SEGREGATION: THEORETICAL APPROACHES

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1 Introduction

Segregation and its measurement has been an object of interest for sociologists since at least the late nineteen forties. More recently, economists as well have become interested in segregation and its effects on the wage gender gap, the educational attainment of minorities and on other socio-economic variables. The concept of segregation, nevertheless, has proved to be quite elusive. Indeed, since the seminal paper of Duncan and Duncan [7], the literature on segregation has generated a plethora of indices. To make some order, Massey and Denton [21] identified five different aspects that could be captured by the concept of segregation, namely, evenness, exposure, concentration, centralization and clustering. Evenness refers to the extent to which the members of the different groups are similarly distributed across units; exposure, to the extent to which members of the minority groups are exposed to members of other groups; concentration, to the

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proportion of space occupied by the members of the minority in the city; centralization, to the degree to which the members of the minority group are located near the center of the city; and finally, clustering refers to the closeness of the units occupied by the members of the minority. They have also classified twenty indices into five categories corresponding to the above aspects. One of the differences among the various indices is that they require qualitatively different data. Thus, while indices of evenness and exposure typically take as an input the raw number of members of each group in each unit, indices of concentration and clustering additionally require geographical data such as the areas of different units and their distances from the center. Similarly, indices of clustering require data on the proximity between the different units. More formally, the domains on which the various indices are defined are not the same which makes any comparison between them absurd.

In this chapter we survey the segregation literature with a focus on axiomatic models. We particularly mention James and Tauber [18], who propose a short list of properties for the evaluation of evenness and exposure indices for the case of two groups, and Reardon and Firebaugh [26] who evaluate several such indices, for the multigroup case. Full characterizations of indices or of families of indices, however, did not appear until Philipson [25]. We will concentrate on indices defined within the traditional model of segregation in which members of different groups, e.g. men and women, or blacks, whites and Hispanics, are located in different locations. These locations could be neighborhoods, schools, or occupations. Recently, however, a number of recent papers, however, have opted for a different model in which members of different groups are located on a network. Two notable exponents of this novel approach are Echenique and Fryer [8], and Ballester and Vorsatz [3]. In Echenique and Fryer [8] the model is given by a weighted directed graph where the nodes represent individuals, the arcs represent the existence of an interaction between the corresponding individuals, and the weight measures the intensity of the interaction. In addition, each individual belongs to a particular ethnic group. The segregation index they propose, called the spectral segregation index, can be thought of as a measure of isolation because it tries to capture the degree to which individuals tend to interact with members of their own group.[1] Ballester and Vorsatz [3] also model a city as a graph, but in their case the nodes represent locations which can contain several individuals belonging to various groups. They complement the graph with a Markov matrix with an absorbing state that describes the probabilities of transitions from one location to another. The segregation index they propose captures the probabilities that in the long run members of the same group end up in the same location.

This chapter is organized as follows. After the basic notation is introduced in Section 2, Section 3 defines the Lorenz segregation ordering on two-group cities and enumerates several properties that it satisfies. Section 4 introduces some well known examples of segregation indices. Section 5 enumerates additional axioms that a segregation index may satisfy and Section 6 states three characterization results. Finally, Section 7 focuses on the case in which the number of groups is variable and formulates another characterization theorem.

2 Notation

Throughout the chapter we use the language of urban ethnic segregation. The definitions and results, however, apply in other contexts as well, including occupational gender segregation, religious segregation in schools, etc.

The initial, and perhaps the most important modeling choice concerns the domain on which the segregation measures are to be defined. Some axioms may characterize a particular ordering over a particular domain, but not over some alternative one. In this part of the chapter we will restrict attention to a domain with a fixed number of racial groups. Furthermore, for ease of exposition we will further restrict attention to two groups only, which will be referred to as blacks and whites. This entails little loss

[1]Isolation is to be understood as the opposite of exposure.

of generality since most of the results generalize (some in a straightforward way) to the many groups case. In Section 7 we will focus on a domain with a variable number of groups.

A *neighborhood* n is characterized by a pair (B_n, W_n) of non-negative real numbers, at least one of which is positive. The pair (B_n, W_n) is the neighborhoods ethnic composition. Namely, the first and second components are the numbers of blacks and whites, respectively, in n. A *city* is a finite collection of neighborhoods, at least one of which has a positive number of blacks and at least one of which has a positive number of whites. Formally, a city is a system $\langle N, (B_n, W_n)_{n\in N}\rangle$ such that $\sum_{n\in N} B_n > 0$ and $\sum_{n\in N} W_n > 0$, where N is the set of neighborhoods and for each $n \in N$, (B_n, W_n) is n's racial composition.

Given a city $\langle N, (B_n, W_n)_{n\in N}\rangle$, we denote by B and W the total numbers of blacks and whites, respectively: $B = \sum_{n\in N} B_n$ and $W = \sum_{n\in N} W_n$. Also, the following notation will be useful.

$$\begin{aligned}
P &= \frac{B}{B+W}\text{: the proportion of blacks in the city}\\
p_n &= \frac{B_n}{B_n+W_n}\text{: the proportion of blacks in neighborhood } n\\
T &= B+W\text{: the total population of the city}\\
T_n &= B_n+W_n\text{: the total population of neighborhood } n\\
b_n &= \frac{B_n}{B}\text{: the proportion of the city's blacks that live in neighborhood } n\\
w_n &= \frac{W_n}{W}\text{: the proportion of the city's whites that live in neighborhood } n.
\end{aligned}$$

For any city $X = \langle N, (B_n, W_n)_{n\in N}\rangle$ and any positive constant α, αX denotes the city that results from multiplying the number of blacks and whites in each neighborhood of X by α, namely, $\alpha X = \langle N, (\alpha B_n, \alpha W_n)_{n\in N}\rangle$. For any two cities $X = \langle N_X, (B_n, W_n)_{n\in N_X}\rangle$ and $Y = \langle N_Y, (B_n, W_n)_{n\in N_Y}\rangle$, with disjoint sets of neighborhoods, $X \circ Y$ denotes the concatenation of the two. Formally, $X \circ Y = \langle N_X \cup N_Y, (B_n, W_n)_{n\in N_X\cup N_Y}\rangle$. For convenience, we will sometimes denote a city $X = \langle N, (B_n, W_n)_{n\in N}\rangle$ simply by $(B_n, W_n)_{n\in N}$.

The *city's ethnic distribution* is given by $(P, 1-P)$ and *neighborhood n's ethnic distribution* is given by $(p_n, 1-p_n)$. Neighborhood n is *representative* of the city if the proportions of the city's blacks and of the city's whites who live in the neighborhood are equal; that is, if $b_n = w_n$.

3 Lorenz ordering

We are interested in ordering cities according to their level of segregation. This task is not an easy one, as is evident from the large number of existing segregation indices. However, there are some instances in which the comparison of cities according to their segregation seems to be straightforward. These instances are identified by axioms. Before we present some of the axioms, we motivate them by means of a particular partial order defined on the class of two-group cities, namely the Lorenz order.

