COMMON-VALUE ALL-PAY AUCTIONS WITH ASYMMETRIC INFORMATION AND BID CAPS

Ezra Einy, Ori Haimanko, Ram Orzach
and Aner Sela

Discussion Paper No. 14-02

September 2014

Monaster Center for
Economic Research
Ben-Gurion University of the Negev
P.O. Box 653
Beer Sheva, Israel

Fax: 972-8-6472941
Tel: 972-8-6472286

Common-Value All-Pay Auctions with Asymmetric Information and Bid Caps

Ezra Einy, Ori Haimanko, Ram Orzach, Aner Sela *

September 16, 2014

Abstract

We study two-player common-value all-pay auctions (contests) with asymmetric information under the assumption that one of the players has an information advantage over his opponent and both players are budget-constrained. We generalize the results for all-pay auctions with complete information, and show that in all-pay auctions with asymmetric information, sufficiently high (but still binding) bid caps do not change the players' expected total effort compared to the benchmark auction without any bid cap. Furthermore, we show that there are bid caps that increase the players' expected total effort compared to the benchmark. Finally, we demonstrate that there are bid caps which may have an unanticipated effect on the players' expected payoffs – one player's information advantage may turn into a disadvantage as far as his equilibrium payoff is concerned.

Keywords*:* Common-value all-pay auctions, asymmetric information, information advantage, bid caps.

JEL Classification: C72*,* D44, D82.

***Ezra Einy, Ori Haimanko, Aner Sela**: Department of Economics, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel; their respective e-mail addresses are: einy@bgu.ac.il, ori@bgu.ac.il, anersela@bgu.ac.il. **Ram Orzach**: Department of Economics, Oakland University, Rochester, MI 48309, USA; orzach@oakland.edu

1 Introduction

In an all-pay auction each player submits a bid (effort) and the player with the highest bid wins the contest, but, independently of success, all players bear the cost of their bids. In the framework of all-pay auctions with complete information, where each player's type (value of winning the contest or ability) is common knowledge,[1] and in the incomplete information framework, where each player's type is private and only the distribution from which the players' types is drawn is common knowledge,[2] it was shown that the behavior of the players may change drastically when constraints are placed on their budget or, alternatively, when bid caps are imposed by the contest designer.

Che and Gale (1998) calculated the bidding equilibrium of a complete information all-pay auction with two bidders who have different valuations for the prize and linear cost functions, and demonstrated that a bid cap can increase the players' total effort. Furthermore, they showed that a sufficiently high (but still binding) bid cap changes the players' strategies, but not their expected efforts nor their probabilities to win the contest. The effect of bid caps on the players' strategies may be different in other frameworks as Gavious, Moldovanu and Sela (2003) demonstrated. They studied symmetric all-pay auctions with incomplete information and showed that, regardless of the number of bidders, if agents have linear or concave cost functions then setting a bid cap is not profitable for a designer who wishes to maximize the average bid. On the other hand, if agents have convex cost functions (i.e., increasing marginal costs), then effectively capping the bids is profitable for a designer facing a sufficiently large number of bidders.

In this study we examine some effects of bid caps in common-value all-pay auctions with asymmetric information. We consider a two-player common-value all-pay auction where the value of winning is the same for all the players in the same state of nature, but the information about which state of nature was realized can be different.[3] This model captures contests in which the value of winning is similar for different contestants, but is not precisely known at the time of making a bid. In our framework, the information of a player about the value of winning is described by a partition of the space of states of nature, which is assumed

[1] See, for example, Hillman and Samet (1987), Hillman and Riley (1989), Baye et al. (1993, 1996) and Siegel (2009).

[2] See, for example, Amann and Leininger (1996), Moldovanu and Sela (2001, 2006) and Moldovanu et al. (2010).

[3] Several researchers used the same framework as ours to analyze common-value second-price auctions and common-value first price auctions (see Einy et al. 2001, 2002, Forges and Orzach 2011, Malueg and Orzach 2012 and Abraham et al. 2012).

to be finite. Jackson (1993) and Vohra (1999) showed that this partition representation is equivalent to the more common Harsanyi-type formulation of Bayesian games.[4]

In our model of asymmetric information we assume that information sets of each player are connected with respect to the value of winning the contest (see Einy et al. 2001, 2002 and Forges and Orzach 2011). This means that if a player's information partition does not enable him to distinguish between two possible values of winning, then he also cannot distinguish between these and all intermediate values. This assumption seems plausible in environments where the information of a player only allows him to put an upper and lower bound on the actual value of winning, without being able to rule out any outcome within the bounds. We also assume that one player has an information advantage over the other, which is reflected by having a finer information partition. Then, without loss of generality, it can be postulated that one player has the trivial information partition (he will be referred to as the uninformed player), and that the information partition of the other player is the finest one possible, enabling him to distinguish between all states of nature (he will be referred as the informed player). We finally assume that there is a cap constraining the players' bids – players can submit bids that are smaller than or equal to the bid cap.

We generalize the result of Che and Gale (1998) on bid caps in all-pay auction with complete information, and show that if the bid cap is sufficiently high (but still binding) then its imposition changes the players' strategies compared to the no-cap benchmark, but does not affect the players' *expected* efforts or their ex-ante chances to win the contest. For sufficiently low bid caps we show that when both players make a bid that is equal to the bid cap in every state of nature, then their expected total effort decreases compared to the no-cap benchmark. However, in common with the all-pay auction under the complete information assumption, a bid cap may still increase the players' expected effort. This happens in an equilibrium in which the uninformed player submits a bid that is equal to the bid cap, but the informed player submits it only in some states of nature while in the other states he stays out.

According to Siegel (2014), in the common-value all-pay auction with asymmetric information without bid caps the expected payoff of the informed player is always higher than that of the uninformed player.[5] Here,

[4]Krishna and Morgan (1997) analyzed the equilibrium strategies of the all-pay auction with interdependent types in the Harsanyi-type formulation of Bayesian games. They assumed that the players' types are affiliated and symmetrically distributed.

[5]See Section 2 of the online appendix to his work.

however, we will show that imposing a bid cap may result in the informed player having a lower expected payoff. In other words, an information advantage may turn into a disadvantage as far as equilibrium payoffs are concerned.

Our work is related to the recent research of Siegel (2014) and Einy et al. (2014). Siegel (2014) studies general asymmetric two-player all-pay auctions with interdependent valuations, where the private information of each player is represented by a finite set of possible types. It can be shown that his framework can accommodate our common value all-pay auctions where one player has an information advantage over his opponent. Einy et al. (2014) study two-player common-value all pay auctions, but assume that no player has an information advantage over his opponent. Both works do not consider bid caps as we do here.

The rest of the paper is organized as follows. In Section 2 we present the model. In Section 3 we characterize the equilibrium in common-value all-pay auctions with high but binding bid caps. In Section 4 we show how an information advantage may turn into a payoff disadvantage when there are bid caps. In Section 5 we demonstrate that bid caps may positively affect the players' total bid. Section 6 concludes.

2 The model

Consider the set $\mathcal{N} = \{1, 2\}$ of two players who compete in an all-pay auction, where the player with the highest bid (≡effort) wins the contest but all the players bear the cost of their bid. We assume that each player can submit any non-negative bid that is lower than or equal to a given budget constraint $d > 0$. The uncertainty in our model is described by a finite set Ω of states of nature, and a probability distribution p over Ω that represents the common prior belief about the realized state of nature (w.l.o.g. $p(\omega) > 0$ for every $\omega \in \Omega$). The uncertain common value of winning the contest is given by a function $v : \Omega \to \mathbb{R}_+$, i.e., if $\omega \in \Omega$ is realized then the value of winning is $v(\omega)$ for every player. The private information of each player $i \in \mathcal{N}$ is described by a partition Π_i of Ω.

A common-value all-pay auction starts by a move of nature that chooses a state ω form Ω according to the distribution p. Each player i is informed of the element $\pi_i(\omega)$ of Π_i which contains ω (thus, $\pi_i(\omega)$ constitutes the information set of player i at ω), and then he chooses an effort $x_i \in [0, d]$. Note that when the players have different information partitions they are *ex-ante* asymmetric.

The utility (payoff) of player $i \in \mathcal{N}$ is given by the function $u_i : \Omega \times \mathbb{R}_+^2 \to \mathbb{R}$ defined as follows:

$$u_i(\omega, x) = \begin{cases} \frac{1}{m(x)} v(\omega) - x_i, & \text{if} \quad x_i = \max\{x_1, x_2\}, \\ \\ -x_i, & \text{if} \quad x_i < \max\{x_1, x_2\}, \end{cases}$$

where $m(x)$ denotes the number of players who exert the highest effort, namely, $m(x) = |i \in N : x_i = \max\{x_1, x_2\}|$. A two-player common-value all-pay auction with incomplete information is fully described by and identified with the collection $G = ((\Omega, p), u_1, u_2, \Pi_1, \Pi_2, d)$.

