NON FIXED-PRICE TRADING RULES IN SINGLE-CROSSING CLASSICAL EXCHANGE ECONOMIES

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Abstract

This paper defines the single-crossing property for two-agent, two-good exchange economies for classical (i.e., continuous, strictly monotonic, and strictly convex) individual preferences. Within this framework and on a rich single-crossing domain, the paper characterizes the family of continuous, strategy-proof and individually rational social choice functions whose range belongs to the interior of the set of feasible allocations. This family is shown to be the class of generalized trading rules. This result highlights the importance of the concavification argument in the characterization of fixed-price trading rules provided by Barberà and Jackson (1995), an argument that does not hold under single-crossing. The paper also shows how several features of abstract single-crossing domains, such as the existence of an ordering over the set of preference relations, can be derived endogenously in economic environments by exploiting the additional structure of classical preferences.

Keywords: social choice, classical preference, single-crossing, concavification.
JEL Classification: D00; D51; D71.

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1 Introduction

The set of rules that allocate resources (or the set social choice functions, SCF in short) amongst a set of agents who have preferences over these resources typically vary with the axioms that these rules are required to satisfy. Consider for instance classical exchange economies (an exchange economy where agents' preferences are classical i.e. preferences are continuous, strictly monotonic and strictly convex). If the SCFs are required to satisfy *strategy-proofness* and *Pareto-efficiency* then Zhou (1991) (for two-agent exchange economies) showed that these rules must be dictatorial. It is well-known that strategy-proofness, Pareto-efficiency and individual-rationality are incompatible in classical exchange economies. Hurwicz (1972) demonstrates this for the case of two-agent and two-good models. Serizawa (2002) extends this result to an arbitrary number of agents and goods. However Barberà and Jackson (1995) show that a strategy-proof and individual-rationality are compatible in classical exchange economies. In particular, they show that the strategy-proof and individually rational SCF defined on the domain of all classical preferences is a *Fixed-Price Trading* or FPT rule. These rules have also been shown to be salient in other mechanism design problems. For instance, Hagerty and Rogerson (1987) consider a bilateral trading model with quasi-linear utility functions for the agents and show that a strategy-proof, individually rational and budget-balanced SCF is an FPT rule.

A critical element in the arguments in Barberà and Jackson (1995) is the use of *concavified* preferences. If a domain admits concavification then it means that at every consumption bundle in the commodity space and for every indifference curve of every preference, there exists a preference relation in the domain that induces an indifference curve that is tangent to the indifference curve of the former preference at the consumption bundle. The domain of all classical preferences admits concavification. However, for the domains that fail to satisfy this property, the restrictions implied by strategy-proofness and individual-rationality are not known. Our goal is to study the constraints implied by strategy-proofness and individual-rationality over the domain of preferences that satisfy the single-Rcrossing property. A domain of classical preferences which satisfies the single-crossing property does not admit concavification. This paper studies SCFs that are defined on classical single-crossing preference domains, for two-agent and two-good exchange economies. The single-crossing property for two goods imply that the indifference curves of any two preference orderings can cut only once. Therefore, indifference curves of two preference orderings from a single-crossing domain will not be tangential to each other at any consumption bundle (in the interior of the consumption space).

We provide examples to show that if the two agents' preferences belong to classical single-crossing domains then there exist strategy-proof and individually rational SCFs that do not belong to the class of FPT rules. Furthermore, we provide a characterization of such SCFs for two-agent and two-good economies that satisfy an additional continuity requirement. Our characterization can be briefly described as follows. An FPT rule requires the range of the SCFs to be piecewise linear with a kink (possibly) at the endowment. We show that for single-crossing domains, the range need not be piecewise linear although it must contain the endowment (assuming both the agents to be endowed with positive amounts of both the goods) and satisfy certain additional properties. We call these rules *Generalized Trading* rules, GT in short. An FPT rule with a connected range is an example of a GT rule. We also show existence of an ordering over the set of single-crossing classical preferences. This ordering is derived endogenously by exploiting the additional structure of classical preferences. The ordering over preferences and strategy-proofness imply a monotonicity property of the SCFs in the agents' own preferences. This monotonicity property is critical in deriving our characterization result.

1.1 Literature Review

The single-crossing property has been extensively used in the mechanism design and contract theory. Classical papers in this regard are Spence (1973), Mirrlees (1971) and Rothschild and Stiglitz (1976). Saporiti and Thomé (2006) show that single-crossing domains allow for strategy-proof SCFs. Saporiti (2009) considers a single-crossing domain with a finite number of alternatives and strict preferences and provides a characterization of the strategy-proof SCFs. Saporiti (2009) finds these rules to be generalized median voter schemes. Caroll (2012) shows that local incentive constraints are sufficient to have strategy-proofness in the rich single-crossing domain that Saporiti (2009) considers. Gershkov et al. (2013) use Saporiti (2009) to construct constrained-efficient optimal mechanism in single-crossing domains. Our results are independent of Saporiti (2009) because our model is different. A more detailed discussion of the relationship between our model and Saporiti (2009) can be found in Section 3.

Barberà and Jackson (2004) consider a model where society's preferences over voting rules satisfy the single-crossing property. However, their objective is to analyze self-stable rather than strategy-proof voting rules. Gans and Smart (1996) study an Arrovian aggregation problem with single-crossing preferences for voters. They show that median voters are decisive in all majority elections between pairs of alternatives. Barberà and Moreno (2011) develop the concept of 'top monotonicity' as a common generalization of single-peakedness and single-crossingness. Corchón and Rueda-Llano (2008) analyze a public-good-private-good production economy where agents' preferences satisfy the single-crossing property. They show the non-existence of smooth strategy-proof, Pareto-efficient SCFs that give strictly positive amount of both goods to the agents.

We also remark that the concavification property of various preference domains has been used extensively in the characterization of strategy-proof SCFs in economic environments. See for instance, Zhou (1991), Serizawa and Weymark (2003), Ju (2003), Serizawa (2006), Hashimoto (2008) and Momi (2013). Goswami et al. (2013), show that some results in the literature on strategy-proofness and Pareto-efficiency are carried over to the domains where concavification is not permitted. However, the present paper demonstrates that a similar conclusion does not obtain when Pareto-efficiency is replaced by individual-rationality. In particular, a strictly larger class than FPT rules will satisfy strategy-proofness and individual-rationality.

This paper is organized as follows. Section 2 sets up the notation, the definition of an FPT rule and the fundamental Barberà and Jackson (1995) characterization result. Section 3 introduces concavification, single-crossing domains and some preliminary but useful results pertaining to this domain. A subsection of this section compares our concept of the single-crossing property with that of Saporiti (2009). Section 4 provides examples of non-FPT rules that are strategy-proof and individually rational and Section 5 contains the main characterization result. Section 6 concludes.

2 Notation and Definitions

Throughout this paper, we will restrict attention to a two-agent, two-good model. We denote the set of agents by $I = \{1, 2\}$ and the two goods by x and y. The set of goods is denoted by M. Each agent i has an endowment ω_i^x and ω_i^y of goods x and y respectively. Let $\omega = ((\omega_1^x, \omega_1^y), (\omega_2^x, \omega_2^y))$ denote the endowment vector. Let $\Omega_j = \sum_{i=1}^{2} \omega_i^j$, $j \in M$ be the total endowment of good j. Let $\Omega = (\Omega_x, \Omega_y)$ denote the total endowment in the economy. Define the set of feasible allocations as,

$$\Delta = \{((x_1, y_1), (x_2, y_2))|\ x_1 + x_2 = \Omega_x \text{ and } y_1 + y_2 = \Omega_y; x_i \geq 0,\ y_i \geq 0, \text{for all } i \in I\}.$$

A preference ordering for agent i, R_i is a complete, reflexive and transitive ordering of the elements of $\Re_+^2$. We say that R_i is *classical* if it is (a) continuous, (b) strictly monotonic in $\Re_{++}^2$ and (c) the upper contour sets are strictly convex in $\Re_{++}^2$ [1]. We consider only classical preferences and denote the set of such preferences by $\mathbb{D}^c$. A preference profile R is an 2-tuple $R \equiv (R_1, R_2) \in [\mathbb{D}^c]^2$. We shall let R_{-i} denote the 1-tuple $R_{-i} \equiv R_j \in \mathbb{D}^c$.

Thus $UC(R_i, (x_i, y_i))$ is the set of commodity bundles that are at least as good as (x_i, y_i) according to R_i and $LC(R_i, (x_i, y_i))$ is the set of commodity bundles that are no better than (x_i, y_i) according to R_i. An indifference curve for preference R_i through a bundle (x_i, y_i) denoted by $IC(R_i, (x_i, y_i))$ is defined as follows: $IC(R_i, (x_i, y_i)) = UC(R_i, (x_i, y_i)) \cap LC(R_i, (x_i, y_i))$. We make the following important remark.

[1] For a preference ordering R_i and a vector $x \in \Re_+^2$, the upper contour set of R_i at x is denoted by $UC(R_i, x)$ and is the set $\{z \in \Re_+^2 | zR_ix\}$. Similarly the lower contour set of R_i at x is denoted by $LC(R_i, x)$ and is the set $\{z \in \Re_+^2 | xR_iz\}$. A preference ordering R_i is continuous if $UC(R_i, x)$ and $LC(R_i, x)$ are both closed for all $x \in \Re_+^2$. A preference ordering R_i is strictly convex if for all $x \in \Re_{++}^2$, $x^{'}, x^{''} \in UC(R_i, x)$ and $x^{'} \neq x^{''}$ implies $\lambda x^{'} + (1-\lambda)x^{''} P_i x$ for all $\lambda \in (0, 1)$. For $x, z \in \Re_+^2$ by $x > z$ we mean $x_k \geq z_k$ for all $k \in M$ and $x_k > z_k$ for some k. A preference ordering is strictly monotonic in $\Re_{++}^2$ if $x > z$ implies xP_iz.

Remark 1 Note that $IC(R_i, (x_i, y_i))$ is a set. But due to the classical properties of the preferences these sets can also be represented as a downward sloping curve in the relevant consumption space. Hence, by $IC(R_i, (x_i, y_i))$ we also denote the indifference curve of the preference R_i that passes through the consumption bundle (x_i, y_i). When a preference ordering R_i is represented by a differentiable utility function, by slope of $IC(R_i, (x_i, y_i))$ at a consumption bundle on $IC(R_i, (x_i, y_i))$ we mean slope of the indifference curve that passes through (x_i, y_i) at the relevant consumption bundle.

An SCF F is a mapping $F : [\mathbb{D}]^2 \to \Delta$ where $\mathbb{D} \subseteq \mathbb{D}^c$. The range of an SCF F is denoted by $\Re_F$. We now introduce some important but standard definitions.

Definition 1 *A social choice function F is* **manipulable** *by agent i at profile R via $R_i^{'} \in \mathbb{D}$ if $F(R_i^{'}, R_{-i}) P_i F(R)$. It is* **strategy-Proof** *if it is not manipulable by any agent at any profile. Equivalently F is strategy-proof if $F_i(R) R_i F_i(R_i^{'}, R_{-i})$ for all $R_i, R_i^{'} \in \mathbb{D}$, for all $R_{-i} \in \mathbb{D}$ and for all $i \in I$.*

In the usual strategic model, an agent's preference ordering is private information and F represents the mechanism designer's objectives. If F is strategy-proof, revealing the private information truthfully is a dominant strategy for each agent.

Definition 2 *An allocation $x \in \Delta$ is* **Pareto-efficient** *at profile R if there does not exist another allocation $x' \in \Delta$ such that $x'_i R_i x_i$ for all $i \in I$ and $x'_j P_j x_j$ for some $j \in I$.*

Let $PE(R)$ denote the collection of Pareto-Efficient allocations at the profile R. Individual-rationality is defined below: it ensures that an agent is not made worse-off relative to his endowment by F.

Definition 3 *Aa social choice function $F : [\mathbb{D}]^2 \to \Delta$ satisfies* **individual-rationality** *with respect to ω, if $F_i(R) R_i(\omega_i^x, \omega_i^y))$ for all i and for all $R \in [\mathbb{D}]^2$.*

Our next goal is to introduce fixed-price trading rules. We closely follow the notation and definitions in Barberà and Jackson (1995). Consider $a \in \Delta$ and let $a_i = (x_i, y_i)$. The next definition illustrates a property of the range of the SCF observed in Barberà and Jackson (1995) which is diagonality. To define the notion of diagonal set, let $a = ((a_1^x, a_1^y), (a_2^x, a_2^y)) = (a_1, a_2)$ and $b = ((b_1^x, b_1^y), (b_2^x, b_2^y)) = (b_1, b_2)$. That is, a_i denotes the 2 dimensional allocation of the goods x and y to agent i under the allocation vector $((a_1^x, a_1^y), (a_2^x, a_2^y))$. Similarly b_i is defined. Consider the following notation: for all $a_i, b_i \in \Re^2$, $a_i > b_i$ if $a_i^x \geq b_i^x$, $a_i^y \geq b_i^y$ and either $a_i^x > b_i^x$ or $a_i^y > b_i^y$. For two distinct allocations a_i and b_i by $a_i \not> b_i$ we mean if $a_i^x > b_i^x$ then $b_i^y > a_i^y$.

Definition 4 *We call two distinct feasible allocations a and b to be* **diagonal allocations** *if $a_i \not> b_i$ and $b_i \not> a_i$ for all $i \in I$. A set $B \subset \Delta$ is* **diagonal** *if for each agent i and for all distinct a and b in B, $a_i \not> b_i$ and $b_i \not> a_i$. We call (a_i^x, a_i^y) and (b_i^x, b_i^y) to be two* **diagonal bundles** *for agent i if $a_i^x > b_i^x (b_i^y > a_i^y)$ then $b_i^y > a_i^y (a_i^x > b_i^x)$.*

The piecewise linear graph drawn in Figure 1 is a diagonal subset of Δ. Agent i's allocation is measured from the origin O_i and agent j's are measured from O_j. Consider two allocations d and d'. Note that $d_i^x > d_i'^x$ and $d_i^y < d_i'^y$. Also $d_j^x < d_j'^x$ and $d_j^y > d_j'^y$. That is d' and d are diagonal allocations.

For any $a, b \in \Delta$, let $\overline{ab} = \{x | \exists \gamma \in [0,1] \text{ s.t. } x = \gamma a + (1-\gamma) b\}$ that is $\overline{ab}$ is the straight line segment that connects a and b. If $c_i \geq \gamma a_i + (1-\gamma) b_i$ for some $\gamma \in [0,1]$, we write $c \otimes_i ab$ that is c lies on or above $\overline{ab}$ relative to agent i's origin.

For any $B \subset \Delta$ and $R \in [\mathbb{D}^c]^2$, let $Top(R_i; B, R_j)$ denote the set of allocations in B that maximize R_i given R_j.[2] A function t_i which is a selection from $Top(R_i; B, R_j)$ is called a *tie-breaking rule.* A tie-breaking rule t_i is j-favorable at $B \subset \Re_F$ if for any R, $t_i(R_i; B, R_j) \neq t_i(R_i; B, R_j')$ only if $t_i(R_i; B, R_j') R_j' t_i(R_i; B, R_j)$. In words, if agent i has multiple tops on a set under R_i and if the choice of the tops varies when agent j changes his announced preference ordering, then it should vary in such a way that agent j cannot manipulate. The following is the definition a fixed-price trading rule as in Barberà and Jackson (1995).

Definition **5** *(***fixed-price trading rules***) A social choice function $F : [\mathbb{D}]^2 \to \Delta$ is a fixed-price trading rule if $\Re_F$ is closed, diagonal and contains ω and there exists an agent i such that the following hold:*

1. *For all distinct a and b in $\Re_F$, either $a \in \overline{\omega b}$, $b \in \overline{\omega a}$ or $\omega \otimes_i \overline{ab}$.*
2. *There exist tie-breaking rules t_i and t_j such that t_i is j-favorable at $\Re_F$ and t_j is i-favorable at $\overline{\omega a} \cap \Re_F$ for all $a \in \Re_F$.*
3. *$F(R) = t_j(R_j; \overline{\omega a} \cap \Re_F, R_i)$, where $a = t_i(R_i; \Re_F, R_j)$.*

[2] We realize that maximums of R_i on B does not depend on R_j; however as we shall see, this notation helps in defining the tie-breaking rule for FPT rules. Later in the paper while describing our results, we will drop R_j from the notation because tie-breaking is not required in our characterization.

