COMMON-VALUE ALL-PAY AUCTIONS WITH ASYMMETRIC INFORMATION

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Abstract

We study two-player common-value all-pay auctions in which the players have ex-ante asymmetric information represented by finite partitions of the set of possible values of winning. We consider two families of such auctions: in the first, one of the players has an information advantage (henceforth, IA) over his opponent, and in the second, no player has an IA over his opponent. We show that there exists a unique equilibrium in auctions with IA and explicitly characterize it; for auctions without IA we find a sufficient condition for the existence of equilibrium in monotonic strategies. We further show that, with or without IA, the ex-ante distribution of equilibrium effort is the same for every player (and hence the players' expected efforts are equal), although their expected payoffs are different. In auctions with IA, the players have the same ex-ante probability of winning, which is not the case in auctions without IA.

Keywords*:* Common-value all-pay auctions, asymmetric information, information advantage.

JEL Classification: C72*,* D44, D82.

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1 Introduction

All-pay auctions are used in diverse areas of economics, such as lobbying in organizations, R&D races, political contests, promotions in labor markets, trade wars, and biological wars of attrition. In an all-pay auction each player submits a "bid" (i.e., exerts effort) and the player with the highest bid wins the contest. However, regardless of who the winner is, each player bears the cost of his bid. All-pay auctions have been studied when players have either private or common values of winning.[1] In this paper we focus on common-value all-pay auctions and consider contests with two ex-ante asymmetrically informed players where the value of winning is identical for both players in the same state of nature, but the information about which state of nature was realized is different. We assume that each player's information is represented by a finite partition of the set of states of nature that can be identified with the set of possible common values, but these partitions are different.[2] When the state of nature is chosen, each player learns which element of partition contains the realized common-value, but the players do not necessarily know the exact value of winning the contest.[3] This model captures situations in which winning a contest is of similar benefit to each contestant, but the precise value of winning, which depends on several random parameters, may be unknown.

In our two-player model of asymmetric information we assume that the information sets of each player are connected with respect to the value of winning the contest (see Einy et al. (2001, 2002) and Forges and Orzach (2011)). This means that if a player's information partition does not enable him to distinguish between two possible values of winning, then he also cannot distinguish between all intermediate values. Connectness seems plausible in environments where the information of a player allows him to put upper and lower bounds on the actual value of winning, without ruling out any outcome within these bounds.

We first study a common-value all-pay auctions where one player has an *information advantage* (hence-

[1] To mention just a few works, all-pay auctions have been considered by, e.g., Hillman and Riley (1989), Baye et al. (1993, 1996), Amann and Leininger (1996), Che and Gale (1998), Moldovanu and Sela (2001, 2006), Siegel (2009) and Moldovanu et al. (2010).

[2] This partition representation is equivalent to the more common Harsanyi-type formulation of Bayesian games (see Jackson (1993) and Vohra (1999))

[3] This framework has been used in several works to analyze common-value second-price auctions (see Einy et al. (2001, 2002), Forges and Orzach (2011), and Abraham et al. (2012)), and common-value first-price auctions (see Malueg and Orzach (2009, 2012)).

forth, IA) over the other, which means that his information partition is finer than that of his opponent. It can be shown that without loss of generality we can assume that one player is completely informed about the state of nature, while the other player is completely uninformed. We establish existence and uniqueness of equilibrium in mixed strategies in this class of contests and provide its complete characterization. Our results show that although the players have asymmetric strategies that yield different expected payoffs, the expected efforts of both players are the same. Moreover, the probability of each player to win the contest in equilibrium is the same. Hence, we find that asymmetry of information between the players does not result in unequal expected efforts or different chances to win the contest, but it does affect the allocation of payoffs between the players. In the unique equilibrium of this model, the expected payoff of the uninformed player is zero, while the expected payoff of the informed payer is positive.

We then examine how the relation between players' information sets affects their expected total effort. We find that maximizing the total effort calls for narrowing the information gap between the players. Specifically, if there are three players (a,b and c) where a has an IA over b who has an IA over c, then the expected total effort in the contest between a and c is necessarily lower than in the contest between b and c. In other words, when the players' information endowments become closer to each other, their total effort grows. Although we analyze two-player (common-value all-pay) auctions, the above results can be generalized to any number of players as long as the players' information partitions can be ranked, namely, as long as in all pairwise comparisons one player has an IA over the other. In such a case, as well as in the complete information all-pay auction with more than two players (see Baye et al. 1996), there will be an equilibrium in which only the two best-informed players participate, while the rest stay out of the contest (i.e., place bids of zero).[4]

We also study common-value all-pay auctions in which, except in the extreme states of nature (corresponding to the lowest and the highest possible values of winning), neither player has an IA over his opponent.[5] For this case, we construct a "candidate" for an equilibrium with monotonic strategies (i.e., strategies where more favorable signals do not lead to lower bids) and find sufficient conditions for the "candidate" to be a true equilibrium. In this model without an IA, we show that as in the IA case, the expected efforts of both players are the same in equilibrium, but, in contrast to the IA case, their probabilities to win

[4] For a complete discussion see Section 4.

[5] Malueg and Orzach (2009) studied the first-price auction with information partitions of this type.

the contest may be distinct. We also show that a player's expected payoff, conditional on an element of his partition, monotonically increases with the values of winning. Thus, higher/lower values in the information set of a player lead to a higher/lower conditional expected payoff.[6]

The most related work to ours is Siegel (2013), who studies asymmetric two-player all-pay auctions with interdependent valuations, where private information of each player is represented by a finite set of possible types. He shows that a unique equilibrium exists in his set-up, and provides an algorithm to calculate the equilibrium strategies. Our equilibrium characterization for common-value all-pay auctions with an IA is akin to the result of Siegel (2013), but we also compare the distributions of equilibrium efforts and payoffs across players and for different information structures, including the case without an IA.

The paper is organized as follows. In Section 2 we present the model. In Section 3 we give a numerical example that demonstrates how to find the equilibrium in a model with an IA. In Section 4, we characterize the equilibrium in the general case with an IA, and analyze the players' expected efforts and payoffs as well as their probabilities of winning. In Section 5 we give a numerical example of equilibrium in a model without an IA. In Section 6 we find sufficient conditions for the existence of equilibrium in the general case without an IA, and explicitly describe one such equilibrium. In Subsection 6.1 we analyze the players' expected efforts and payoffs in a model without an IA. Section 7 concludes. Some of the proofs are in the Appendix.