In all-pay auctions without a bid cap there is usually no equilibrium in pure strategies, and thus our attention will be given first to mixed strategy equilibria. A (mixed) strategy of player i is a function $F_i : \Omega \times [0, d] \to [0, 1]$, such that for every $\omega \in \Omega$, $F_i\,(,\cdot)$ is a cumulative distribution function (c.d.f.) of i's effort on $[0, d]$, and for all $x \in \mathbb{R}_+$, $F_i\,(\cdot, x)$ is a Π_i-measurable function (that is, $F_i\,(\cdot, x)$ is constant on every element of Π_i). Slightly abusing notation, for any $\pi_i \in \Pi_i$ we will denote the constant value of $F_i\,(\cdot, x)$ on π_i by $F_i\,(\pi_i, x)$, whenever convenient.

If player i's effort choice given π_i is deterministic (pure), i.e., if the distribution represented by $F_i\,(\pi_i, \cdot)$ is supported on some $y \in [0, d]$, we will identify between $F_i\,(\pi_i, \cdot)$ and y wherever appropriate. Furthermore, if i's strategy F_i is pure, i.e., if i's effort choice is deterministic in every $\omega \in \Omega$, we will identify between F_i and a function $\mathbf{x}_i : \Omega \to [0, d]$ that represents i's state-dependent (and Π_i-measurable) effort choice.

Given a strategy profile $F = (F_1, F_2)$, denote by $E_i(F)$ the expected payoff of player i when players use that strategy profile, i.e.,

$$E_i(F) \equiv E(\int_0^d \int_0^d u_i(\cdot, (x_1, x_2)) dF_1(\cdot, x_1) dF_2(\cdot, x_2)).$$

For $\pi_i \in \Pi_i$, $E_i(\pi_i, F)$ will denote the conditional expected payoff of player i given his information set π_i, i.e.,

$$E_i(\pi_i, F) \equiv E(\left[\int_0^d \int_0^d u_i(\cdot, (x_1, x_2)) dF_1(\cdot, x_1) dF_2(\cdot, x_2)\right] \mid \pi_i).$$

A profile $F^* = (F_1^*, F_2^*)$ of mixed strategies constitutes a (Bayesian Nash) equilibrium in the common-value all-pay auction G if for every player i, and every mixed strategy F_i of that player, the following inequality holds:

$$E_i(F^*) \geq E_i(F_i, F_{-i}^*),$$

where $-i$ denotes i's rival.

We will assume throughout that player 2 has an information advantage over player 1. In our framework, where the players' private information is represented by information partitions, the information advantage of player 2 over player 1 is tantamount to 2's partition Π_2 being finer than 1's partition Π_1. Let us write Ω as an indexed sequence,

$$\Omega = \{\omega_1, ..., \omega_n\}. \tag{1}$$

It can be easily shown (see Remark 1 in the Appendix) that as far as equilibrium analysis is concerned, the information advantage assumption can be reduced to the postulate that

$$\Pi_1 = \{\Omega\} \text{ and } \Pi_2 = \{\{\omega_1\}, \{\omega_1\}, ..., \{\omega_n\}\}, \tag{2}$$

i.e., that player 1 has no information on the realized state of nature (other than the common prior distribution p, and thus he has the trivial information partition), while player 2 knows the realized state precisely (and thus his information partition is the finest one possible, that consists of singletons).

In addition to (1) and (2), the following notation will be used throughout:

$$v_i = v(\omega_i) \text{ and } p_i = p(\omega_i) > 0$$

for every $i = 1, ..., n$. We will assume that the possible values of winning are distinct, and thus, w.l.o.g., strictly ranked as follows:

$$0 < v_1 < v_2 < ... < v_n.$$

3 The effect of high (but binding) bid caps

In this section we will consider the effects on equilibrium efforts of a high but still binding bid cap, compared to the no-cap benchmark. Throughout the section we will assume that the bid cap d satisfies the following:

$$\sum_{j=1}^{n-1} p_j v_j + \frac{1}{2} p_n v_n < d \leq \sum_{j=1}^{n} p_j v_j. \tag{3}$$

Without the bid cap ($d = \infty$) the contest has a unique equilibrium in mixed strategies[6], each of which is supported on $\left[0, \sum_{j=1}^{n} p_j v_j\right]$; we will refer to this contest as the *no-cap benchmark*. Thus, the upper

[6]See Section 2 in the online appendix to Siegel (2014).

boundary of the range of d in (3) is the *minimal* non-binding bid cap, under which the equilibrium strategies in the no-cap benchmark remain such when the efforts are capped. All lower caps, $d < \sum_{j=1}^{n} p_j v_j$, would lead to equilibrium strategies that are different from those of the no-cap benchmark. Accordingly, the right-hand inequality in (3) means that d is binding (unless it holds as equality), while the left-hand inequality means that d is nonetheless high.

In what follows, we describe a mixed strategy profile (F_1^*, F_2^*) of the all-pay auction with a bid cap d satisfying (3) that will turn out to be an equilibrium.

Let $x_0 \equiv 0$, and for each $i = 1, ..., n$, set

$$x_i \equiv \sum_{j=1}^{i} p_j v_j. \tag{4}$$

Thus, $x_0 < x_1 < ... < x_n$. In addition, let

$$\widetilde{x} = 2d - \sum_{j=1}^{n} p_j v_j. \tag{5}$$

Condition (3) on the cap d implies that

$$x_{n-1} < \widetilde{x} \le d \le x_n$$

(with all inequalities being *strict* if and only if $d < \sum_{j=1}^{n} p_j v_j$).

Consider now a function F_1^* on $[0, d]$ which is given by

$$F_1^*(x) = \begin{cases} \frac{x}{v_i} + \sum_{j=1}^{i-1} p_j \left[1 - \frac{v_j}{v_i}\right], & \text{if } x \in [x_{i-1}, x_i) \text{ for } i = 1, ..., n-1, \\ \frac{x}{v_n} + \sum_{j=1}^{n-1} p_j \left[1 - \frac{v_j}{v_n}\right], & \text{if } x \in [x_{n-1}, \widetilde{x}), \\ \frac{\widetilde{x}}{v_n} + \sum_{j=1}^{n-1} p_j \left[1 - \frac{v_j}{v_n}\right], & \text{if } x \in [\widetilde{x}, d), \\ 1, & \text{if } x = d. \end{cases}$$

It is easy to see that F_1^* is well defined, strictly increasing on $[0, \widetilde{x})$, continuous on $[0, d)$, and constant on $[\widetilde{x}, d)$. Moreover, $F_1^*(0) = 0$ and $F_1^*(d) = 1$. Thus, F_1^* is a c.d.f. of a probability distribution with full support on $[0, \widetilde{x}] \cup \{d\}$ (and a unique atom at d) if $d < \sum_{j=1}^{n} p_j v_j$, and a continuous distribution with full support on $[0, \widetilde{x}]$ if $d = \sum_{j=1}^{n} p_j v_j$ $(= \widetilde{x} = x_n)$. As the function F_1^* is state-independent, it can be viewed as a mixed strategy of the uninformed player 1.

Next, for each $i = 1, ..., n-1$, consider a function $F_2^*(\omega_i, \cdot)$ on $[0, d]$ which is given by

$$F_2^*(\omega_i, x) = \begin{cases} 0, & \text{if } x < x_{i-1}, \\ \frac{x - \sum_{j=1}^{i-1} p_j v_j}{p_i v_i}, & \text{if } x \in [x_{i-1}, x_i), \\ 1, & \text{if } x \geq x_i, \end{cases}$$

and for $i = n$, a function $F_2^*(\omega_n, \cdot)$ given by

$$F_2^*(\omega_n, x) = \begin{cases} 0, & \text{if } x < x_{n-1}, \\ \frac{x - \sum_{j=1}^{n-1} p_j v_j}{p_n v_n}, & \text{if } x \in [x_{n-1}, \widetilde{x}), \\ \frac{\widetilde{x} - \sum_{j=1}^{n-1} p_j v_j}{p_n v_n}, & \text{if } x \in [\widetilde{x}, d), \\ 1, & \text{if } x = d. \end{cases}$$

Note that for $i = 1, ..., n-1$, $F_2^*(\omega_i, \cdot)$ is well defined, strictly increasing and continuous on $[x_{i-1}, x_i]$, and $F_2^*(\omega_i, x_{i-1}) = 0$ and $F_2^*(\omega_i, x_i) = 1$. Thus, $F_2^*(\omega_i, \cdot)$ is a c.d.f. of a continuous probability distribution with full support on $[x_{i-1}, x_i]$. In addition, the function $F_2^*(\omega_n, \cdot)$ is well defined, strictly increasing on $[x_{n-1}, \widetilde{x})$, continuous on $[0, d)$, and constant on $[\widetilde{x}, d)$. Moreover, $F_2^*(\omega_n, x_{n-1}) = 0$ and $F_2^*(\omega_n, d) = 1$. Thus, $F_2^*(\omega_n, \cdot)$ is a c.d.f. of a probability distribution with full support on $[x_{n-1}, \widetilde{x}] \cup \{d\}$ (and a unique atom at d) if $d < \sum_{j=1}^{n} p_j v_j$, and a continuous distribution with full support on $[x_{n-1}, \widetilde{x}]$ if $d = \sum_{j=1}^{n} p_j v_j$ $(= \widetilde{x} = x_n)$. In particular, F_2^* constitutes a mixed strategy of player 2.