Let $X = \langle N, (B_n, W_n)_{n \in N} \rangle$ be a city, and let $\phi : \{1, 2, \ldots, |N|\} \to N$ be an ordering of the neighborhoods such that $i \leq j \Rightarrow p_{\phi(i)} \geq p_{\phi(j)}$. Namely, ϕ orders neighborhoods in a non-increasing manner according to their proportion of blacks. Also let $\beta_0 = \omega_0 = 0$, and for $i = 1, 2, \ldots, |N|$, let $\beta_i = \beta_{i-1} + b_{\phi(i)}$ and $\omega_i = \omega_{i-1} + w_{\phi(i)}$. That is, β_i is the proportion of blacks that reside in the i neighborhoods with the highest proportions of blacks. Similarly, ω_i is the proportion of whites that reside in these same neighborhoods. *The Lorenz curve of* X is the graph that is obtained by plotting the points (β_i, ω_i) and connecting the dots.[2] Note that the line segment that connects the points $(\beta_{i-1}, \omega_{i-1})$ and (β_i, ω_i) has a slope of $w_{\phi(i)}/b_{\phi(i)}$. It can be checked that $b_i/(b_i + w_i) \geq b_j/(b_j + w_j) \iff p_i \geq p_j$. Therefore, the Lorenz curve is invariant to the choice of ordering ϕ as long as it satisfies $i \leq j \Rightarrow p_{\phi(i)} \geq p_{\phi(j)}$. Figure 1 illustrates the Lorenz curve of a city with three neighborhoods.

[2]The cities' Lorenz curves are usually referred to as segregation curves. Note that the segregation curve of X is the standard Lorenz curve of an income distribution in which blacks play the role of individuals and whites play the role of income.

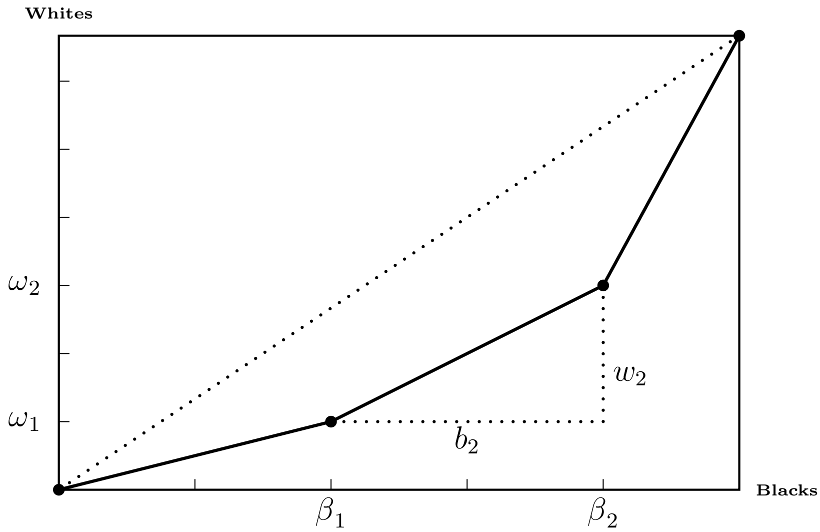

Figure 1: The Lorenz curve of a three-neighborhood city.

Based on the Lorenz curves described above we can now define the Lorenz partial order.

Definition 1 Let X and Y be two cities. We say that Y is more segregated than X according to the Lorenz criterion, denoted by $Y \succcurlyeq_L X$, if the Lorenz curve of Y is nowhere above the Lorenz curve of X.

Figure 2 depicts the Lorenz curves of two cities, X and Y, the latter being more segregated than the former according to the Lorenz criterion. The relation "being more segregated than, according to the Lorenz criterion" is an example of a segregation order. A segregation order is a binary relation that is used to compare cities according to their respective levels of segregation. More formally, if we denote the set of all cities with two ethnic groups by $\mathcal{C}_2$, *a segregation order*, $\succcurlyeq$, is a reflexive and transitive binary relation on $\mathcal{C}_2$. We interpret $X \succcurlyeq Y$ to mean that "city X is at least as segregated as city Y according to $\succcurlyeq$." The relations $\sim$ and $\succ$ are derived from $\succcurlyeq$ in the usual way.[3] Much

[3] That is, $X \sim Y$ if both $X \succcurlyeq Y$ and $Y \succcurlyeq X$; $X \succ Y$ if $X \succcurlyeq Y$ but not $Y \succcurlyeq X$.

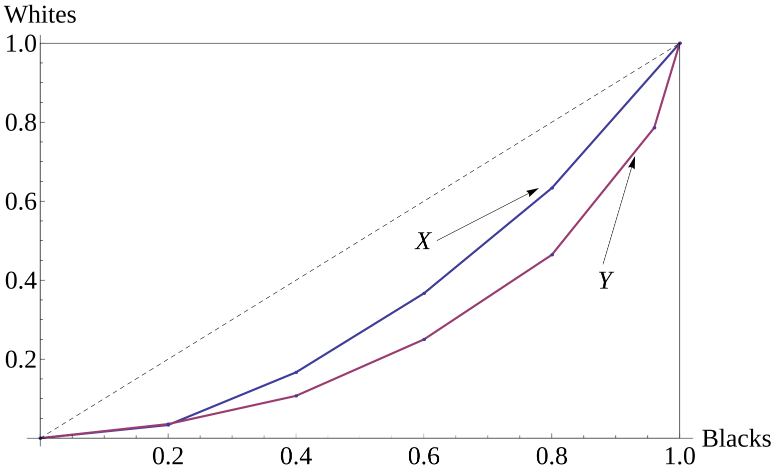

Figure 2: Y is more segregated than X according to the Lorenz criterion.

of the segregation literature is interested in identifying segregation orders that satisfy desirable properties. In order to motivate them, we next analyze some of the properties that the Lorenz order satisfies.

3.1 Properties of the Lorenz partial order

We say that two cities, $X = \langle N, (B_n, W_n)_{n \in N} \rangle$ and $X' = \langle N', (B'_m, W'_m)_{m \in N'} \rangle$, are *neighborhood-equivalent* if there is a one-to-one mapping $\varphi : N \to N'$ such that for all $n \in N$, $(B_n, W_n) = (B'_{\varphi(n)}, W'_{\varphi(n)})$. That is, two neighborhood-equivalent cities are, up to renaming of neighborhoods, identical. Table 1 provides an example of two equivalent cities.

	X			Y		
	A	B	C	D	E	F
Blacks	30	50	120	120	30	50
Whites	120	50	30	30	120	50

Table 1: Equivalent cities.

It is clear that two neighborhood-equivalent cities have the same Lorenz curves. Therefore, the Lorenz order satisfies the following axiom.

Neighborhood-Anonymity (N-ANON) An order $\succcurlyeq$ on $\mathcal{C}_2$ satisfies neighborhood-anonymity if for any two neighborhood-equivalent cities X and Y we have that $X \sim Y$.

Neighborhood-anonymity allows us to represent any city $\langle N, (B_n, W_n)_{n\in N}\rangle$ graphically by drawing the collection of points $(B_n, W_n)_{n\in N}$. Figure 3 depicts a city with three neighborhoods. The rectangle represents the city's total population of blacks and whites, (B, W). The slope of the dotted line is the city's ratio of whites over blacks. It can be seen that blacks are over-represented in one of the neighborhoods and under-represented in the other two.

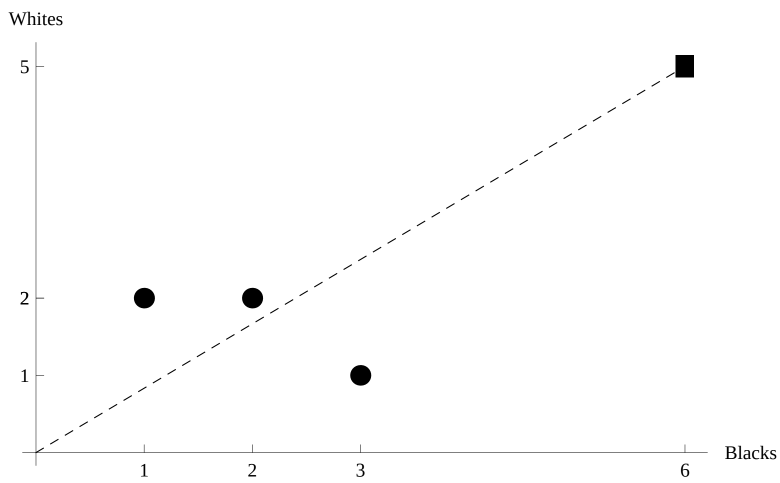

Figure 3: A city with three neighborhoods:$\langle(3,1),(2,2),(1,2)\rangle$.