2 The model

Consider the set $\mathcal{N} = \{1, 2, ..., N\}$ of $N \geq 2$ players who compete in an all-pay auction where the player with the highest effort (output) wins the contest, but all the players bear the cost of their effort. The uncertainty in our model is described by a finite set Ω of states of nature and a probability distribution p over Ω which can be interpreted as the common prior belief about the realized state of nature (w.l.o.g. $p(\omega) > 0$ for every $\omega \in \Omega$). A function $v : \Omega \to \mathbb{R}_+$ represents the common value of winning the contest, i.e., if $\omega \in \Omega$ is realized then the value of winning is $v(\omega)$ for every player.

[6] Unlike the model with IA, when there is no IA the information structure does not give advantage to any player when their expected payoffs are concerned, as it can be easily seen that the expected payoff of a player may be higher or lower than that of his opponent.

The private information of each player $i \in \mathcal{N}$ is described by a partition Π_i of Ω. We assume that each Π_i is *connected* with respect to the common value function v, i.e., for every element $\pi_i \in \Pi_i$, if $\omega_1, \omega_2 \in \pi_i$ and $\omega \in \Omega$ satisfy $v(\omega_1) \leq v(\omega) \leq v(\omega_2)$, then $\omega \in \pi_i$.[7]

A common-value all-pay auction starts when nature chooses a state ω form Ω according to the distribution p. Each player $i \in \mathcal{N}$ is informed of the element $\pi_i(\omega)$ of Π_i which contains ω (thus, $\pi_i(\omega)$ constitutes the information set of player i at ω), and then he chooses an effort $x_i \in \mathbb{R}_+$. The players will typically have different information partitions, and thus are ex-ante asymmetric.

The utility (payoff) of player $i \in \mathcal{N}$ is given by the function $u_i : \Omega \times \mathbb{R}_+^N \to \mathbb{R}$ as follows:

$$u_i(\omega, x) = \begin{cases} \frac{1}{m(x)} v(\omega) - x_i, & \text{if} \quad x_i = \max\{x_k\}_{k \in \mathcal{N}}, \\ -x_i, & \text{if} \quad x_i < \max\{x_k\}_{k \in \mathcal{N}}, \end{cases}$$

where $m(x)$ denotes the number of players who exert the highest effort, namely, $m(x) = |i \in N : x_i = \max\{x_k\}_{k \in \mathcal{N}}|$. A common-value all-pay auction with differential information is fully described by and identified with the collection $G = (N, (\Omega, p), \{u_i\}_{i \in \mathcal{N}}, \{\Pi_i\}_{i \in \mathcal{N}})$.

In all-pay auctions, there is usually no equilibrium in pure strategies. Thus our attention will be given to mixed strategy equilibria. A mixed strategy of player $i \in \mathcal{N}$ is a function $F_i : \Omega \times \mathbb{R}_+ \to [0, 1]$, such that for every $\omega \in \Omega$, $F_i\,(, \cdot)$ is a cumulative distribution function (c.d.f.) on $\mathbb{R}_+$, and for all $x \in \mathbb{R}_+$, $F_i\,(\cdot, x)$ is a Π_i-measurable function (that is, $F_i\,(\cdot, x)$ is constant on every element of Π_i). Slightly abusing notation, for any $\pi_i \in \Pi_i$ we will denote the constant value of $F_i\,(\cdot, x)$ on π_i by $F_i\,(\pi_i, x)$, whenever convenient. If player i plays a pure strategy given π_i, i.e., if the distribution represented by $F_i\,(\pi_i, \cdot)$ is supported on some $y \in \mathbb{R}_+$, we will identify between $F_i\,(\pi_i, \cdot)$ and y wherever appropriate.

Given a mixed strategy profile $F = (F_1, ..., F_N)$, denote by $E_i(F)$ the expected payoff of player i when players use that strategy profile, i.e.,

$$E_i(F) \equiv E(\int_0^\infty ... \int_0^\infty u_i(\cdot, (x_1, ..., x_N)) dF_1(\cdot, x_1), ..., dF_N(\cdot, x_N)).$$

[7] It is worth noting that our analysis remains valid if Ω is an infinite set of states of nature provided the partitions are finite. To see this, simply replace Ω with a finite Ω', which is the coarsest partition of Ω that refines all $\{\Pi_n\}_{n \in \mathcal{N}}$, and, for each $\pi \in \Omega'$, let the value of winning at π, $v(\pi)$, be equal to the conditional expectation $E(v(\cdot) \mid \pi)$.

For $\pi_i \in \Pi_i$, $E_i(\pi_i, F)$ will denote the conditional expected payoff of player i given his information set π_i, i.e.,

$$E_i(\pi_i, F) \equiv E(\left[\int_0^\infty ... \int_0^\infty u_i(\cdot, (x_1, ..., x_N))dF_1(\cdot, x_1), ..., dF_i(\cdot, x_i), ..., dF_N(\cdot, x_N)\right] \mid \pi_i).$$

An N-tuple of mixed strategies $F^* = (F_1^*, ..., F_N^*)$ constitutes a (Bayesian Nash) equilibrium in the common-value all-pay auction G if for every player i, and every mixed strategy F_i of that player, the following inequality holds:

$$E_i(F^*) \geq E_i(F_1^*, ..., F_i, ..., F_N^*).$$

3 Information advantage: Example 1

We begin with a simple example to illustrate the players' behavior in our model with an IA. Consider a common-value all-pay auction with two players. Assume that there are three states of nature such that in state ω_i the value of winning is $v(\omega_i) = i$ with probability of $p_i = \frac{1}{3}$, for $i = 1, 2, 3$. Player 1 knows *only* the prior distribution p, and hence he has the trivial information partition, $\Pi_1 = \{\{\omega_1, \omega_2, \omega_3\}\}$, while player 2 is completely informed of the value of winning, hence $\Pi_2 = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}\}$ partitions Ω into singletons.