Proposition 1 *The strategy profile (F_1^*, F_2^*) is a mixed strategy Bayesian Nash equilibrium in a common value all-pay auction with a bid cap satisfying (3).*

Proof. Note that

$$E_2(\{\omega_i\}, F_1^*, x) = v_i F_1^*(x) - x = v_i F_1^*(x_{i-1}) - x_{i-1} = E_2(\{\omega_i\}, F_1^*, x_{i-1})$$

for every $i = 1, ..., n-1$ and $x \in [x_{i-1}, x_i)$, or $i = n$ and $x \in [x_{n-1}, \widetilde{x})$. Thus, given that ω_i was realized, the informed player 2 is indifferent between all efforts in the interval $[x_{i-1}, x_i)$ if $i = 1, ..., n-1$, and is indifferent between all efforts in the interval $[x_{n-1}, \widetilde{x})$ if $i = n$, provided that his rival acts according to F_1^*.

Since the slopes of the function $v_i F_1^*(x) - x = E_2(\{\omega_i\}, F_1^*, x)$ are positive when $0 \leq x < x_{i-1}$ and negative when $\min(x_i, \widetilde{x}) < x < d$, the set of player 2's pure best responses (given ω_i) *when the bids are constrained to be less then* d contains the interval $[x_{i-1}, x_i)$ if $i = 1, ..., n-1$, and contains $[x_{n-1}, \widetilde{x})$ if $i = n$.

Now note that the following holds for $x = d$:

$$
\begin{aligned}
E_2(\{\omega_i\}, F_1^*, d) &= v_i(F_1^*(\widetilde{x}) + \frac{1 - F_1^*(\widetilde{x})}{2}) - d = v_i(\frac{1 + F_1^*(\widetilde{x})}{2}) - d \\
&= \frac{v_i}{v_n}(\frac{1}{2}v_n + \frac{2d - \sum_{j=1}^{n} p_j v_j + \sum_{j=1}^{n-1} p_j [v_n - v_j]}{2}) - d \\
&= \frac{d(v_i - v_n)}{v_n} - \frac{v_i}{v_n}(\sum_{j=1}^{n-1} p_j v_j - \frac{1}{2}v_n(1 - p_n + \sum_{j=1}^{n-1} p_j)) \\
&= d\frac{(v_i - v_n)}{v_n} + \frac{v_i}{v_n}\sum_{j=1}^{n-1} p_j [v_n - v_j].
\end{aligned} \tag{6}
$$

Thus, if $i < n$, by using (6) and (3) we obtain

$$
\begin{aligned}
E_2(\{\omega_i\}, F_1^*, d) &\leq \frac{1}{v_n}(\sum_{j=1}^{n-1} p_j v_j + \frac{1}{2}p_n v_n)(v_i - v_n) + \frac{v_i}{v_n}\sum_{j=1}^{n-1} p_j [v_n - v_j] \\
&= (v_i - v_n)\frac{1}{2}p_n + \sum_{j=1}^{n-1} p_j(v_i - v_j).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
E_2(\{\omega_i\}, F_1^*, x_{i-1}) &= v_i F_1^*(x_{i-1}) - x_{i-1} \\
&= v_i(\frac{x_{i-1}}{v_i} + \sum_{j=1}^{i-1} p_j \frac{(v_i - v_j)}{v_i}) - x_{i-1} \\
&= \sum_{j=1}^{i-1} p_j(v_i - v_j),
\end{aligned} \tag{7}
$$

and hence

$$
\begin{aligned}
& E_2(\{\omega_i\}, F_1^*, d) - E_2(\{\omega_i\}, F_1^*, x_i) \\
\leq\ & (v_i - v_n)\frac{1}{2}p_n + \sum_{j=i+1}^{n-1} p_j(v_i - v_j) < 0.
\end{aligned}
$$

It follows that the set of player 2's pure best responses (given ω_i) contains the interval $[x_{i-1}, x_i)$ if $i = 1, ..., n-1$. Furthermore, if $i = n$, then it follows from (6) and (7) (the latter does not require $i < n$) that

$$
E_2(\{\omega_n\}, F_1^*, d) = \sum_{j=1}^{n-1} p_j [v_n - v_j] = E_2(\{\omega_n\}, F_1^*, x_{n-1});
$$

hence the set of player 2's pure best responses (given ω_n) contains the set $[x_{n-1}, \widetilde{x}) \cup \{d\}$.

Therefore, the mixed strategy F_2^* of player 2 is his best response against player 1's strategy F_1^*, since it has been shown that for each $i = 1, ..., n$, the distribution corresponding to $F_2^*(\omega_i, \cdot)$ is supported on a set of 2's pure best responses to F_1^* conditional on ω_i.

Next, observe that

$$E_1(x, F_2^*) = \sum_{j=1}^{i-1} p_j v_j + p_i v_i F_2^*(\omega_i, x) - x = 0 \tag{8}$$

for every $i = 1, ..., n-1$ and $x \in [x_{i-1}, x_i)$, or $i = n$ and $x \in [x_{n-1}, \widetilde{x})$. Note also that

$$\begin{aligned} E_1(d, F_2^*) &= \sum_{i=1}^{n-1} p_i v_i + p_n v_n (F_2^*(\omega_n, \widetilde{x}) + \frac{1 - F_2^*(\omega_n, \widetilde{x})}{2}) - d \\ &= \sum_{i=1}^{n-1} p_i v_i + p_n v_n (\frac{\widetilde{x} - \sum_{j=1}^{n-1} p_j v_j}{2 p_n v_n} + \frac{1}{2}) - d = 0. \end{aligned} \tag{9}$$

It follows from (8) and (9) that player 1 is (in expectation) indifferent between all efforts in $[0, \widetilde{x}) \cup \{d\}$ (and is obviously worse off when efforts are outside $[0, \widetilde{x}) \cup \{d\}$) provided his rival 2 acts according to F_2^*. Thus the mixed strategy F_1^* of player 1 is his best response against player 2's strategy F_2^*, since it has been shown that the distribution corresponding to F_1^* is supported on a set of 1's pure best responses to F_2^*.

We conclude that (F_1^*, F_2^*) is indeed a mixed strategy equilibrium. ∎

Proposition 1 can be strengthened. Call mixed strategy F_2 of player 2 *monotone* if, for every $i = 1, ..., n$, there is a set $A_i \subset [0, d]$ such that the distribution corresponding to $F_2^*(\omega_i, \cdot)$ is supported on A_i, and the sets $A_1, ..., A_n$ are "ordered" on $[0, d]$ according to their index: if $1 \leq i < j \leq n$ and $x \in A_i$, $y \in A_j$, then $x \leq y$. This means that the bids of player 2 are (weakly) increasing in the value v_i of winning (which he knows by assumption). Note that the strategy F_2^* of player 2 is monotone.

Proposition 2 *The mixed strategy equilibrium (F_1^*, F_2^*) of Proposition 1 is the* unique *Bayesian Nash equilibrium in a common value all-pay auction with a bid cap satisfying (3) in which the strategy of the informed player 2 is monotone.*

Proof. See the Appendix. ∎

The inclusion of the no-cap benchmark contest (corresponding to the case of $d = \sum_{j=1}^{n} p_j v_j$, that was studied in Siegel (2014)) in our framework allows direct comparisons of expected equilibrium efforts and of ex-ante probabilities of winning in equilibrium in the scenario with and without caps. According to the next proposition, sufficiently high but binding caps make no difference as far as the expected effort is concerned:

Proposition 3 *The expected effort that each player exerts in the unique equilibrium of a common-value all-pay auctions with a bid cap d satisfying (3) is independent of d, and in particular is equal to the effort exerted in the no-cap benchmark auction.*

Proof. Player 1's expected effort when he uses F_1^* is

$$TE_1 = \int_{[x_0,x_{n-1})} x dF_1^*(x) + \int_{[x_{n-1},\widetilde{x})} x dF_1^*(x) + d\cdot(1 - F_1^*(\widetilde{x})) \tag{10}$$

$$= \sum_{i=1}^{n-1}\int_{x_{i-1}}^{x_i}\frac{x}{v_i}dx + \int_{x_{n-1}}^{\widetilde{x}}\frac{x}{v_n}dx + d\cdot(1 - \frac{\widetilde{x}}{v_n} - \sum_{j=1}^{n-1}p_j\left[1-\frac{v_j}{v_n}\right])$$

$$= \sum_{i=1}^{n-1}\frac{x_i^2 - x_{i-1}^2}{2v_i} + \frac{(2d-\sum_{j=1}^{n}p_jv_j)^2 - x_{n-1}^2}{2v_n} + d\cdot(1 - \frac{2d-\sum_{j=1}^{n}p_jv_j}{v_n} - \sum_{j=1}^{n-1}p_j\left[1-\frac{v_j}{v_n}\right]) \tag{11}$$

$$= \sum_{i=1}^{n}\frac{x_i^2 - x_{i-1}^2}{2v_i} = \sum_{i=1}^{n}p_i\left(\sum_{j=1}^{i-1}p_jv_j + \frac{1}{2}p_iv_i\right),$$

and player 2's expected effort when he uses F_2^* is

$$TE_2 = \sum_{i=1}^{n-1}p_i\int_{[x_{i-1},x_i)} x dF_2^*(\omega_i, x) + p_n\cdot(\int_{[x_{n-1},\widetilde{x})} x dF_2^*(\omega_n, x) + d\cdot(1 - F_2^*(\omega_n,\widetilde{x})) \tag{12}$$

$$= \sum_{i=1}^{n-1}p_i\int_{x_{i-1}}^{x_i}\frac{x}{p_iv_i}dx + p_n\cdot(\int_{x_{n-1}}^{\widetilde{x}}\frac{x}{p_nv_n}dx + d\cdot(1 - \frac{\widetilde{x}-\sum_{j=1}^{n-1}p_jv_j}{p_nv_n})$$

$$= \sum_{i=1}^{n-1}\int_{x_{i-1}}^{x_i}\frac{x}{v_i}dx + \int_{x_{n-1}}^{\widetilde{x}}\frac{x}{v_n}dx + d\cdot(p_n - \frac{2d-\sum_{j=1}^{n}p_jv_j - \sum_{j=1}^{n-1}p_jv_j}{v_n}) \tag{13}$$

$$= \sum_{i=1}^{n}p_i\left(\sum_{j=1}^{i-1}p_jv_j + \frac{1}{2}p_iv_i\right)$$

(the last equality is obtained just as in the computation of TE_1 above, since expression (13) is identical to expression (11)). Thus, both TE_1 and TE_2 are independent of d. ■