Consider now the cities $X = \langle N, (B_n, W_n)_{n\in N}\rangle$ and $Y = \left\langle N, (\alpha B_n, \beta W_n)_{n\in N}\right\rangle$, the latter being obtained from X by multiplying the number of X's blacks by $\alpha > 0$ and the number of X's whites by $\beta > 0$. Table 2 depicts an example of two such cities.

	X			Y		
	A	B	C	D	E	F
Blacks	3	5	12	30	50	120
Whites	120	50	30	60	25	15

Table 2: Doubly scaled cities.

Since for all n, neighborhood n in both cities contain the same proportions b_n and w_n of the total number of blacks and whites, respectively, X and Y have the same Lorenz curve. Therefore, the Lorenz order satisfies the following axiom.

Composition Invariance (CI) Let $X = \langle N, (B_n, W_n)_{n\in N}\rangle$ be a city and let $Y = \left\langle N\,(\alpha B_n, \beta W_n)_{n\in N}\right\rangle$ be the city that is obtained from X by multiplying the number of agents of a given group by the same nonzero factor in all neighborhoods. An order $\succcurlyeq$ on $\mathcal{C}_2$ satisfies composition invariance if for any such cities we have $X \sim Y$.

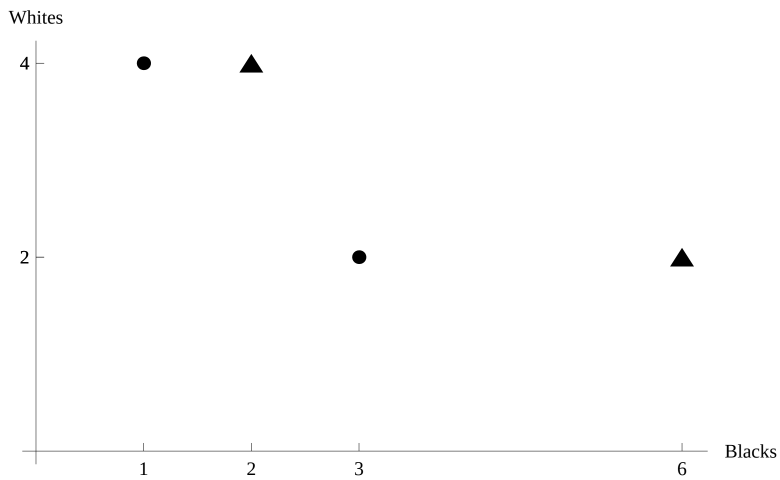

Figure 4: Two cities. One is obtained from the other by scaling down the number of blacks.

Figure 4 depicts two cities, one of which has, for each of its neighborhoods, the same number of whites and half the number of blacks as the other one. According to Composition Invariance, these two cities are equally segregated.

	X			Y			
	A	B	C	A_1	A_2	B	C
Blacks	30	50	120	20	10	50	120
Whites	120	50	30	80	40	50	30

Table 3: Splitting a neighborhood and keeping its ethnic distribution.

Now let $X = \langle N, (B_n, W_n)_{n \in N} \rangle$ be a city and consider the city Y that is obtained from X by splitting a particular neighborhood n into two neighborhoods n_1 and n_2 with the same ethnic distribution. Namely, $(B_{n_1}, W_{n_1}) = (\alpha B_n, \alpha W_n)$ and $(B_{n_2}, W_{n_2}) = ((1-\alpha)B_n, (1-\alpha)W_n)$ for some $\alpha \in (0,1)$. Table 3 illustrates two such cities.

Then, we have $b_n = b_{n_1} + b_{n_2}$, $w_n = w_{n_1} + w_{n_2}$, and $p_{n_1} = p_{n_2} = p_n$. Therefore, both X and Y have the same Lorenz curve, as depicted in Figure 5. Indeed, the segment that represents neighborhood n in X is split into two segments representing neighborhoods n_1 and n_2 in Y.

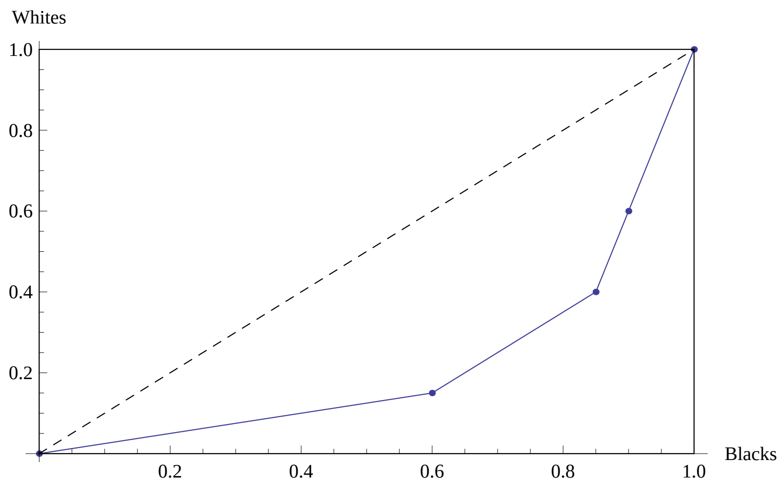

Figure 5: The two Lorenz curves.

As a result, the Lorenz order satisfies the following axiom.

Organizational Equivalence (OE) Let $X \in \mathcal{C}_2$ be a city and let (B_n, W_n) be a neighborhood of X. Also let Y be the city that results from X by dividing (B_n, W_n) into two neighborhoods, (B_{n_1}, W_{n_1}) and (B_{n_2}, W_{n_2}), with the same ethnic distribution. Namely, $p_{n_1} = p_{n_2}$. An order $\succcurlyeq$ on $\mathcal{C}_2$ satisfies organizational equivalence if for any such cities we have $X \sim Y$.

In order to motivate the next axiom, consider a city $X = \langle N, (B_n, W_n)_{n \in N} \rangle$. Let $i, j \in N$ be two neighborhoods such that $0 < p_i < 1$ and $p_j \leq p_i$. That is, neighborhood i contains both blacks and whites, but has proportionally less whites than neighborhood j.

Let $\varepsilon \in (0, W_i]$, and let Y be the city that is obtained from X by moving ε whites from neighborhood i to neighborhood j. That is, $Y = \langle N, (B'_n, W'_n)_{n \in N} \rangle$ in which $(B'_i, W'_i) = (B_i, W_i - \varepsilon)$, $(B'_j, W'_j) = (B_j, W_j + \varepsilon)$, and $(B'_n, W'_n) = (B_n, W_n)$ for all $n \neq i, j$. Then we have that

$$\begin{aligned} w_n/b_n &= w'_n/b'_n \text{ for all } n \neq i, j \\ w'_i/b'_i &< w_i/b_i \leq w_j/b_j < w'_j/b'_j. \end{aligned}$$

Figure 6 depicts the outcome of transferring white individuals from a neighborhood with relatively few whites to another one with relatively more whites.

It can be checked that the Lorenz curve of X lies nowhere below the Lorenz curve of Y while it is not true that the Lorenz curve of Y lies nowhere below the Lorenz curve of X. (See Figure 7 for an example of the effect of a transfer of white residents on the Lorenz curve.) Therefore, the Lorenz order satisfies the following axiom.