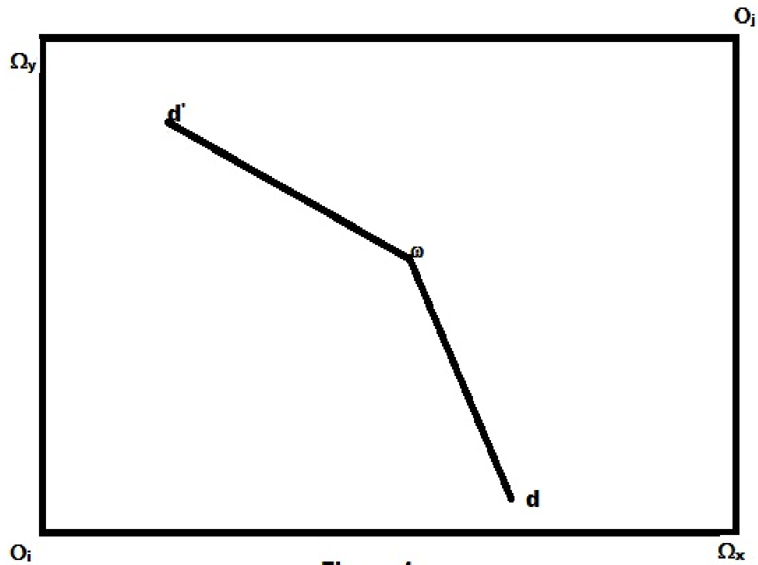

Figure 1

The piecewise linear graph in Figure 1 can be supported as the range of an FPT rule, for an SCF defined over the classical domain. This graph is closed, diagonal and also contains the endowment. The first property in the definition of FPT says that if any two feasible allocations a and b lie on the same side of the endowment then ω, a and b are collinear. This condition is satisfied by the graph in Figure 1. Since the indifference curves of classical preferences are strictly convex, both agents' preferences are single-peaked on each side of the endowment. In fact agent i's preferences are single-peaked on the whole graph. Note that agent j may have multiple tops on the graph, one top on each side of the endowment. Let the SCF be the median of the endowment, agent i's top and agent j's top (in the case of multiple tops of agent j consider the top of agent j which lies on the same side of the endowment on which agent i's top lies) then such a rule will be strategy-proof, individually-rational and satisfies all the properties of an FPT rule. In order to define median define an order on the piecewise linear graph. Let a, b be in the piecewise linear graph. Fix an agent i. Define the order $>_i$ to be $a >_i b$ if and only if $a_i^x > b_i^x$. The second property has no role in this example since the set $\overline{d'\omega} \cup \overline{d\omega}$ that we want to sustain as the range is connected. The third property is satisfied because the SCF chooses the (relevant) median. In Figure 5 the graph with two end points, one being the endowment cannot be sustained as range of an FPT rule because allocations to the right of ω are not collinear.

Theorem 1 *Barberà and Jackson (1995) Let $F : [\mathbb{D}^c]^2 \to \Delta$ be a social choice function. The social choice function F is strategy-proof and individually rational if and only if it is a fixed-price trading rule.*[3]

We make a remark about the implication of Theorem 1.

[3] Barberà and Jackson (1995) prove an FPT result for an arbitrary number of agents by imposing further assumptions on the SCF.

Remark **2** An implication of Theorem 1 is that the set of SCFs defined on the classical domain that are strategy-proof, individually rational and Pareto-efficient is empty. Serizawa (2002) proved this negative result for classical homothetic domains. However, there are positive results in certain restricted environments. When agents' preferences belong to the domain of Leontief preferences, Nicolò (2004) shows the existence of strategy-proof, Pareto-efficient and individually rational SCFs in two-good, two-agent exchange economies. We discuss Nicolò (2004) in Observation 4. In the context of housing market allocation problems from Roth and Postlewaite (1977) and Ma (1994) it follows that there exist strategy-proof, Pareto-efficient and individually rational allocation rules.

We end this section by introducing further notation that will be used subsequently. Consider a feasible allocation $(x^{'},y^{'}) = ((x^{'}_1,y^{'}_1),(x^{'}_2,y^{'}_2))$. We let $FIQ_i(x^{'},y^{'})$, $SEQ_i(x^{'},y^{'})$ $THQ_i(x^{'},y^{'})$ and $FOQ_i(x^{'},y^{'})$ denote the first, second, third and fourth quadrants of $(x^{'},y^{'})$ from i's perspective. Specifically, $FIQ_i(x^{'},y^{'}) = \{(x,y)|x_i \geq x^{'}_i \text{ and } y_i \geq y^{'}_i)\}$; $SEQ_i(x^{'},y^{'}) = \{(x,y)|x_i \leq x^{'}_i \text{ and } y_i \geq y^{'}_i)\}$, $THQ_i(x^{'},y^{'}) = \{(x,y)|x_i \leq x^{'}_i \text{ and } y_i \leq y^{'}_i)\}$ and $FOQ_i(x^{'},y^{'}) = \{(x,y)|x_i \geq x^{'}_i \text{ and } y_i \leq y^{'}_i)\}$.

Let $a,b \in \Delta$, then $[ab] = \{B \subset \Delta | B \text{ is diagonal, connected and } a,b \in B\}$. Note that $\overline{ab} \in [ab]$ i.e. $\overline{ab}$ is diagonal. A typical element of $[ab]$ will be denoted by $\widetilde{ab}$.

The notation *int* B refers to the interior of the set B.

3 Single-Crossing Domains and Concavification

In this section we first introduce the notion of classical single-crossing preference domains and analyze some of its properties. We then compare this notion of single-crossing with the one in Saporiti (2009). We shall also demonstrate that the domains of classical single-crossing preferences do not admit concavification.

The single-crossing property requires that two different indifference curves of two distinct preference orderings are never tangential to each other, that is either they cut each other from above or from below. In other words, this property rules out the situations in Panel A and allows the situation in Panel B of Figure 2. The single-crossing property is defined formally below.

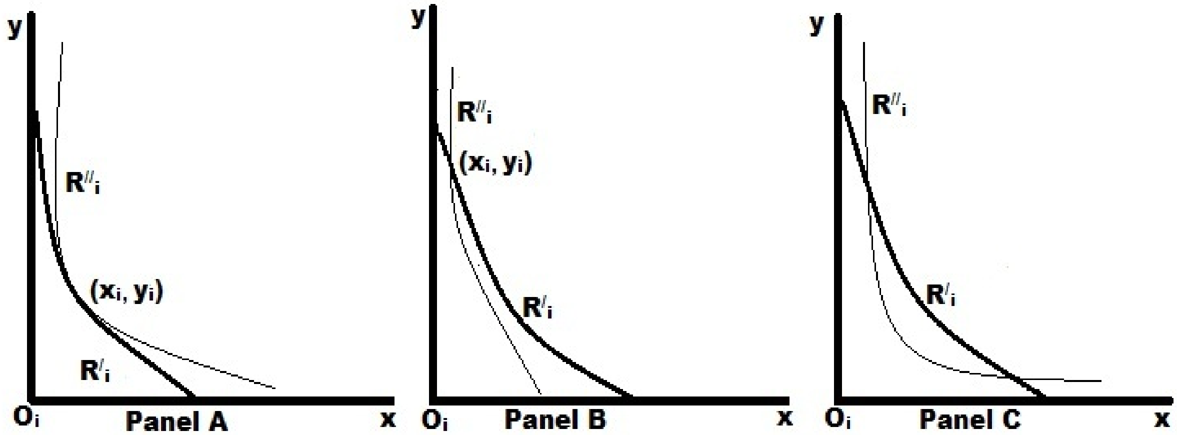

Figure 2

Definition 6 *The domain $\mathbb{D}^s$ of classical preferences admits the single-crossing property if for all distinct $R_i^{'}$, $R_i^{''}$ in $\mathbb{D}^s$ and all $(x_i, y_i) \in \Re^2_{++}$,*

1. *$UC(R_i^{'}, (x_i, y_i)) \subsetneq UC(R_i^{''}, (x_i, y_i))$ and $UC(R_i^{''}, (x_i, y_i)) \subsetneq UC(R_i^{'}, (x_i, y_i))$.*
2. *$(x_i, y_i) \neq (x_i^{'}, y_i^{'})$ and $IC(R_i^{'}, (x_i, y_i)) = IC(R_i^{'}, (x_i^{'}, y_i^{'}))$ imply $IC(R_i^{''}, (x_i, y_i)) \neq IC(R_i^{''}, (x_i^{'}, y_i^{'}))$*

In other words, single-crossing rules out the cases in Panel A and C in Figure 2. The indifference curves in Panel B of Figure 2 satisfies the single crossing property. Note that in Panel B of Figure 2 the indifference curve of the preference ordering $R_i^{''}$ cuts the indifference curve of $R_i^{'}$ at (x_i, y_i) from above. We formalize this in the following definition.

In the following definition by $IC(R_i, (x_i, y_i))$ we mean the indifference curve of preference R_i that passes through the consumption bundle (x_i, y_i), (see Remark 1).

Definition 7 *Let $\mathbb{D}^s$ be a domain of classical single-crossing preferences and let $R_i^{'}$, $R_i^{''}$ are from $\mathbb{D}^s$. Let $(x_i, y_i) \in \Re^2_{++}$ and let for two distinct bundles (x_i^{*}, y_i^{*}) and (x_i^{**}, y_i^{**}), $\{(x_i, y_i)\} = IC(R_i^{'}, (x_i^{*}, y_i^{*})) \cap IC(R_i^{'}, (x_i^{**}, y_i^{**}))$. We say $R_i^{''}$ cuts $R_i^{'}$ from above at (x_i, y_i) or $R_i^{'}$ cuts $R_i^{''}$ from below at (x_i, y_i)*

if for all $(x_i^{'}, y_i^{'})$ with $x_i^{'} < x_i, y_i^{'} > y_i, IC(R_i^{''}, (x_i, y_i))$ lies above $IC(R_i^{'}, (x_i, y_i))$ and

if for all $(x_i^{'}, y_i^{'})$ with $x_i^{'} > x_i, y_i^{'} < y_i, IC(R_i^{''}, (x_i, y_i))$ lies below $IC(R_i^{'}, (x_i, y_i))$.

In Panel B of Figure 2, $R_i^{''}$ cuts $R_i^{'}$ from above at (x_i, y_i) (or $R_i^{'}$ cuts $R_i^{''}$ from below at (x_i, y_i)). This means that in the interior of the second quadrant of (x_i, y_i) the indifference curve of the preference ordering R_i'' that passes through (x_i, y_i) lies above R_i'. Analogously, in the interior of the fourth quadrant of (x_i, y_i) the indifference curve of the preference ordering R_i'' that passes through (x_i, y_i) lies below R_i'. We record condition 2 in Definition 6 as an observation for future reference.

Observation **1** Let R_i' and R_i'' be distinct and from a classical single-crossing domain $\mathbb{D}^s$. Then there do not exist distinct $(x_i, y_i), (x_i', y_i') \in \Re^2_{++}$ such that

$$IC(R_i', (x_i, y_i)) = IC(R_i', (x_i', y_i')) \text{ and } IC(R_i'', (x_i, y_i)) = IC(R_i'', (x_i', y_i')).$$

We now provide some examples of single-crossing domains.

Example **1** Consider the preferences represented by the utility functions of the form

$$u_i(x_i, y_i; \theta_i) = \theta_i \sqrt{x_i} + y_i, \quad \theta_i > 0$$

We claim that this domain is single-crossing. Note that preferences represented by this class of utility functions are classical and smooth. The slope of an indifference curve of a preference in this class is given by $\frac{-\theta_i}{2\sqrt{x_i}}$ for all $x_i > 0$. Thus, at a given bundle (x_i, y_i), the slope of an indifference curve is described uniquely by the parameter θ_i. Suppose, $\theta_i' > \theta_i''$. Consider indifference curves $IC(\theta_i', (x_i, y_i))$ and $IC(\theta_i'', (x_i, y_i))$. Notice that the absolute value of the slope of $IC(\theta_i', (x_i, y_i))$ is higher than $IC(\theta_i'', (x_i, y_i))$ at (x_i, y_i). Hence the condition 1 or 2 in Definition 6 are satisfied.

In Example 1, preferences are quasi-linear. We also provide an example of a domain consisting of homothetic preferences that is single-crossing.

Example **2** Consider preferences represented by utility functions of the form

$$u_i(x_i, y_i; \alpha_i) = x_i^{\alpha_i} y_i, \quad \alpha_i > 0.$$

By definition of single-crossing we do not have to consider consumption bundles where either $x_i = 0$ or $y_i = 0$. For all other consumption bundles the slope of an indifference curve is given by $\frac{\alpha_i y_i}{x_i}$. If $\alpha_i' \neq \alpha_i''$ then the slopes are different at any (x_i, y_i). Hence the condition 1 or 2 in Definition 6 are satisfied.

In the next subsection we describe some important properties of single-crossing domains that are useful in our characterization.

3.1 Some Properties of Classical Single-crossing Domains

We first obtain some restrictions on preferences over triples implied by the single-crossing property.

For any $a, b, c \in \Delta$, define the following sets of preference orderings over a, b, c.[4]

$$\mathbb{D}^1(\{a, b, c\}) = \{cP_ibP_ia,\ bP_icP_ia,\ bP_iaP_ic,\ aP_ibP_ic,\ bI_iaP_ic,\ bP_icI_ia,\ cI_ibP_ia\}$$

$$\mathbb{D}^2(\{a, b, c\}) = \{cP_ibP_ia,\ aP_ibP_ic,\ cI_ibI_ia\}$$

We will show that if a domain is single-crossing, then three diagonal allocations can be chosen in such a way that all possible preference orderings over these allocations are either from $\mathbb{D}^1(\{a, b, c\})$ for one agent and from $\mathbb{D}^2(\{a, b, c\})$ for the other.

Proposition 1 *Consider any classical single-crossing domain $\mathbb{D}^s$. There exist three diagonal allocations a, b, c in Δ such that $R_i|_{\{a,b,c\}} \in \mathbb{D}^1(\{a, b, c\})$ and $R_j|_{\{a,b,c\}} \in \mathbb{D}^2(\{a, b, c\})$, $i \neq j$ for all $(R_i, R_j) \in [\mathbb{D}^s]^2$.*[5]

Proof: We follow the proof in Figure 3.[6]

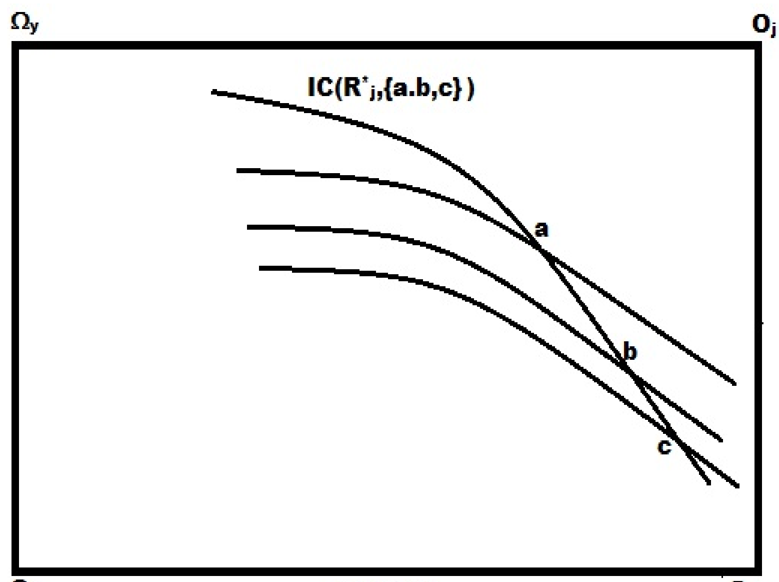

Figure 3

[4]For any $a, b \in \Delta$, aP_ib is written for a_i is preferred to b_i by agent i with preferences R_i while aI_ib means that agent i is indifferent between a_i and b_i under R_i

[5]The notation $R_i|_{\{a,b,c\}}$ refers to R_i restricted to $\{a, b, c\}$.

[6]The indifference curves in Figure 3 appear to be smooth.

Pick an agent j and a preference ordering $R_j^* \in \mathbb{D}^s$. Pick $a, b, c \in int\ \Delta$ that lie on an indifference curve under R_j^*. Label this indifference curve as $IC(R_j^*, \{a, b, c\})$. Note that a, b and c are diagonal. Furthermore, since preferences are classical cP_iaP_ib, aP_icP_ib, cI_ibI_ia, cP_ibI_ia, cI_iaP_ib, and aP_icI_ib are ruled out for any $R_i \in \mathbb{D}^s$.