The proof of Proposition 3 reveals that the players' ex-ante expected efforts are the same. The expected *total* effort is therefore

$$TE = 2\sum_{i=1}^{n}p_i\left(\sum_{j=1}^{i-1}p_jv_j + \frac{1}{2}p_iv_i\right), \tag{14}$$

and, just as the individual's expected effort, it is *independent of the bid cap* d (provided (3) holds). This generalizes the result of Che and Gale (1998) on the effect of high but binding caps in all-pay auctions with complete information (which, in our framework, is captured by the information structure where the set Ω consists of just one state of nature).

The fact that the expected efforts of both players are identical (though not their equality across d) can also be obtained as a corollary of the following observation, which has already been noted by Siegel (2014) for the no-cap benchmark auction. It turns out that the ex-ante distribution of equilibrium effort is identical for both players:

Proposition 4 *In the unique equilibrium* (F_1^*, F_2^*) *of a common-value all-pay auctions with a bid cap* d *satisfying (3), the* ex ante *distribution of equilibrium effort of player 2 is identical to* F_1^**. In particular, each player wins with (ex-ante) probability* $\frac{1}{2}$*.*

Proof. Denote by $F_2(x)$ the ex-ante probability that player 2 exerts an effort that is smaller than or equal to x when acting according to his strategy F_2^*. For every $i = 1, ..., n-1$ and $x \in [x_{i-1}, x_i]$, or $i = n$ and $x \in [x_{n-1}, \widetilde{x})$,

$$\begin{aligned} F_2(x) &= \sum_{j=1}^{i-1} p_j + p_i F_2^*(\omega_i, x) = \sum_{j=1}^{i-1} p_j + p_i \cdot \frac{x - \sum_{j=1}^{i-1} p_j v_j}{p_i v_i} \\ &= \frac{x}{v_i} + \sum_{j=1}^{i-1} p_j \left[1 - \frac{v_j}{v_i}\right] = F_1^*(x). \end{aligned}$$

When $x \in [\widetilde{x}, d)$,

$$\begin{aligned} F_2(x) &= \sum_{j=1}^{n-1} p_j + p_n F_2^*(\omega_i, \widetilde{x}) = \sum_{j=1}^{n-1} p_j + p_n \cdot \frac{\widetilde{x} - \sum_{j=1}^{n-1} p_j v_j}{p_n v_n} \\ &= \frac{\widetilde{x}}{v_n} + \sum_{j=1}^{n-1} p_j \left[1 - \frac{v_j}{v_n}\right] = F_1^*(x), \end{aligned}$$

and clearly $F_2(d) = F_1^*(d)$. Thus, the ex-ante distribution of equilibrium effort of player 2 is identical to the distribution of equilibrium effort of player 1.

Since player 1's strategy F_1^* is state-independent, his chance to win only depends on the *ex-ante* distribution of his rival efforts, i.e., on F_2. But it has been shown that $F_2 = F_1^*$, and hence the (ex-ante) probability to win is $\frac{1}{2}$ for each player. ■

4 Possible disadvantage of information advantage

With sufficiently high but still binding bid caps (described in the previous section) the expected payoff of the uninformed player is zero in equilibrium, while his informed rival's expected payoff is positive. The following example demonstrates the effect of lower bid caps on the players' equilibrium strategies, and, in particular, shows that for certain bid caps the expected payoff to the uninformed player 1 is higher than for the informed player 2.

Example 1 *Assume that* $n = 3$*, and that in state* ω_i *the value of winning is* $v(\omega_i) = i$ *with probability of* $p_i = \frac{1}{3}, i = 1, 2, 3$*. Assume also that the players have the same bid cap* $d = \frac{5}{6}$*. Then the following constitutes a pure strategy Bayesian equilibrium: player 1's bid is independent of the state of nature,* $\mathbf{x}_1^* \equiv \frac{5}{6}$*, and player 2's state-dependent bid is given by*

$$\mathbf{x}_2^*(\omega) = \begin{cases} 0, & \text{if } \omega = \omega_1, \\ \frac{5}{6}, & \text{if } \omega \neq \omega_1 \end{cases}.$$

The expected payoff of player 1 is then

$$E_1 = \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot \frac{1}{2} \cdot 2 + \frac{1}{3} \cdot \frac{1}{2} \cdot 3 - \frac{5}{6} = \frac{1}{3},$$

and the expected payoff of player 2 is

$$E_2 = \frac{1}{3} \cdot (\frac{1}{2} \cdot 2 - \frac{5}{6}) + \frac{1}{3} \cdot (\frac{1}{2} \cdot 3 - \frac{5}{6}) = \frac{5}{18}.$$

Thus, the expected payoff of the uninformed player 1 is higher than that of the informed player 2.

In the next proposition we describe some sufficient conditions (extending Example 1) under which this unusual result of a higher expected payoff to the uninformed player can be obtained.

Proposition 5 *Consider a common-value all-pay auction with a bid cap* d*. Suppose that there exists* $1 \leq j \leq n - 1$ *such that*

(i)

$$\frac{1}{2} \sum_{m=j+1}^{n} p_m v_m \geq d, \tag{15}$$

(ii)

$$\frac{v_j}{2} \leq d \leq \frac{v_{j+1}}{2}, \tag{16}$$

and

(iii)

$$\sum_{m=1}^{j} p_m (v_m - d) \geq 0. \tag{17}$$

Then there exists a pure strategy Bayesian Nash equilibrium in which the expected payoff of the uninformed player (player 1) is higher than that of the informed player (player 2).

Proof. Consider a pure strategy profile $(\mathbf{x}_1^*, \mathbf{x}_2^*)$ in which

$$\mathbf{x}_1^* \equiv d, \tag{18}$$

and

$$\mathbf{x}_2^*(\omega_k) = \begin{cases} 0, & \text{for } \ k = 1, ..., j, \\ d, & \text{for } k = j+1, ..., n \end{cases}. \tag{19}$$

The expected payoff of player 2 when $(\mathbf{x}_1^*, \mathbf{x}_2^*)$ is played is

$$E_2 \left(\mathbf{x}_1^*, \mathbf{x}_2^*\right) = \sum_{m=j+1}^{n} p_m(\frac{1}{2}v_m - d), \tag{20}$$

and the expected payoff of player 1 is

$$E_1 \left(\mathbf{x}_1^*, \mathbf{x}_2^*\right) = \sum_{m=1}^{j} p_m(v_m - d) + \sum_{m=j+1}^{n} p_m(\frac{1}{2}v_m - d). \tag{21}$$

We will now check that $(\mathbf{x}_1^*, \mathbf{x}_2^*)$ is an equilibrium. If player 1 submits a bid of $0 \leq x_1 < d$, then by (15)

$$E_1 \left(x_1, \mathbf{x}_2^*\right) \leq \sum_{m=1}^{j} p_m v_m - x_1 \leq E_1 \left(\mathbf{x}_1^*, \mathbf{x}_2^*\right).$$

Thus $\mathbf{x}_1^*$ is player 1's best response to $\mathbf{x}_2^*$.

