EQUILIBRIUM EXISTENCE IN TWO-PLAYER CONTESTS WITHOUT ABSOLUTE CONTINUITY OF INFORMATION

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Equilibrium Existence in Two-player Contests Without Absolute Continuity of Information

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Abstract

We prove the existence of a behavioral-strategy Bayesian Nash equilibrium, without assuming absolute continuity of information, in two-player common-value contests where each player's probability to win is continuous in efforts outside the zero-effort profile and non-decreasing in his own effort. In particular, equilibrium exists even if both players have a continuum of interdependent information types without joint density.

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Key words: Tullock contests, Bayesian Nash Equilibrium, equilibrium existence, zero-sum games, absolute continuity of information, continuum of types, joint density.

1 Introduction

Tullock contests and their variants are widely used in modeling R&D races, political contests, and rent-seeking and lobbying activities.[1] In recent years, systematic progress has been made in establishing equilibrium existence in contests with incomplete information for general classes of information structure and endowments, including both the discrete setting (as in Einy et al. (2015) and Ewerhart and Quartieri (2020)) and the setting with a continuum of information types (as in Ewerhart (2014)

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[1]See Tullock (1980), and Corchón (2007) for an extensive survey.

and Haimanko (2021)). In common with nearly all results on Bayesian Nash equilibrium existence,[2] absolute continuity of information is essential for the existence results in those works. Introduced by Milgrom and Weber (1986), absolute continuity of information requires the joint distribution of the players' types to be absolutely continuous with respect to the product of its marginal distributions. While always in effect in a discrete setting, this condition is not innocuous when there is a continuum of information types: in applications, players' types are usually assumed to be independent, or to have joint density, in order for information to be absolutely continuous.[3]

It is easy to encounter scenarios with a continuum of types where the absolute continuity condition does not hold. As in "purely atomic" games of Hellman and Levy (2017), each player's type may have a continuous distribution, but reveal, at each realization, a *finite set* of possible values of another player's type rather than yield a continuous posterior distribution.[4] Some representations of information advantage can in particular have such a structure: one player's type may fully reveal the type of his rival, whereas the rival's type may only point at finitely many possibilities and give away their distribution.[5] In this note we show that in a sizeable class of contests absolute continuity of information is not, in fact, required for the existence of a Bayesian Nash equilibrium, and hence information structures as in above scenarios, *inter alia*, can be admitted in these contests.

The contests in our class have two players, and the value for winning is common (given any realization of information types). These unquestionably restrictive features bring with them a significant technical benefit, somewhat hidden from view. Namely, they make a contest strategically equivalent to a two-person zero-sum game. Indeed, modify each player's payoff by adding to it the cost incurred by his rival. This does

[2]See, e.g., Carbonell-Nicolau and McLean (2018) for a survey of results on equilibrium existence in Bayesian games.

[3]Under such assumptions, the information is indeed absolutely continuous by Proposition 3 in Milgrom and Weber (1986).

[4]Consider, for instance, players 1 and 2, such that player 1's type X_1 has the uniform distribution on $[0,1]$, and player's 2 type X_2 is determined by a fair lottery over $\left\{X_1, \frac{1}{2}X_1\right\}$. Then, knowing X_1 reveals the two equiprobable values for X_2; knowing X_2 fully reveals X_1 if $X_2 > \frac{1}{2}$, and discloses that X_1 is either X_2 or $2X_2$ with equal probability if $X_2 \leq \frac{1}{2}$.

[5]An example is obtained by the following modification of X_2 in the preceding footnote: let $X_2 := 2X_1$ if $X_1 \leq \frac{1}{2}$, and $X_2 := 2X_1 - 1$ otherwise. Then knowing X_1 fully reveals X_2, but knowing X_2 only discloses that X_1 is either $\frac{1}{2}X_2$ or $\frac{1}{2}X_2 + \frac{1}{2}$ with equal probability.

not affect strategic considerations but now the sum of (modified) payoffs is equal to the common value for winning, which is independent of players' choices of effort. The expected (modified) payoffs add up to the expected common value, making the game constant-sum, and hence, after a shift of scale, zero-sum.