Other preference orderings are not permissible because indifference curves cannot intersect. Therefore, $R_i|_{\{a,b,c\}} \in \mathbb{D}^1(\{a, b, c\})$ for all $R_i \in \mathbb{D}^s$.

The single-crossing property rules out indifference between any pair in $\{a, b, c\}$ for any $R_j \in \mathbb{D}^s$ unless all the three alternatives are indifferent. If cP_jaP_jb then the indifference curve passing through a under R_j must intersect $IC(R_j^*, \{a, b, c\})$ twice. However, this would contradict Observation 1. Finally, note that bP_jaP_jc and bP_jcP_ja are ruled out. This can happen only if an indifference curve of these two orderings cuts $IC(R_j^*, \{a, b, c\})$ between a and b and between b and c. This would again violate Observation 1. Therefore, $R_j|_{\{a,b,c\}} \in \mathbb{D}^2(\{a, b, c\})$ for all $R_j \in \mathbb{D}^s$ as required. ■

Remark **3** The above proposition reveals that over a triple of allocations in the interior of an Edgeworth box, the single-crossing property imposes for the case of classical preferences a restriction on the possible orderings over the triple. In particular, single-crossing preferences impose more restriction on the possible preference ordering over the triple for one of the agents. Bordes et al. (1995) show that the classical preferences in general, impose more restrictions of one agent relative to the other agent, on the possible preference orderings over triples. For instance, Bordes et al. (1995) show if over a triple one agent's preferences admit all possible preference orderings, then for the other agent some orderings are ruled out.

Proposition 1 specifies the restriction on preferences arising due to the single-crossing property. However, all preferences in $\mathbb{D}^1(\{a, b, c\}$ and $\mathbb{D}^2(\{a, b, c\})$ need not be present in an arbitrary single-crossing domain. We will impose a *richness* requirement on the single-crossing domains that ensures that we can pick the a, b, c triple in such a way that all preferences in $\mathbb{D}^1(\{a, b, c\}$ and $\mathbb{D}^2(\{a, b, c\})$ are present in the domain. The definition of richness is given below.

Definition **8** *The classical single-crossing domain* $\mathbb{D}^s$ *is* **rich** *if for all diagonal bundles* $(x_i^{'}, y_i^{'}), (x_i^{''}, y_i^{''}) \in \Re^2_{++}$, *there exist* $R_i \in \mathbb{D}^s$ *such that* $IC(R_i, (x_i^{'}, y_i^{'})) = IC(R_i, (x_i^{''}, y_i^{''}))$.

A single-crossing domain is rich if any two diagonal bundles can be joined by an indifference curve. We show that the domains specified in Examples 1 and 2 earlier are rich.

Example 1 (**Continued**) Consider the domain introduced in Example 1. We have already seen that this domain is a single-crossing domain. Let $(x_i^{'}, y_i^{'}), (x_i^{''}, y_i^{''}) \in \Re^2_+$ be such that $x_i^{'} > x_i^{''}$ and $y_i^{''} > y_i^{'}$. Set $\theta_i = \frac{y_i^{''} - y_i^{'}}{\sqrt{x_i^{'}} - \sqrt{x_i^{''}}}$. Note that $\theta_i > 0$ and $\theta_i\sqrt{x_i^{'}} + y_i^{'} = \theta_i\sqrt{x_i^{''}} + y_i^{''}$. Hence $(x_i^{'}, y_i^{'})$ and $(x_i^{''}, y_i^{''})$ lie on the indifference curve corresponding to θ_i. Therefore the domain is rich.

Example 2 (**Continued**) Consider the domain introduced in Example 2. We have already seen that this is a single-crossing domain. Let $(x_i^{'}, y_i^{'}), (x_i^{''}, y_i^{''}) \in \Re^2_{++}$ be such that $x_i^{'} > x_i^{''}$ and $y_i^{''} > y_i^{'}$. Set $\alpha_i = \frac{\ln(y_i^{''})-\ln(y_i^{'})}{\ln(x_i^{'})-\ln(x_i^{''})}$. Since ln is a strictly increasing function, $\alpha_i > 0$. Therefore, $\alpha_i \ln(x_i^{'}) + \ln(y_i^{'}) = \alpha_i \ln(x_i^{''}) + \ln(y_i^{''})$ or equivalently, $(x_i^{'})^{\alpha_i} y_i^{'} = (x_i^{''})^{\alpha_i} y_i^{''}$. Hence, $(x_i^{'}, y_i^{'})$ and $(x_i^{''}, y_i^{''})$ are on the same indifference curve corresponding to α_i. Therefore this domain is rich.

We now define maximal single-crossing domains. It says that a domain of single-crossing preferences is maximal if any superset of this domain is not single-crossing.

Definition **9** *Let $\mathbb{D}$ be a domain of classical single-crossing preferences. We say $\mathbb{D}$ is* **maximal single-crossing** *if it is not possible to add any preference to $\mathbb{D}$ and still have a single-crossing domain.*

An important implication of richness is that a rich single-crossing domain is also a maximal single-crossing domain. We formalize this below.

Proposition **2** *Let $\mathbb{D}^*$ and $\mathbb{D}^s$ be two classical single-crossing domains. Let $\mathbb{D}^s$ be rich. Then either $\mathbb{D}^* \cup \mathbb{D}^s$ is not a single-crossing domain or $\mathbb{D}^* \subset \mathbb{D}^s$.*

Proof: Pick an arbitrary $R_i \in \mathbb{D}^* \setminus \mathbb{D}^s$. Pick an arbitrary indifference curve of R_i and $(x_i, y_i), (x_i^{'}, y_i^{'}) \in \Re^2_{++}$ such that $IC(R_i, (x_i, y_i)) = IC(R_i, (x_i^{'}, y_i^{'}))$. Since $\mathbb{D}^s$ is rich, there exists $R_i^{'} \in \mathbb{D}^s$, such that $IC(R_i^{'}, (x_i, y_i)) = IC(R_i^{'}, (x_i^{'}, y_i^{'}))$. If $IC(R_i, (x_i, y_i)) \neq IC(R_i^{'}, (x_i, y_i))$, then we contradict Observation 1, i.e. in this case a situation as in Panel C of Figure 2 emerges. Since $R_i \neq R_i^{'}$, if on the other hand $IC(R_i, (x_i, y_i)) = IC(R_i^{'}, (x_i, y_i)) = IC(R_i, (x_i^{'}, y_i^{'})) = IC(R_i^{'}, (x_i^{'}, y_i^{'}))$, then R_i and $R_i^{'}$ cannot be elements of a single-crossing domain (1 in Definition 6), that is $\mathbb{D}^* \cup \mathbb{D}^s$ is not single-crossing. ■

In the next Lemma we show that if a domain of single-crossing preferences is rich, we can find allocations a, b, c with $c >_i b >_i a$ (as in Figure 3)[7], such that all preferences ordering from $\mathbb{D}^1(\{a, b, c\})$ and $\mathbb{D}^2(\{a, b, c\})$ are induced.

Lemma **1** *Let the classical single-crossing domain $\mathbb{D}^s$ be rich. Then there exists $a, b, c \in \Delta$ with $c >_i b >_i a$ such that $\{R_i|_{\{a,b,c\}} | R_i \in \mathbb{D}^s\} = \mathbb{D}^1(\{a, b, c\})$ and $\{R_j|_{\{a,b,c\}} | R_j \in \mathbb{D}^s\} = \mathbb{D}^2(\{a, b, c\})$.*

[7] Let a and b be two diagonal allocations. Fix an agent i. We say $a >_i b$ if $a_i^x >_i b_i^x$ (or $a_i^y <_i b_i^y$). Let $\{a^1, a^2, a^3\}$ be a set of diagonal allocations. We define median$\{a^1, a^2, a^3\} \in \{a^1, a^2, a^3\}$ to be the median of $\{a^1, a^2, a^3\}$ if $\left|\{a^j | a^j \geq_i \text{median}\{a^1, a^2, a^3\}\}\right| \geq 2$ and $\left|\{a^j | \text{median}\{a^1, a^2, a^3\} \geq_i a^j\}\right| \geq 2$. The usual definition of single-peaked preferences apply on a diagonal set according to this order. For instance, if a diagonal set is a straight line segment and lies in the interior of Δ, then all the allocations on it can be sustained as tops for some preference ordering from $\mathbb{D}^s$ (this follows from arguments similar to Proposition 4). Also, on both sides of the top, the allocation nearer (note that under the order, distance between two allocations can be defined in the Euclidean sense) to it are preferred to the one which is further.

Proof: Choose R_j^* and fix an indifference curve. Now pick R_i and an indifference curve of R_i such that it is tangent to the chosen indifference curve of R_j^* in the interior of Δ. Label the tangency point as b. Then choose a and c on the indifference curve of R_j^* such that bP_icI_ia. Also choose $R_j^{'}$ such that $IC(R_j^{'}, b)$ cuts $IC(R_j^*, b)$ from above. By richness, such an $R_j^{'}$ exists. Since the domain is single-crossing, we have $aP_j^{'}bP_j^{'}c$.[8] Also choose $R_j^{''}$ such that $IC(R_j^{''}, b)$ cuts $IC(R_j^*, b)$ from below. This will result in $cP_j^{''}bP_j^{''}a$. Hence, $\{R_j|_{\{a,b,c\}}|R_j \in \mathbb{D}^s\} = \mathbb{D}^2(\{a, b, c\})$.

Using the richness of $\mathbb{D}^s$ we can show that $\{R_i|_{\{a,b,c\}}|R_i \in \mathbb{D}^s\} = \mathbb{D}^1(\{a, b, c\})$. For instance, choose $R_i^{'}$ whose indifference curve passes through a and some allocation between b and c. This results in $bP_i^{'}aP_i^{'}c$. Choose $R_i^{''}$ to be such that agent i is indifferent between a and b. This gives $bI_i^{''}aP_i^{''}c$. Similarly, all other preference orderings in $\mathbb{D}^1(\{a, b, c\})$ can be constructed.

■

We will use the sets $\mathbb{D}^1(\{a, b, c\})$ and $\mathbb{D}^2(\{a, b, c\})$ to construct SCFs that are not FPT rules.

Our next goal is to show that an order relation can be defined on a classical single-crossing domain $\mathbb{D}^s$. The next Proposition is crucial for that purpose.

PROPOSITION **3** *Consider any classical single-crossing domain $\mathbb{D}^s$. Let $\bar{R}_i, \tilde{R}_i \in \mathbb{D}^s$ and $(x_i^*, y_i^*) \in \Re^2_{++}$. If $IC(\tilde{R}_i, (x_i^*, y_i^*))$ cuts $IC(\bar{R}_i, (x_i^*, y_i^*))$ from above at (x_i^*, y_i^*), then $IC(\tilde{R}_i, (x_i, y_i))$ cuts $IC(\bar{R}_i, (x_i, y_i))$ from above at all $(x_i, y_i) \in \Re^2_{++}$.*

Proof: We prove the Proposition in four steps.

Step 1: We show that an indifference curve of $\bar{R}_i$ that cuts the indifference curve of $\tilde{R}_i$ at (x_i^*, y_i^*) from below must cut $\tilde{R}_i$ from below at all $(x_i, y_i) \in \Re^2_{++} \cap IC(\tilde{R}_i, (x_i^*, y_i^*))$. By way of contradiction, suppose that an indifference curve of $\bar{R}_i$ cuts the indifference curve of $\tilde{R}_i$ that contains (x_i^*, y_i^*) from above at $(x_i^{'}, y_i^{'})$. Suppose $(x_i^{'}, y_i^{'})$ is in $SEQ_i((x_i^*, y_i^*))$. Hence, $IC(\bar{R}_i, (x_i^{'}, y_i^{'})) \cap UC(\bar{R}_i, (x_i^*, y_i^*)) \cap LC(\tilde{R}_i, (x_i^*, y_i^*)) \neq \emptyset$. Since indifference curves of an ordering cannot intersect, $IC(\bar{R}_i, (x_i^{'}, y_i^{'}))$ and $IC(\tilde{R}_i, (x_i^*, y_i^*))$ must cut twice, contradicting Observation 1. We reach the same contradiction if $(x_i^{'}, y_i^{'})$ belongs to $FOQ_i((x_i^*, y_i^*))$.

Step 2: Let $\mathbb{E} = \{(x_i, y_i)|y_i = y_i^*, x_i \geq 0\}$. We will show that the indifference curves of $\tilde{R}_i$ must cut the indifference curves of $\bar{R}_i$ at all the bundles in $\mathbb{E}$ from above. For the purpose of contradiction, consider $(x_i^{**}, y_i^*) \in \mathbb{E}$ such that $x_i^{**} > x_i^*$ and $IC(\tilde{R}_i, (x_i^{**}, y_i^*))$ cuts $IC(\bar{R}_i, (x_i^{**}, y_i^*))$ from below. Figure 4 is helpful in understanding the argument that follows. In the figure, the darker indifference curves represent $\tilde{R}_i$ and the lighter ones $\bar{R}_i$.

Note that if $IC(\bar{R}_i, (x_i, y_i^*))$ cuts $IC(\tilde{R}_i, (x_i, y_i^*))$ from below (resp.above) at (x_i, y_i^*) then Step 1 and the continuity of preferences imply that there is a neighborhood $N(x_i) \subset \mathbb{E}$ of

[8]Note that we are using the order over a, b and c for this inference.

x_i such that $IC(\bar{R}_i, (\bar{x}_i, y_i^*))$ cuts $IC(\tilde{R}_i, (\bar{x}_i, y_i^*))$ from below (resp.above) for all $\bar{x}_i \in N(x_i)$. Call this the "openness property". Let $x_i^{'}$ be the largest number for which $IC(\bar{R}_i, (x_i, y_i^*))$ cuts $IC(\tilde{R}_i, (x_i, y_i^*))$ from below for all $x_i \in [x_i^*, x_i^{'})$. By the openness property and our assumption that $IC(\bar{R}_i, (x_i^{**}, y_i^*))$ cuts $IC(\tilde{R}_i, (x_i^{**}, y_i^*))$ from above such $x_i^{'}$ is well defined. Note that openness property implies that $IC(\bar{R}_i, (x_i^{'}, y_i^*))$ does not cut $IC(\tilde{R}_i, (x_i^{'}, y_i^*))$ at $(x_i^{'}, y_i^*)$ from below but it cannot cut from above either otherwise the openness property will be violated. An analogous argument can be used when $x_i^{**} < x_i^*$.

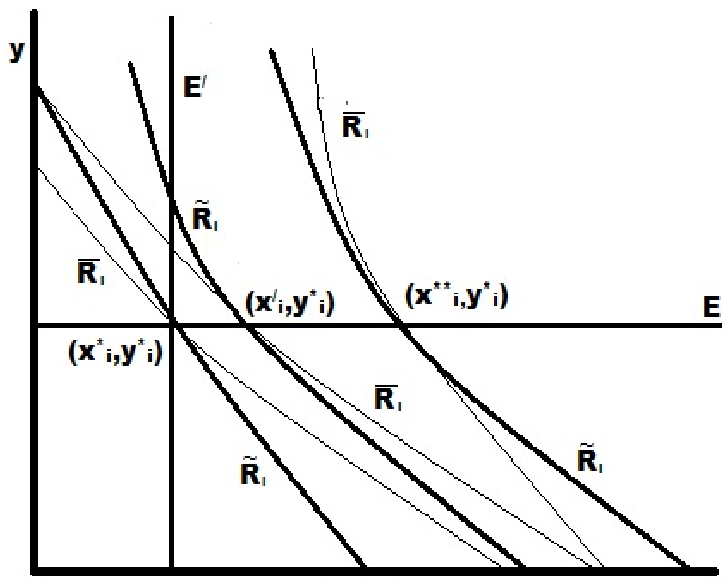

Figure 4

Step 3: Consider the subset $\mathbb{E}^{'} = \{(x_i, y_i) | x_i = x_i^*, y_i \geq 0\}$. Using the same argument as above it follows that indifference curves of $\tilde{R}_i$ must cut the indifference curves of $\bar{R}_i$ at all the bundles of $\mathbb{E}^{'}$ from above.

Step 4: From Step 2 and Step 3 it follows that along every horizontal and vertical line, the indifference curves of $\tilde{R}_i$ must cut the indifference curves of $\bar{R}_i$ at all the bundles from from above. This establishes the Proposition. ■

Let $R_i^{'}, R_i^{''} \in \mathbb{D}^s$. We say $R_i^{'} \succ R_i^{''}$ if the indifference curves of $R_i^{'}$ cut the indifference curves of $R_i^{''}$ from above at all bundles. Proposition 3 ensures that the order $\succ$ is well-defined. However, we want to emphasize that the use of classical preferences plays an important role in the validity of this order.