The W Transfer Principle (WT) For any city $X = \langle N, (B_n, W_n)_{n \in N} \rangle$, let $i, j \in N$ be two neighborhoods such that $B_i W_i > 0$ and

$$\frac{W_i}{B_i + W_i} \leq \frac{W_j}{B_j + W_j}.$$

Also let $\varepsilon \in (0, W_i]$, and Y be the city that is obtained from X by moving ε whites from neighborhood i to neighborhood j. A segregation order $\succcurlyeq$ on $\mathcal{C}_2$ satisfies the W Transfer Principle if for any such cities we have that$Y \succ X$.

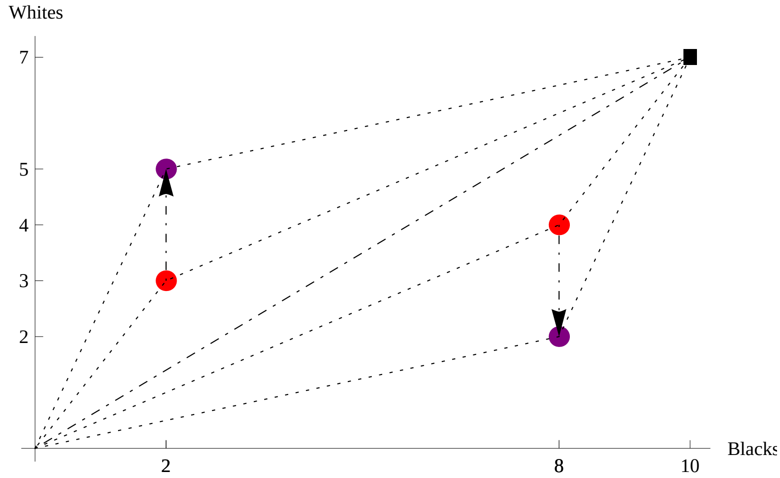

Figure 6: Transferring whites from a neighborhood with a relatively low percentage of whites to one with a relatively high percentage of whites.

Analogously, the Lorenz order satisfies the following axiom.[4]

The B Transfer Principle (BT) For any city $X = \langle N, (B_n, W_n)_{n \in N} \rangle$, let $i, j \in N$ be two neighborhoods such that $B_i W_i > 0$ and

$$\frac{B_i}{B_i + W_i} \leq \frac{B_j}{B_j + W_j}.$$

Also let $\varepsilon \in (0, B_i]$, and Y be the city that is obtained from X by moving ε blacks from neighborhood i to neighborhood j. A segregation order $\succcurlyeq$ on $\mathcal{C}_2$ satisfies the B Transfer Principle if for any such cities we have $Y \succ X$.

We can summarize the two axioms in the following.

The Transfer Principle (T) A segregation order $\succcurlyeq$ on $\mathcal{C}_2$ satisfies the transfer principle if it satisfies both WT and BT.

[4]The two transfer principles are not equivalent. When one of the neighborhoods has no blacks, blacks cannot be transferred from it. Therefore, in this case only the W Transfer Principle is not vacuous.

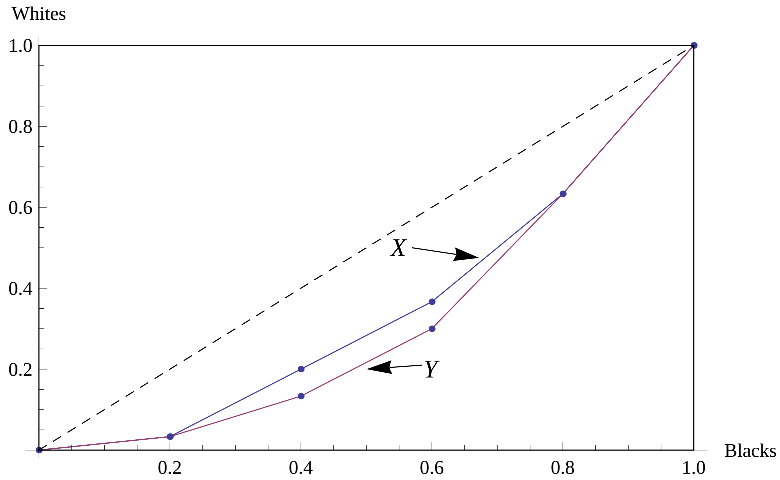

Figure 7: Lorenz curves before ($X = \{(30,5),(30,25),(30,25),(30,40),(30,55)\}$) and after the transfer ($Y = (30,5),(30,15),(30,25),(30,50),(30,55)$).

The Lorenz order satisfies the above four axioms, but it may not be the only ordering of cities that does this. It turns out that any order that satisfies the above four axioms must be consistent with the Lorenz order, and conversely, any order that is consistent with the Lorenz ordering must satisfy the four axioms. This result, which is stated in the following Proposition, was demonstrated by Hutchens [14] for the case in which all neighborhoods have an identical number of blacks or an identical number of whites. It was also mentioned, without proof, in James and Taeuber [18, (page 19)], and in Hutchens [15]. For a proof, see Lasso de la Vega and Volij [20].

Proposition 1 *Let $\succcurlyeq$ be an order on $\mathcal{C}_2$. It satisfies Neighborhood-Anonymity, Composition Invariance, Organizational Equivalence and the Transfer Principle if and only if for all two cities $X, Y \in \mathcal{C}_2$ the following implications hold:*

$$Y \succ_L X \quad \Rightarrow \quad Y \succ X \tag{1}$$

$$Y \sim_L X \quad \Rightarrow \quad Y \sim X. \tag{2}$$

4 Segregation orderings and their measures

As we have seen, the above four axioms let us identify a partial order among cities. But the literature on segregation is interested in identifying reasonable *complete* orderings, so that *any* two cities could be compared according to their respective segregation levels. Complete segregation orders are usually represented by segregation indices, which are functions that assign to each city a nonnegative number that is meant to capture its level of segregation. Given a segregation index S, the associated segregation order is defined by $X \succeq Y \Leftrightarrow S(X) \geq S(Y)$. Clearly, a segregation order may be represented by more than one index.

Additional axioms to the ones presented above have been proposed in order to identify reasonable segregation orderings. Before we introduce them, we first list a number of segregation indices that have been widely used to measure segregation.

4.1 Examples of Segregation Indices

The following is one of the most widely-used indices of segregation.

The Index of Dissimilarity. It is defined as:

$$D(X) = \sum_{n \in N : b_n > w_n} (b_n - w_n) \,. \tag{3}$$

This index was introduced to the literature by Jahn et al. [17]. It assigns to each city the proportion of either blacks or whites that would need to be relocated in order to obtain perfect integration. For example, if $b_n > w_n$, one needs to remove a proportion $b_n - w_n$ of the city's blacks from neighborhood n for the neighborhood to be representative, and if $b_n < w_n$, one needs to add a proportion $w_n - b_n$ of the city's blacks to neighborhood n for the neighborhood to be representative. The index of dissimilarity can be represented graphically as the maximum distance between the Lorenz curve and the 45 degree line. Since two Lorenz curves may have the same maximum distance to the 45 degree line with one being below the other, the index of dissimilarity is not consistent with the Lorenz ordering. Indeed, as we shall see below, it does not satisfy the transfer principle. Karmel and MaClachlan [19] proposed a slight modification to the dissimilarity index that takes into account the city's ethnic distribution. It is given by

$$2P(1-P)D(X).$$

This index calculates the number of people that would need to be relocated in order to obtain perfect integration, keeping each of the neighborhoods' population unchanged, and normalizes it so that the index ranges between zero and one. Like the Dissimilarity index, the Karmel and MaClachlan index is not consistent with the Lorenz ordering. Since the value of the index depends on the city's ethnic distribution it does not satisfy composition invariance.