If player 2 unilaterally deviates from $\mathbf{x}_2^*$ to a strategy $\mathbf{x}_2$ with $0 < \mathbf{x}_2\left(\omega_k\right) = \varepsilon \leq d$ for some $1 \leq k \leq j$, then

$$\begin{aligned} E_2 \left(\left\{\omega_k\right\}, \mathbf{x}_1^*, \mathbf{x}_2\right) &\leq \max\left(-\varepsilon, \frac{v_k}{2} - d\right) \\ &\leq 0 = E_2 \left(\left\{\omega_k\right\}, \mathbf{x}_1^*, \mathbf{x}_2^*\right), \end{aligned}$$

where the second inequality is implied by (16). But, also by (16), if player 2 unilaterally deviates from $\mathbf{x}_2^*$ to a strategy $\mathbf{x}_2$ with $0 \leq \mathbf{x}_2\left(\omega_k\right) = \varepsilon < d$ for some $j+1 \leq k \leq n$, then

$$\begin{aligned} E_2 \left(\left\{\omega_k\right\}, \mathbf{x}_1^*, \mathbf{x}_2\right) &= -\varepsilon \\ &\leq 0 \leq \frac{v_k}{2} - d = E_2 \left(\left\{\omega_k\right\}, \mathbf{x}_1^*, \mathbf{x}_2^*\right). \end{aligned}$$

By taking expectation over ω_k, it follows that

$$E_2 \left(\mathbf{x}_1^*, \mathbf{x}_2\right) \leq E_2 \left(\mathbf{x}_1^*, \mathbf{x}_2^*\right)$$

for *any* pure strategy $\mathbf{x}_2$ of player 2 that obeys his bid cap, and thus $\mathbf{x}_2^*$ is player 2's best response to $\mathbf{x}_1^*$.

We conclude that $(\mathbf{x}_1^*, \mathbf{x}_2^*)$ is an equilibrium. By comparing the players' expected payoffs given by (20) and (21), we obtain by (17) that the expected payoff of player 1 is higher than that of player 2. *Q.E.D.* ■

5 The advantage of bid caps

Che and Gale (1998) have shown that low bid caps may lead to higher efforts by players. This occurs in an equilibrium where both players make bids that are equal to the bid cap. We will first examine whether this can also happen in our common-value all-pay auctions with incomplete information.

Assume that both players make a bid that is equal to the bid cap in all the states of nature:

$$\begin{aligned}\mathbf{x}_1^* &\equiv d,\\ \mathbf{x}_2^*(\omega_j) &= d, \;\; j = 1, ..., n.\end{aligned}$$

The players' expected payoffs when $(\mathbf{x}_1^*, \mathbf{x}_2^*)$ is played are:

$$E_1\left(\mathbf{x}_1^*, \mathbf{x}_2^*\right) = \sum_{j=1}^{n} \frac{p_j v_j}{2} - d,$$

and, for $j = 1, 2, ..., n$,

$$E_2\left(\{\omega_j\}, \mathbf{x}_1^*, \mathbf{x}_2^*\right) = \frac{v_j}{2} - d.$$

The necessary and sufficient condition for $(\mathbf{x}_1^*, \mathbf{x}_2^*)$ to be an equilibrium is $d \leq \frac{v_1}{2}$, since then both players have non-negative expected payoffs in all states of nature, and therefore none of them has an incentive to deviate from the above strategies. The total effort in this equilibrium is equal to $2d$ $(\leq v_1)$. On the other hand, in the common-value all-pay auction without a bid cap of Siegel (2014) (which is equivalent to the assumption that $d = \sum_{j=1}^{n} p_j v_j$, considered among other cases in Section 3), the players' expected total effort is given by (14), and hence

$$TE = 2\sum_{i=1}^{n} p_i \left(\sum_{j=1}^{i-1} p_j v_j + \frac{1}{2} p_i v_i \right) > 2\sum_{i=1}^{n} p_i \left(\sum_{j=1}^{i-1} p_j v_1 + \frac{1}{2} p_i v_1 \right) = v_1.$$

Thus, a simple adaptation of the bid that is equal to the bid cap from the complete information case does not increase the expected total effort. The following result, however, demonstrates that a bid cap may increase the players' expected effort compared to the no-cap benchmark in some equilibria.

Proposition 6 *The players' expected total effort in a common-value all-pay auction with a bid cap may be higher than in the same auction without a bid cap.*

Proof. Consider a common-value all-pay auction with a bid cap d. Suppose that there are two states of nature (i.e., $n = 2$) that occur with probabilities (p_1, p_2), and that

$$\frac{v_1}{2} \leq d \leq \frac{p_2 v_2}{2}.$$

As in Proposition 5 there is a pure strategy Bayesian Nash equilibrium $(\mathbf{x}_1^*, \mathbf{x}_2^*)$, in which

$$\mathbf{x}_1^* \equiv d,$$

and

$$\mathbf{x}_2^*(\omega_i) = \begin{cases} 0, & \text{for } i = 1, \\ d, & \text{for } i = 2 \end{cases}.$$

The players' expected total effort is then

$$\widehat{TE} = (1 + p_2)d.$$

On the other hand, by (14), in the common-value all-pay auction without a bid cap the player' expected total effort is given by

$$TE = v_1(p_1^2 + 2p_1p_2) + p_2^2 v_2.$$

Since $\frac{(1+p_2)p_2}{2} > p_2^2$ for all $p_2 < \frac{1}{2}$, we obtain that for such p_2, d that is close to $\frac{p_2 v_2}{2}$, and a sufficiently low value of v_1, $\widehat{TE} > TE$. *Q.E.D.* ■

6 Concluding remarks

In this paper we generalize some of the results of Che and Gale (1998), who studied all-pay auctions with bid caps under the complete information assumption. We consider common-value all-pay auctions with asymmetric information and show that high (but still binding) levels of bid caps do not change the players' expected efforts, but other levels of bid caps may drastically affect them, and in particular increase the total expected effort compared to the no-cap benchmark. We also show that if players face bid caps, the information advantage might become a disadvantage as far as equilibrium payoffs are concerned in that the player with the information advantage may have a lower expected payoff than his opponent. This unusual result implies that by imposing bid caps, the contest designer can control the relation between the players' expected payoffs.

We provide a partial characterization of the equilibrium strategies for high (but binding) levels of bid caps. It is not clear, however, whether equilibrium exists, or is unique, for other levels of bid caps. The study of all-pay auctions under asymmetric information with and without bid caps remains an important endeavor, and much more work is needed for a complete analysis.

7 Appendix

Remark 1

As far as equilibrium analysis is concerned, the assumption that player 2 has an information advantage over player 1 can be reduced to (2). Indeed, in the general case of Π_2 being finer than Π_1 note the following. Given $\pi_1 \in \Pi_1$, the event π_1 is common knowledge at any $\omega \in \pi_1$. Thus, the equilibrium analysis can be carried out separately for each $\pi_1 \in \Pi_1$, as conditional on the occurrence of π_1 the auction G can be viewed as a distinct common-value all-pay auction G', where the set of states of nature is $\Omega' = \pi_1$ and the conditional distribution $p(\cdot \mid \pi_1)$ serves as the common prior distribution p'. In G', player 1 has the trivial information partition, $\Pi_1' = \{\Omega'\}$.

Thus, we may w.l.o.g. assume that $\Pi_1 = \{\Omega\}$ in the original auction G. Now consider another common-value all-pay auction G'' in which the set of states of nature $\Omega'' \equiv \Pi_2$ consists of information sets of player 2 in G; the common value function v'' is given by $v''(\pi_2) = E(v \mid \pi_2)$ for every $\pi_2 \in \Omega''$; player 2 has the full information partition, consisting of all singleton subsets of Ω'', i.e., $\Pi_2'' = \{\{\pi_2\} \mid \pi_2 \in \Omega''\}$; and player 1 has the trivial information partition $\Pi_1'' = \{\Omega''\}$. Naturally, every mixed strategy $F_2(\cdot,\cdot)$ of player 2 in G corresponds to a unique mixed strategy $F_2''(\cdot,\cdot)$ of 2 in G'' that is given by $F_2''(\pi_2,\cdot) = F_2(\omega,\cdot)$

for every $\pi_2 \in \Omega''$ and every $\omega \in \pi_2$ (which is well-defined because of Π_2-measurability of F_2 in its first coordinate), and the mixed strategies of player 1 in G and G'' are identical as they are state-independent. The correspondence $(F_1, F_2) \leftrightarrow (F_1, F_2'')$ between mixed strategy-profiles in G and G'' obviously preserves both players' expected payoffs. Thus, G and G'' are payoff-equivalent, and hence identical in terms of equilibrium analysis. However, G'' has the property that player 1 has no information on the realized state of nature while player 2 is completely informed of it. Thus, in studying equilibria in two-player common-value all-pay auctions, attention can be confined w.l.o.g. to auctions with the latter property.

Proof of Proposition 2

Fix an equilibrium (F_1, F_2) in the auction G, in which F_2 is monotone. We will prove that $(F_1, F_2) = (F_1^*, F_2^*)$.

In what follows, for $k = 1, 2$ and $\omega \in \Omega$, $F_k(\omega, \cdot)$ will be treated either as a *probability measure* on $[0, d]$, or as the corresponding *c.d.f.*, sometimes in the same context. Accordingly, for $A \subset [0, d]$, $F_k(\omega, A)$ will stand for the probability that the equilibrium effort of player k, conditional on ω, belongs of the set A, and for $x \in [0, d]$, $F_k(\omega, x)$ will stand for the probability that the equilibrium effort of player k, conditional on ω, is less or equal to x. Also, as F_1 is state-independent, $F_1(\omega, \cdot)$ will be shortened to $F_1(\cdot)$, whenever convenient.