Observation **2** Consider two preference orderings defined as follows. Let $(x_1, y_1), (x_2, y_2)$ be in $\Re^2_+$.

1. $(x_1, y_1) \succeq^{'} (x_2, y_2)$ iff $[x_1 > x_2$ or $(x_1 = x_2$ and $y_1 \geq y_2)]$

2. $(x_1, y_1) \succeq'' (x_2, y_2)$ iff $[x_1 < x_2$ or $(x_1 = x_2$ then $y_1 \leq y_2)]$

None of these preferences are classical, in fact $\succeq''$ is not even monotonic and it is well known that $\succeq'$ is not continuous. Since $\succeq'$ is monotonic and $\succeq''$ is not, the first condition in Definition 6 is satisfied at all consumption bundles in $\Re^2_{++}$. Under both the orderings (x_1, y_1) and (x_2, y_2) are indifferent if and only if $(x_1, y_1) = (x_2, y_2)$, that is the indifference sets are singletons. Hence, the second condition in Definition 6 is vacuously satisfied. But also note that since indifference sets are singletons vacuously $\succeq''$ cuts $\succeq'$ from above and $\succeq'$ cuts $\succeq''$ from above at all consumption bundles in $\Re^2_{++}$. Hence, we cannot derive a linear order on the set $\{\succeq', \succeq''\}$ using the two conditions listed in Definition 6.

The next Lemma describes an important set theoretic feature of a classical single-crossing domain which is rich.

Definition **10** *Let $\mathbb{D}^s$ be a classical single-crossing domain. We say that $\mathbb{D}^s$ is a* **linear continuum** *if the following conditions hold.*

1. *If $R_i'' \succ R_i'$, then there exists R_i''' such that $R_i'' \succ R_i''' \succ R_i'$.*
2. *The domain $\mathbb{D}^s$ has the least upper bound property.*[9]

Lemma **2** *If the classical single-crossing domain $\mathbb{D}^s$ is rich, then it is a linear continuum.*

Proof: Let $R_i'' \succ R_i'$ and let (x_i, y_i) be arbitrary. Choose $(x_i', y_i') \in int\ UC(R_i', (x_i, y_i)) \cap int\ LC(R_i'', (x_i, y_i))$. Note that (x_i, y_i) and (x_i', y_i') are diagonal. Since $\mathbb{D}^s$ is rich, there exists R_i''' such that it has an indifference curve that passes through these two bundles. Note that since $\mathbb{D}^s$ is a single-crossing domain, it will cut the indifference curve of R_i' from above and cut the indifference curve of R_i'' from below at (x_i, y_i). Hence, $R_i'' \succ R_i''' \succ R_i'$.

Now we show that $\mathbb{D}^s$ has the least upper bound property. Choose an allocation $(x_i, y_i) \gg (0, 0)$. Let $C_\delta((x_i, y_i)) \subset \Re^2_{++}$ be a circle centered at (x_i, y_i) with radius δ. Observe that for all $R_i \in \mathbb{D}^s$, there exists an indifference curve passing thorough (x_i, y_i). Let $\mathbb{A}$ be the arc of $C_\delta((x_i, y_i))$ in $SEQ_i((x_i, y_i))$ excluding the end points. Let the arc intersect the vertical axis through (x_i, y_i) at (x_i, y_i^*) and let it intersect the horizontal axis at (x_i^*, y_i). Note that by the single crossing property, for any two diagonal bundles there is a unique $R_i \in \mathbb{D}^s$ with an indifference curve containing them.

Hence, we obtain a bijection $G : \mathbb{D}^s \to \mathbb{A}$ where $G(R_i') = (x_i', y_i')$ if $IC(R_i', (x_i, y_i)) = IC(R_i', (x_i', y_i'))$. Since the open interval (x_i^*, x_i) has the least upper bound property, so does $\mathbb{D}^s$.

■

[9]A ordered set S is said to have the least upper bound property if every bounded subset of S has the supremum in S.

From now on, we will assume that the topology on $\mathbb{D}^s$ is the order topology generated by the collection of open intervals of the form $(R_i', R_i'') = \{R_i | R_i'' \succ R_i \succ R_i'\}$. Since $\mathbb{D}^s$ is a linear continuum, $\mathbb{D}^s$ is connected in this order topology.[10] It follows that if F is a continuous SCF defined on $\mathbb{D}^s$, then $\Re_F$ is connected.

According the next Proposition, any interior allocation in Δ can be sustained as a Pareto-efficient allocation for some preference profile.

Proposition 4 *Let the classical single-crossing domain $\mathbb{D}^s$ be rich. Consider an arbitrary allocation $((x_i, y_i), (x_j, y_j))$ in the interior of Δ. Then there exists $(R_i, R_j) \in [\mathbb{D}^s]^2$ such that $((x_i, y_i), (x_j, y_j)) \in PE(R_i, R_j)$.*

Proof: Since $(x, y) = ((x_i, y_i), (x_j, y_j))$ is in the interior of Δ we can choose an $R_j \in \mathbb{D}^s$ such that $(x^1, y^1) \in IC(R_j, (x, y))$ in the interior of Δ, such that (x^1, y^1) is in the $SEQ_i((x, y))$. Choose R_i^1 such that $IC(R_i^1, (x^1, y^1)) = IC(R_i^1, (x, y))$. Let $(x^2, y^2) \in PE(R_i^1, R_j)$ on $IC(R_j, (x, y))$. Then choose R_i^2 such that $IC(R_i^2, (x_i^2, y_i^2)) = IC(R_i^2, (x_i, y_i))$. Note that $R_i^2 \succ R_i^1$. In this way, we construct an increasing sequence of preferences $\{R_i^k\}_{k=1}^{\infty}$ and an associated sequence of Pareto-efficient allocations for the profiles $\{(R_i^k, R_j)\}_{k=1}^{\infty}$ on $IC(R_j, (x, y))$ such that Euclidean distance between (x, y) and the Pareto-efficient allocations monotonically converge to zero. Now note that $\{R_i^k\}_{k=1}^{\infty}$ is bounded above by any $R_i \in \mathbb{D}^s$ which cuts $IC(R_j, (x, y))$ from above at (x, y).[11]

Since $\mathbb{D}^s$ has the least upper bounded property, $\{R_i^k\}_{k=1}^{\infty}$ converges to its supremum. Since the Euclidean distance between (x, y) and $\{(x^k, y^k)\}_{k=1}^{\infty}$ monotonically converges to zero, the supremum must be the R_i such that $(x, y) \in PE(R_i, R_j)$.

■

Remark 4 From Goswami et al. (2013) we know that allocating a zero amount of good x and positive amount of good y to an agent does not correspond to a Pareto-efficient allocation for the single-crossing domain in Example 1. Therefore, Proposition 4 is not true for non-interior allocations.

Since Saporiti (2009) also characterizes strategy-proof SCFs, before we proceed further it is important to understand the similarities and differences between our model and the model in Saporiti (2009). The next subsection deals with this aspect.

3.2 Discussion

In this subsection, we compare our model with that of Saporiti (2009). He assumes a finite set of alternatives with cardinality at least three. He assumes strict orderings and that the domain is an ordered set. His definition of the single-crossing property is as follows.

[10] See Munkres (2005), page 169.

[11] Note that such an upper bound need not exist if either the indifference curves are not strictly convex or the allocation that is considered is not in the interior of Δ.

Definition **11** *(Saporiti (2009)) A set of preferences (strict) $\mathbb{L}$ exhibits the single-crossing property on the set of alternatives X if there is a linear order $>$ on X and a liner order $\succ$ on $\mathbb{L}$ such that for all $a, b \in X$ and for all $P_i, P_i^{'} \in \mathbb{L}$*

$$(SC1)\ [b > a, P_i^{'} \succ P_i \ and\ bP_ia] \Rightarrow bP_i^{'}a$$

and

$$(SC2)\ [b > a, P_i^{'} \succ P_i \ and\ aP_i^{'}b] \Rightarrow aP_ib.$$

We remark below on the differences and the similarities between our model and his.

- Since we are concerned with allocations in an Edgeworth box, we do not assume finiteness of the set of alternatives. The assumption of strict preferences is also inappropriate in our model because it rules out classical preferences.
- Our notion of the single-crossing property is consistent with Definition 11. Consider $SC(1)$. Suppose $b > a$, bP_ia and $IC(R_i^{'}, b)$ cuts $IC(R_i, b)$ from above in the sense of our definition. We then have $bP_i^{'}a$. Similarly for $SC(2)$, if $b > a$, aP_ib and $IC(R_i, a)$ cuts $IC(R_i^{'}, a)$ from below we have aP_ib.
- Due to the specific characteristics of the classical preferences we have been able to derive an order on single-crossing preferences. Hence, classical preferences which are important in many economic environments, are important examples of Saporiti (2009)'s definition of single-crossing preferences.

The following observation points out to the importance of classical single-crossing preferences when we compare our model with that of Saporiti (2009).

Observation **3** The consistency of the notion of single-crossing in this paper with the one defined in Saporiti (2009) discussed in this section depends on the fact that the preferences considered in this paper are classical. Consider $a = (x_i^{'}, y_i^{'}), b = (x_i^{''}, y_i^{''})$ in $\Re^2_{++}$. Let $a > b$ that is $x_i^{'} > x_i^{''}$ and $y_i^{'} > y_i^{''}$. Let $R_i^{'}$ and $R_i^{''}$ be two preference relations such that the indifference curves are strictly concave from the origin. Also let the indifference curves of $R_i^{''}$ cuts the indifference curves of $R_i^{'}$ from above, so that $R_i^{''} \succ R_i^{'}$. Let $R_i^{'}$ is strictly monotonic and $R_i^{''}$ increases in the direction of the origin. Hence we have $a > b$, $R_i^{''} \succ R_i^{'}$, $bP_i^{''}a$, but $aP_i^{'}b$. Hence, $(SC2)$ is violated. Note that $R_i^{''}$ is not monotonic and none of the preferences are strictly convex.

We conclude this section with a note that single-crossing domains do not satisfy concavification. First we state the notion of concavification as defined in Barberà and Jackson (1995).

Definition **12** *Let* $R_i^{'}$ *be a preference ordering and let* $(x_i, y_i) \in \Re^2_+$*. The preference ordering* $R_i^{''}$ *is a* **concavification** *of* $R_i^{'}$ *at* (x_i, y_i) *if*

(i) $UC(R_i^{''}, (x_i, y_i)) \subset UC(R_i^{'}, (x_i, y_i))$ *and*

(ii) $(x_i^{'}, y_i^{'}) \in UC(R_i^{''}, (x_i, y_i))$ *and* $(x_i^{'}, y_i^{'}) \neq (x_i, y_i) \Rightarrow (x_i^{'}, y_i^{'}) P_i^{'} (x_i, y_i)$.

In Figure 2 Panel A, $R_i^{''}$ is a concavification of $R_i^{'}$ at (x_i, y_i). The indifference curve of $R_i^{''}$ touches the indifference curve of $R_i^{'}$ at (x_i, y_i) and lies strictly above it at all other bundles. In Panel B, neither $R_i^{''}$ is a concavification of $R_i^{'}$ nor $R_i^{'}$ a concavification of $R_i^{''}$ at (x_i, y_i). In other words, single-crossing does not allow concavification.

Definition **13** *The domain* $\mathbb{D} \subseteq \mathbb{D}^c$ **admits concavification** *if for all* $R_i \in \mathbb{D}$ *and* $a_i \in \Re^2_{++}$*, there exists* $\tilde{R}_i \in \mathbb{D}$ *that is a concavification of* R_i *at* a_i*.*

The following remark tells us why the single-crossing property has been defined for the consumption bundles in the interior of the consumption space.

Remark **5** For the bundles on the boundary of the consumption space, concavification and single-crossing property can co-exist. Therefore, only the consumption bundles in the interior of the consumption space have been considered in Definition 6.

The domain $\mathbb{D}^c$ is an example of a domain that admits concavification. An example of a "smaller" domain that also admits concavification is provided below.

Example **3** Consider the domain represented by the utility function

$$u_i(x_i, y_i; \theta_i, \alpha_i) = \theta_i x_i^{\alpha_i} + y_i, \quad \theta_i > 0 \text{ and } 0 < \alpha_i < 1.$$

Note that this domain consists of classical preferences; moreover all preferences in the domain are smooth. Fix an utility function $(\theta_i^{'}, \alpha_i^{'})$ and a consumption bundle (x_i^*, y_i^*). If $x_i^* = 0$ then $(\theta_i^{''}, \alpha_i^{'})$ is a concavification of $(\theta_i^{'}, \alpha_i^{'})$ at (x_i^*, y_i^*), where $\theta_i^{''} < \theta_i^{'}$. Now consider the case where $x_i^* > 0$. Then the absolute value of the slope of an indifference curve of the utility function $(\theta_i^{'}, \alpha_i)$ at (x_i^*, y_i^*) is $\theta_i^{'}\alpha_i(x_i^*)^{\alpha_i - 1}$.

Note that, $\lim_{\alpha_i \to 0} \theta_i^{'}\alpha_i(x_i^*)^{(\alpha_i-1)} = 0$. Therefore, we can choose $\alpha_i^{''} < \alpha_i^{'}$ such that $\theta_i^{'}\alpha_i^{''}(x_i^*)^{(\alpha_i^{''}-1)} < \theta_i^{'}\alpha_i^{'}(x_i^*)^{(\alpha_i^{'}-1)}$. Then choose $\theta_i^{''} > \theta_i^{'}$ such that $\theta_i^{''}\alpha_i^{''}(x_i^*)^{(\alpha_i^{''}-1)} = \theta_i^{'}\alpha_i^{'}(x_i^*)^{(\alpha_i^{'}-1)}$. We claim that $(\theta_i^{''}, \alpha_i^{''})$ is a concavification of $(\theta_i^{'}, \alpha_i^{'})$ at (x_i^*, y_i^*). Since $\theta_i^{'}\alpha_i^{'}(x_i^*)^{(\alpha_i^{'}-1)} = \theta_i^{''}\alpha_i^{''}(x_i^*)^{(\alpha_i^{''}-1)}$, slopes of the two utility functions at (x_i^*, y_i^*) are equal. We can write the equality as $\frac{\theta_i^{'}\alpha_i^{'}}{\theta_i^{''}\alpha_i^{''}} = \frac{1}{(x_i^*)^{(\alpha_i^{'}-\alpha_i^{''})}}$. Since $(\alpha_i^{'} - \alpha_i^{''}) > 0$, if $x_i < x_i^*$ then $\theta_i^{'}\alpha_i^{'}x_i^{(\alpha_i^{'}-1)} < \theta_i^{''}\alpha_i^{''}x_i^{(\alpha_i^{''}-1)}$ and if $x_i > x_i^*$ then $\theta_i^{'}\alpha_i^{'}x_i^{(\alpha_i^{'}-1)} > \theta_i^{''}\alpha_i^{''}x_i^{(\alpha_i^{''}-1)}$. This establishes our claim.

We conclude this section with the following definition single-crossing classical exchange economies.

DEFINITION **14** *An exchange economy i.e.* $\{I, M, \omega, (R_1, R_2)\}$ *is a* **single-crossing classical exchange economy** *if* R_1 *and* R_2 *are classical single-crossing preferences.*

In the next section we provide examples of SCFs in single-crossing domains that do not belong to the class of FPT rules.

4 Non Fixed-price Trading Rules

In this section, we give examples of SCFs that are strategy-proof and individually rational but are not FPT rules. These examples show that the Barberà and Jackson (1995) characterization does not hold over single-crossing domains and highlights the role of concavification in their arguments. In addition, various features of these examples will illustrate the role of rich domains and continuity of the SCFs, the properties that we require for our characterization.

EXAMPLE **4** We consider a rich single-crossing domain $\mathbb{D}^s$. In this example allocations a, b and c are as shown in Figure 3. Preferences over this triple for agents 1 and 2 belong to $(\mathbb{D}^1(\{a, b, c\})$ and $\mathbb{D}^2(\{a, b, c\})$ (Proposition 1). Let the endowment ω be the allocation a. The SCF, that depends only on the preference restricted to $\{a, b, c\}$, is depicted in Table 1 below.

	cP_2bP_2a	aP_2bP_2c	cI_2bI_2a
cP_1bP_1a	c	a	c
bP_1cP_1a	b	a	b
bP_1aP_1c	b	a	b
aP_1bP_1c	a	a	a
bI_1aP_1c	b	a	b
bP_1cI_1a	b	a	b
cI_1bP_1a	b	a	b

Table 1: A Non FPT Rule

It is easily verified that the SCF in Table 1 is strategy-proof, individually rational and has three elements in the range. It is not a FPT rule because all the allocations in the range lie on one side of the endowment but are not collinear. The range contains the endowment.