The Gini Index. It is defined as:

$$G(X) = \frac{1}{2} \sum_{n \in N} \sum_{m \in N} |b_n w_m - b_m w_n| \,. \tag{4}$$

This index is adapted from the income inequality index of the same name. As in the case of the Index of Dissimilarity, this index is also related to the Lorenz curve. Indeed, it can be shown that $G(X)$ equals twice the area between the Lorenz curve and the 45 degree line.

The next two indices are related to the concept of entropy. The entropy of a random variable is the expected number of bytes needed to transmit the value of its realization. In the case of a two-value random variable with distribution $(q, 1-q)$, its entropy is given by

$$h(q, 1-q) = q \log_2 \left(\frac{1}{q}\right) + (1-q) \log_2 \left(\frac{1}{1-q}\right).$$

The entropy of a random variable is a measure of the uncertainty contained in it.

One can interpret a city's ethnic distribution as the distribution of a random variable, i.e., the ethnicity of a city's resident. Therefore, the entropy of a city's ethnic distribution

$(P, 1-P)$ represents the uncertainty concerning the ethnicity of a randomly chosen city resident. Similarly, the entropy of a neighborhood's ethnic distribution, $(p_n, 1-p_n)$, represents the uncertainty concerning the ethnicity of a randomly chosen individual, conditional on knowing that he belongs to that neighborhood. Although the entropy of a given neighborhood may be higher or lower than the entropy of the whole city, the entropy of the latter is at least as high as the average entropy of its neighborhoods.[5] The next two indices compare the entropy of a city's ethnic distribution with the average entropy of its neighborhoods.

The Mutual Information Index. It is defined as

$$MI(X) = h(P, 1-P) - \sum_{n \in N} \frac{T_n}{T} h\left(p_n, 1-p_n\right) \tag{5}$$

where h is the entropy function.

The Mutual Information is the average reduction in entropy that results from learning the neighborhood in which a randomly chosen individual lives. This index was first proposed by Theil [30] and has been applied by Fuchs [13] and Mora and Ruiz-Castillo [22, 24]. In the case of two ethnic groups, many of its properties, have been pointed out by Mora and Ruiz-Castillo [23].

The Entropy Index. It is defined as

$$H(X) = \sum_{n \in N} \frac{T_n}{T} \frac{h(P, 1-P) - h\left(p_n, 1-p_n\right)}{h(P, 1-P)}. \tag{6}$$

Note that $H(X) = MI(X)/h(P, 1-P)$.

The Entropy Index is the average decrease in entropy that results from learning the neighborhood, relative to the whole city's entropy. It was proposed by Theil [31] and Theil and Finizza [32].

The next index has been used in several applications.

[5] This is just a restatement of the fact that the entropy function h is concave.

Index of Isolation. It is given by:

$$J(X) = \frac{\left(\sum_{n\in N} b_n p_n\right) - P}{1-P}. \tag{7}$$

This index calculates the gap between the average of the neighborhoods' proportions of blacks (weighted by the neighborhood's fraction of the city's blacks) and the city's overall proportion of blacks, and normalizes it so that it ranges between zero and one. It turns out that this index is symmetric with respect to ethnic groups. Therefore, it also equals the normalized gap between the average of the neighborhoods' proportions of whites (weighted by the neighborhood's fraction of the city's whites) and the city's overall proportion of whites. This index was originally proposed by Bell [4]. James and Taeuber [18] refer to J as the variance ratio index. Massey and Denton [21] call it the correlation ratio. Reardon and Firebaugh [26] call it the index of Normalized Exposure. A variant of this index was used by Cutler, Glaeser, and Vigdor [6] to measure the evolution of segregation in American cities.

The following family of indices, called *generalized entropy measures*, was introduced and characterized by Hutchens [16]. It is defined for $\beta \in (0,1)$ as follows:

$$O_\beta = 1 - \left[\sum_{n\in N} w_n^{1-\beta} b_n^\beta\right].$$

When $\beta = 1/2$, the corresponding generalized entropy measure, sometimes called the *square root index*, is

$$O_{1/2} = 1 - \sum_{n\in N} \sqrt{w_n b_n}.$$

A closely related family of segregation measures is the *Atkinson's family* which was introduced by James and Tauber [18]. For $\beta \in (0,1)$ it is defined as follows:

$$A_\beta = 1 - \frac{P}{1-P}\left[\frac{1}{PT}\sum_{n\in N} T_n (1-p_n)^{1-\beta} p_n^\beta\right]^{\frac{1}{1-\beta}}.$$

By routine substitutions this expression can be rewritten as

$$A_\beta = 1 - \left[\sum_{n\in N} w_n^{1-\beta} b_n^\beta\right]^{\frac{1}{1-\beta}}.$$

It is readily seen that measure A_β is obtained from O_β by means of a monotonic transformation. Indeed,

$$O_\beta = 1 - (1 - A_\beta)^{1-\beta}.$$

Consequently, for any $\beta \in (0, 1)$, the indices A_β and O_β represent the same segregation ordering. In particular, the symmetric Atkinson measure $A_{1/2}$ represents the same segregation order as the Square Root index $O_{1/2}$.

5 More Axioms

Since there is such a great variety of indices that are available for researchers to measure segregation it is worth asking if any are more desirable than others? In this section we discuss a number of properties to help answer this question.[6]

The next axiom is similar to the Transfer Principle in that it states circumstances under which the overall city's segregation increases. Specifically, it states that if a neighborhood is split into two neighborhoods with different ethnic distributions, then segregation must increase. Formally,

Neighborhood Division Property (NDP) Let $X \in \mathcal{C}_2$ be a city and let n be a neighborhood of X. Also let Y be the city that results from dividing n into two neighborhoods, n_1 and n_2. If n_1 and n_2 have different ethnic distributions (i.e., $p_{n_1} \neq p_{n_2}$), then $Y \succ X$.

Frankel and Volij (2011) show that Organizational Equivalence and the Transfer Principle imply the Neighborhood Division Property. The next claim, the proof of which

[6]With some abuse of language, we will say that a segregation index satisfies a property if its induced segregation order does.

can be found in the appendix, shows that, assuming Organizational Equivalence, the Transfer Principle and the Neighborhood Division Property are in fact equivalent. One advantage of the Neighborhood Division Property over the Transfer Principle, however, is that while the former is naturally extended to cities with more than two groups, the latter is not.

Claim 1 *Let $\succcurlyeq$ be an order on $\mathcal{C}_2$ that satisfies Organizational Equivalence. Then, it satisfies the Transfer Principle if and only if it satisfies the Neighborhood Division Property.*

The next axiom states that segregation does not depend on which ethnic group is called black and which one is called white.

Group Symmetry (GS) Segregation in a city is unaffected by relabeling the ethnic groups: $\langle N, (B_n, W_n)_{n\in N}\rangle \sim \langle N, (W_n, B_n)_{n\in N}\rangle$.

This axiom is satisfied by all the indices presented in section 4.1 except for the Atkinson indices with parameter $\beta \neq 1/2$. These latter indices view segregation not as a feature of the city as a whole but as the relation between the city and its minority group which is given a different weight than the one received by the other group.

The next axiom states that adjoining the same set of neighborhoods to two cities with the same population and ethnic distribution preserves their order.

Independence (IND) Let X and Y be two cities with the same number of blacks and whites. Then $X \succcurlyeq Y$ if and only if $X \circ Z \succcurlyeq Y \circ Z$, for all cities Z.