The zero-sum feature is known to be conducive to equilibrium existence without assuming absolute continuity of information – that assumption is not needed for the classical result of Mamer and Schilling (1986) that guarantees existence of a saddle point in distributional strategies in a large class of zero-sum two-person Bayesian games in which the payoff functions are separately continuous in the players' actions. This existence result is not directly applicable to Tullock contests, however, because the payoffs in these contests – with the player's probability of winning being the ratio between his effort and the total effort in the simplest version – are separately discontinuous at the zero effort profile.[6] We will deal with the discontinuity issue by also admitting contests with a positive lower bound on efforts. First, it will be shown that equilibrium in behavioral strategies exists in contests constrained by a positive effort floor. That constraint ensures that the expected payoffs are separately continuous in the two strategies,[7] without the need to assume absolute continuity of information, and equilibrium existence follows from Sion's (1958) minimax theorem because of the aforementioned zero-sum strategic nature of the contest. It will be then shown that a limit point of a sequence of equilibria in constrained contests, as the effort floor is allowed to drop to zero, is an equilibrium in the unconstrained contest.[8]

Except for allowing non-absolutely continuous information, our class of contests is a two-player common-value version of the class considered in Haimanko (2021). It is very general in terms of conditions on the contest success function, requiring only two features that are inspired by the specific functional form of probabilities of winning in Tullock contests: we need the player's probability of winning to be

[6] Also, on a more technical level, Mamer and Schilling's (1986) metrizability assumption on type sets is unduely restrictive. We do not utilize their result also for this reason.

[7] We will use the weak topology of Balder (1988) on behavioral strategy sets, in which they are compact; all forthcoming references to continuity and limits of strategies are with respect to this topology.

[8] Similar "limit" approaches have been employed in nearly all recent results on equilibrium existence in contests (see Einy et al. (2013), Ewerhart (2014), Ewerhart and Quartieri (2020) and Haimanko (2021)).

continuous with respect to all efforts whenever the effort profile is non-zero, and to be non-decreasing in the player's own effort. Our equilibrium existence result generalizes Haimanko's (2021) corresponding result[9] when there are only two players, by removing the need to assume absolute continuity of information. If, in addition to the above two properties of the player's probability of winning, that probability is also strictly concave in the player's own effort and equal to 1 if the player is the only one exerting positive effort, and if the costs are concave – making the contest what was termed as *generalized (concave) Tullock contest* in Einy et al. (2015) and Haimanko (2021) – then any equilibrium must be in pure strategies, as was observed in Haimanko (2021). Thus, pure-strategy equilibrium in particular exist in two-player common-value Tullock lotteries and Tullock contests with concave impact functions, for all, not necessarily absolutely continuous, information structures.

The paper is organized as follows. Section 2 presents our class of contests and recalls the concepts of Bayesian behavioral strategies and equilibrium. Section 3 explains the strategic equivalence of our contests to zero-sum games and states the equilibrium existence result, which is proved in Section 4. Section 5 concludes.

2 Two-player common-value contests with incomplete information

2.1 The model

We adopt Haimanko's (2021) model of contest, and adjust it to reflect the two-player common-value setting but discard the assumption of absolute continuity of information. In our set-up, two players, $i = 1, 2$, compete for a prize. The information endowment of each i is given by a measurable type-space $(T_i, \mathcal{T}_i)$ that is countably generated. The players have a common prior probability p on the product space $(T, \mathcal{T}) := (T_1 \times T_2, \mathcal{T}_1 \otimes \mathcal{T}_2)$ of type-profiles. No relation is assumed between p and the product $p_1 \otimes p_2$ of its marginals, and hence information need not be absolutely continuous in the sense of Milgrom and Weber (1986).

Upon privately observing their respective types, players simultaneously choose

[9] See Proposition 1 therein. The other equilibrium existence results of Haimanko (2021) are not in the common-value setting.

their effort levels from a bounded interval $[0, M]$. The common value for the prize given by a a function $V : T \to \mathbb{R}_+$, i.e., if $t \in T$ is the realized type profile then value is $V(t) \geq 0$. The type-dependent cost of effort of player i is described by $c_i : T \times [0, M] \to \mathbb{R}_+$. The following assumptions are made on the functions V and c_i :

(i) V is $\mathcal{T}$-measurable and c_i is $\mathcal{T} \otimes \mathcal{B}([0, M])$-measurable;[10]

(ii) V and $c_i(\cdot, M)$ are bounded;

(iii) for any $t \in T$, the functions $c_i(t, \cdot)$ are non-decreasing and continuous.

The prize is awarded according to a $\mathcal{T} \otimes \mathcal{B}\left([0, M]^2\right)$-measurable *success function* $\rho : T \times [0, M]^2 \to [0, 1]^2$, where $\rho_1 + \rho_2 \equiv 1$. That is, for each effort profile $x \in [0, M]^2$, $\rho_i(x)$ is the probability that player i will be the recipient of the prize if x is realized.