It can be easily verified that the corresponding common-value all-pay auction does not have an equilibrium in pure strategies. However, there does exist a mixed strategy equilibrium. In this equilibrium, player 1's mixed strategy F_1^* is a state-independent c.d.f. given by

$$F_1^*(x) = \begin{cases} 0, & \text{if } x < 0, \\ x, & \text{if } 0 \leq x \leq \frac{1}{3}, \\ \frac{x}{2} + \frac{1}{6}, & \text{if } \frac{1}{3} < x \leq 1, \\ \frac{x}{3} + \frac{1}{3}, & \text{if } 1 < x \leq 2, \\ 1, & \text{if } 2 < x. \end{cases}$$

Player 2's mixed strategy F_2^* does depend on the state of nature (of which he is informed):

$$F_2^*(\omega_1, x) = \begin{cases} 0, & \text{if } x < 0, \\ 3x, & \text{if } 0 \leq x \leq \frac{1}{3}, \\ 1, & \text{if } x > \frac{1}{3}, \end{cases}$$

$$F_2^*(\omega_2, x) = \begin{cases} 0, & \text{if } x \leq \frac{1}{3}, \\ \frac{3}{2}x - \frac{1}{2}, & \text{if } \frac{1}{3} < x \leq 1, \\ 1, & \text{if } x > 1, \end{cases}$$

$$F_2^*(\omega_3, x) = \begin{cases} 0, & \text{if } x < 1, \\ x - 1, & \text{if } 1 \leq x \leq 2, \\ 1, & \text{if } x > 2. \end{cases}$$

In order to see that the above strategies are in equilibrium, note that, given player 2's mixed strategy F_2^*, player 1's expected payoff, if he exerts effort $x \in [1, 2]$, is

$$E_1(x, F_2^*) = \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 2 + \frac{1}{3} \cdot 3 \cdot (x - 1) - x = 0$$

When $x \in [\frac{1}{3}, 1]$,

$$E_1(x, F_2^*) = \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 2 \cdot (\frac{3}{2}x - \frac{1}{2}) - x = 0,$$

and when $x \in [0, \frac{1}{3}]$,

$$E_1(x, F_2^*) = \frac{1}{3} \cdot 1 \cdot (3x) - x = 0.$$

As any effort above 2 would result in a negative expected payoff, $[1, 2]$ is the set of player 1's pure strategy best responses to to F_2^*, and in particular his mixed strategy F_1^* is a best response to F_2^* as it results in an expected payoff of zero.

Now, fix payer 1's mixed strategy F_1^*, and assume that ω_3 is the realized state of nature. If player 2 exerts effort $x \in [1, 2]$, then his conditional expected payoff is

$$E_2(\{\omega_3\}, F_1^*, x) = 3 \cdot (\frac{x}{3} + \frac{1}{3}) - x = 1.$$

If he exerts $x \in [\frac{1}{3}, 1)$ or $x \in [0, \frac{1}{3}]$, correspondingly, his expected payoff is

$$E_2(\{\omega_3\}, F_1^*, x) = 3 \cdot (\frac{x}{2} + \frac{1}{6}) - x = \frac{x}{2} + \frac{1}{2} < 1,$$

or

$$E_2(\{\omega_3\}, F_1^*, x) = 3 \cdot x - x = 2x < 1,$$

and thus, conditional on the realization of ω_3, $[1, 2]$ is the set of player 2's pure strategy best responses to F_1^*. In particular, conditional on ω_3, $F_2^*(\omega_3, \cdot)$ is a mixed strategy best response to F_1^*.

If ω_2 is the realized state, by exerting $x \in [\frac{1}{3}, 1]$ player 2 obtains the expected payoff

$$E_2(\{\omega_2\}, F_1^*, x) = 2 \cdot (\frac{x}{2} + \frac{1}{6}) - x = \frac{1}{3}.$$

As before, it can be seen that all effort levels outside $[\frac{1}{3}, 1]$ lead to a lower expected payoff, and thus conditional on ω_2, $F_2^*(\omega_2, \cdot)$ is a mixed strategy best response to F_1^*.

If ω_1 is the realized state, by exerting $x \in [0, \frac{1}{3}]$ player 2, in expectation, obtains

$$E_2(\{\omega_1\}, x) = 1 \cdot x - x = 0,$$

while effort levels outside $[0, \frac{1}{3}]$ lead to negative expected payoffs. Thus, also conditional on ω_1, $F_2^*(\omega_1, \cdot)$ is a mixed strategy best response to F_1^*. We conclude that F_2^* is a best response of player 2 also w.r.t. the unconditional expected payoff. Hence, the pair (F_1^*, F_2^*) is a mixed strategy equilibrium. The expected payoff of player 2 is then

$$E_2(F_1^*, F_2^*) = \frac{1}{3}(E_2(\{\omega_1\}, F_1^*, F_2^*) + E_2(\{\omega_2\}, F_1^*, F_2^*) + E_2(\{\omega_3\}, F_1^*, F_2^*)) = \frac{4}{9}.$$

In the next section, we characterize the players' mixed-strategy equilibrium in a general two-player common-value all-pay auction with an IA, and prove its uniqueness.

4 Information advantage: Equilibrium analysis

We consider first the case of two players, where player 2 has an IA over player 1 (i.e., information partition Π_2 of player 1 is finer than Π_1). This model will be called common-value all-pay auction with an IA. Let us write Ω as an indexed sequence $\{\omega_i\}_{i=1}^n$. It can be easily shown that as far as the equilibrium analysis is concerned, the IA assumption can be reduced to the postulate that $\Pi_1 = \{\Omega\}$ and $\Pi_2 = \{\{\omega_1\}, \{\omega_1\}, ..., \{\omega_n\}\}$.

Now, for each state of nature $\omega_i \in \Omega$, denote

$$v_i = v(\omega_i) \text{ and } p_i = p(\omega_i) > 0. \tag{1}$$

Assume that the possible values are positive and strictly ranked as

$$0 < v_1 < v_2 < ... < v_n. \tag{2}$$

In what follows, we describe a mixed strategy equilibrium (F_1^*, F_2^*) of the all-pay auction with an IA.

Let $x_0 \equiv 0$, and

$$x_i \equiv \sum_{j=1}^{i} p_j v_j \tag{3}$$

for each $i = 1, ..., n$. Thus, $x_0 < x_1 < ... < x_n$. Consider a function F_1^* on $\mathbb{R}_+$ given by

$$F_1^*(x) = \begin{cases} 0, & \text{if } x < 0, \\ \frac{x}{v_i} + \sum_{j=1}^{i-1} p_j \left[1 - \frac{v_j}{v_i}\right], & \text{if } x \in [x_{i-1}, x_i] \text{ for } i = 1, ..., n, \\ 1, & \text{if } x > x_n. \end{cases} \tag{4}$$

It is easy to see that F_1^* is well defined, strictly increasing on $[x_0, x_n]$, and continuous. Moreover, $F_1^*(x_0) = 0$ and $F_1^*(x_n) = 1$. Thus, F_1^* is a c.d.f. of a continuous probability distribution supported on the interval $[x_0, x_n]$. (Such a distribution is obtained by assigning probability p_i to each interval $[x_{i-1}, x_i]$, randomly choosing an interval, and then selecting a point w.r.t. the uniform distribution on the chosen interval). Being that the function F_1^*, is state-independent, it can be viewed as a mixed strategy of the uninformed player 1.