Figure 8 illustrates the independence axiom. Two cities are depicted, one consisting of the two neighborhoods that are denoted by small dots, and the other of the two neighborhoods that are denoted by large dots. Note that these two cities have the same number of blacks and whites. Independence requires that no matter how these two cities

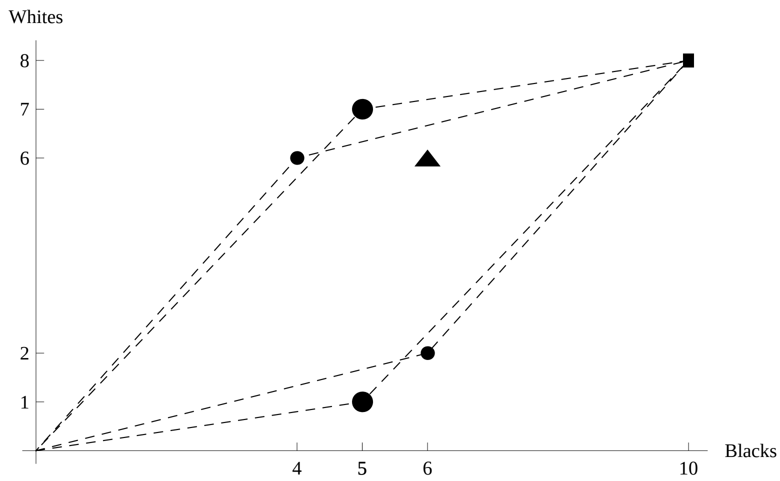

Figure 8: IND: Adding the triangular neighborhood to the small-dot and large-dot cities does not affect their segregation ranking.

are ranked by the segregation order, the annexation of the neighborhood denoted by a triangle to both of them does not affect their ranking.

The Mutual Information index as well as the whole family of Atkinson indices satisfy independence. The Gini and the Dissimilarity indices do not.

Independence is a strong axiom in the sense that it requires order preservation whenever any set of of neighborhoods is adjoined to the existing ones. The next axiom weakens Independence in that order preservation is required only if the set of neighborhoods that is added, Z, has the same ethnic distribution as the two existing cities.

Weak Independence (WIND) Let X Y and Z be three cities. Suppose all three of them have the same proportion of blacks, and that X and Y have the same total populations. Then $X \succcurlyeq Y$ if and only if $X \circ Z \succcurlyeq Y \circ Z$.

Figure 9 illustrates Weak Independence. Two cities are depicted, one consisting of the two small-dot neighborhoods and the other of the two large-dot neighborhoods. A hypothetical isolated neighborhood is also depicted there by a triangle. This neighborhood has the same proportion of blacks and whites as the other two cities. Weak

Independence requires that however these two cities are ranked, the annexation of the black neighborhood to both of them does not affect the ranking. If the annexed neighborhood did not have the same ethnic distribution as the other two cities, then Weak Independence (as opposed to Independence) would not have required anything.

Although the Dissimilarity index does not satisfy Independence, it does satisfy Weak Independence. The Gini index, however, does not satisfy this weaker axiom.

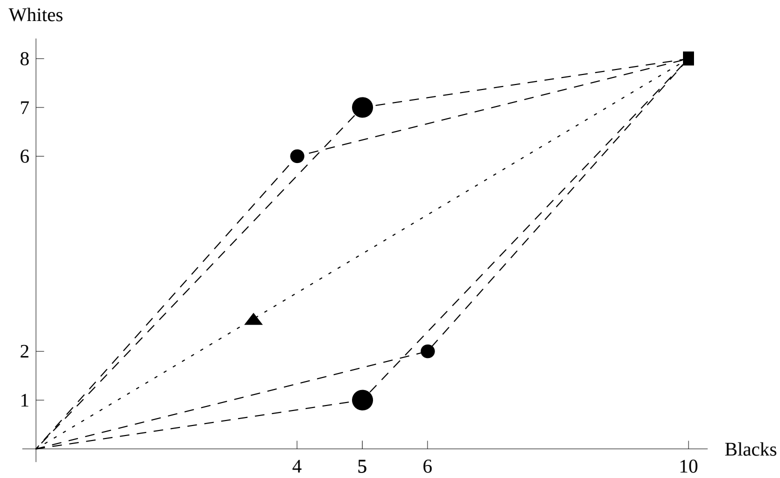

Figure 9: WIND: Adding the triangular neighborhood to the small-dot and large-dot cities does not affect their segregation ranking.

For the same reason that not all preference relations can be represented by a utility function, not all segregation orders can be represented by a segregation index. Furthermore, the segregation literature is usually interested not just in segregation indices but in continuous ones.[7] In order to guarantee that a continuous representation exists, a continuity axiom is often needed. The following one, which is satisfied by all the indices introduced so far, requires that certain similar cities must have similar segregation levels.

Continuity (C) For any cities X Y and Z, where X and Y have the same proportion

[7]Any city with $|N|$ neighborhoods can be seen as a point in a $\mathbb{R}^{2|N|}$ Euclidean space.

of blacks and the same total population, the sets

$$\{c \in [0,1] : cX \circ (1-c)Y \succcurlyeq Z\} \text{ and } \{c \in [0,1] : Z \succcurlyeq cX \circ (1-c)Y\}$$

are closed.

All the axioms presented thus far impose conditions on the segregation order. The literature also offers axioms which impose conditions directly on segregation indices. One example is the next one, which turns out to be closely related to IND.

Aggregation (AGG) An index S is Aggregative if there is a function F such that $S(X \circ Y) = F(S(X), S(Y), B(X), W(X), B(Y), W(Y))$, where F is a continuous function, strictly increasing in its first and second arguments.

The next claim, which is proved in the appendix, shows that Independence and Aggregation are to some extent equivalent axioms.

Claim 2 *If an index S satisfies Aggregation then it also satisfies Independence. Furthermore, a continuous index that satisfies Independence satisfies Aggregation as well.*

6 Implications of the Axioms

In this section we present two results that show the implications of the axioms presented in Section 5.[8] As stated before, Neighborhood-Anonymity, the Transfer Principle and Organizational Equivalence are minimal requirements that any segregation ordering should satisfy. Furthermore, Composition Invariance is a requirement that cannot be violated if the dimension of evenness is to be captured by the ordering. Group Symmetry is an uncontroversial axiom if we want to be color blind, namely, if segregation is to be independent of the name of the ethnic groups. Continuity is a technical requirement

[8]From now on, we restrict attention to complete segregation orders.

which is also uncontroversial; similar cities should have similar segregation levels. The following result, which can be found in Frankel and Volij [9], shows that adding Weak Independence to the above axioms yields a very convenient additive representation.

Theorem 1 *The segregation ordering $\succcurlyeq$ satisfies Neighborhood-Anonymity, Group Symmetry, Composition Invariance, the Transfer Principle, Weak Independence, Organizational Equivalence, and Continuity, if and only if there is a function $f : [0,1]\times[0,1] \to I\!R$ with the following properties:*

1. *For all cities X and Y,*

$$X \succcurlyeq Y \text{ if and only if } \sum_{i\in N(X)} f(b_i, w_i) \geq \sum_{j\in N(Y)} f(b_j, w_j).$$

2. *f is symmetric, homogeneous of degree 1, and strictly convex on the simplex $\Delta = \{(b,w) \in [0,1] : b + w = 1\}$.*

In addition, the function $f(c, 1-c)$ is unique up to a positive affine transformation. That is, f and g both satisfy properties 1 and 2 if and only if there are constants $\alpha \in (0,\infty)$ and $\beta \in I\!R$, such that

$$f(c,1-c) = \alpha g(c,1-c) + \beta \qquad \forall c \in [0,1].$$

Hutchens [15] proved a similar result by replacing Weak Independence and Continuity with the requirement of Additivity. This axiom requires the ordering to be represented by an index of the form $\sum_{i\in N} f(b_i, w_i)$ for some continuous function f.

As Theorem 1 shows, there are many indices that satisfy Weak Independence, Composition Invariance and Continuity, along with the basic axioms of Group Symmetry, Anonymity, Organizational Equivalence, and the Transfer Principle. The next theorem, proved by Hutchens [16], shows that by strengthening the weak independence axiom one obtains a full characterization of a single ordering.