Denote by $\mathbf{0} \in \mathbb{R}^2$ the zero vector. We impose the following conditions on ρ, for every $t \in T$:

(iv) $\rho_1(t, \cdot)$ is continuous on $[0, M]^2 \setminus \{\mathbf{0}\}$;

(v) $\rho_1(t, (x_1, x_2))$ is non-decreasing in x_1 for a fixed x_2, any non-increasing in x_2 for a fixed x_1.

Definition 1. An incomplete-information *contest* is given by the collection $G = (N, \{(T_i, \mathcal{T}_i)\}_{i=1}^2, p, V, \{c_i\}_{i=1}^2, \rho)$ of the above-described attributes, such that (**i**)–(**v**) are satisfied. If the allowable efforts of both players are additionally constrained to lie in an interval $[m, M]$, where $0 < m < M$, the resulting contest will be denoted by $G(m)$ and called a *constrained contest*, with an additional convention that $G(0)$ refers to G.

For any realized type profile $t \in T$ and any effort profile $x \in [0, M]^2$, the payoff of each player i in a contest G is given by his expected share of the prize net of his cost of effort, namely,

$$u_i(t, x) = \rho_i(t, x) \cdot V(t) - c_i(t, x_i); \tag{1}$$

u_i is clearly $\mathcal{T} \otimes \mathcal{B}\left([0, M]^2\right)$-measurable and bounded.

[10] Here and henceforth, given a Borel set $S \subset \mathbb{R}^m$ for some $m \geq 1$, $\mathcal{B}(S)$ will denote the σ-algebra of Borel subsets of S. The measurability of real-valued functions will be required w.r.t. the Borel σ-algebra on their stated range.

2.2 Bayesian strategies and equilibrium

Given $0 \leq m < M$, a *pure (Bayesian) strategy* of player i in a contest $G(m)$ is a $\mathcal{T}_i$-measurable function $s_i : T_i \to [m, M]$; that is, upon learning his type t_i, the player chooses effort $s_i(t_i)$. The more general concept of a behavioral strategy allows randomness in the type-dependent choice of effort: a *behavioral strategy* of i in $G(m)$ is a mapping $\sigma_i : T_i \times \mathcal{B}([m, M]) \to [0, 1]$, such that $\sigma_i(t_i, \cdot)$ is a probability measure on $[m, M]$ for every $t_i \in T_i$ and $\sigma_i(\cdot, A)$ is $\mathcal{T}_i$-measurable for every $A \in \mathcal{B}([m, M])$. Any pure strategy is canonically identifiable with a behavioral strategy.[11]

We denote by $\Sigma_i(m)$ the set of behavioral strategies of player i in $G(m)$, and note that any $\sigma_i \in \Sigma_i(m)$ for $m > 0$ can be viewed as a strategy in $\Sigma_i(0)$ in an obvious fashion; henceforth, $\Sigma_i(m)$ for $m > 0$ will be regarded as a subset of $\Sigma_i(0)$. The product set $\Sigma(m) = \Sigma_1(m) \times \Sigma_2(m)$ contains the corresponding strategy profiles. For any $\sigma = (\sigma_1, \sigma_2) \in \Sigma(0)$, the expected payoff of player $i = 1, 2$ is given by[12]

$$U_i(\sigma) = \int_T \int_{[0,M]^2} u_i(t, x)\, \sigma_1(t_1, dx_1)\sigma_2(t_2, dx_2)p(dt). \tag{2}$$

Definition 2. For any $0 \leq m < M$, a behavioral strategy profile $\sigma^* = (\sigma_1^*, \sigma_2^*) \in \Sigma(m)$ constitutes a *Bayesian Nash equilibrium* (or *BNE*, for short) of a contest $G(m)$ if

$$U_1(\sigma^*) \geq U_1(\sigma_1, \sigma_2^*) \text{ and } U_2(\sigma^*) \geq U_2(\sigma_1^*, \sigma_2) \tag{3}$$

for any $\sigma_1 \in \Sigma_1(m)$ and $\sigma_2 \in \Sigma_2(m)$. If σ^* consists of pure strategies then it is a *pure-strategy* BNE.