Note next that

$$\begin{aligned} E_2(\{\omega_i\}, F_1^*, x) &= v_i F_1^*(x) - x \qquad (5) \\ &= v_i F_1^*(x_{i-1}) - x_{i-1} = E_2(\{\omega_i\}, F_1^*, x_{i-1}) \qquad (6) \end{aligned}$$

for every $x \in [x_{i-1}, x_i]$, and $i = 1, ..., n$. Thus, given that ω_i was realized, the informed player 2 is indifferent between all efforts in the interval $[x_{i-1}, x_i]$, provided that his rival acts according to F_1^*. Since the slopes of the function $v_i F_1^*(x) - x$ are positive when $x < x_{i-1}$ and negative when $x > x_{i-1}$, the set of player 2's pure strategy best responses is the interval $[x_{i-1}, x_i]$.

Now, for each $i = 1, ..., n$, consider a function $F_2^*(\omega_i, \cdot)$ on $\mathbb{R}_+$ given by

$$F_2^*(\omega_i, x) = \begin{cases} 0, & \text{if } x < x_{i-1}, \\ \frac{x - \sum_{j=1}^{i-1} p_j v_j}{p_i v_i}, & \text{if } x \in [x_{i-1}, x_i], \\ 1, & \text{if } x > x_i \end{cases} . \tag{7}$$

Note that the function $F_2^*(\omega_i, \cdot)$ is well defined, strictly increasing on $[x_{i-1}, x_i]$, continuous, $F_2^*(\omega_i, x_{i-1}) = 0$ and $F_2^*(\omega_i, x_i) = 1$. Thus, $F_2^*(\omega_i, \cdot)$ is a c.d.f. of a probability distribution supported on $[x_{i-1}, x_i]$, and, in particular, F_2^* constitutes a mixed strategy of player 2. Moreover,

$$E_1(x, F_2^*) = \sum_{j=1}^{i-1} p_j v_j + p_i v_i F_2^*(\omega_i, x) - x = 0 \tag{8}$$

for every $x \in [x_{i-1}, x_i]$. Thus, player 1 is (in expectation) indifferent between all efforts in $[x_0, x_n]$ (and is obviously worse off when efforts are outside $[x_0, x_n]$) provided his rival 2 acts according to F_2^*. We conclude that (F_1^*, F_2^*) is a mixed strategy equilibrium. It turns out that it is the only one:

Proposition 1 *Mixed strategy equilibrium (F_1^*, F_2^*) described above is the unique equilibrium in G.*

Proof. See Appendix. ■

Thus far, we have assumed that there are only two players. This entails no loss of generality in the following sense. Suppose that there are $N > 2$ players, such that the players' information endowments are ranked as follows: player 2 has an IA over player 1, and player 1 has an IA over or the same information endowment as players $3, ..., N$. Let (F_1^*, F_2^*) be the unique equilibrium in the contest between 1 and 2 (which exists by Proposition 1 and footnote 4). We claim that in the contest between $1, 2, ..., N$, strategy profile $(F_1^*, F_2^*, 0, 0, ...0)$ constitutes an equilibrium. That is, all but the two players with the best information submit bids of zero which means that they are effectively staying out of the contest, while players 1 and 2 behave as if they were engaged in a two-player contest. This will ensure that any N-player contest in which information endowments are ranked possesses a reduction to the two-player case.

In order to see that $(F_1^*, F_2^*, 0, 0, ...0)$ is an equilibrium, note first that players 1 and 2 have no incentive to unilaterally deviate from their strategies in $(F_1^*, F_2^*, 0, 0, ...0)$, since their payoffs are identical to those in a two-player contest where such deviations are not profitable in expectation. Note next that if any of the remaining players (say, player 3) had a profitable deviation F_3 from bid 0, we would have had

$$E_3(F_1^*, F_2^*, F_3, 0, ..., 0) > E_3(F_1^*, F_2^*, 0, 0, ..., 0) = 0,$$

and hence

$$E_3(F_1^*, F_2^*, F_3, 0, ..., 0) > 0.$$

Since player 1 has an IA over (or the same information as) player 3, F_3 is also a Bayesian strategy of player 1. As F_1^* is 1's best response to $(F_1^*, F_2^*, 0, 0, ...0)$, it follows that

$$\begin{aligned} E_1(F_1^*, F_2^*, 0, 0, ...0) &\geq E_1(F_3, F_2^*, 0, 0, ..., 0) \\ &= E_3(0, F_2^*, F_3, 0, ..., 0) \\ &\geq E_3(F_1^*, F_2^*, F_3, 0, ..., 0) > 0. \end{aligned}$$

Thus

$$E_1(F_1^*, F_2^*, 0, 0, ...0) > 0,$$

and in particular $E_1(F_1^*, F_2^*) > 0$ in the two-player contest between 1 and 2. However, it follows from (8), Proposition 1, and footnote 4 that the expected payoff to player 1 in the unique equilibrium is zero, a contradiction. We conclude that players $3, .., N$ cannot unilaterally deviate from bid 0 and make a profit, and hence that $(F_1^*, F_2^*, 0, 0, ...0)$ is an equilibrium of the N-player contest.

We have shown that the equilibrium strategies in a two-payer common-value all-pay auction are determined uniquely. From (8) it follows that the expected equilibrium payoff of player 1 is

$$E_1(F_1^*, F_2^*) = 0. \tag{9}$$

It follows also from (5)-(6) that player 2's expected payoff is

$$\begin{aligned} E_2(F_1^*, F_2^*) &= \sum_{i=1}^{n} p_i(v_i F_1(x_{i-1}) - x_{i-1}) \\ &= \sum_{i=1}^{n} p_i(\sum_{j=1}^{i-1} p_j(v_i - v_j)). \end{aligned} \tag{10}$$

The equilibrium strategies F_1^*, F_2^* of the two players are quite different. Among other distinctions, F_2^* is state-dependent, while F_1^* is not. However, both players have the same *ex-ante* distribution of the effort they make. Indeed, for every $i = 1, 2, ..., n$, and every $x \in [x_{i-1}, x_i]$ (where x_i is given by (3)), the ex-ante probability $F_2(x)$ that player 2 exerts an effort that is smaller than or equal to x according to his strategy F_2^* is given by

$$\begin{aligned} F_2(x) &= \sum_{j=1}^{i-1} p_j + p_i F_2^*(\omega_i, x) = \sum_{j=1}^{i-1} p_j + p_i \cdot \frac{x - \sum_{j=1}^{i-1} p_j v_j}{p_i v_i} \\ &= \frac{x}{v_i} + \sum_{j=1}^{i-1} p_j \left[1 - \frac{v_j}{v_i}\right] = F_1^*(x). \end{aligned}$$

Thus, the ex-ante distribution of equilibrium effort is identical for both players. This fact leads to the following proposition.