Theorem 2 *Let* $S : \mathcal{C}_2 \to I\!R$ *be a continuous aggregative segregation index that satisfies Neighborhood-Anonymity, Group Symmetry, Composition Invariance, the Transfer Principle, Organizational Equivalence. Then,* S *is a strictly increasing transformation of the square root index,* $O_{1/2}$*.*

In fact, Frankel and Volij [11, Theorem 2] show that Continuity is not needed:

Theorem 3 *The ordering represented by the Symmetric Atkinson index,* $A_{1/2}$*, is the only one that satisfies Neighborhood-Anonymity, Organizational Equivalence, Neighborhood Division Property, Composition Invariance, Group Symmetry, and Independence on* $\mathcal{C}_2$*.*

Table 4 provides a summary of the satisfied properties, or those that fail to be satisfied by the indices presented in Section 4.1. For proofs, see Frankel and Volij [10]

		N-Anon	OE	CI	T	WIND	GS	IND	NDP	CONT
1	Symmetric Atkinson: $A_{1/2}(X)$	√	√	√	√	√	√	√	√	√
2	Asymmetric Atkinson: $A_{\mathbf{w}}(X)$	√	√	√	√	√	×	√	√	√
6	Mutual Information: $M(X)$	√	√	×	√	√	√	√	√	√
11	Dissimilarity: $D(X)$	√	√	√	×	√	√	×	×	√
13	Gini Index: $G(X)$	√	√	√	√	×	√	×	√	√
14	Entropy Index: $H(X)$	√	√	×	√	√	√	×	√	√
15	Isolation: $J(X)$	√	√	×	√	√	√	√	√	√

Table 4: Which Indices Violate Which Axioms? A "√" means that the axiom is satisfied while an "×" indicates that it is not.

7 Variable number of groups

As we mentioned before, most of the axioms can be extended to segregation orderings defined on any class with a fixed number of groups. Moreover, both Theorem 1 and

Theorem 3 can be extended to these classes. However, if one is interested to characterize a segregation order on the whole class of cities, an axiom that restricts the order when it compares cities with different numbers of groups is required. In this section we will present one such axiom, and state a characterization of the Mutual Information index that is based on it.

Before we introduce the new axiom, we need to define an extended class of cities that allows for a variable number of ethnic groups.

Definition 2 A *city* is a system, $\left\langle N, G, \left((T_n^g)_{g\in G}\right)_{n\in N}\right\rangle$, where N is a finite nonempty set of neighborhoods, G is a finite nonempty set of ethnic groups, and for each ethnic group $g \in G$ and neighborhood $n \in N$, T_n^g is a nonnegative real number that represents the number of members of ethnic group g that reside in neighborhood n.

We denote by $\mathcal{C}$ the class of all cities.

As before, $T_n = \sum_{g\in G} T_n^g$ denotes the total number of people located in neighborhood n, and $T = \sum_{n\in N} T_n$ denotes the city's total population. The total number of members of ethnic group g is denoted by T^g, and the proportion of group g members in the city is denoted by P^g. The city's ethnic distribution is given by the list $(P^g)_{g\in G}$. Similarly, the proportion of neighborhood n's population that belongs to ethnic group g is denoted by p_n^g, and neighborhood n's ethnic distribution by $(p_n^g)_{g\in G}$. The entropy of a distribution $(q^g)_{g\in G}$ is now given by

$$h((q^g)_{g\in G}) = \sum_{g\in G} q^g \log_2\left(\frac{1}{q^g}\right).$$

The extension of the Mutual Information index to the class of all cities is given by the function $M : \mathcal{C} \to \mathbb{R}$ defined by

$$M(X) = h\left((P^g)_{g\in G}\right) - \sum_{n\in N} \frac{T^n}{T} h\left((p_n^g)_{g\in G}\right).$$

In words, the Mutual Information of a city is the difference between the entropy of the city's ethnic distribution and the average entropy of its neighborhoods' ethnic distributions. Alternatively, it is the average reduction of entropy that results from the knowledge of the neighborhood where a randomly chosen individual is located.

As in the case of two ethnic groups, we would like segregation orderings defined on the class of all cities to be invariant to the renaming of both neighborhoods and ethnic groups. For this purpose, it is convenient to identify cities that are equivalent, up to the renaming of groups and neighborhoods. We say that two cities, $X = \left\langle \mathbf{N}, \mathbf{G}, \left((T_n^g)_{g \in G}\right)_{n \in N} \right\rangle$ and $X' = \left\langle N', \mathbf{G}', \left((T_n'^g)_{g \in G'}\right)_{n \in N'} \right\rangle$, are *equivalent* if there are one-to-one mappings $\phi : N \to N'$ and $\psi : G \to G'$ such that for all $n \in N$ and $g \in \mathbf{G}$, $(T_n^g) = (T_{\phi(n)}^{\psi(g)})$. The following axiom states that only the racial composition of neighborhoods and not their names matter.

Anonymity (ANON) A segregation ordering satisfies anonymity if any two equivalent cities are equally segregated.

Anonymity is the conjunction of the Neighborhood-Anonymity and Group Symmetry axioms defined in Sections 3 and 5.

We are now ready to state the above-mentioned axiom. It states that splitting a group, say blacks, into two subgroups, say black females and black males, while keeping their distribution across neighborhoods constant, should not affect segregation. Formally,

Group Division Property (GDP) Let $X \in \mathcal{C}$ be a city in which the set of ethnic groups is G. Let X' be the result of partitioning some group $g \in G$ into two subgroups, g_1 and g_2, such that the two subgroups have the same distribution across neighborhoods; namely, $T_n^{g_1}/T_n^g$ is independent of n, and thus equals T^{g_1}/T^g. Then $X' \sim X$.

One of the main axioms used in previous results is CI, which is an essential axiom of any index that wants to capture the dimension of evenness. However, there are some segregation orders, in particular those that intend to capture the dimension of isolation, that fail to satisfy CI. To measure isolation, a weaker requirement is enough.

Scale Invariance (SI) Let X be a city and let $Y = \alpha X$ be the city that is obtained from X by multiplying the number of residents by a positive factor α. An order $\succcurlyeq$ on $\mathcal{C}$ satisfies scale invariance if for any such cities we have $X \sim Y$.

Scale invariance states that for the purpose of measuring segregation one does not need to know whether people of the various ethnic groups are measured in units, tens, or thousands, etc., as long as they are measured in the same units.

The axioms of Independence, Organizational Equivalence and Continuity are generalized for the case of a variable number of ethnic groups in a straightforward way.

The next result, proved in Frankel and Volij [12], provides a characterization of the Mutual Information order on the class of all cities.

Theorem 4 *An ordering on $\mathcal{C}$ satisfies Scale Invariance, Independence, Organizational Equivalence, the Neighborhood Division Property, the Group Division Property, Anonymity,and Continuity if and only if it is represented by the Mutual Information index.*

Not all the indices we have discussed in Section 4.1 can be generalized to the many-group case in a straightforward way. However, the symmetric Atkinson index, the Gini index and the Entropy index can.[9] The information that appears in Table 4 is still valid for the multigroup version of these indices, except of course for the transfer principle. We should mention, however, that none of the above-mentioned indices satisfy the Group Division property. See Frankel and Volij [12] for details.

Finally, let us point out a relationship between the Mutual Information index and the informativeness of information structures. A city can be interpreted as an information structure where neighborhoods are signals that provide information about the ethnicity of a randomly chosen city resident. Consequently, we can adopt Blackwell's [5] partial order, which ranks information structures according to their informativeness, and use it

[9]For generalizations of the Dissimilarity index and the Isolation index, see Reardon and Firebaugh [26].

to partially order cities. It turns out that since the Mutual Information index does not satisfy Composition Invariance, it is not consistent with Blackwell's order. However, as argued in Frankel and Volij [12], if one restricts comparisons to pairs of cities with the same ethnic distribution, the Mutual Information index is consistent with Blackwell's ordering.