2.3 A special case: two-player generalized Tullock contests

Tullock (1980) considered a family of success functions that ascribe probabilities of winning in proportion to "impacts" of individual efforts: for a given $r > 0$, the success function $\rho = \rho^r$ in an r-*Tullock contest* is defined, for each $x \in [0, M]^2 \setminus \{\mathbf{0}\}$, by the ratio $\rho_1^r(x) = \dfrac{x_1^r}{x_1^r + x_2^r}$, with constant marginal costs of effort. The case of $r > 1$ represents "increasing returns to aggressive bidding" (see Baye *et al.* (1994)) because

[11] Specifically, a pure strategy s_i is identified with $\sigma_i^{s_i} \in \Sigma_i(m)$ for which $\sigma_i^{s_i}(t_i, \cdot)$ is the Dirac measure concentrated on $s_i(t_i)$.

[12] If $\sigma = (s_1, s_2)$ is a pure-strategy profile then the expected payoff is simply $U_i(\sigma) = \int_T u_i(t, (s_1(t_1), s_2(t_2)))\, p(dt)$.

that is when the impact function x_i^r is convex. In the complementary case of $r \leq 1$, the returns to effort are non-increasing also at the level of the probability of winning: the ith component of the r-Tullock success function is then *strictly concave* in i's own effort x_i whenever his rival's effort is positive. The property of strict concavity is preserved when, for each player i, a general effort-impact function $g_i : T \times \mathbb{R}_+ \to \mathbb{R}_+$ is allowed (assumed to be strictly increasing, continuous, concave, and with $g_i(\cdot, 0) \equiv 0$), in which case ρ is defined, for any $t \in T$ and $x \in [0, M]^2 \setminus \{\mathbf{0}\}$, by

$$\rho_1(t, x) = \frac{g_1(t, x_1)}{g_1(t, x_1) + g_2(t, x_2)}. \tag{4}$$

The specification in (4) generalizes Szidarovszky and Okuguchi (1997) model, who assumed that the impact functions are in addition twice differentiable. We now present a further extension, *generalized (concave) Tullock contests* – based on identically called contests considered in Einy et al. (2015) and Haimanko (2021) outside the two-player common-value framework – that satisfy in addition to (**i**)–(**v**) of Section 2.1 the following requirements, for any $t \in T$:

(vi) $\rho_1(t, (x_1, x_2))$ is *strictly concave* in x_1 for a fixed $x_2 > 0$ and *strictly convex* in x_2 for a fixed $x_1 > 0$;

(**vii**) the functions $c_i(t, \cdot)$ are *convex* and *strictly increasing*;

(**viii**) $\rho_1(t, (x_1, 0)) = 1$ and $\rho_1(t, (0, x_2)) = 0$ for any $0 < x_1, x_2 \leq M$ (that is, a player receives the prize with certainty if only his effort is positive).[13]

The r-Tullock contests with $r \leq 1$, and more general contests that satisfy (4), fall within the domain of generalized Tullock contests. But the category of generalized Tullock contests extends much further. For instance, as noted in Haimanko (2021), ρ in a generalized Tullock contest may be a convex combination of several contest functions, each of the form (4) for a distinct set of impact specifications, corresponding to the case where the winning is determined via *one* of several criteria whose choice is perceived to be random *a priori*. As far as BNE existence in generalized Tullock contests is concerned, only pure strategies are relevant:

Fact 1 (Haimanko (2021)). For any $0 \leq m < M$, if a generalized Tullock contest

[13]For our purpose, it would in fact be enough to assume, as in Haimanko (2021), that there exist $\overline{\rho}_1 \in (\frac{1}{2}, 1]$ and $\underline{\rho}_1 \in [0, \frac{1}{2})$ such that $\rho_1(t, (x_1, 0)) \equiv \overline{\rho}_1$ and $\rho_1(t, (0, x_2)) \equiv \underline{\rho}_1$. Generalized Tullock contest would thereby contain cases where the prize is given, with some probability, w.r.t. extraneous probability distribution, unrelated to efforts.

$G(m)$ has a BNE, then it has a pure-strategy BNE.[14]

3 BNE existence

The Milgrom and Weber (1986) condition of absolutely continuous information, which is common to nearly all BNE existence results with a continuum of information types, turns out to be fully dispensable in regard to the contests that we consider. Our existence result is mainly driven by the fact that these contests are strategically equivalent to (two-player) *zero-sum* Bayesian games.[15] The reason for this equivalence is the following. Modify the payoffs u_i in (1) by adding to u_i the cost incurred by the other player and subtracting half the value, i.e., let

$$h_1(t,x) := [\rho_1(t,x) \cdot V(t) - c_1(t,x_1)] + c_2(t,x_2) - \frac{1}{2}V(t)$$

and

$$h_2(t,x) := [\rho_2(t,x) \cdot V(t) - c_2(t,x_2)] + c_1(t,x_1) - \frac{1}{2}V(t)$$

for every $t \in T$ and $x \in [0,M]^2$; clearly,

$$h_1 + h_2 \equiv 0. \tag{5}$$

Also modify the expected payoffs accordingly, based on (2), i.e., let

$$H_i(\sigma) = \int_T \int_{[0,M]^2} h_i(t,x)\, \sigma_1(t_1, dx_1)\sigma_2(t_2, dx_2)p(dt).$$