Notice that $F_k(\cdot, \{c\}) \equiv 0$ for any effort $c \in (0, d)$ and $k = 1, 2$. Indeed, if $F_k(\omega, \{c\}) > 0$ for some $c \in (0, d)$, k and ω, then $F_{-k}(\omega', (c - \varepsilon, c]) = 0$ for the other player $-k$ and every $\omega' \in \Omega$, and some sufficiently small $\varepsilon > 0$. But then k would be strictly better off by shifting the probability from c to $c - \frac{\varepsilon}{2}$, a contradiction to F_k being an equilibrium strategy. Thus, $F_1(\cdot), F_2(\omega, \cdot)$ are non-atomic on $(0, d)$ for every $\omega \in \Omega$. Notice also that there is no interval $(a, b) \subset (0, d)$ on which in some state of nature *only one* player places a positive probability according to his equilibrium strategy. Indeed, otherwise there would exist $a' > 0$ such that only one player places a positive probability on (a', b), and it would then be profitable for that player to deviate (in at least one state of nature, if this is the informed player 2) by shifting a positive probability from (a', b) to a'.

Suppose now that there is a (non-degenerate) interval $(a, b) \subset (0, d)$ such that $F_1((a, b)) = 0$ (and thus $F_2(\omega, (a, b)) = 0$ for every $\omega \in \Omega$, by the previous paragraph), but $F_1([b, d)) > 0$. By increasing

$b < d$ if necessary, it can also be assumed that (a, b) is maximal with respect to this property, i.e., that $F_1([b, b+\varepsilon)) > 0$ for every small enough $\varepsilon > 0$. However, the expected payoff of player 1 at $\frac{a+b}{2}$ is strictly bigger than his payoff for any effort in $[b, b+\varepsilon)$, if $\varepsilon > 0$ is small enough. This contradicts the assumption that F_1 is an equilibrium strategy, and shows that there exists no interval (a, b) as above. Non-existence of such (a, b) together with non-atomicity of $F_1(\cdot)$ on $(0, d)$ imply that there exist $0 \leq \beta \leq d$ such that $F_1(\cdot)$ is supported on $[0, \beta]_- \cup \{d\}$ (where $[0, \beta]_-$ denotes the interval $[0, \beta)$ if $\beta > 0$ and $[0, 0]_- = \{0\}$), and *either* the restriction of $F_1(\cdot)$ to $[0, \beta]_-$ has *full support* on it, *or* $F_1(\cdot)$ is concentrated on d (i.e., $F_1(\{d\}) = 1$).

Note also that for every $\omega \in \Omega$, $F_2(\omega, \cdot)$ must also be supported on $[0, \beta]_- \cup \{d\}$ (though there need not be full support on $[0, \beta]_-$) if $F_1(\{d\}) < 1$, and on $\{0\} \cup \{d\}$ if $F_1(\{d\}) = 1$, since otherwise there would be an open subinterval of $(0, d)$ where only player 2 places positive probability, and this was ruled out.

Observe next that $F_1(\cdot)$ cannot be concentrated on d. For otherwise each $F_2(\omega_i, \cdot)$ would be supported on $\{0\} \cup \{d\}$. It cannot be that *all* $F_2(\omega_i, \cdot)$ are concentrated 0, or *all* $F_2(\omega_i, \cdot)$ are concentrated on d (in the first case, player 1 would have a profitable deviation to a positive effort near 0, and in the second case player 1 would have a profitable deviation to 0 since his expected payoff under F_1 would be negative, as follow from assumption (3)). Thus, since the strategy F_2 is, by assumption, monotonic, there exists i' such that $F_2(\omega_{i'}, \cdot)$ is supported on $\{0\} \cup \{d\}$, whereas $F_2(\omega_i, \cdot)$ is concentrated on 0 for $i < i'$and $F_2(\omega_i, \cdot)$ is concentrated on d for $i > i'$. If $i' \neq n$, however, F_1 (which is concentrated on d) gives 1 a negative expected payoff by assumption (3) on d, and thus cannot be a best response. We conclude that $i' = n$. But then, by lowering his effort from d to a positive effort near zero, player 1 will save almost the entire cost of effort d, while his loss of utility from winning will not exceed $\frac{1}{2} p_n v_n$. By assumption (3), this will be a profitable deviation, in contradiction to F_1 being an equilibrium strategy. Thus, F_1 cannot be concentrated on d, and consequently the restriction of $F_1(\cdot)$ to $[0, \beta]_-$ has full support on it.

Note further that the interval $[0, \beta]_-$ is non-degenerate, i.e., that $0 < \beta$. Indeed, if $\beta = 0$ then both equilibrium strategies prescribe mixtures of effort 0 and effort d. It was shown that $F_1(\cdot)$ is not concentrated on d, and thus it is supported on $\{0\} \cup \{d\}$ with $F_1(\{0\}) > 0$. Then the only strategy F_2 of 2 that may constitute a best response to F_1 would prescribe d with probability 1 at every $\omega \in \Omega$, and then F_1 must be concentrated on 0 (since choosing d will give 1 a negative expected payoff by assumption (3) on d). However, no strategy of 2 can be a best response to such F_1, a contradiction. We conclude that, indeed, $\beta > 0$, and the interval $[0, \beta]_- = [0, \beta)$ is non-degenerate.

In keeping with our earlier notation, for $0 \leq a \leq b \leq d$ denote by $[a, b]_-$ the interval $[a, b)$ if $b > 0$ and the set $\{0\}$ if $a = b = 0$. Given $i = 1, ..., n$, we will now show that there is a (possibly empty or degenerate) subinterval $[a_i, b_i]_-$ of $[0, \beta)$ such that $F_2(\omega_i, \cdot)$ is supported on $[a_i, b_i]_- \cup \{d\}$, and, if $F_2(\omega_i, \{d\}) < 1$, the restriction of $F_2(\omega_i, \cdot)$ to $[0, \beta)$ has *full support* on $[a_i, b_i]_-$. If $F_2(\omega_i, \cdot)$ is supported on $\{0\} \cup \{d\}$, the claim obviously holds for $0 < a_i = b_i$ (an empty $[a_i, b_i]_-$), or $a_i = b_i = 0$. Assume now that $F_2(\omega_i, \cdot)$ is not supported on $\{0\} \cup \{d\}$, but the claim does not hold. Then there are $0 < a < b < \beta$ such that $F_2(\omega_i, (a, b)) = 0$, but $F_2(\omega_i, [0, a]) > 0$ and $F_2(\omega_i, [b, \beta)) > 0$. Since $F_1((a, b)) > 0$, there must be $j \neq i$ such that $F_2(\omega_j, (a, b)) > 0$. Assume that $i < j$ (the opposite case is treated similarly). Then there are $x \in [b, \beta)$ and $y \in (a, b)$ such that

$$\begin{aligned} v_i F^1(x) - x &= E_2\left(\{\omega_i\}, F^1, x\right) && (22) \\ &\geq E_2\left(\{\omega_i\}, F^1, y\right) = v_i F^1(y) - y && (23) \end{aligned}$$

and

$$\begin{aligned} v_j F^1(x) - x &= E_2\left(\{\omega_j\}, F^1, x\right) && (24) \\ &\leq E_2\left(\{\omega_j\}, F^1, y\right) = v_j F^1(y) - y. && (25) \end{aligned}$$

But $x > y$, and therefore

$$(v_j - v_i) F^1(x) > (v_j - v_i) F^1(y) \tag{26}$$

since $v_i < v_j$ and the c.d.f. F^1 is strictly increasing on $[0, \beta)$ (as the restriction of this distribution to $[0, \beta)$ has full support). Adding (26) to the inequality in (22)-(23) contradicts the inequality obtained in (24)-(25), and therefore no such (a, b) exists. Consequently, the restriction of each $F_2(\omega_i, \cdot)$ to $[0, \beta)$ has full support on some $[a_i, b_i]_-$.

If there is $1 \leq i \leq n$ such that $F_2(\omega_i, \{d\}) > 0$, denote by i' the smallest index with this property. Since F_2 is a monotone strategy, $F_2(\omega_i, \cdot)$ is concentrated on d for every $i > i'$, and hence, if $i' < n$, $F_2(\omega_n, \cdot)$ is concentrated on d. Thus

$$E_1(d', F_2) \leq \sum_{j=1}^{n-1} p_j v_j + \frac{1}{2} p_n v_n - d' < 0$$

by assumption (3) for all efforts $d' \leq d$ that are sufficiently close to d. Consequently, the equilibrium strategy F^1 of player 1 will not have support in some left-hand neighborhood of d. Thus player 2 will not exert effort d at ω_n with positive probability, a contradiction to $F_2(\omega_n, \cdot)$ being concentrated on d. Thus, even if i' is defined, it must be equal to n, implying that $F_2(\omega_i, \{d\}) = 0$ for every $i = 1, ..., n-1$. Thus, the distribution $F_2(\omega_i, \cdot)$ has full support on $[a_i, b_i]_\_ \subset [0, \beta)$ for $i = 1, ..., n-1$ (and in particular each $[a_i, b_i]_\_$ is non-empty). As $F_2(\omega_n, \cdot)$ cannot be concentrated on d, $[a_n, b_n]_\_$ is also non-empty. It furthermore follows from monotonicity of F_2 that, if $i < j$, then $[a_i, b_i]_\_$ lies below $[a_j, b_j]_\_$ (or coincides with $[a_j, b_j]_\_$, if $[a_j, b_j]_\_ = \{0\}$).