8 Concluding comments

The axiomatic approach, by emphasizing the essential properties that the various measures share and those on which they differ, can help researchers select the segregation indices that best fit their purposes. This chapter has surveyed some of what we believe are the most interesting theoretical results that employ this approach.

We first noted that the Lorenz order satisfies the four basic properties of neighborhood-anonymity, composition invariance, organizational equivalence and the transfer principle. We further saw that every segregation order that satisfies these properties must agree with the Lorenz order. Therefore, the Lorenz order is the common denominator of all segregation orders that satisfy them.

We later showed that adding independence to the above list of axioms results in a full characterization of the segregation order that is represented by the symmetric Atkinson index, also known as the Square Root index. We also saw that by weakening the independence axiom and by adding a continuity requirement one obtains a family of segregation measures that have a convenient additive representation.

Finally, we demonstrated that for the class of a variable number of ethnic groups, weakening composition invariance and adding the group division property results in the characterization of the Mutual Information index.

9 Appendix

Proof of Claim 1. Let $\succcurlyeq$ be a segregation order that satisfies OE and T. We will show that it satisfies NDP as well. Let X be a city and let n be a neighborhood of X. Let Y be the city that results from dividing n into two neighborhoods, n_1 and n_2. Assume that n_1 and n_2 don't have the same ethnic distributions. Further assume, without loss of generality, that the proportion of blacks is higher in n_1 than in n_2; namely, $B_{n_1}W_{n_2} > B_{n_2}W_{n_1}$. Neighborhood n in city X can be written $(B_n, W_n) = (B_{n_1} + B_{n_2}, W_{n_1} + W_{n_2})$. Let $\alpha = \frac{B_{n_1}+W_{n_1}}{B_n+W_n}$ and let X' be the city that results from X by splitting neighborhood n into the following two neighborhoods: $n_1' = \alpha (B_n, W_n)$ and $n_2' = (1-\alpha)(B_n, W_n)$. By organizational equivalence, $X \sim X'$. Since $B_{n_1}W_{n_2} > B_{n_2}W_{n_1}$ we have

$$B_{n_1} > \alpha B_n.$$

Transfer $B_{n_1} - \alpha B_n > 0$ blacks from from n_2' to n_1', (since $B_{n_1} - \alpha B_n < (1-\alpha)B_n$, this can be done). Further transfer the same amount of whites from n_1' to n_2'. The city that results is Y. By the transfer principle, this operation strictly raises segregation; namely, $Y \succ X' \sim X$, so by transitivity, $Y \succ X$.

We will now show that NDP implies WT. The proof that it also implies BT is analogous and is left to the reader. Let $X = \langle N, (B_n, W_n)_{n\in N}\rangle$, and let $i, j \in N$ be two neighborhoods such that $B_iW_i > 0$ and

$$B_iW_j \geq B_jW_i$$

Let $\varepsilon \in (0, W_i]$, and let Y be the city that is obtained from X by moving ε whites from neighborhood i to neighborhood j. That is, $Y = \langle N, (B_n', W_n')_{n\in N}\rangle$ in which $(B_i', W_i') = (B_i, W_i - \varepsilon)$, $(B_j', W_j') = (B_j, W_j + \varepsilon)$, and $(B_n', W_n') = (B_n, W_n)$ for all $n \neq i, j$. We need to show that $Y \succ X$. If $B_j = 0$, then Y is the result of splitting neighborhood (B_i, W_i) into $(B_i, W_i - \varepsilon)$ and $(0, \varepsilon)$ and then merging $(0, \varepsilon)$ with $(0, W_j)$. By NDP, the splitting operation increases segregation, and by OE, the merging of two neighborhoods with the same proportion of whites leaves segregation unchanged. Therefore, $Y \succ X$.

If $B_j > 0$, define the following values:

$$\begin{aligned}\alpha &= \frac{W_j+\varepsilon}{B_j}\\ \beta &= \frac{W_i-\varepsilon}{B_i}\\ \gamma &= \frac{\varepsilon}{\alpha-\beta} = \frac{\varepsilon B_i B_j}{B_i(W_j+\varepsilon)-B_j(W_i-\varepsilon)}\end{aligned}$$

Since $W_jB_i \geq W_iB_j$, we have that $\gamma, \beta, \alpha > 0$. Split (B_i, W_i) into $(B_i-\gamma, W_i-\alpha\gamma)$ and $(\gamma, \alpha\gamma)$. Similarly, split (B_j, W_j) into $(B_j-\gamma, W_j-\beta\gamma)$ and $(\gamma, \beta\gamma)$. (This can be done because $\gamma < \min\{B_i, B_j\}$. Indeed,

$$\gamma = \frac{\varepsilon B_i B_j}{B_i(W_j+\varepsilon)-B_j(W_i-\varepsilon)} < \frac{B_iB_j}{B_i+B_j}$$

Therefore, $\gamma(B_i+B_j) < B_iB_j$ or equivalently, which implies that $\gamma < B_i$ and $\gamma < B_j$.) By NDP the resulting city is more segregated. Now merge $(B_i-\gamma, W_i-\alpha\gamma)$ with $(\gamma, \beta\gamma)$ and also merge $(B_j-\gamma, W_j-\beta\gamma)$ with $(\gamma, \alpha\gamma)$. Since

$$\begin{aligned}\frac{W_i-\alpha\gamma}{B_i-\gamma} &= \beta\\ \frac{W_j-\beta\gamma}{B_j-\gamma} &= \alpha\end{aligned}$$

by OE this merger does not affect segregation. The resulting pair of neighborhoods is

$$(B_i, W_i-\alpha\gamma+\beta\gamma) \text{ and } (B_j, W_j-\beta\gamma+\alpha\gamma)$$

which happen to be $(B_i, W_i-\varepsilon)$ and $(B_j, W_j+\varepsilon)$, respectively. ■

Proof of Claim 2. (Based on Frankel and Volij [12]) Let X and Y be two cities with the same number of blacks and the same number of whites and let Z be any city.

$$\begin{aligned}X\circ Z &\succcurlyeq Y\circ Z\\ &\Leftrightarrow S(X\circ Z)\geq S(Y\circ Z)\\ &\Leftrightarrow F\left(S\left(X\right),S\left(Z\right),B,W,B\left(Z\right),W\left(Z\right)\right)\geq F\left(S\left(Y\right),S\left(Z\right),B,W,B\left(Z\right),W\left(Z\right)\right)\\ &\Leftrightarrow S\left(X\right)\geq S\left(Y\right)\\ &\Leftrightarrow X\succcurlyeq Y.\end{aligned}$$

Conversely, assume that $\succcurlyeq$ satisfies IND and that is represented by a continuous index S. Define $F : \mathbb{R}^6 \to \mathbb{R}$ by

$$F(s_x, s_y, b_x, w_x, b_y, w_y) = S(X \circ Y)$$

where X is a city with $S(X) = s_x$, $B(X) = b_x$, $W(X) = w_x$ and Y is a city with $S(Y) = s_y$, $B(Y) = b_y$, $W(Y) = w_y$. First note that F is well-defined. Indeed, if X' and Y' are cities such that $X \sim X'$, $Y \sim Y'$, $B(X) = B(X')$, $W(X) = W(X')$, $B(Y) = B(Y')$, $W(Y) = W(Y')$, then by IND applied twice,

$$S(X \circ Y) = S(X' \circ Y) = S(X' \circ Y').$$

Second, note that by IND, F is increasing in its first two arguments. Third, since S is continuous, so is F. ■

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