Proposition 2 *In the unique equilibrium* (F_1^*, F_2^*) *of every two-player common-value all-pay auction with an IA*

(i) each player has (ex-ante) probability $\frac{1}{2}$ *to win;*

and

(ii) both players exert the same expected effort

$$\sum_{i=1}^{n} p_i \left(\sum_{j=1}^{i-1} p_j v_j + \frac{1}{2} p_i v_i \right).$$

Proof. It was shown above that the players have ex-ante identical (and, obviously, independent) distributions of efforts, and hence, as claimed in (i), each wins the contest with the same probability. It also follows that the expected efforts of both players are equal. Calculating the expected payoff for player 1 (using (4)) leads to the formula claimed in (ii):

$$EE_1 = \int_{x_0}^{x_n} x dF_1^* (x) = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \frac{x}{v_i} dx = \sum_{i=1}^{n} \frac{x_i^2 - x_{i-1}^2}{2v_i} = \sum_{i=1}^{n} p_i \left(\sum_{j=1}^{i-1} p_j v_j + \frac{1}{2} p_i v_i \right).$$

Q.E.D. ■

According to Proposition 2, the asymmetry in information does not affect the ratio of the two players' expected efforts, as the expected efforts are equal. However, the asymmetric information does affect the players' expected *total* effort.

4.1 Comparative results

We have just shown that the expected payoff of player 1, over whom player 2 has an IA, is zero in equilibrium. We will now examine how the *extent* of IA affects the expected payoff of player 2. Assume, as before, that $\Pi_1 = \{\Omega\}$ and $\Pi_2 = \{\{\omega_1\}, \{\omega_1\}, ..., \{\omega_n\}\}$. Also consider an additional player 2' with an "intermediate" connected information partition Π_2', which is a strict coarsening of Π_2 and a strict refinement of Π_1. Then we have the following comparative result.

Proposition 3 *In a two-player common-value all-pay auction with an IA, the expected payoff of player 2 (when he competes against player 1) is higher than the expected payoff of player 2' (when he competes against player 1).*

Proof. By (10), the expected payoff of player 2, when he competes against player 1, is given in equilibrium by

$$E_2 = \sum_{i=1}^{n} p_i (\sum_{k=1}^{i-1} p_k (v_i - v_k)).$$

Regarding player 2', assume first that Π_2' is different from Π_2 only in that player 2' cannot distinguish between the states ω_j and ω_{j+1}, for some $1 \leq j < n$. Thus, $\Pi_2' = \{\{\omega_1\}, \{\omega_2\}, ...\{\omega_{j-1}\}, \{\omega_j, \omega_{j+1}\}, \{\omega_{j+2}\}, ...\{\omega_n\}\}$. The auction in which player 2' competes against player 1 is amenable to our previous analysis, but with a minor modification: the set of states of nature must be redefined as $\Omega' = (\Omega \backslash \{\omega_j, \omega_{j+1}\}) \cup \{\omega_{j,j+1}\}$, where the new state $\omega_{j,j+1}$ is the amalgamation of ω_j and ω_{j+1}, occurring with probability $p_{j,j+1} = p_j + p_{j+1}$ and having the common value of $v_{j,j+1} = \frac{p_j}{p_j + p_{j+1}} v_j + \frac{p_{j+1}}{p_j + p_{j+1}} v_{j+1}$. In this modified contest (payoff-equivalent to the original), player 1 has the trivial information, while player 2' has the finest possible information partition. Applying (10) to this contest, the expected payoff of player 2 is equilibrium is given by

$$\begin{aligned} E_2' &= \sum_{i=1}^{j-1} p_i (\sum_{k=1}^{i-1} p_j (v_i - v_j)) \\ &\quad + p_{j,j+1} \sum_{k=1}^{j-1} p_k (v_{j,j+1} - v_k) \\ &\quad + \sum_{i=j+2}^{n} p_i (\sum_{k \leq i-1, k \neq j, k \neq j+1} p_k (v_i - v_k) + p_{j,j+1} (v_i - v_{j,j+1})). \end{aligned}$$

Then we have

$$E_2 - E_2' = p_j p_{j+1} (v_{j+1} - v_j) + \sum_{i=j+2}^{n} p_i (\sum_{k=j}^{j+1} p_k (v_i - v_k) - p_{j,j+1} (v_i - v_{j,j+1})).$$

Since $p_j p_{j+1} (v_{j+1} - v_j) > 0$ and $\sum_{i=j+2}^{n} p_i (\sum_{k=j}^{j+1} p_k (v_i - v_k) - p_{j,j+1} (v_i - v_{j,j+1})) = 0$ we obtain that $E_2 - E_2' > 0$.

We have thus shown that player 2' obtains in expectation less than player 2 (when competing against 1 in a two-player auction) if Π_2' is a connected partition which is a strict coarsening of Π_2 with $|\Pi_2'| = |\Pi_2| - 1$. Inductively, the claim can be extended to *any* connected partition Π_2' with $|\Pi_2'| < n$. *Q.E.D.* ■

The next result shows that there is an opposite relation between the players' total expected payoff and their total expected effort.

Proposition 4 *In a two-player common-value all-pay auction with an IA, the expected total effort when player 2 competes against player 1 is lower than the expected total effort when player 2' competes against player 1.*

Proof. In every common-value all-pay auction, the relation between the players' expected total effort and their expected total payoff is

$$\text{Expected total effort} \;=\; \text{Expected reward} \;-\; \text{Expected total payoff}$$

Since the expected payoff of player 1 when he competes against player 2 or against 2' is zero (see (9)), in both auctions

$$\text{Expected total effort} = \text{Expected reward} - \text{Expected payoff of player 2 (or, 2')}$$

By Proposition 3, the expected payoff of player 2 is higher than that of player 2' (when competing against player 1). On the other hand, both contests clearly have the same expected reward, $E(v)$. Thus, the expected total effort when player 2 competes against player 1 is lower than when player 2' competes against player 1. *Q.E.D.* ■

The above propositions demonstrate that increasing asymmetry between players in a two-player common-value all-pay auction with an IA has a positive effect on the expected payoff of the player with an IA, and a negative effect on the expected total effort.