Thus, the intervals $\left\{[a_i, b_i]_\_\right\}_{i=1}^n$ are disjoint (barring the set $\{0\}$), and "ordered" according to the index i on the interval $[0, \beta)$. Moreover, $\cup_{i=1}^n [a_i, b_i]_\_ = [0, \beta)$, since otherwise there would be a "gap" (a, b) on which only player 1 places positive probability, which is impossible as we have seen earlier. It follows that there are points $0 = y_0 \leq y_1 \leq ... < y_n \equiv \beta$ such that $[a_i, b_i]_\_ = [y_{i-1}, y_i]_\_$ for every $i = 1, 2, ..., n$, i.e., for $i = 1, ..., n-1$, $F_2(\omega_i, \cdot)$ has full support on $[y_{i-1}, y_i]_\_$, and the restrictions of $F^1(\cdot)$ and $F_2(\omega_n, \cdot)$ to $[0, \beta)$ have full support on $[0, y_n)$ and $[y_{n-1}, y_n)$, respectively. We denote by i_0 the smallest integer with $y_{i_0} > 0$.[7]

Since $F^1(\cdot)$ has full support on $[0, y_n)$ (when restricted to $[0, y_n)$) and $F_2(\omega, \cdot)$ has no atoms in $(0, y_n)$, player 1 is indifferent between any two efforts in $(0, y_n)$. Thus, the following equality must hold for every $i = i_0, ..., n$ and every positive $x \in [y_{i-1}, y_i)$:

$$\sum_{j=1}^{i-1} p_j v_j + p_i v_i F_2(\omega_i, x) - x = E_1(x, F_2) = \lim_{y \searrow 0} E_1(y, F_2) \equiv e_1 \geq 0.$$

In particular,

$$F_2(\omega_i, x) = \frac{x - \sum_{j=1}^{i-1} p_j v_j + e_1}{p_i v_i} \tag{27}$$

for every $i = i_0, ..., n$ and every $x \in [y_{i-1}, y_i)$. Since, for $i = i_0 + 1, ..., n$, $F_2(\omega_i, \cdot)$ has full support (after restriction to $[0, \beta)$ for $i = n$) on $[y_{i-1}, y_i) \subset (0, \beta)$, we have $F_2(\omega_i, y_{i-1}) = 0$, and thus

$$y_i = \sum_{j=1}^{i} p_j v_j - e_1 \tag{28}$$

for every $i = i_0, ..., n-1$.

[7] Since each interval $[y_{i-1}, y_i]_\_$ is either non-degenerate or $\{0\}$, $0 = y_0 = ... = y_{i_0-1} < y_{i_0} < ... < y_n$.

Since, for $i = i_0, ..., n$, $F_2(\omega_i, \cdot)$ has full support (after restriction to $[0, \beta)$ for $i = n$) on $[y_{i-1}, y_i)$ and $F_1(\cdot)$ has no atoms (except, possibly, at 0) on $[0, \beta)$, player 2 is indifferent between all positive efforts in $[y_{i-1}, y_i)$. Thus, the following equality must hold for every positive $x \in [y_{i-1}, y_i)$:

$$\begin{aligned} v_i F_1(x) - x &= E_2(\{\omega_i\}, F_1, x) \\ &= \lim_{y \searrow y_{i-1}} E_2(\{\omega_i\}, F_1, y) = v_i F_1(y_{i-1}) - y_{i-1}, \end{aligned}$$

and in particular

$$F_1(x) = \frac{x}{v_i} + F_1(y_{i-1}) - \frac{y_{i-1}}{v_i}. \tag{29}$$

The rest of the proof will separately consider the following cases.

Case 1: Assume that $e_1 > 0$ *or* $i_0 > 1$.

The assumption of Case 1 implies that either $F_2(\omega_{i_0-1}, \cdot)$ is concentrated on 0 (if $i_0 > 1$) or $F_2(\omega_1, \{0\}) > 0$ by (27) (if $i_0 = 1$ but $e_1 > 0$). Thus $F_1(\cdot)$ has no atom at 0 (otherwise shifting mass from 0 to a sightly higher effort would constitute a profitable deviation), and hence $F_1(0) = 0$. Using this, (28) and (29), we obtain

$$F_1(x) = \frac{x + e_1 - \sum_{j=1}^{i} p_j v_j}{v_i} + \sum_{j=i_0+1}^{i} p_j + \frac{\sum_{j=1}^{i_0} p_j v_j - e_1}{v_{i_0}} \tag{30}$$

for every $i = i_0, ..., n$, and every positive $x \in [y_{i-1}, y_i)$.

Case 1.1: Assume that $F_1(\{d\}) > 0$.

It is a corollary of this assumption that $F_2(\omega_n, \{d\}) > 0$ (since otherwise for some small $\varepsilon > 0$ and every i we would have $F_2(\omega_i, (d - \varepsilon, d]) = 0$, and then player 1 could profitably shift probability from d to a lower effort). Note further that having an atom at d by both $F_1(\cdot)$ and $F_2(\omega_n, \cdot)$ implies that these distributions place no probability on an open interval $(d - \varepsilon, d)$ for some small $\varepsilon > 0$, and hence

$$\beta < d, \tag{31}$$

and $F_2(\omega_n, \{d\}) = 1 - F_2(\omega_n, \beta)$.[8] As $F_2(\omega_n, \cdot)$ has no atom at β, $F_2(\omega_n, \beta)$ is given by (27) for $x = \beta$ and $i = n$. Since $F_1(\{d\}) > 0$, the expected payoff of player 1 from choosing d must be equal to the expected

[8] In the LHS F_2 is viewed as a probability measure, and in the RHS as a c.d.f.

payoff from choosing effort levels in $(0,\beta)$, i.e.,

$$\begin{aligned}
e_1 &= \lim_{y\searrow 0} E_1(y,F_2) = E_1(d,F_2)\\
&= \sum_{j=1}^{n-1} p_j v_j + p_n v_n \left[F_2(\omega_n,\beta) + \frac{1}{2}\left(1 - F_2(\omega_n,\beta)\right)\right] - d\\
&= \sum_{j=1}^{n-1} p_j v_j + \frac{1}{2} p_n v_n \left[F_2(\omega_n,\beta)+1\right] - d\\
&= \sum_{j=1}^{n-1} p_j v_j + \frac{1}{2} p_n v_n \left[\frac{\beta - \sum_{j=1}^{n-1} p_j v_j + e_1}{p_n v_n} + 1\right] - d\\
&= \frac{1}{2}\sum_{j=1}^{n} p_j v_j + \frac{1}{2}\beta + \frac{1}{2} e_1 - d.
\end{aligned}$$

Therefore

$$\beta = 2d - \sum_{j=1}^{n} p_j v_j + e_1. \tag{32}$$

Since $F_2(\omega_n,\cdot)$ has full support on $[y_{n-1},\beta)\cup\{d\}$ (as we have seen, d is chosen with positive probability), the conditional expected payoff of player 2 at ω_n from choosing d must be equal to his conditional expected payoff from choosing effort levels close to (but smaller than) β, i.e.,

$$\begin{aligned}
v_n F_1(\beta) - \beta &= \lim_{x\nearrow\beta} E_2(\{\omega_n\},F_1,x)\\
&= E_2(\{\omega_n\},F_1,d) = v_n\left[F_1(\beta) + \frac{1}{2}\left(1 - F_1(\beta)\right)\right] - d\\
&= \frac{v_n}{2}\left[1 + F_1(\beta)\right] - d.
\end{aligned}$$

It follows, by using (30) and (32) for $x = \beta$ and $i = n$,[9] that

$$\begin{aligned}
0 &= \frac{v_n}{2}\left[1 - F_1(\beta)\right] + \beta - d\\
&= \frac{v_n}{2}\left[1 - \frac{\left(2d - \sum_{j=1}^{n} p_j v_j + e_1\right) + e_1 - \sum_{j=1}^{n} p_j v_j}{v_n} - \sum_{j=i_0+1}^{n} p_j - \frac{\sum_{j=1}^{i_0} p_j v_j - e_1}{v_{i_0}}\right]\\
&\quad + \left(2d - \sum_{j=1}^{n} p_j v_j + e_1\right) - d\\
&= \frac{v_n}{2}\left[1 - \sum_{j=i_0+1}^{n} p_j - \frac{\sum_{j=1}^{i_0} p_j v_j - e_1}{v_{i_0}}\right]\\
&= \frac{v_n}{2}\left[\sum_{j=1}^{i_0} p_j\left(1 - \frac{v_j}{v_{i_0}}\right) + \frac{e_1}{v_{i_0}}\right].
\end{aligned}$$

[9] Distribution $F_1(\cdot)$ does not have an atom at β, as follows from (31), and hence formula (30) can be used for $x = \beta$ although it is stated for $x < y_n = \beta$.

But the expression in brackets is positive under the assumptions of Case 1, a contradiction. Thus, the properties assumed in Case 1 and Case 1.1 cannot hold jointly in equilibrium.

Case 1.2: Assume that $F_1(\{d\}) = 0$.