Consider a preference domain where aP_ibP_ic is not admissible for both $i = 1, 2$. The domain remains single-crossing although it is no longer rich. The SCF over the restricted domain is still strategy-proof and individually rational. However, the endowment is no longer in its range. This example in conjunction with our characterization result makes it clear that richness of the domain is critical in ensuring that the endowment lies in the range of the SCF.

The next example shows that agents' preferences need not be single-peaked on the range on each side of the endowment.

Example **5** Consider a rich domain of classical single-crossing preferences. The SCF is shown in Figure 5.

Consider a preference R_j^* such that $IC(R_j^*, \omega) = IC(R_j^*, d)$. We will define a strategy-proof and individually rational SCF F such that $\Re_F = \widetilde{\omega d}$, where $\widetilde{\omega d} \subset IC(R_j^*, \omega)$. By the single-crossing property, $Top(R_j, \widetilde{\omega d}) \in \{\omega, d\}$ for all $R_j \neq R_j^*$.

Define the SCF F as follows:

$$F(R_i, R_j) = \begin{cases} \omega, & \text{if } \omega \in Top(R_j, \widetilde{\omega d}); \\ x, & \text{if } x = Top(R_i, \widetilde{\omega d}) \text{ and } d = Top(R_j, \widetilde{\omega d}) \end{cases}$$

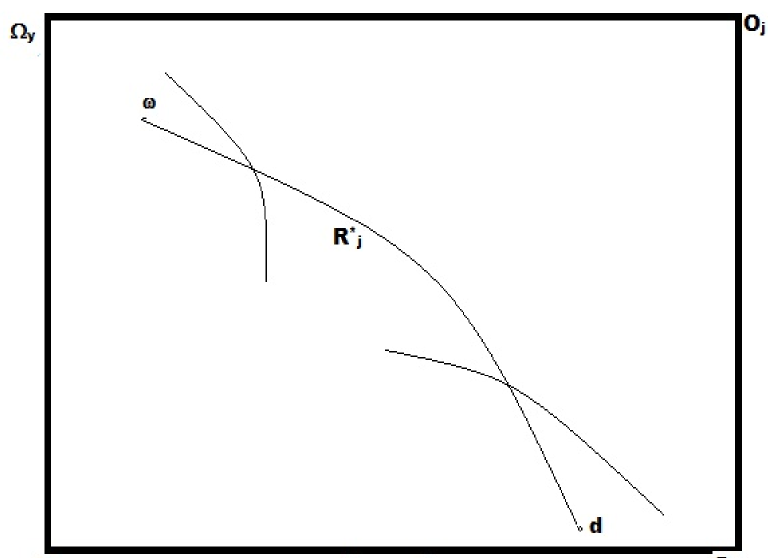

Figure 5

Since the domain of F is rich there exists R_i such that $x = Top(R_i, \widetilde{\omega d})$ for all $x \in \widetilde{\omega d}$. To see this note that the range drawn is in the interior of the Edgeworth box. Arguing in the same way as in Proposition 4 we can show that there exists R_i such that $x = Top(R_i, \widetilde{\omega d})$ for all $x \in \widetilde{\omega d}$.

We claim that F is strategy-proof. Note that agent i does not have any incentive to deviate when the outcome is ω because he cannot change the outcome by changing his announcement and when the outcome is other than ω he gets his best outcome.

Agent j is not going to change her announcement when the outcome is ω because she is getting her best allocation. When the outcome is not ω she can change it to ω only. But if $d = Top(R_j, \widetilde{\omega d})$, then ω is the worst allocation for agent j in $\widetilde{\omega d}$ because of the single-crossing property. This SCF is also individually rational because an allocation other than ω is chosen only when both the agents are better-off relative to ω.

Observe that F is not continuous. To see this choose a profile (R_i^1, R_j^1) such that $d = Top(R_i, \widetilde{\omega d})$ for all i. Such a profile exists by richness and since $d \in int\ \Delta$.[12] Also consider R_j^2 such that $\omega = Top(R_j, \widetilde{\omega d})$. According to the construction of the order $\succ$ on the domain of preferences, $R_j^2 \succ R_j^1$. By definition $F(R_i^1, R_j^1) = d$ and $F(R_i^1, R_j^2) = \omega$. Also note that $F(R_i^1, R_j) \in \{\omega, d\}$ for all R_j. Hence continuity for agent j is violated when agent i's preference is fixed at R_i^1.

We will show in our characterization result that if continuity is additionally imposed on the SCF, then agents' preferences over the range are single-peaked on each side of the endowment. In the next section we discuss our characterization result.

5 A Characterization Result

In this section, we will provide a characterization of the SCFs defined on the rich single-crossing domains that are strategy-proof, individually rational and continuous. We call this class of trading rules to be *Generalized Trading* rules. We define such rules as follows.

Definition 15 *Let $F : [\mathbb{D}]^2 \to \Delta$ be an SCF, where $\mathbb{D} \subseteq \mathbb{D}^c$. We say that F is a* **generalized trading (GT)** *rule if the following conditions hold:*

1. $\omega \in \Re_F$.
2. *$\Re_F$ is diagonal.*
3. *Agent preferences restricted to $\Re_F$ are single-peaked on each side of the endowment.*
4. *There exists an agent i whose preferences are single-peaked on $\Re_F$ such that,*

$$F(R_i, R_j) = \begin{cases} median\{Top(R_i, \Re_F), Top(R_j, SEQ_i(\omega) \cap \Re_F), \omega)\}, \\ \quad \text{if } i\text{'s peak is in } SEQ_i(\omega) \cap \Re_F; \\ median\{Top(R_i, \Re_F), Top(R_j, FOQ_i(\omega) \cap \Re_F), \omega\}, \\ \quad \text{if } i\text{'s peak is in } FOQ_i(\omega) \cap \Re_F. \end{cases}$$

From Definition 15, it follows that F is a GT rule then $\omega \in \Re_F$. The range of a GT rule is diagonal and agent preferences are single-peaked on both sides of the endowment. However, the range of a GT rule need not be piece-wise linear.

Saporiti (2009) shows that if an SCF is strategy-proof, anonymous and unanimous, then the SCF must be a *peak* rule. For two agent economies, a peak rule ensures that $F(R_i, R_j) \in$ median$\{Top(R_i, \Re_F), Top(R_j, \Re_F), \tau)\}$, where $\tau = Top(R_*, \Re_F)$ for some $R_* \in \mathbb{D}^s$. GT rules

[12] If d is not an interior allocations in Δ, then this may not be true. We will discuss about this later.

are a little different from the peak rules but both are similar in spirit. These two classes of rules are similar because both of them choose median of *some* tops (or peaks). In a GT rule the median need not be the median of tops of both the agents in the range. In a GT rule only one agent needs to have single-peaked preference in the entire range. The alternative that cannot be manipulated i.e. τ in Saporiti (2009) is ω in a GT rule. If the range is a straight line then both the agents have single-peaked preference in the entire range and in this case a GT rule is also a peak rule. However, it does not mean that a GT rule is a generalization of a peak rule or a peak rule is a generalization of a GT rule. A peak rule is not a generalization of a GT rule because a GT rule needs only one agent to have a single-peaked preference on the entire range. A GT rule is not a generalization of a peak rule because GT rules have been defined only for a particular kind of non-manipulable alternative which is ω.

Example 6 is an example of a GT rule. In this example the range of the SCF is not piece-wise linear but the SCF is continuous.

Example **6** Let agent i's preferences be given by utility functions of the form $u_i(x_i, y_i; \theta_i) = \theta_i\sqrt{x_i} + y_i$ with $\theta_i > 0$ and $i = 1, 2$. In Example 1, we have shown that this is a rich domain of classical single-crossing preferences.

Let $\Omega_x = 8$ and $\Omega_y = 4$. Let $\omega^1 = (4, 4 - 4^{\frac{2}{3}})$ and $\omega^2 = (4, 4^{\frac{2}{3}})$.

Let $B = \{((x_1, y_1)(x_2, y_2))|y_1 + x_1^{\frac{2}{3}} = 4\}$.

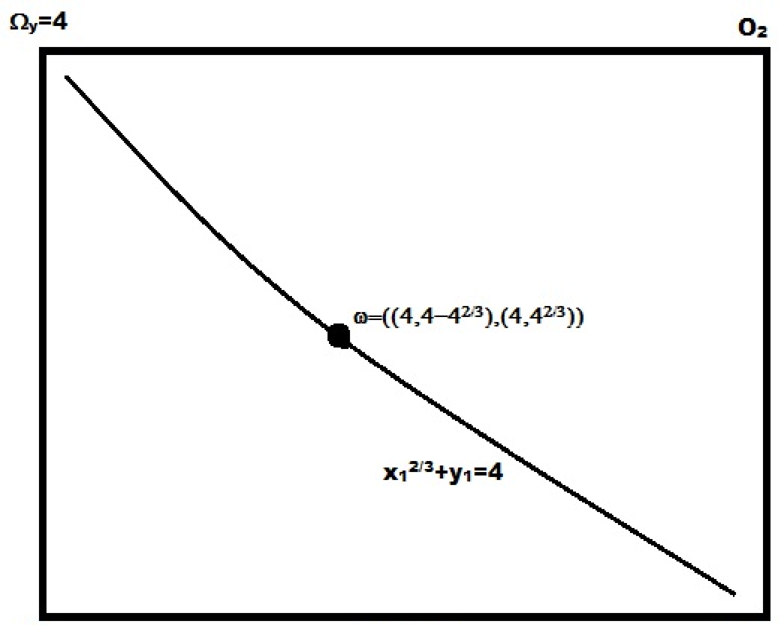

Figure 6

Let $B^{'} = B \cap \{((x_1, y_1), (8 - x_1, 4 - y_1))|0.1 \leq x_1 \leq 7.9 \text{ and } 0.1 \leq y_1 \leq 3.9\}$. The set $B^{'}$ is depicted in Figure 6. By our choice of the domain and $B^{'}$, we can find θ_1 such that

$\left(\left(\frac{3}{4}\theta_1\right)^6, 4-\left(\frac{3}{4}\theta_1\right)^4\right)$ solves the problem $\text{Max}_{(0.1\leq x_1\leq 7.9, 0.1\leq y_1\leq 3.9)}\theta_1\sqrt{x_1}+y_1$ s.t. $x_1^{\frac{2}{3}}+y_1=4.$[13] Otherwise the solution is one of the end points where $x_1 = 0.1$ or $x_1 = 7.9$. Note that $Top(\theta_1, B^{'})$ is unique for every θ_1, also $Top(\theta_1, B^{'})$ is a continuous function of θ_1. Since $0 < \theta_1 < \infty$ we can find the relevant θ_1 satisfying $\left(\left(\frac{3}{4}\theta_1\right)^6, 4-\left(\frac{3}{4}\theta_1\right)^4\right)$ for all the consumption bundles along $B^{'}$. Hence, given that the domain satisfies the single-crossing property the preference must exhibit single-peakedness on $B^{'}$. To see this note that by the single-crossing property, for any θ_1 we cannot have the situation such that $a \neq b$, $\widetilde{ab} \subseteq B^{'}$ and $\widetilde{ab} \subseteq LC(\theta_1, a) = LC(\theta_1, b)$.

For agent 2 the optimization problem is,

$$\text{Max}_{(0.1\leq x_2\leq 7.9, 0.1\leq y_2\leq 3.9)}\theta_2\sqrt{x_2}+y_2 \text{ s.t. } (8-x_2)^{\frac{2}{3}}+4-y_2=4.$$

The first order condition is given by the equation, $\theta_2 = \frac{4}{3}\frac{\sqrt{x_2}}{(8-x_2)^{\frac{1}{3}}}$. Note that the derivative in the right hand side of this equation is $\frac{2}{3(8-x_2)^{\frac{1}{3}}\sqrt{x_2}} + \frac{4\sqrt{x_2}}{9(8-x_2)^{\frac{4}{3}}}$. It is continuous and strictly positive for all $x_2 \in (0.1, 7.9)$ and hence by the Inverse Function Theorem, the solution of x_2 is a continuous function of θ_2. The first order condition of optimization problem indeed results in the maximum because from agent 2's origin the constraint $B^{'}$ is strictly concave. Otherwise the solution is one of the end points where $x_2 = 0.1$ or $x_2 = 7.9$.

Hence, $Top(\theta_2, B^{'})$ is unique for every θ_2 and a continuous function of θ_2.

Define an SCF F as follows,

$$F(\theta_1, \theta_2) = \text{median}\{Top(\theta_1, B^{'}), Top(\theta_2, B^{'}), \omega\}.$$

By construction, $\Re_F = B^{'}$. Also the SCF F is strategy-proof and individually rational. Since $Top(\theta_i, B^{'})$ is continuous for $i = 1, 2$, F is also continuous.

Remark **6** Example 6 implies that the continuity of the SCF need not result in the piecewise linearity of an SCF.

Now we state our main characterization result.

Theorem **2** *Let $\omega \in int\ \Delta$, the classical single-crossing domain $\mathbb{D}^s$ be rich and $F : [\mathbb{D}^s]^2 \rightarrow \Delta$ be a continuous SCF such that $\Re_F \subseteq int\ \Delta$ and $\Re_F$ is a closed set. Then F is strategy-proof and individually rational if and only if it is a GT rule.*

[13] For any relevant θ_1 the solution given by $(\frac{3\theta_1}{4})^6$ indeed corresponds to the maximum. To see this note that the maximization problem can be equivalently written as Max $\theta_1\sqrt{x_1} - x_1^{\frac{2}{3}}$. The first order condition is $\frac{\theta_1}{2\sqrt{x_1^*}} = \frac{2}{3}\frac{1}{(x_1^*)^{\frac{1}{3}}}$, which can be equivalently written as $\frac{3\theta_1}{4} = (x_1^*)^{\frac{1}{6}}$. Since $\frac{\sqrt{x_1}}{x_1^{\frac{1}{3}}} = x_1^{\frac{1}{6}}$ is an increasing function of x_1, $\frac{\theta_1}{2\sqrt{x_1}} > \frac{2}{3}\frac{1}{x_1^{\frac{1}{3}}}$ for $x_1 < x_1^*$ and $\frac{\theta_1}{2\sqrt{x_1}} < \frac{2}{3}\frac{1}{x_1^{\frac{1}{3}}}$ for $x_1 > x_1^*$. Hence, for the relevant θ_1 the family of functions $\theta_1\sqrt{x_1}+y_1$ concavify $x_1^{\frac{2}{3}}+y$ at the consumption bundles along $B^{'}$. Therefore, for relevant $\theta_1 s$ the solution given by $(\frac{3}{4}\theta_1)^6$ indeed correspond to the maximum.

REMARK 7 Our result implies that the Barberà and Jackson (1995) result is robust to domain contractions. On the contrary, Sprumont (1995) shows that the Barberà and Jackson (1995) result is not robust to domain expansions. Sprumont (1995) considers the problem of allocating two private goods between two agents whose preferences are continuous, convex (i.e. not strictly convex hence a domain expansion) and strictly increasing. Sprumont (1995) shows that every strategy-proof SCF that is continuous in the preferences must 'let one agent choose his best bundle from some exogenous set'. In other words Sprumont (1995) obtains a dictatorship result by expanding the domain of the SCFs.

Note that we provide a characterization of GT rules under the assumptions that $\Re_F \subseteq int\ \Delta$ and that $\Re_F$ is closed. In Barberà and Jackson (1995) the range has been proved to be closed. Closedness of the range is a result of strategy-proofness in their model. They show that limit points must be in the range. Otherwise strategy-proofness will be violated. In order to show this, Barberà and Jackson (1995) first show that there exists a preference ordering in the domain such that a limit point is uniquely preferred to the all other allocations on the closure of the range. The largeness of the domain of classical preferences is an important reason that given a limit point such a preference ordering exists. A classical, rich, single-crossing domain is not so large. Therefore, it may be difficult to use such techniques to show closedness of the range of an SCF. We explain this further in Example 7.