5 No information advantage: Example 2

We now illustrate by a simple example the behavior of players in our model when no player possesses IA. Consider a common-value all-pay auction with two players, and assume that there are three states of nature. For $i = 1, 2, 3$, state ω_i occurs with probability $p_i = \frac{1}{3}$ and the value for the prize in ω_i is $v(\omega_i) = i$. Player 1's information partition is $\Pi_1 = \{\{\omega_1\}, \{\omega_2, \omega_3\}\}$, while player 2's information partition is $\Pi_2 = \{\{\omega_1, \omega_2\}, \{\omega_3\}\}$.

It can be easily verified that the corresponding common-value all-pay auction does not have an equilibrium in pure strategies. However, there does exist a mixed strategy equilibrium. In this equilibrium, player 1's mixed strategy F_1^{**} is a state-dependent c.d.f. given by

$$F_1^{**}(\{\omega_1\},x)=\begin{cases}0, & \text{if } x<0,\\ 2x, & \text{if } 0\le x\le \frac{1}{2},\\ 1, & \text{if } x>\frac{1}{2}\end{cases}$$

and

$$F_1^{**}(\{\omega_2,\omega_3\},x)=\begin{cases}0, & \text{if } x<\frac{1}{2},\\ x-\frac{1}{2}, & \text{if } \frac{1}{2}\le x\le 1,\\ \frac{x}{3}+\frac{1}{6}, & \text{if } 1<x\le \frac{5}{2},\\ 1, & \text{if } x>\frac{5}{2}.\end{cases}$$

Player 2's mixed strategy F_2^{**} is a state-dependent c.d.f. given by

$$F_2^{**}(\{\omega_1,\omega_2\},x)=\begin{cases}0, & \text{if } x<0,\\ x, & \text{if } 0\le x\le 1,\\ 1, & \text{if } x>1\end{cases}$$

and

$$F_2^{**}(\{\omega_3\},x)=\begin{cases}0, & \text{if } x<1,\\ \frac{2}{3}x-\frac{2}{3}, & \text{if } 1\le x\le \frac{5}{2},\\ 1, & \text{if } x>\frac{5}{2}.\end{cases}$$

In order to see that the above strategies are an equilibrium, note that when player 2 uses mixed strategy F_2^{**}, player 1's expected payoff conditional on the event $\{\omega_1\}$ is

$$E_1(\{\omega_1\},x,F_2^{**})=1\cdot x-x=0$$

for any effort $x\in[0,\frac{1}{2}]$. It is easy to see that efforts above $\frac{1}{2}$ would result in a non-positive conditional expected payoff to player 1. Thus, any effort in $[0,\frac{1}{2}]$ is a best response of player 1 to F_2^{**} conditional on $\{\omega_1\}$. Furthermore, player 1's expected payoff conditional on the event $\{\omega_2,\omega_3\}$ is as follows: when 1 exerts effort $x\in(1,\frac{5}{2}]$,

$$E_1(\{\omega_2,\omega_3\},x,F_2^{**})=\frac{1}{2}\cdot 2+\frac{1}{2}\cdot 3\cdot(\frac{2}{3}x-\frac{2}{3})-x=0,$$

and when 1 exerts effort $x\in[\frac{1}{2},1]$,

$$E_1(\{\omega_2,\omega_3\},x,F_2^{**})=\frac{1}{2}\cdot 2\cdot x-x=0.$$

It is easy to see that exerting efforts below $\frac{1}{2}$ or above $\frac{5}{2}$ would lead to non-positive conditional expected payoffs, and thus any effort in $[\frac{1}{2}, \frac{5}{2}]$ is a best response of player 1 to F_2^{**} conditional on $\{\omega_2, \omega_3\}$. Thus the mixed strategy F_1^{**} (supported on $[0, \frac{1}{2}]$ given $\{\omega_1\}$ and on $[\frac{1}{2}, \frac{5}{2}]$ given $\{\omega_2, \omega_3\}$) is an (unconditional) best response of player 1 to F_2^{**}.

Similarly, when payer 1 uses mixed strategy F_1^{**}, player 2's expected payoff conditional on the event $\{\omega_1, \omega_2\}$ payoff is as follows: when 2 exerts effort $x \in (\frac{1}{2}, 1]$,

$$E_2(\{\omega_1, \omega_2\}, F_1^{**}, x) = \frac{1}{2} + \frac{1}{2} \cdot 2 \cdot (x - \frac{1}{2}) - x = 0,$$

and when 1 exerts effort $x \in [0, \frac{1}{2}]$,

$$E_2(\{\omega_1, \omega_2\}, F_1^{**}, x) = \frac{1}{2} \cdot 1 \cdot 2x - x = 0.$$

It is easy to see that any effort above 1 would result in a negative expected payoff. Also, conditional on the event $\{\omega_3\}$, the expected payoff of player 2 when he exerts effort $x \in [1, \frac{5}{2}]$ is

$$E_2(\{\omega_3\}, x), F_1^{**}, x) = 3 \cdot (\frac{x}{3} + \frac{1}{6}) - x = \frac{1}{2}.$$

Any effort above $\frac{5}{2}$ or below 1 would lead to a conditional expected payoff smaller than $\frac{1}{2}$. Thus the mixed strategy F_2^{**} (supported on $[0, 1]$ given $\{\omega_1, \omega_2\}$ and on $[1, \frac{5}{2}]$ given $\{\omega_3\}$) is an (unconditional) best response of player 2 to F_1^{**}.