Then maximizing $H_i(\sigma)$ in the variable σ_i is equivalent to maximizing $U_i(\sigma)$, and also

$$H_1(\sigma) + H_2(\sigma) = 0 \tag{6}$$

for every $\sigma \in \Sigma(0)$.

Several obvious implications of conditions (**i**) – (**v**) and (5) on the modified payoff function of player 1 are stated below.

Fact 2. The modified payoff $h_1 : T \times [0,M]^2 \to \mathbb{R}$ has the following properties:
(**a**) h_1 is $\mathcal{T} \otimes \mathcal{B}([0,M]^2)$-measurable and bounded;

[14] See Fact 2 and its proof in Haimanko (2021).

[15] In the context of two-player all-pay auctions with common value, an equivalence to zero-sum game has been already observed by, e.g., Pavlov (2013).

($\mathbf{b}$) $h_1(t,\cdot)$ is continuous on $[0,M]^2\setminus\{\mathbf{0}\}$ for any $t\in T$;

($\mathbf{c}$) for any $t\in T$ and any $x\in[0,M]^2$, $\lim_{y_1\to x_1+} h_1(t,(y_1,x_2)) \geq h_1(t,x)$ and $\lim_{y_2\to x_2+} h_1(t,(x_1,y_2)) \leq h_1(t,x)$.[16]

From now on, only modified payoffs will be considered in (constrained or unconstrained) contests, and thereby the contests will be viewed as zero-sum games. Thus, as usual, $\sigma^*\in\Sigma(m)$ is a BNE of a contest $G(m)$ if and only it is a *saddle point* (with σ_i^* being an *optimal strategy* of player i, for $i=1,2$), namely,

$$H_1(\sigma_1^*,\sigma_2) \geq \mathbf{val}(G(m)) \geq H_1(\sigma_1,\sigma_2^*) \tag{7}$$

for any $\sigma_1\in\Sigma_1(m)$ and $\sigma_2\in\Sigma_2(m)$, with a uniquely determined *value* $\mathbf{val}(G(m))$ of the game $G(m)$.

BNE existence without absolute continuity of information has been known, due to Mamer and Schilling (1986), for quite general separately continuous zero-sum games. However, our contests have payoffs that are discontinuous at the zero effort profile. Our theorem, stated next, will therefore be established in two stages. We will first confine attention to constrained contests with positive lower bound m on efforts, where the payoff functions are continuous by Fact 2($\mathbf{b}$), and show that the players possess optimal strategies.[17] We will then consider limits of those optimal strategies as the lower bound m drops to zero, and prove that these limits are optimal strategies in unconstrained contests.

Theorem. For any $0\leq m<M$, contest $G(m)$ possesses a BNE. If, moreover, $G(m)$ is a generalized Tullock contest, then it possesses a pure-strategy BNE.

Our theorem generalizes Proposition 1 in Haimanko (2021) by dropping the absolute continuity of information imposed there on common-value contests, at a price of allowing just two players. In the last Section 5 we offer some comments on the necessity of the main assumptions underlying our setting.

[16] Statement ($\mathbf{b}$) clearly supersedes ($\mathbf{c}$), except when $x=\mathbf{0}$.

[17] BNE existence in constrained contests will be established without an appeal to the result of Mamer and Schilling (1986) because, unlike those authors, we do not assume that the type sets are metric spaces.

4 The proof

4.1 Part 1: Preliminary notions

For any fixed $m \in [0, M)$, we endow the behavioral strategy set $\Sigma_i(m)$ of each player i with the weak topology of Balder (1988), as follows. Recall that p_i denotes the marginal distribution induced by p on $(T_i, \mathcal{T}_i)$. A $\mathcal{T}_i \otimes \mathcal{B}([m, M])$-measurable function $g : T_i \times [m, M] \to \mathbb{R}$ is called *Carathéodory integrand* if $g(t_i, \cdot)$ is continuous for every $t_i \in T_i$, and there exists a p_i-integrable function φ on T_i such that $|g(t_i, x_i)| \leq \varphi(t_i)$ for every $(t_i, x_i) \in T_i \times [m, M]$. The weak topology on $\Sigma_i(m)$ is the coarsest topology in which, for every Carathéodory integrand $g : T_i \times [m, M] \to \mathbb{R}$, the functional $I_g^m : \Sigma_i(m) \to \mathbb{R}$ that is given for any $\sigma_i \in \Sigma_i(m)$ by

$$I_g^m(\sigma_i) = \int_{T_i} \int_{[m,M]} g(t_i, x_i) \sigma_i(t_i, dx_i) p_i(dt_i)$$

is continuous. As observed in Haimanko (2021), this topology is metrizable because $(T_i, \mathcal{T}_i)$ is countably generated.[18]