EXAMPLE 7 Consider the domain in Example 6. The slope of an indifference curve of an utility function from this domain is $-\frac{\theta_i}{2\sqrt{x_i}}$. Note that $\lim_{x_i \to 0} -\frac{\theta_i}{2\sqrt{x_i}} = -\infty$. Consider any straight line $y_i = -mx_i + c$, where $m > 0, c > 0$. The consumption bundle $(0, c)$ cannot be sustained as a top on this straight line for any $\theta_i > 0$. This is because for any $\theta_i > 0$, there exists $\epsilon > 0$ such that $-\frac{\theta_i}{2\sqrt{x_i}} < -m$ for all $x_i \in (0, \epsilon)$. The question we ask is whether it is possible to support the set $\mathbb{B} = \{(x, y) = ((x_i, y_i), (x_j, y_j)) | (x, y) \in \Delta$ and $y_i = -mx_i + c$ and $((o, c), (x_j, y_j)) \in \Delta\}$ as the range of a GT rule. Note that according to definition of a GT rule there must exist a θ_i such that $(0, c) = Top(\theta_i, \mathbb{B})$. But as we have seen, this cannot happen.

REMARK 8 Example 7 implies that if a limit point of $\Re_F$ lies on the boundary of Δ then it need not be attained as a maximum under any preference from $\mathbb{D}^s$. Hence, we cannot prove the range of an SCF defined on a single-crossing domain to be closed. Hence, we assume the 'closed' feature of SCFs that are characterized in Barberà and Jackson (1995). Also we want $\omega \in \Re_F$. Due to the same reason we also assume $\omega \in int\ \Delta$. Using these two assumption of closedness and interiority, we show that if an SCF defined on a classical, rich single-crossing domain is strategy-proof, individually rational and continuous then it must be a GT rule.

A GT rule by definition is strategy-proof and individually rational. The other direction of the proof of this result is contained in Appendix 7. In the next paragraph, we provide an overview of the steps involved in the proof.

Since the domain that we consider is connected, continuity of the SCF implies that the range is also connected. We show that the range contains the endowment (Lemma 3) and that it is diagonal (Proposition 5). Then we argue that the agents' preferences are single-peaked on the both sides of the endowment (Lemma 7 and Remark 10). Next we show that at least one of the agents' preferences are single-peaked on the entire range (Lemma 10). Then in the next step we prove the claim in Theorem 2. With the help of continuity of the SCF, richness and the single-crossing property we can only show that the range is diagonal, but we cannot prove it to be piecewise linear. Therefore single-peakedness on the range does not follow immediately. We use continuity and richness of the domain to conclude that both the agents' preferences are single-peaked on both sides the endowment and for at least one agent preferences are single-peaked on the entire range.

We make the following observation about a characterization result concerning strategy-proof, Pareto-efficient and individually rational SCF on a domain which is not classical.

Observation **4** Nicolò (2004) examines a two-good, two-agent exchange economy with the domain of Leontief preferences.[14] He shows that the range of a strategy-proof and individually rational SCF must consist of a set of diagonal allocations containing the endowment. Our result is intermediate between the Nicolò (2004) and Barberà and Jackson (1995) results in the sense that the range of an SCF on a rich single-crossing domain (satisfying continuity) must satisfy some restrictions, but can be more general than being piece-wise linear. It must however contain the endowment, as in the other cases. Furthermore, agent preferences restricted to the range must be single-peaked on each side of the endowment.

6 Conclusion

In this paper, we have formulated the concept of rich single-crossing domains for two-good exchange economies. Using some regularity conditions we have characterized the class of strategy-proof, individually rational and continuous SCFs and identified them to be the class of Generalized Trading rules. This class is wider than the class of Fixed-Price Trading rules identified in Barberà and Jackson (1995). However, the class is not big enough relative to the class of FPT rules. This is because, other than piecewise linearity all other properties of the FPT rules hold in the restricted domain of single-crossing preferences also. An important question for further research is whether the GT rule result can be extended to the case of an arbitrary number of agents with additional assumptions such as anonymity and non-bossiness as in Barberà and Jackson (1995).

[14] Observe that these preferences are not classical.

7 Appendix A

We prove the sufficient part of Theorem 2. Good x is measured along the horizontal axis and Good y is measured along the vertical axis.

Proof: We first show that the endowment is in the range. This result does not require either strategy-proofness or continuity.

Lemma **3** *Let the classical single-crossing domain* $\mathbb{D}^s$ *be rich. Let* $\omega_i \gg (0,0)$ *for all* i. *Let the SCF* $F : [\mathbb{D}^s]^2 \to \Delta$ *be individually rational. Then* $\omega \in \Re_F$.

Proof: Since $\mathbb{D}^s$ is rich, we know from Proposition 4 that there exists a profile (R_i, R_j) such that $\omega \in PE(R_i, R_j)$. Note that indifference curves are strictly convex. Therefore, for any allocation other than ω at the profile (R_i, R_j), F will violate individual-rationality for at least one agent. Hence, $F(R_i, R_j) = \omega$ that is $\omega \in \Re_F$. ■

We now establish a monotonicity result with reference to the order on $\mathbb{D}^s$ defined earlier.

Lemma **4** *Let the classical single-crossing domain* $\mathbb{D}^s$ *be rich. Let* $F : [\mathbb{D}^s]^2 \to \Delta$ *be a strategy-proof SCF. If* $R_i^{'} \succ R_i^{''}$, *then* $x_i(R_i^{'}, R_j) \geq x_i(R_i^{''}, R_j)$.

Proof: By strategy-proofness, $F_i(R_i^{'}, R_j) \in LC(R_i^{''}, F_i(R_i^{''}, R_j)) \cap UC(R_i^{'}, F_i(R_i^{''}, R_j))$. Hence, $x_i(R_i^{'}, R_j) \geq x_i(R_i^{''}, R_j)$. ■

Next we show that if an SCF is continuous, strategy-proof and individually rational, then its range is a diagonal subset of Δ.

Proposition **5** *Let the classical single-crossing domain* $\mathbb{D}^s$ *be rich. If the SCF* $F : [\mathbb{D}^s]^2 \to \Delta$ *is strategy-proof, individually rational and continuous, then* $\Re_F$ *is diagonal.*

Proof:

For the sake of contradiction assume that $F_i(\tilde{R}_i, \tilde{R}_j) > F_i(\bar{R}_i, \bar{R}_j)$ as shown in Figure $A.1$ [15]. Note that by strategy-proofness $\bar{R}_k \neq \tilde{R}_k$ for all $k \in I$. We prove that if $\tilde{R}_i \succ \bar{R}_i$ then $\tilde{R}_j \succ \bar{R}_j$. We use Lemma 4 twice. Let $\tilde{R}_i \succ \bar{R}_i$. Then by Lemma 4 it follows that $x_i(\tilde{R}_i, \bar{R}_j) \geq x_i(\bar{R}_i, \bar{R}_j)$. If $F(\tilde{R}_i, \bar{R}_j) \in THQ_i(F(\tilde{R}_i, \tilde{R}_j))$ then agent j will manipulate at the profile $(\tilde{R}_i, \tilde{R}_j)$ via $\bar{R}_j$. Therefore, $F(\tilde{R}_i, \bar{R}_j) \in int\ FOQ_i(F(\tilde{R}_i, \tilde{R}_j))$. This means that $x_j(\tilde{R}_i, \tilde{R}_j) > x_j(\tilde{R}_i, \bar{R}_j)$. Hence by Lemma 4 $\tilde{R}_j \succ \bar{R}_j$.

Similarly, if $\bar{R}_i \succ \tilde{R}_i$ then $\bar{R}_j \succ \tilde{R}_j$. Therefore, without loss of generality we assume that $\tilde{R}_i \succ \bar{R}_i$. Also, note by individual-rationality, either $\omega \in int\ FOQ_i(F(\tilde{R}_i, \tilde{R}_j))$ or $\omega \in int\ SEQ_i(F(\bar{R}_i, \bar{R}_j))$.

Lemma 5 and 6 are important intermediate steps to prove this proposition.

[15] For $a, b \in \Re^2$, $a > b$ means $a_k \geq b_k$ for all $k \in M$ and $a_k > b_k$ for at least one k.

Lemma 5 *Let $F_i(\tilde{R}_i, \tilde{R}_j) > F_i(\bar{R}_i, \bar{R}_j)$. There exists a sequence of profiles such that, $(\tilde{R}_i, \tilde{R}_j) \succ\succ (R_i^{k+1}, R_j^{k+1}) \succ\succ (R_i^k, R_j^k) \succ\succ (\bar{R}_i, \bar{R}_j)$, $(R_i^k, R_j^k) \to (\tilde{R}_i, \tilde{R}_j)$ (resp. $(\tilde{R}_i, \tilde{R}_j) \succ\succ (R_i^k, R_j^k) \succ\succ (R_i^{k+1}, R_j^{k+1}) \succ\succ (\bar{R}_i, \bar{R}_j)$, $(R_i^k, R_j^k) \to (\bar{R}_i, \bar{R}_j)$) for any k (Figure A.1).*

Proof: The construction of the sequence in the Lemma follows from Steps 1, 2 and 3.

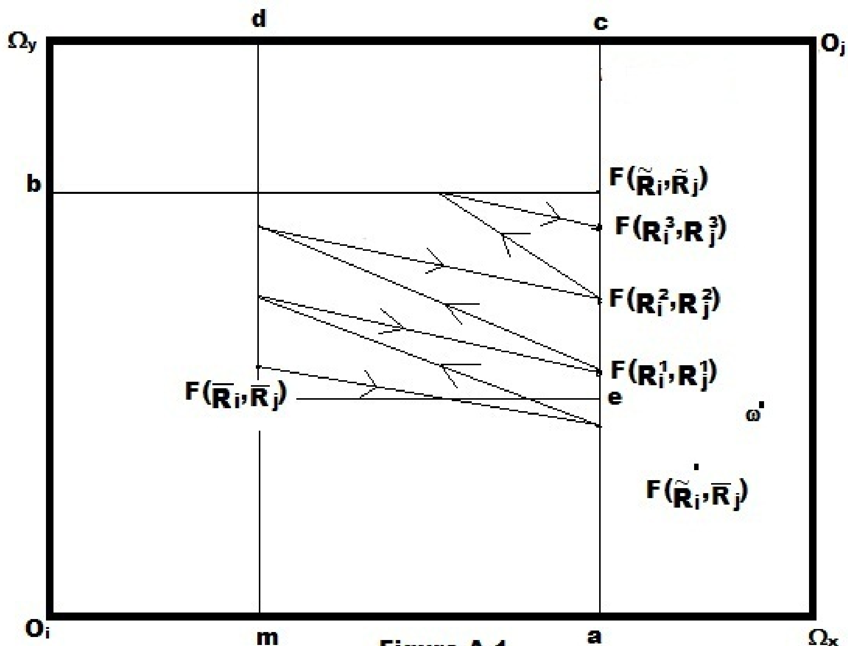

Figure A.1

We show the construction of the increasing sequence. The construction of the decreasing sequence is similar to the construction of the increasing sequence. Consider Figure $A.1$ and the set $\mathbb{A}$.

$\mathbb{A} = \{(R_i, R_j) | x_i(R_i, R_j) = a_i^x, a_i^y \leq y_i(R_i, R_j) \leq y_i(\tilde{R}_i, \tilde{R}_j)$, and $(\tilde{R}_i, \tilde{R}_j) \succeq (R_i, R_j) \succeq (\bar{R}_i, \bar{R}_j)\}$.

Step 1 shows that there exists profiles in $\mathbb{A}$ other than $(\tilde{R}_i, \tilde{R}_j)$.

Step 1: There exists $(R_i, R_j) \in \mathbb{A}$ such that $(\tilde{R}_i, \tilde{R}_j) \succ\succ (R_i, R_j) \succ\succ (\bar{R}_i, \bar{R}_j)$ with $F(R_i, R_j) \neq F(\tilde{R}_i, \tilde{R}_j)$ and $F(R_i, R_j) \neq F(\bar{R}_i, \bar{R}_j)$.

Proof of Step 1: Since F is strategy-proof $F(\tilde{R}_i, \bar{R}_j) \in int\ FOQ_i(F(\tilde{R}_i, \tilde{R}_j))$. By continuity of F there exists R_i^* such that $\tilde{R}_i \succ R_i^* \succ \bar{R}_i$ and $F(R_i^*, \bar{R}_j)$ lies on the line segment joining a and $F(\tilde{R}_i, \tilde{R}_j)$. Also note that by strategy-proofness of i, $F(R_i^*, \bar{R}_j) \neq F(\tilde{R}_i, \tilde{R}_j)$[16]. [**]

[16] Let $F_i^x(R_i, R_j)$ and $F_i^y(R_i, R_j)$ denote the allocation of good x and y to agent i according to the SCF F at the profile (R_i, R_j). Consider the situation $F_i^x(\bar{R}_i, \bar{R}_j) = F_i^x(\tilde{R}_i, \tilde{R}_j)$ and $F_i^y(\bar{R}_i, \bar{R}_j) < F_i^y(\tilde{R}_i, \tilde{R}_j)$. By Lemma 4 $F(\bar{R}_i, R_j) \in SEQ_i(F(\bar{R}_i, \bar{R}_j))$ for all $R_j \succ \bar{R}_j$. By continuity and strategy-proofness there exists R_j such that $\tilde{R}_j \succ R_j \succ \bar{R}_j$ and $F(\bar{R}_i, R_j) \in int\ THQ_i(F(\tilde{R}_i, \tilde{R}_j))$. Hence, the positions of $F(\bar{R}_i, \bar{R}_j)$ and $F(\tilde{R}_i, \tilde{R}_j)$ are without the loss of generality.

Since F is continuous, $F(R_i^*, R_j) \neq F(R_i^*, \bar{R}_j)$ for some $R_j \in (\bar{R}_j, \tilde{R}_j)$. If $F(R_i^*, R_j) = F(R_i^*, \bar{R}_j)$ for all $R_j \in (\bar{R}_j, \tilde{R}_j)$ then by continuity of F, $F(R_i^*, \tilde{R}_j) = F(R_i^*, \bar{R}_j)$. Then agent i will manipulate F at $(R_i^*, \tilde{R}_j)$ via $\tilde{R}_i$ because $F(\tilde{R}_i, \tilde{R}_j) P_i^* F(R_i^*, \tilde{R}_j) = F(R_i^*, \bar{R}_j)$. By Lemma 4 choose $R_j^{**} \in (\bar{R}_j, \tilde{R}_j)$ such that $F(R_i^*, R_j^{**}) \in int\ THQ_i(F(\tilde{R}_i, \tilde{R}_j))$. Note that $F_i(\tilde{R}_i, \tilde{R}_j) > F_i(R_i^*, R_j^{**})$. Therefore, by [**] there exists R_i^{**} such that $\tilde{R}_i \succ R_i^{**} \succ R_i^*$ and $F(R_i^{**}, R_j^{**})$ lies on the line segment joining a and $F(\tilde{R}_i, \tilde{R}_j)$. Hence, again by strategy-proofness of i, $F(R_i^{**}, R_j^{**}) \neq F(\tilde{R}_i, \tilde{R}_j)$. Hence Step 1 is established.

Step 2: Let (R_i', R_j') be such that $(\tilde{R}_i, \tilde{R}_j) \succ\succ (R_i', R_j')$ and $(R_i', R_j') \in \mathbb{A}$ with $F(R_i', R_j') \neq F(\tilde{R}_i, \tilde{R}_j)$. Then there exists (R_i'', R_j'') such that $(\tilde{R}_i, \tilde{R}_j) \succ\succ (R_i'', R_j'') \succ\succ (R_i', R_j')$ and $(R_i'', R_j'') \in \mathbb{A}$ with $F(R_i'', R_j'') \neq F(\tilde{R}_i, \tilde{R}_j)$.

Proof of Step 2: The proof follows immediately by repeating the a rguments that has been used to prove Step 1. First increase agent j's preference and then increase agent i's.

Step 3: Consider a profile (R_i^*, R_j^*) such that $(\tilde{R}_i, \tilde{R}_j) \succ\succ (R_i^*, R_j^*) \succ\succ (\bar{R}_i, \bar{R}_j)$. There exists (R_i, R_j) such that $(\tilde{R}_i, \tilde{R}_j) \succ\succ (R_i, R_j) \succ\succ (R_i^*, R_j^*)$ and $(R_i, R_j) \in \mathbb{A}$.

Proof of Step 3: Since F is continuous $\mathbb{A}$ is a closed set in the order topology. Therefore set $\mathbb{A}$ contains all its limit points. Therefore by Step 2 the proof follows.

Now consider any neighborhood around $(\tilde{R}_i, \tilde{R}_j)$. By Step 3 there exists (R_i, R_j) in this neighborhood such that $(\tilde{R}_i, \tilde{R}_j) \succ\succ (R_i, R_j)$ and $(R_i, R_j) \in \mathbb{A}$. This proves the existence of the desired increasing sequence. By decreasing R_is and R_js we obtain the desired decreasing sequence.