Hence, the pair $F^{**} = (F_1^{**}, F_2^{**})$ is a mixed strategy equilibrium. The expected payoff of the players in F^{**} are

$$\begin{aligned} E_1(F_1^{**}, F_2^{**}) &= 0, \\ E_2(F_1^{**}, F_2^{**}) &= \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}. \end{aligned}$$

The ex-ante probability of player 2 to win is given by

$$\begin{aligned} P_2 &= \frac{1}{3}[\int_0^{\frac{1}{2}} \left(\int_0^x 2ds\right) 1dx + \int_{\frac{1}{2}}^1 1dx] \\ &\quad + \frac{1}{3} \int_{\frac{1}{2}}^1 \left(\int_{\frac{1}{2}}^x 1ds\right) 1dx \\ &\quad + \frac{1}{3}[\int_1^{\frac{5}{2}} \left(\int_1^x \frac{1}{3}ds\right) \frac{2}{3}dx + \int_{\frac{1}{2}}^1 1ds] \\ &= \frac{13}{24} \end{aligned}$$

Thus, the ex-ante probabilities of players 1 and 2 to win differ from $\frac{1}{2}$ (they are $\frac{11}{24}$ and $\frac{13}{24}$, respectively), and thus part (i) of Proposition 2 does not apply to common-value all-pay auctions without an IA.

In the next section we will present a sufficient condition for the existence of a mixed-strategy equilibrium, and describe the monotonic equilibrium strategies in a general two-player common-value all-pay auction without an IA.

6 No information advantage: Equilibrium analysis

We assume that players 1 and 2 have the following information partitions of $\Omega = \{\omega_1, \omega_2, ..., \omega_n\}$ (where $n > 1$ is an odd number):

$$\begin{aligned} \Pi_1 &= \{\{\omega_1\}, \{\omega_2, \omega_3\}, \{\omega_4, \omega_5\}, ..., \{\omega_{n-1}, \omega_n\}\} \\ \Pi_2 &= \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, ..., \{\omega_{n-2}, \omega_{n-1}\}, \{\omega_n\}\} \end{aligned} \tag{11}$$

We will use throughout notation (1) and assumption (2) of section 4.

Before we describe equilibrium strategies of the players, it is important to emphasize that the all-pay auction with the above information structure (which will be referred to as a common-value all-pay auction *without* an IA), is strategically equivalent to a large family of all-pay auctions with more general information partitions. For this purpose we make the following definition:

Definition 1 *Partitions* Π_1 *and* Π_2 *are* overlapping *if for any* $\pi_1 \in \Pi_1$ *and* $\pi_2 \in \Pi_2$ *the following holds:*

$$\begin{aligned} \omega_1, \omega_n &\notin \pi_1 \Longrightarrow \pi_1 \nsubseteq \pi_2 \\ \omega_1, \omega_n &\notin \pi_2 \Longrightarrow \pi_2 \nsubseteq \pi_1 \end{aligned}$$

This feature means that except in the extreme states of nature (containing the lowest and the highest values of winning), after observing their own signals neither player has an IA over his opponent. It can then be shown (see Malueg and Orzach (2009)) that every common-value all-pay auction with information partitions $\Pi_i, i = 1, 2$ which are connected and overlapping, is strategically equivalent to a common-value all-pay auction with information partitions given by (11).[8]

In order to describe equilibrium strategies we introduce the following notations. Denote

$$\pi^i \equiv \begin{cases} \{\omega_i, \omega_{i+1}\}, & \text{if } i = 1, 2, ..., n-1, \\ \{\omega_1\}, & \text{if } i = 0, \\ \{\omega_n\}, & \text{if } i = n. \end{cases}$$

Thus, Π_1 consists of the sets π^i for every even integer $0 \leq i \leq n-1$, and Π_2 consists of the sets π^i for every odd integer $1 \leq i \leq n$. Also, for every $i = 1, ..., n-1$, let

$$p^{i,i+1} \equiv p(\omega_i \mid \pi^i) \ (= \frac{p_i}{p_i + p_{i+1}})$$

and

$$p^{i+1,i} \equiv 1 - p^{i,i+1}$$

be the conditional probabilities of the states ω_i and ω_{i+1} given the event π^i. Additionally, set

$$p^{n,n+1} = p^{1,0} \equiv 1, \ p^{n+1,n} = p^{0,1} \equiv 0.$$

In what follows, we describe a mixed strategy equilibrium (F_1^{**}, F_2^{**}) of the all-pay auction. Let $x_0 = 0$, and for every $i = 1, 2..., n$ set

$$x_i \equiv \sum_{j=1}^{i} p^{j,j-1} p^{j,j+1} v_j.$$

Given $\pi^0 = \{\omega_1\}$, player 1's mixed strategy is

$$F_1^{**}\left(\pi^0, x\right) = \begin{cases} 0, & \text{if } x < 0, \\ \frac{x}{p^{1,2} v_1}, & \text{if } 0 \leq x \leq x_1, \\ 1, & \text{if } x > x_1. \end{cases} \tag{12}$$

Note that the function $F_1^{**}\left(\pi^0, \cdot\right)$ is well defined, strictly increasing on $[x_0, x_1]$, continuous, $F_1^{**}\left(\pi^0, x_0\right) = 0$ and $F_1^{**}\left(\pi^0, x_1\right) = 1$. Thus, $F_1^{**}\left(\pi^0, \cdot\right)$ is a c.d.f. of a continuous probability distribution supported on $[x_0, x_1]$.

[8] Malueg and Orzach (2009) provided other examples depicting environments that can be transformed into the setting analyzed in this paper.

For $i = 1, 2, ..., n-1$, given $\pi^i = \{\omega_i, \omega_{i+1}\}$ and assuming that $\pi^i \in \Pi_j$ for player j, the mixed strategy of player j is

$$F_j^{**}\left(\pi^i, x\right) = \begin{cases} 0, & \text{if } x < x_{i-1}, \\ \frac{x - x_{i-1}}{p^{i,i-1} v_i}, & \text{if } x_{i-1} \le x \le x_i, \\ \frac{x + p^{i+1,i+2} v_{i+1} - x_{i+1}}{p^{i+1,i+2} v_{i+1}}, & \text{if } x_i < x \le x_{i+1}, \\ 1, & \text{if } x > x_{i+1}. \end{cases} \tag{13}$$

The function $F_j^{*}\left(\pi^i, \cdot\right)$ is well defined, strictly increasing on $[x_{i-1}, x_{i+1}]$, continuous, $F_j^{**}\left(\pi^i, x_{i-1}\right) = 0$, $F_j^{**}\left(\pi^i, x_{i+1}\right) = 1$ and $F_j^{**}\left(\pi^i, x_i\right) = p^{i,i+1}$. Thus, $F_j^{**}\left(\pi^i, \cdot\right)$ is a c.d.f. of a continuous probability distribution supported on $[x_{i-1}, x_{i+1}]$.