4.2 Part 2: Contests with $m > 0$

Here we consider a constrained contest $G(m)$ with $0 < m < M$. We first show that player 1's expected payoff function $H_1(\sigma_1, \sigma_2)$ is weakly continuous in each strategy separately. To this end, choose a regular conditional probability distribution $p(\cdot \mid t_1)$ on $(T_2, \mathcal{T}_2)$ (that exists as a consequence of, e.g., Theorem 10.2.2 in Dudley (2003)), and, for a fixed $\sigma_2 \in \Sigma_2(m)$, define a $\mathcal{T}_1 \otimes \mathcal{B}([m, M])$-measurable and bounded function $\widehat{h}_{1,\sigma_2} : T_1 \times [m, M] \to \mathbb{R}$ by

$$\widehat{h}_{1,\sigma_2}(t_1, x_1) := \int_{T_2} \int_{[m,M]} h_1(t, (x_1, x_2)) \sigma_2(t_2, dx_2) p(dt_2 \mid t_1) \tag{8}$$

for every $(t_1, x_1) \in T_1 \times [m, M]$. Since $h_1(t, x)$ is bounded and continuous in $x \in [m, M]^2$ for a fixed t by Fact 2 (**a,b**), $\widehat{h}_{1,\sigma_2}(t_1, \cdot)$ is continuous on $[m, M]$ by the bounded convergence theorem for every $t_1 \in T_1$, and hence $\widehat{h}_{1,\sigma_2}$ is a Carathéodory

[18] See Part 1 in Section 4.4 therein.

integrand on $T_1 \times [m, M]$. Furthermore,

$$
\begin{aligned}
& H_1(\sigma_1, \sigma_2) && (9) \\
= & \int_T \int_{[m,M]^2} h_1(t, (x_1, x_2))\, \sigma_1(t_1, dx_1)\sigma_2(t_2, dx_2)p(dt) && (10) \\
= & \int_{T_1} \int_{[m,M]} \left[\int_{T_2} \int_{[m,M]} h_1(t, (x_1, x_2))\, \sigma_2(t_2, dx_2)p(dt_2 \mid t_1) \right] \sigma_1(t_1, dx_1)p_1(dt_1) && (11) \\
= & \int_{T_1} \int_{[m,M]} \widehat{h}_{1,\sigma_2}(t_1, x_1)\, \sigma_1(t_1, dx_1)p_1(dt_1) = I^m_{\widehat{h}_{1,\sigma_2}}(\sigma_1). && (12)
\end{aligned}
$$

Thus, $H_1(\sigma_1, \sigma_2) = I^m_{\widehat{h}_{1,\sigma_2}}(\sigma_1)$, which shows that $H_1(\sigma_1, \sigma_2)$ is continuous in 1's own strategy σ_1 by the definition of the weak topology on $\Sigma_1(m)$ in Part 1 of the proof. Similarly, $H_2(\sigma_1, \sigma_2)$ is weakly continuous in player 2's own strategy σ_2. Since $H_2 = -H_1$ by (6), H_1 is indeed weakly continuous in each strategy separately.

As H_1 is also bilinear, and each strategy space $\Sigma_i(m)$ is convex and compact in the weak topology, by Sion's (1958) minimax theorem

$$\inf_{\sigma_2 \in \Sigma_2(m)} \sup_{\sigma_1 \in \Sigma_1(m)} H_1(\sigma_1, \sigma_2) = \sup_{\sigma_1 \in \Sigma_1(m)} \inf_{\sigma_2 \in \Sigma_2(m)} H_1(\sigma_1, \sigma_2). \tag{13}$$