■

Observation **5** Note that by the construction of the sequence in Lemma 5 for all k $F(R_i^k, R_j^k) \neq F(\tilde{R}_i, \tilde{R}_j)$. To show this, let $F(R_i^{k-1}, R_j^{k-1}) \in \mathbb{A}$ and $F(R_i^{k-1}, R_j^{k-1}) \neq F(\tilde{R}_i, \tilde{R}_j)$. Note that $F(R_i^{k-1}, R_j^{k-1})$ and $F(\tilde{R}_i, \tilde{R}_j)$ are non diagonal. Therefore, by Step 1 of Lemma 5 we obtain (R_i^k, R_j^k) such that $F(R_i^k, R_j^k) \in \mathbb{A}$ and $F(R_i^k, R_j^k) \neq F(\tilde{R}_i, \tilde{R}_j)$.

Lemma **6** *Let $F_i(\tilde{R}_i, \tilde{R}_j) > F_i(\bar{R}_i, \bar{R}_j)$. For some $R_j \succ \tilde{R}_j$ (resp. $\bar{R}_j \succ R_j$), $F(\tilde{R}_i, R_j) \in int\ SEQ_i(F(\tilde{R}_i, \tilde{R}_j))$ (resp. $F(\bar{R}_i, R_j) \in int\ FOQ_i(F(\bar{R}_i, \bar{R}_j))$).*

Proof: Assume for the sake of contradiction that this does not happen. Then for all $R_j \succ \tilde{R}_j$, $F(\tilde{R}_i, R_j) = F(\tilde{R}_i, \tilde{R}_j)$. Consider $\tilde{\tilde{R}}_j \succ \tilde{R}_j$. Note that $(\tilde{R}_i, \tilde{\tilde{R}}_j) \succ\succ (\bar{R}_i, \bar{R}_j)$. By Lemma 5, we can construct an increasing sequence of profiles (R_i^k, R_j^k) with $F(R_i^k, R_j^k) \in THQ_i(F(\tilde{R}_i, \tilde{R}_j)) = THQ_i(F(\tilde{R}_i, \tilde{\tilde{R}}_j))$, $F(\tilde{R}_i, \tilde{\tilde{R}}_j) > F(R_i^k, R_j^k)$ and the sequence converges to $(\tilde{R}_i, \tilde{\tilde{R}}_j)$. Observe that for k large enough we can choose R_j^k such that $\tilde{\tilde{R}}_j \succ R_j^k \succ \tilde{R}_j$ and $F(\tilde{R}_i, \tilde{\tilde{R}}_j) > F(R_i^k, R_j^k)$. But $F(\tilde{R}_i, R_j^k) = F(\tilde{R}_i, \tilde{R}_j) P_i^k F(R_i^k, R_j^k)$. Hence agent i will manipulate F at (R_i^k, R_j^k) via $\tilde{R}_i$.

■

Back to the proof of the proposition. By $\overline{F(\bar{R}_i, \bar{R}_j)e}$, $\overline{ec}$ and $\overline{F(\bar{R}_i, \bar{R}_j)m}$ we denote the straight lines that joins $F(\bar{R}_i, \bar{R}_j)$ and e, e and c and $F(\bar{R}_i, \bar{R}_j)$ and m respectively. This is an abuse of notation because none of these line segments are diagonal (Figure A.1). We consider two cases.

Case 1: $\omega \in int\ FOQ_i(F(\tilde{R}_i, \tilde{R}_j))$.

By Lemma 6 choose $R_j^1 \succ \tilde{R}_j$ such that $F(\tilde{R}_i, R_j^1) \in int\ SEQ_i(F(\tilde{R}_i, \tilde{R}_j)) \cap FIQ_i(F(\bar{R}_i, \bar{R}_j))$. By individual-rationality of agent i, $F(R_i, R_j^1) \in FOQ_i(c)$ for some R_i where $R_i \succ \tilde{R}_i$. By continuity there exists $R_i^1 \succ \tilde{R}_i$ such that $F(R_i^1, R_j^1) \in \overline{F(\bar{R}_i, \bar{R}_j))e}$ or $F(R_i^1, R_j^1) \in \overline{ec}$. Note that $(R_i^1, R_j^1) \succ\succ (\bar{R}_i, \bar{R}_j)$ and $F_i(R_i^1, R_j^1) > F_i(\bar{R}_i, \bar{R}_j)$. By using Lemma 5 again we find a sequence of increasing profiles that converges to (R_i^1, R_j^1). Then use Lemma 6, individual-rationality and continuity to find $(R_i^2, R_j^2) \succ\succ (R_i^1, R_j^1)$ such that $F(R_i^2, R_j^2) \in \overline{F(\bar{R}_i, \bar{R}_j))e}$ or $F(R_i^2, R_j^2) \in \overline{ec}$. Continuation of this process results in (R_i^*, R_j^*) high enough such that $F(R_i^*, R_j^*) \in \overline{F(\bar{R}_i, \bar{R}_j))e}$ or $F(R_i^*, R_j^*) \in \overline{ec}$ and $\omega P_i^* F(R_i^*, R_j^*)$. Hence individual-rationality of agent i is violated at R_i^*.

Case 2: $\omega \in int\ SEQ_i(F(\bar{R}_i, \bar{R}_j))$.

By Lemma 6 choose R_j^1 such that $\bar{R}_j \succ R_j^1$ and $F(\bar{R}_i, R_j^1) \in int\ FOQ_i(F(\bar{R}_1, \bar{R}_2)) \cap THQ_i(F(\tilde{R}_i, \tilde{R}_j))$.

As in Case 1, by continuity and individual-rationality of agent i there exists R_i^1 such that $\bar{R}_i \succ R_i^1$ and $F(R_i^1, R_j^1) \in \overline{F(\bar{R}_i, \bar{R}_j))m}$ or $F(R_i^1, R_j^1) \in \overline{F(\bar{R}_i, \bar{R}_j)e}$. Note that $(\tilde{R}_i, \tilde{R}_j) \succ\succ (R_i^1, R_j^1)$ and $F(\tilde{R}_i, \tilde{R}_j) > F(R_i^1, R_j^1)$. Using Lemma 5, Lemma 6, individual-rationality and continuity repeatedly we find R_i^* and R_j^* low enough such that $F(R_i^*, R_j^*) \in \overline{F(\bar{R}_i, \bar{R}_j)m}$ or $F(R_i^*, R_j^*) \in \overline{F(\bar{R}_i, \bar{R}_j)e}$ and individual-rationality of agent i is violated since $\omega \in int\ SEQ_i(F(\bar{R}_i, \bar{R}_j))$.

Hence, by contradiction the proof of the Proposition follows.

■

REMARK **9** Since the rich classical single-crossing domain $\mathbb{D}^s$ is connected in the order topology and F is continuous, $\Re_F$ is connected. Hence, the diagonal property of $\Re_F$ implies that $\Re_F$ can be written as $\widetilde{d'\omega} \cup \widetilde{\omega d}$ where $d' \in SEQ_i(\omega)$ and $d \in FOQ_i(\omega)$, for some i.

The following Lemma demonstrates that agent preferences are single-peaked on each side of the endowment.

LEMMA **7** *Let the classical single-crossing domain $\mathbb{D}^s$ be rich. Let the SCF $F : [\mathbb{D}^s]^2 \to \Delta$ be strategy-proof, individually rational and continuous. Let $\Re_F \subseteq int\ \Delta$ and let $\Re_F$ be closed. Let $\Re_F = \widetilde{d'\omega} \cup \widetilde{\omega d}$. Then agent preferences are single-peaked on $\Re_F \cap int\ SEQ_i(\omega)$ and $\Re_F \cap int\ FOQ_i(\omega)$.*

Proof: First we show that d and $d^{'}$ are attained as tops in $\widetilde{\omega d}$ and $\widetilde{d^{'}\omega}$ respectively. Without the loss of generality consider d. Since $d \in \Re_F$ there exists $(R_i^{'}, R_j^{'})$ such that $F(R_i^{'}, R_j^{'}) = d$. Let there exists $y \in \widetilde{\omega d}$ such that $yP_i^{'}d$. By strategy-proofness of i, there does not exist R_i such that $F(R_i, R_j^{'}) = y$. Hence, by continuity for all R_i such that $R_i^{'} \succ R_i$, $F(R_i, R_j^{'}) \in \widetilde{yd} \setminus \{y\}$.

Since $\Re_F$ is diagonal by richness there exists $R_i^{''}$ (small according to the order $\succ$) such that $\omega P_i^{''} b$ for all $b \in \widetilde{yd}$. Since $F(R_i^{''}, R_j^{'}) \in \widetilde{yd} \setminus \{y\}$, individual-rationality of agent i will be violated at $R_i^{''}$.

Hence, we have shown $d \in Top(R_i^{'}, \widetilde{\omega d})$. By richness there exists $R_i \succ R_i^{'}$ such that $d = Top(R_i, \widetilde{\omega d})$.

Now we prove the Lemma in four steps. In the first step we show that any allocation such that $c \in \widetilde{d^{'}\omega} \setminus \{\omega\}$ or $c \in \widetilde{\omega d} \setminus \{\omega\}$ can be sustained as a top in $\widetilde{d^{'}\omega}$ or $\widetilde{\omega d}$ respectively. In the second step we show that no agent has isolated tops on any side of the endowment. In the third step we show that both agents' preferences have unique top on both sides of the endowment. In the fourth step we show that preferences are in fact, single-peaked on both sides of the endowment.

Note that both $\widetilde{d^{'}\omega}$ and $\widetilde{\omega d}$ are compact. Since, preferences are continuous, on each of these segments a maximum exists under all R_i. In Step 1 we show that all the allocations in each side of ω can be sustained as a top under some R_i.

Step 1: Let $c \in \widetilde{d^{'}\omega} \setminus \{\omega\}$ or $c \in \widetilde{\omega d} \setminus \{\omega\}$. Then there exists R_i such that $c \in Top(R_i, \widetilde{d^{'}\omega})$ or $c \in Top(R_i, \widetilde{\omega d})$.

Proof of Step 1: Consider $c \in \widetilde{\omega d} \setminus \{\omega\}$ and $c \neq d$. We know that $F(R_i^{'}, R_j^{'}) = d$. Since $\Re_F$ is diagonal, by richness there exists R_i^* such that $\omega P_i^* b$ for all $b \in \widetilde{cd}$. Therefore, by individual-rationality of i at R_i^*, $F(R_i^*, R_j^{'}) \in SEQ_i(c)$. Hence by continuity of F there exists $R_i^{''}$ such that $F(R_i^{''}, R_j^{'}) = c$. If there exists $y \in \widetilde{cd}$ such that $yP_i^{''}c$ then agent i will manipulate F at $(R_i^{''}, R_i^{'})$ via R_i^{**} where $F(R_i^{**}, R_i^{'}) = y$. Such R_i^{**} exists because $F(R_i^{'}, R_j^{'}) = d$, $F(R_i^{''}, R_j^{'}) = c$ and F is continuous.

Therefore, let $y \in \widetilde{\omega c}$ such that $yP_i^{''}c$. If $y = \omega$ then individual-rationality of i at $R_i^{''}$ is violated. Therefore, let $y \neq \omega$. By Lemma 4 for all R_i such that $R_i^{''} \succ R_i$, $F(R_i, R_j^{'}) \in SEQ_i(c)$. Since $\Re_F$ is diagonal by richness there exists R_i^* (small according to the order $\succ$) such that $\omega P_i^* b$ for all $b \in \widetilde{yd}$. But then by continuity, there exists R_i^{**} such that $F(R_i^{**}, R_j^{'}) = y$. Hence, i will manipulate F at $(R_i^{''}, R_i^{'})$ via R_i^{**}.

Step 2: Consider an agent i and $\widetilde{\omega d}$. Let $a, b \in Top(R_i, \widetilde{\omega d})$ such that $b_i^x > a_i^x$ and $b_i^y < a_i^y$. Consider $x_i \in [a_i^x, b_i^x]$ and $y_i \in [b_i^y, a_i^y]$ such that $((x_i, y_i), (\omega - x_i, \omega - y_i)) \in \widetilde{\omega d}$. Then, $(x_i, y_i) \in Top(R_i, \widetilde{\omega d})$.

Proof of Step 2: We will prove Step 2 by contradiction. Suppose F has isolated plateaus on $\widetilde{\omega d}$ at some R_i^* (Figure A.2).

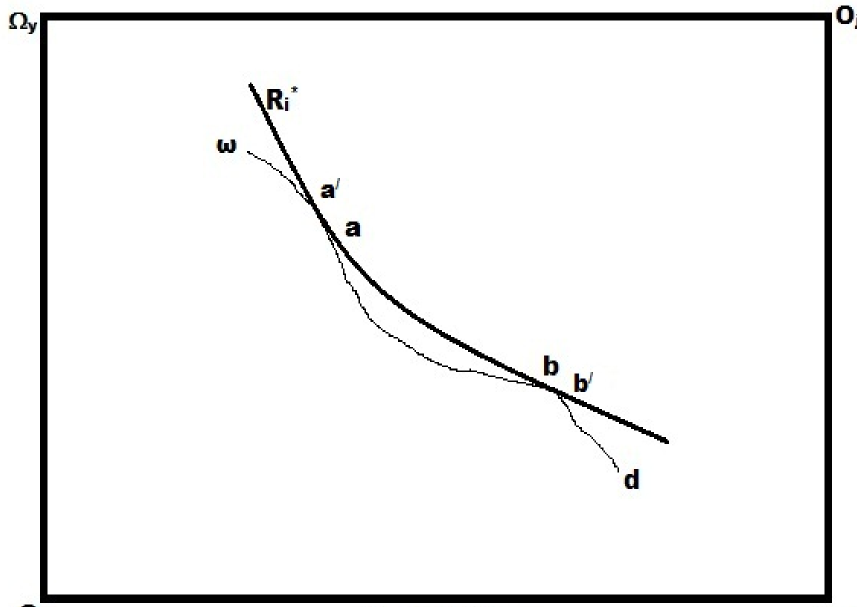

Figure A.2

Let $\widetilde{a'a}$ and $\widetilde{bb'}$ be two plateaus under R_i^*. Pick an allocation c between a and b such that $a \neq c \neq b$. By Step 1 there exists a preference R_i such that $c \in Top(R_i, \widetilde{\omega d} \setminus \{\omega\})$. Since $c \in int\ LC(R_i^*, a)$, R_i and R_i^* cuts twice. This is a contradiction to the single-crossing property.

Step 3. Consider agent i, R_i and $\widetilde{\omega d}$. Then $Top(R_i, \widetilde{\omega d})$ is unique.

Proof of Step 3: It follows from Step 2 that agent preferences have unique plateaus on both sides of ω. We show that preferences admit unique maximal element on both sides of ω. Let $\widetilde{a'a}$ be a plateau for some preference R_i^*.

Let $c \in \widetilde{a'a}$ and $a' \neq c \neq a$. We have $F(R_i', R_j') = d$. By richness we can find R_i^{**} such that $wP_i^{**}b$ for all $b \in \widetilde{cd}$. Hence, by individual-rationality of i at R_i^{**}, $F(R_i^{**}, R_j') \in int\ SEQ_i(c)$. Therefore, by continuity, there exists R_i'' such that $F(R_i'', R_j') = c$.

From the single-crossing property it follows that either $a'P_i''c$ or $aP_i''c$. Consider $a'P_i''c$. If $a' = \omega$ then individual-rationality of i is violated at R_i''. If $a' \neq \omega$ then we can choose R_i^{***} (small according to the order $\succ$) such that $wP_i^{***}b$ for all $b \in \widetilde{a'd}$. By individual-rationality of agent i at R_i^{***}, $F(R_i^{***}, R_j') \in int\ SEQ_i(a')$. By continuity there exists R_i^{iv} such that $F(R_i^{iv}, R_j') = a'$. Then agent i will manipulate F at (R_i'', R_j') via R_i^{iv}.

Now let $aP_i''c$. We have $F(R_i', R_j') = d$. Hence, by strategy-proofness of i $a \neq d$. Choose R_i^{***} such that $wP_i^{***}b$ for all $b \in \widetilde{ad}$. By individual-rationality of i at R_i^{***} $F(R_i^{***}, R_j') \in intSEQ_i(a)$. Therefore, by continuity there exists R_i^v such that $F(R_i^v, R_j') = a$. Hence agent i will manipulate F at (R_i'', R_j') via R_i^v.