This assumptions implies that $F_2(\omega_n, \{d\}) = 0$ as well (since otherwise for some small $\varepsilon > 0$ we would have $F_1((d-\varepsilon, d]) = 0$, and then player 2 could profitably shift mass from d to a lower effort at ω_n). Thus, $F_1(\cdot)$ and $F_2(\omega_i, \cdot)$ for each $i = i_0, ..., n$ are supported on $[0, \beta)$, and non-atomic on $(0, \beta)$. In particular, formulas (27) and (30) hold for $x = y_n = \beta$ and $i = n$, and

$$F_2(\omega_n, \beta) = F_1(\beta) = 1. \tag{33}$$

It follows from (27) and (33) that

$$\beta = y_n = \sum_{j=1}^{n} p_j v_j - e_1.$$

Thus

$$\begin{aligned}
1 &= F_1(\beta) = \frac{\beta + e_1 - \sum_{j=1}^{n} p_j v_j}{v_i} + \sum_{j=i_0+1}^{n} p_j + \frac{\sum_{j=1}^{i_0} p_j v_j - e_1}{v_{i_0}} \\
&= \frac{\left(\sum_{j=1}^{n} p_j v_j - e_1\right) + e_1 - \sum_{j=1}^{n} p_j v_j}{v_i} + \sum_{j=i_0+1}^{n} p_j + \frac{\sum_{j=1}^{i_0} p_j v_j - e_1}{v_{i_0}} \\
&= \sum_{j=i_0+1}^{n} p_j + \frac{\sum_{j=1}^{i_0} p_j v_j - e_1}{v_{i_0}} \\
&= \sum_{j=i_0+1}^{n} p_j + \sum_{j=1}^{i_0} p_j \frac{v_j}{v_{i_0}} - \frac{e_1}{v_{i_0}}.
\end{aligned}$$

The last expression is smaller than 1, however, under the assumptions of Case 1, a contradiction.

We conclude that the assumptions of Case 1 *cannot hold in equilibrium*, and consider next the complimentary Case 2:

Case 2: Assume that $i_0 = 1$ and $e_1 = 0$.

Given the assumptions, it follows that (27), (28), and (29) can be rewritten in the following way:

$$y_i = \sum_{j=1}^{i} p_j v_j \ (= x_i) \tag{34}$$

for every $i = 1, ..., n-1$ (where x_i was defined in (4)), and

$$F_2(\omega_i, x) = \frac{x - \sum_{j=1}^{i-1} p_j v_j}{p_i v_i}, \tag{35}$$

$$F_1(x) = \frac{x}{v_i} + F_1(y_{i-1}) - \frac{y_{i-1}}{v_i} \tag{36}$$

for every $i = 1, ..., n$ and every $x \in [y_{i-1}, y_i)$.

Denote $e_2 \equiv F_1(y_0) \geq 0$. From (36) and (34) we obtain

$$F_1(x) = \frac{x}{v_i} + \sum_{j=1}^{i-1} p_j \left[1 - \frac{v_j}{v_i}\right] + e_2 \tag{37}$$

for every $i = 1, ..., n$ and every $x \in [y_{i-1}, y_i)$.

Case 2.1: Assume that $F_1(\{d\}) > 0$.

Arguing as in Case 1.1 (but substituting $e_1 = 0$), we obtain

$$y_n = \beta = 2d - \sum_{j=1}^{n} p_j v_j \ (= \widetilde{x}), \tag{38}$$

where $\widetilde{x}$ was defined in (5), and

$$v_n F_1(\beta) - \beta = \frac{v_n}{2} [1 + F_1(\beta)] - d.$$

Using these equalities and (37), it follows that

$$\begin{aligned} 0 &= \frac{v_n}{2} [1 - F_1(\beta)] + \beta - d \\ &= \frac{v_n}{2} \left[1 - \frac{2d - \sum_{j=1}^{n} p_j v_j}{v_n} - \sum_{j=1}^{n-1} p_j \left[1 - \frac{v_j}{v_n}\right] - e_2\right] \\ &\quad + \left(2d - \sum_{j=1}^{n} p_j v_j\right) - d = -\frac{v_n}{2} e_2, \end{aligned}$$

and we conclude that $e_2 = 0$. This turns (37) into

$$F_1(x) = \frac{x}{v_i} + \sum_{j=1}^{i-1} p_j \left[1 - \frac{v_j}{v_i}\right] \tag{39}$$

for every $i = 1, ..., n$ and every $x \in [y_{i-1}, y_i)$.

Due to (34), (35), (38) and (39), the equilibrium strategy profile (F_1, F_2) is identical to (F_1^*, F_2^*) of Proposition 1.

Case 2.2: Assume that $F_1(\{d\}) = 0$.

Arguing as in Case 1.2 (but substituting $e_1 = 0$ and $i_0 = 1$), we obtain

$$F_2(\omega_n, \beta) = F_1(\beta) = 1 \tag{40}$$

and

$$\beta = y_n = \sum_{j=1}^{n} p_j v_j. \tag{41}$$

(Since $\beta \leq d \leq \sum_{j=1}^{n} p_j v_j$, (41) shows that Case 2.2 can only occur when $d = \sum_{j=1}^{n} p_j v_j$.) It follows from (37), (40), and (41) that

$$1 = \frac{\sum_{j=1}^{n} p_j v_j}{v_n} + \sum_{j=1}^{n-1} p_j \left[1 - \frac{v_j}{v_n}\right] + e_2 = 1 + e_2,$$

and hence $e_2 = 0$. Thus, (39) also holds in Case 2.2, for every $i = 1, ..., n$ and every $x \in [y_{i-1}, y_i)$. It follows from this, together with (34), (35), and (38), that the equilibrium strategy profile (F_1, F_2) is identical to (F_1^*, F_2^*) of Proposition 1 in the case of $d = \sum_{j=1}^{n} p_j v_j$. ■

References

[1] Abraham, I., Athey, S., Babaiof, M., Grubb, M.: Peaches, Lemons, and Cookies: Designing auction markets with dispersed information. Working paper, Harvard University (2014)

[2] Amman, E., Leininger,W.: Asymmetric all-pay auctions with incomplete information: the two-player case. *Games and Economic Behavior* 14, 1-18 (1996)

[3] Baye, M. R., Kovenock, D., de Vries, C. G.: Rigging the lobbying process: an application of the all-pay auction. *American Economic Review* 83, 289-294 (1993)

[4] Baye, M., Kovenock, D., de Vries, C.: The all-pay auction with complete information. *Economic Theory* 8, 291-305 (1996)

[5] Che, Y-K., Gale, I.: Caps on political lobbying. *American Economic Review* 88(3), 643-651 (1998)

[6] Einy, E., Haimanko, O., Orzach, R., Sela, A.: Dominant strategies, superior information, and winner's curse in second-price auctions. *International Journal of Game Theory* 30, 405-419 (2001)

[7] Einy, E., Haimanko, O., Orzach, R., Sela, A.: Dominance solvability of second-price auctions with differential information. *Journal of Mathematical Economics* 37, 247-258 (2002)

[8] Einy, E., Haimanko, O., Orzach, R., Sela, A.: Common-value all-pay auctions with asymmetric information. Working paper, Ben-Gurion University (2014).

[9] Forges, F., Orzach, R.: Core-stable rings in second price auctions with common values. *Journal of Mathematical Economics* 47, 760-767 (2011)

[10] Gavious, A., Moldovanu, B., Sela, A.: Bid costs and endogenous bid caps. *Rand Journal of Economics* 33(4), 709-722 (2003)

[11] Hillman, A., Riley, J.: Politically contestable rents and transfers. *Economics and Politics* 1, 17-40 (1989)

[12] Hillman, A., Samet, D.: Dissipation of contestable rents by small numbers of contenders. *Public Choice* 54(1), 63-82 (1987)

[13] Jackson, M.: Bayesian implementation. *Econometrica* 59, 461-477 (1993)

[14] Krishna, V., Morgan, J.: An analysis of the war of attrition and the all-pay auction. *Journal of Economic Theory* 72(2), 343-362 (1997)

[15] Malueg, D., Orzach, R.: Revenue comparison in common-value auctions: two examples. *Economics Letters* 105, 177-180 (2009)

[16] Malueg, D., Orzach, R.: Equilibrium and revenue in a family of common-value first-price auctions with differential information. *International Journal of Game Theory* 41(2), 219-254 (2012)

[17] Milgrom, P., Weber, R.: A theory of auctions and competitive bidding. *Econometrica* 50(5), 1089-1122 (1982)

[18] Moldovanu, B., Sela, A.: The optimal allocation of prizes in contests. *American Economic Review* 91(3), 542-558 (2001)

[19] Moldovanu, B., Sela, A.: Contest architecture. *Journal of Economic Theory* 126(1), 70-97 (2006)

[20] Moldovanu, B., Sela, A., Shi, X.: Carrots and sticks: prizes and punishments in contests. *Economic Inquiry* 50(2), 453-462 (2012)

[21] Siegel, R.: All-pay contests. *Econometrica* 77 (1), 71-92 (2009)

[22] Siegel, R.: Asymmetric contests with interdependent valuations. *Journal of Economic Theory*, forthcoming (2014)

[23] Vohra, R.: Incomplete information, incentive compatibility and the core. *Journal of Economic Theory* 54, 429-447 (1999)