By the separate weak continuity of H_1, $\sup_{\sigma_1 \in \Sigma_1(m)} H_1(\sigma_1, \sigma_2)$ is weakly lower semi-continuous in σ_2 and $\inf_{\sigma_2 \in \Sigma_2(m)} H_1(\sigma_1, \sigma_2)$ is weakly upper semi-continuous in σ_1. Hence, the infimum in $\inf_{\sigma_2 \in \Sigma_2(m)} \sup_{\sigma_1 \in \Sigma_1(m)} H_1(\sigma_1, \sigma_2)$ is attained at a strategy that is optimal for player 2 and the supremum in $\sup_{\sigma_1 \in \Sigma_1(m)} \inf_{\sigma_2 \in \Sigma_2(m)} H_1(\sigma_1, \sigma_2)$ is attained at a strategy that is optimal for player 1. The expression in (13) is the value of the contest, $\mathbf{val}(G(m))$, and the saddle point that has been found is a BNE of $G(m)$ as remarked in Section 3.

Finally, if $G(m)$ is a generalized Tullock contest, existence of a pure-strategy BNE now follows from Fact 1.

4.3 Part 3: Contests with $m = 0$

We now consider an unconstrained contest, $G = G(0)$. Pick a sequence $\{m_k\}_{k=1}^{\infty} \subset (0, M)$ with $\lim_{k\to\infty} m_k = 0$, and a sequence $\{(\sigma_1^k, \sigma_2^k)\}_{k=1}^{\infty}$ where $\sigma_i^k \in \Sigma_i(m_k)$ ($\subset \Sigma_i(0)$) is an optimal strategy for player i ($i = 1, 2$) in $G(m_k)$ for each k (the existence of such strategies was shown in Part 2). Since $\Sigma_i(0)$ is metrizable and compact in the weak topology for each player i, $\{\sigma_i^k\}_{k=1}^{\infty}$ has a subsequence that converges to some $\sigma_i^* \in \Sigma_i(0)$; it can be assumed w.l.o.g. that, for every player i, $\lim_{k\to\infty} \sigma_i^k = \sigma_i^*$ in the weak topology on $\Sigma_i(0)$, and that the bounded sequence $\{\mathbf{val}(G(m_k))\}_{k=1}^{\infty}$ converges to a limit $\mathbf{v}$. Our aim is to show that the profile of limit strategies $\sigma^* \in \Sigma(0)$ is a saddle point in the unconstrained contest $G(0)$, with $\mathbf{v} = \mathbf{val}(G(0))$.

First, fix a strategy $\sigma_2 \in \Sigma_2(0)$ that also belongs to $\Sigma_2(m)$ for some $0 < m < M$. By arguing as in Part 2 of the proof,[19] the function $\widehat{h}_{1,\sigma_2}$ (that can be defined on the entire $T_1 \times [0, M]$ by (8)) is a Carathéodory integrand on $T_1 \times [0, M]$, and, replacing m by 0 in equations (9)-(12), we also have $H_1(\sigma_1, \sigma_2) = I^0_{\widehat{h}_{1,\sigma_2}}(\sigma_1)$. Since $\lim_{k\to\infty} \sigma_1^k = \sigma_1^*$ in the weak topology on $\Sigma_1(0)$,

$$\lim_{k\to\infty} H_1\left(\sigma_1^k, \sigma_2\right) = \lim_{k\to\infty} I^0_{\widehat{h}_{1,\sigma_2}}(\sigma_1^k) = I^0_{\widehat{h}_{1,\sigma_2}}(\sigma_1^*) = H_1(\sigma_1^*, \sigma_2). \tag{14}$$

But σ_1^k is an optimal strategy for player 1 in every $G(m_k)$ with $m_k < m$, and that strategy therefore guarantees him the expected payoff of at least $\mathbf{val}(G(m_k))$ against σ_2, by (7). Hence,

$$\liminf_{k\to\infty} H_1\left(\sigma_1^k, \sigma_2\right) \geq \lim_{k\to\infty} \mathbf{val}(G(m_k)) = \mathbf{v}. \tag{15}$$

From (14) and (15),

$$H_1(\sigma_1^*, \sigma_2) \geq \mathbf{v}. \tag{16}$$

Now consider *any* strategy $\sigma_2 \in \Sigma_2(0)$. For any $0 < m < M$, let $\sigma_{2,m} \in \Sigma_2(m)$ ($\subset \Sigma_2(0)$) be the strategy satisfying $\sigma_{2,m}(t_2, [m, a]) = \sigma_2(t_2, [0, a])$ for any $a \in [m, M]$ and $t_2 \in T_2$.[20] It follows from Fact 2(**c**) that

