^1 \left[g'\left(n\frac{1-\beta}{\beta}\left(vG(v) - \int_0^v G(s)ds\right)\right)\frac{vG(v) - \int_0^v G(s)ds}{R_{lin}}\right] f(v)dv \end{aligned}$$

Similarly, differentiating the designer's expected payoff in the case of a prize subsidy is

$$\frac{\partial R_{ps}}{\partial \theta} = n\int_0^1 \left[g'\left((v+\theta)G(v) - \int_0^v G(s)ds\right)G(v)\right] f(v)dv - g'(\theta).$$

Thus, when θ approaches zero we get

$$\lim_{\theta\to 0}\frac{\partial\left(R_{cs}-R_{ps}\right)}{\partial\theta}=\frac{n}{R_{lin}}\int_0^1\left[g'\left(vG(v)-\int_0^v G(s)ds\right)\left(vG(v)-\int_0^v G(s)ds-G(v)R_{lin}\right)\right]f(v)dv. \quad (26)$$

It can be easily verified that there is $0<v^*<1$ such that $h(v)=vG(v)-\int_0^v G(s)ds-G(v)R_{lin}$ is decreasing for $0<v<v^*$ and is increasing for $v^*\leq v\leq 1$. In particular, there is $0<v^{**}<1$ such that $h(v)<0$ for $0<v<v^{**}$, and $h(v)\geq 0$ for $v^{**}\leq v\leq 1$. Thus, if g is concave such that $g'\left(vG(v)-\int_0^v G(s)ds\right)$ is a decreasing function, we obtain that

$$\begin{aligned}&\int_0^1\left[g'\left(vG(v)-\int_0^v G(s)ds\right)\left(vG(v)-\int_0^v G(s)ds-G(v)R_{lin}\right)\right]f(v)dv\\ \leq\ &\int_0^1\left[g'\left(v^{**}G(v^{**})-\int_0^{v^{**}} G(s)ds\right)\left(vG(v)-\int_0^v G(s)ds-G(v)R_{lin}\right)\right]f(v)dv.\end{aligned}$$

Thus, by (26), we have

$$\begin{aligned}\frac{\partial\left(R_{cs}-R_{ps}\right)}{\partial\theta}|_{\theta=0}\ &\leq\ g'\left(v^{**}G(v^{**})-\int_0^{v^{**}} G(s)ds\right)\frac{n}{R_{lin}}\int_0^1\left[\left(\left(vG(v)-\int_0^v G(s)ds\right)-G(v)R_{lin}\right)\right]f(v)dv\\ &=\ g'\left(v^{**}G(v^{**})-\int_0^{v^{**}} G(s)ds\right)\frac{n}{R_{lin}}\left[R_{lin}-R_{lin}\right]=0,\end{aligned}$$

That is, since $R_{cs}(0)=R_{ps}(0)$ we obtain that if g is concave then for sufficiently small θ, $R_{cs}(\theta)-R_{ps}(\theta)\leq 0$. On the other hand, if g is convex then for sufficiently small θ, $R_{cs}(\theta)-R_{ps}(\theta)\geq 0$. $\square$

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