We have proved that on both sides of the endowment each agent i has a unique top. However this is not enough to conclude that agent preferences are single-peaked. We now show this.

Step 4: The preferences of any agent i exhibit single-peakedness over $\widetilde{d'\omega} \setminus \{\omega\}$ and $\widetilde{\omega d} \setminus \{\omega\}$.

Suppose R_i^* is not single-peaked but has a unique top on $\widetilde{\omega d}$. Therefore, there exists an indifference curve of the preference R_i^* and allocations a and b, $a \neq b$ in $\widetilde{\omega d}$ such that $\widetilde{ab} \subseteq LC(R_i^*, a) = LC(R_i^*, b)$. But by Step 1 this will violate the single-crossing property.

■

The following Lemma demonstrates a relationship between the end elements of a diagonal set and single-crossing preferences.

Lemma **8** *Let the classical single-crossing domain $\mathbb{D}^s$ be rich. Let $\Re_F \subseteq int\ \Delta$ be closed with $\Re_F = \widetilde{d'\omega} \cup \widetilde{\omega d}$. Then there exist profiles (R_i^*, R_j^*) and (R_i^{**}, R_j^{**}) such that,*

1. *dP_i^*b and dP_j^*b for all $b \in \Re_F$, $d'P_i^{**}b$ and $d'P_j^{**}b$ for all $b \in \Re_F$.*
2. *We can choose (R_i^*, R_j^*) and (R_i^{**}, R_j^{**}) in such a way that for all i, $\omega P_i^* b$ for all $b \in \widetilde{d'\omega} \setminus \{\omega\}$ and $\omega P_i^{**} b$ for all $b \in \widetilde{\omega d} \setminus \{\omega\}$.*

Proof: Without the loss of generality consider d. We have shown in the proof of Lemma 7 that there exists R_i such that $d = Top(R_i, \widetilde{\omega d})$. Since $\Re_F$ is diagonal by richness 1 follows.

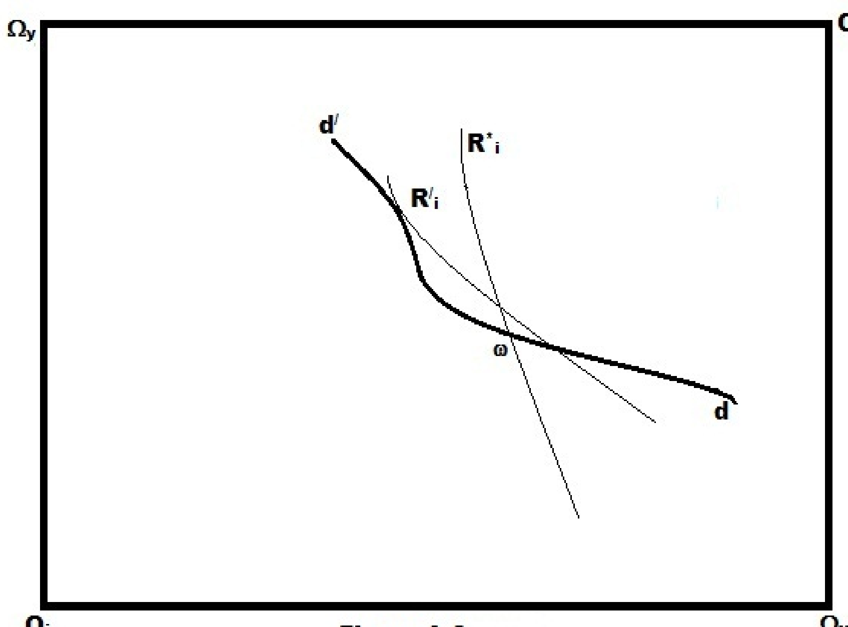

Figure A.3

However note that the existence of R_i' such that $d = Top(R_i', \Re_F)$ does not mean that $\omega P_i' b$ for all $b \in \widetilde{d'\omega} \setminus \{\omega\}$. It is possible that there exists $b \in \widetilde{d'\omega}$ such that $bR_i'\omega$ as shown in Figure A.3.

By richness there exists R_j^* such that $IC(R_j^*, \omega) = IC(R_j^*, d')$. Now recall from Lemma 7 that for all $b \in \widetilde{d'\omega} \setminus \{\omega\}$, $b = Top(R_j, \widetilde{d'\omega})$ for some R_j. Hence, by Lemma 7, $\widetilde{d'\omega} \subseteq UC(R_j^*, \omega)$, otherwise the single-crossing property will be violated. Hence, by applying richness the desired R_i^* is obtained.

■

Remark **10** By Lemma 7 all the allocations on $\widetilde{d'\omega} \setminus \{\omega\}$ are supported as singele-peaked tops. Hence R_i^* obtained in Lemma 8 exhibits single-peakedness on $\widetilde{d'\omega}$ with $\omega = Top(R_i^*, \widetilde{d'\omega})$. In other words, by Lemma 7 and Lemma 8 both the agents exhibit single-peakedness on $\widetilde{d'\omega}$ and $\widetilde{\omega d}$.

The following property of a closed range is useful.

Lemma **9** *Let the classical single-crossing domain $\mathbb{D}^s$ be rich. Let the SCF $F : [\mathbb{D}^s]^2 \to \Delta$ be strategy-proof, individually rational and continuous. Let $\Re_F \subseteq int\ \Delta$ be closed and $\Re_F = \widetilde{d'\omega} \cup \widetilde{\omega d}$. If the profiles (R_i^*, R_j^*) and (R_i^{**}, R_j^{**}) are such that they satisfy* (1) *and* (2) *in Lemma 8, then $F(R_i^*, R_j^*) = d$ and $F(R_i^{**}, R_j^{**}) = d'$.*

Proof: Let R_i^* and R_i^* satisfy (1) in Lemma 8. Since d is in the range of F, there exists (R_i, R_j) such that $F(R_i, R_j) = d$. Since d is the best element on the range of F at R_i^* , by strategy-proofness, $F(R_i^*, R_j) = d$. Similaraly, we have $F(R_i^*, R_j^*) = d$. ■

We now show that at least one agent's preferences exhibit single-peakednes on $\Re_F$.

Lemma **10** *Let the classical single-crossing domain $\mathbb{D}^s$ be rich. Let the SCF $F : [\mathbb{D}^s]^2 \to \Delta$ be strategy-proof, individually rational and continuous. Let $\Re_F \subseteq int\ \Delta$ be closed $\Re_F = \widetilde{d'\omega} \cup \widetilde{\omega d}$. Then there exists an agent i such that his preferences are single-peaked on $\Re_F$.*

Proof: By Lemma 7 agent preferences are single-peaked on both sides of the endowment. Suppose agent j has a preference R_j^* such that $a \in int\ \widetilde{\omega d}$ and $b \in int\ \widetilde{d'\omega}$ are tops on $\widetilde{\omega d}$ and $\widetilde{d'\omega}$ respectively under R_j^* (Figure A.4).

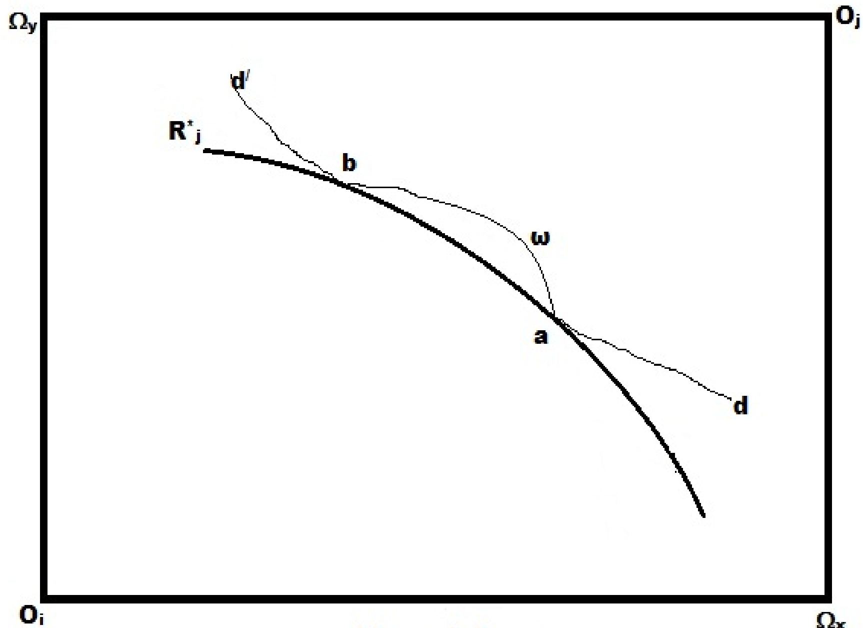

Figure A.4

Consider the profiles (R_i^1, R_j^1) and (R_i^2, R_j^2) such that they satisfy (1) and (2) in Lemma 8. By Lemma 9 $F(R_i^1, R_j^1) = d$ and $F(R_i^2, R_j^2) = d'$.

By individual-rationality of agent i, $F(R_i^1, R_j^*) \in \widetilde{\omega d}$ and $F(R_i^2, R_j^*) \in \widetilde{d'\omega}$. By individual-rationality of j, $F(R_i^1, R_j^2) = \omega$ and $F(R_i^2, R_j^1) = \omega$. Hence now we have, $F(R_i^1, R_j^1) = d$, $F(R_i^1, R_j^2) = \omega$ and $F(R_i^1, R_j^*) \in \widetilde{\omega d}$. Since agent j's preferences are single-peaked on each side of the endowment, continuity and strategy-proofness imply $F(R_i^1, R_j^*) = a$. Similarly, $F(R_i^2, R_j^*) = b$. Note that $R_i^1 \succ R_i^2$. By continuity there exists R_i^* such that $F(R_i^*, R_j^*) = \omega$. By strategy-proofness of agent i at the profile (R_i^*, R_j^*), $\omega \in Top(R_i^*, \widetilde{\omega a})$ and $\omega \in Top(R_i^*, \widetilde{b\omega})$. But by Lemma 7 and the single-crossing property $\omega = Top(R_i^*, \widetilde{\omega a})$ and $\omega = Top(R_i^*, \widetilde{b\omega})$. Again by Lemma 7 and the single-crossing property $\omega = Top(R_i^*, \widetilde{\omega d})$ and $\omega = Top(R_i^*, \widetilde{d'\omega})$. Therefore, $\omega = Top(R_i^*, \Re_F)$. By Lemma 7 agent i's preferences are also single-peaked on both sides of the endowment. Since $\omega = Top(R_i^*, \Re_F)$ by the single-crossing property, agent i's preferences exhibit single-peakedness on $\Re_F$.

■

From the preceding Lemma it follows that there exists an agent i such that for some R_i^*, $Top(R_i^*, \Re_F) = \omega$. Hence, $Top(R_i, \Re_F) \in \widetilde{\omega d}$ if $R_i \succ R_i^*$ and $Top(R_i, \Re_F) \in \widetilde{d'\omega}$ if $R_i^* \succ R_i$. Therefore, by individual-rationality for all R_j, $F(R_i, R_j) \in \widetilde{\omega d}$ if $R_i \succ R_i^*$ and $F(R_i, R_j) \in \widetilde{d'\omega}$ if $R_i^* \succ R_i$. By individual-rationality, $F(R_i^*, R_j) = \omega$ for all R_j. Now consider the following partition of $[\mathbb{D}^s]^2$: $K(1) = \{(R_i, R_j) | R_i = R_i^*\}$, $K(2) = \{(R_i, R_j) | R_i \succ R_i^*\}$ and $K(3) = \{(R_i, R_j) | R_i^* \succ R_i\}$. Hence, it follows that if $(R_i, R_j) \in K(2)$ then $F(R_i, R_j) \in \widetilde{\omega d}$ and if $(R_i, R_j) \in K(3)$ then $F(R_i, R_j) \in \widetilde{d'\omega}$. Now note that in both $\widetilde{\omega d}$ and $\widetilde{d'\omega}$ preferences of agent j are also single-peaked.

Now consider $K(1) \cup K(2)$. We know by individual-rationality of F that if $(R_i, R_j) \in K(2)$ then $F(R_i, R_j) \in \widetilde{\omega d}$. Now we know that
$F(R_i, R_j) =$

$$\min\Big\{a_\emptyset, \max\{Top(R_i, \Re_F), a_{\{i\}}\}, \max\{Top(R_j, \widetilde{\omega d}), a_{\{j\}}\}, \max\{Top(R_i, \Re_F), Top(R_j, \widetilde{\omega d}), a_{\{i,j\}}\}\Big\}.$$

From Theorem 2 in Barberà and Jackson (1994) if $s \subseteq s' \subseteq \{i, j\}$ then $a_s \geq a_{s'}$, where a_s is an extended real number for all $s \subseteq \{i, j\}$. [***]

Claim 1: $a_\emptyset \geq d$ and $a_{\{i,j\}} \leq \omega$.

Proof of Claim 1: Let for the sake of contradiction $a_\emptyset < d$. From Lemma 8 and Lemma 9 we can choose a profile such that under F,

$$\min\Big\{a_\emptyset, \max\{d, a_{\{i\}}\}, \max\{d, a_{\{j\}}\}, \max\{d, d, a_{\{i,j\}}\}\Big\} = d.$$

By [***], $\min\Big\{a_\emptyset, d, d, d\Big\} < d$, which is a contradiction to the above equality.

Now, let for the sake of contradiction $a_{\{i,j\}} > \omega$. By individual-rationality of F from our earlier discussion we can choose a profile such that,

$$\min\Big\{a_\emptyset, \max\{\omega, a_i\}, \max\{\omega, a_j\}, \max\{\omega, \omega, a_{i,j}\}\Big\} = \omega.$$

By [***] $\min\Big\{a_\emptyset, a_{\{i\}}, a_{\{j\}}, a_{\{i,j\}}\Big\} > \omega$, which is a contradiction to the above equality. This establishes Claim 1.

Since $F(R_i, R_j) \in \widetilde{\omega d}$ for all $(R_i, R_j) \in K(1) \cup K(2)$, the definition of min-max function in conjunction with Claim 1 imply, $a_\emptyset = d$ and $a_{\{i,j\}} = \omega$.

Claim 2: $a_{\{i\}} = \omega$ and $a_{\{j\}} = \omega$.

Proof of Claim 2: Let for the sake of contradiction $a_{\{i\}} > \omega$. We know that by individual-rationality of F, $\min\Big\{d, \max\{d, a_{\{i\}}\}, \max\{\omega, a_{\{j\}}\}, \max\{d, \omega, \omega\}\Big\} = \omega$. Hence, $a_{\{j\}} = \omega$. Again by, individual-rationality of F, $\min\Big\{d, \max\{\omega, a_{\{i\}}\}, \max\{d, \omega\}, \max\{\omega, d, \omega\}\Big\} = \omega$, i.e. $\min\Big\{d, \max\{\omega, a_{\{i\}}\}, d, d\Big\} = \omega$, which is a contradiction.

Let for the sake of contradiction $a_{\{j\}} > \omega$. By individual-rationality of F,

$$\min\Big\{d, \max\{d, \omega\}, \max\{\omega, a_{\{j\}}\}, \max\{d, \omega, \omega\}\Big\} = \omega.$$

This is a contradiction since, $\max\{\omega, a_{\{j\}}\} > \omega$. This establishes the Claim 2.

Hence, from the claims above it follows that $a_\emptyset = d$ and $a_s = \omega$ for $s \subseteq \{i, j\}$ and $s \neq \emptyset$. Therefore,

$F(R_i, R_j) =$

$$\min\Big\{d, \max\{Top(R_i, \Re_F), \omega\}, \max\{Top(R_j, \widetilde{\omega d}), \omega\}, \max\{Top(R_i, \Re_F), Top(R_j, \widetilde{\omega d}), \omega\}\Big\}.$$

Since none of the tops can be higher than d so we can write,

$F(R_i, R_j) =$

$$\min\Big\{\max\{Top(R_i, \Re_F), \omega\}, \max\{Top(R_j, \widetilde{\omega d}), \omega\}, \max\{Top(R_i, \Re_F), Top(R_j, \widetilde{\omega d}), \omega\}\Big\}.$$

$$\text{Equivalently, } F(R_i, R_j) = \text{median}\Big\{Top(R_i, \Re_F), Top(R_j, \widetilde{\omega d}), \omega\Big\}.$$

Analogously, if $(R_i, R_j) \in K(1) \cup K(3)$ then

$$F(R_i, R_j) = \text{median}\Big\{Top(R_i, \Re_F), Top(R_j, \widetilde{d'\omega}), \omega\Big\}.$$

This completes the proof of Theorem 2. ■

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