$$\lim_{m\to 0+} h_1(t, (x_1, \max\{x_2, m\})) \leq h_1(t, x). \tag{17}$$

Hence,

$$\begin{aligned}
\limsup_{m\to 0+} H_1(\sigma_1^*, \sigma_{2,m}) &= \limsup_{m\to 0+} \int_T \int_{[0,M]^2} h_1(t, (x_1, \max\{x_2, m\}))\, \sigma_1^*(t_1, dx_1)\sigma_2(t_2, dx_2)p(dt) \\
\text{(by Fatou's lemma)} &\leq \int_T \int_{[0,M]^2} \limsup_{m\to 0+} h_1(t, (x_1, \max\{x_2, m\}))\, \sigma_1^*(t_1, dx_1)\sigma_2(t_2, dx_2)p(dt) \\
\text{(by (17))} &\leq \int_T \int_{[0,M]^2} h_1(t, (x_1, x_2))\, \sigma_1^*(t_1, dx_1)\sigma_2(t_2, dx_2)p(dt) = H_1(\sigma_1^*, \sigma_2),
\end{aligned}$$

showing that

$$\limsup_{m\to 0+} H_1(\sigma_1^*, \sigma_{2,m}) \leq H_1(\sigma_1^*, \sigma_1). \tag{18}$$

[19] Now one must use the fact that $h_1(t, x)$ is bounded and continuous in $x \in [0, M] \times [m, M]$ for a fixed t.

[20] In probabilistic terms, if X_2 is a $\sigma_2(t_2, \cdot)$-distributed random variable on $[0, M]$, then $Y_2 := \max\{X_2, m\}$ is $\sigma_{2,m}(t_2, \cdot)$-distributed.

Since (16) holds for any strategy of player 2 that belongs to $\Sigma_2(m)$ for some $0 < m < M$,

$$\mathbf{v} \leq \lim \inf_{m \to 0+} H_1(\sigma_1^*, \sigma_{2,m}),$$

and, by using (18), we conclude that

$$H_1(\sigma_1^*, \sigma_2) \geq \mathbf{v} \tag{19}$$

for any $\sigma_2 \in \Sigma_2\,(0)$.

Similarly to (19), it can be shown that $H_2(\sigma_1, \sigma_2^*) \geq -\mathbf{v}$ (and hence that $H_1(\sigma_1, \sigma_2^*) \leq \mathbf{v}$) for any $\sigma_1 \in \Sigma_1\,(0)$. Thus, according to (7), $\sigma^* = (\sigma_1^*, \sigma_2^*)$ is a saddle point of $G(0)$, which is a BNE.

Finally, if $G(0)$ is a generalized Tullock contest, existence of a pure-strategy BNE follows from Fact 1.

5 Concluding remarks

Leeway exists for relaxing some of our assumptions. For instance, instead of a universal cap M on efforts, player- and type-dependent caps can be introduced as was done in Section 5.1 of Haimanko (2021). Also, as far as the first part of the Theorem (on BNE existence in behavioral strategies) is concerned, the monotonicity condition ($\mathbf{v}$) may be replaced by a far more permissive requirement that only addresses the behavior of ρ_1 at $\mathbf{0}$, stipulating that $\lim\inf_{y_1 \to 0+} \rho_1\,(t, (y_1, 0)) \geq \rho_1\,(t, \mathbf{0})$ and $\lim\sup_{y_2 \to 0+} \rho_1\,(t, (0, y_2)) \leq \rho_1\,(t, \mathbf{0})$. These inequalities are all that is needed to establish the property stated in Fact 2($\mathbf{c}$);[21] however, as it is hard to envisage applications in which the probability of a player to win fails to be non-decreasing in his effort, we chose to state the model with an explicit requirement ($\mathbf{v}$).

It is not at present clear whether other central features of the model can be lightened in a meaningful way. The assumption that there are just two players and that the value is common, as well as an (implicit) requirement that the prize is given with certainty to one of the players and never withheld, are essential for the contest to be strategically equivalent to a two-person zero-sum game. The zero-sum game structure is the one that, due to Sion's minimax theorem, allows replacing full or

[21] In the first inequality in Fact 2($\mathbf{c}$), lim would then need to be replaced by lim inf, and in the second inequality – by lim sup.

partial continuity of (expected) payoffs in all strategies jointly[22] by separate continuity as a precondition for BNE existence. The standard use of the absolute continuity of information condition is precisely for establishing full[23] or partial[24] mode of continuity of expected payoffs, but here only separate continuity is needed, which makes absolute continuity of information superfluous and leads to our result.

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[22] See Reny (1999).

[23] See, e.g., Milgrom and Weber (1986) and Balder (1988).

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