Finally, given $\pi^n = \{\omega_n\}$, player 2's mixed strategy is

$$F_2^{**}(\pi^n, x) = \begin{cases} 0, & \text{if } x < x_{n-1}, \\ \frac{x - x_{n-1}}{p^{n,n-1} v_n}, & \text{if } x_{n-1} \le x \le x_n, \\ 1, & \text{if } x > x_n\ . \end{cases} \tag{14}$$

The function $F_2^{**}\left(\pi^n, \cdot\right)$ is well defined, strictly increasing on $[x_{n-1}, x_n]$, continuous, $F_2^{**}\left(\pi^n, x_{n-1}\right) = 0$ and $F_2^{**}\left(\pi^n, x_n\right) = 1$. Thus, $F_2^{**}\left(\pi^n, \cdot\right)$ is a c.d.f. of a probability distribution supported on $[x_{n-1}, x_n]$.

Proposition 5 *Suppose that*

$$p^{i+1,i+2} v_{i+1} - v_i \ge 0 \textit{ for every } i = 1, 2, ..., n-1. \tag{15}$$

Then strategy profile $F^{**} = (F_1^{**}, F_2^{**})$ described in (12), (13), and (14) is a mixed strategy equilibrium of G.[9]

Proof. See Appendix. ■

6.1 No information advantage: other results

In this subsection we derive some comparative results about the players' expected payoffs and efforts in equilibrium F^{**} (described in (12), (13), and (14) in section 6).

[9] Note that (15) can be assumed to hold only for $i = 1, 2, ..., n-2$. As for $i = n-1$, the inequality $p^{n,n+1} v_n - v_{n-1} = v_n - v_{n-1} > 0$ holds trivially by (2).

The proof of Proposition 5 (given in the Appendix) provides an explicit formula for expected equilibrium payoffs of the players conditional on each event π^i, for $i = 0, ..., n$. If j is the player for whom $\pi^i \in \Pi_j$, it follows from (27), (28), and (29) that[10]

$$E_j(\pi^i, F^{**}) = \sum_{k=1}^{i-1} p^{k,k+1}(p^{k+1,k+2}v_{k+1} - v_k), \tag{16}$$

when the sum in (16) is defined as 0 if $i \leq 1$. This yields the following immediate result that compares the players' conditional expected payoffs.

Proposition 6 *Under condition (15), for every $i = 1, ..., n$, and player j such that $\pi^i \in \Pi_j$,*[11]

$$E_j(\pi^i, F^{**}) - E_{-j}(\pi^{i-1}, F^{**}) = p^{i-1,i}(p^{i,i+1}v_i - v_{i-1}) \geq 0.$$

It follows, in particular, that the expected payoff of each player j, conditional on $\pi^i \in \Pi_j$, is increasing in i:

$$E_1(\pi^0, F^{**}) \leq E_1(\pi^2, F^{**}) \leq ... \leq E_1(\pi^{n-1}, F^{**})$$

and

$$E_2(\pi^1, F^{**}) \leq E_2(\pi^3, F^{**}) \leq ... \leq E_2(\pi^n, F^{**}).$$

Furthermore, for every $i = 2, 4, ..., n-1$[12]

$$E_1(\pi_1(\omega_i), F^{**}) \geq E_2(\pi_2(\omega_i), F^{**})$$

and for every $i = 1, 3, ..., n$

$$E_1(\pi_1(\omega_i), F^{**}) \leq E_2(\pi_2(\omega_i), F^{**}).$$

The next result shows that although the players are asymmetrically informed (i.e., have different information partitions), their ex-ante distributions of equilibrium effort are identical.

Proposition 7 *In equilibrium F^{**}, the expected efforts of both players are the same.*

[10] We additionally use the fact that, by definition, $F_j^{**}(\pi^i, \cdot)$ is supported on $[x_{i-1}, x_{i+1}]$ if $1 \leq i \leq n-1$ and $\pi^i \in \Pi_j$, $F_1^{**}(\pi^0, \cdot)$ is supported on $[x_0, x_1]$, and $F_2^{**}(\pi^n, \cdot)$ is supported on $[x_{n-1}, x_n]$.

[11] As in the proof of Proposition 5, we use the convention that when $i = 1$, $v_{i-1} = v_0$ is defined as 0.

[12] Recall that $\pi_j(\omega_i) \in \Pi_j$ denotes the element of Π_j that contains ω_i.

Proof. We will show that both players have the same *ex-ante* distribution of effort in equilibrium. This will imply that the expected efforts of both players are equal.

Let $x_{i-1} \leq x \leq x_i$, for $i = 2, ..., n-1$, and let j be the player for whom $\pi^i \in \Pi_j$. Then the ex-ante probability that player j exerts an effort smaller than or equal to x is

$$\begin{aligned}
\overline{F}_j^{**}(x) &= \sum_{k=1}^{i-1} p_k + (p_i + p_{i+1}) F_j^{**}\left(\pi^i, x\right) \\
&= \sum_{k=1}^{i-1} p_k + (p_i + p_{i+1}) \frac{x - x_{i-1}}{p^{i,i-1} v_i} \\
&= \sum_{k=1}^{i-1} p_k + \frac{(p_i + p_{i+1})(p_i + p_{i-1})(x - x_{i-1})}{p_i v_i}.
\end{aligned} \tag{17}$$

The ex-ante probability that j's rival, player $-j$, exerts an effort smaller than or equal to x is then

$$\begin{aligned}
\overline{F}_{-j}^{**}(x) &= \sum_{k=1}^{i-2} p_k + (p_{i-1} + p_i) F_{-j}^{**}\left(\pi^{i-1}, x\right) \\
&= \sum_{k=1}^{i-2} p_k + (p_{i-1} + p_i) \frac{x + p^{i,i+1} v_i - x_i}{p^{i,i+1} v_i} \\
&= \sum_{k=1}^{i-1} p_k + \frac{(p_i + p_{i+1})(p_i + p_{i-1})(x - x_{i-1})}{p_i v_i} \\
&= \overline{F}_j^{**}(x).
\end{aligned} \tag{18}$$

With the convention that $\sum_{k=1}^{i-2} p_k = \sum_{k=1}^{i-1} p_k = 0$ when $i < 2$ and that $p_0 = p_{n+1} = 0$, (17) and (18) also hold for $i = 1$ and $i = n$.

Since we showed that $\overline{F}_1^{**}(x) = \overline{F}_2^{**}(x)$ for every $x \in [0, x_n]$, and since obviously $\overline{F}_1^{**}(x) = \overline{F}_2^{**}(x) = 1$ for $x > x_n$, the ex-ante distributions of effort in equilibrium F^{**} are identical for both players. ■

7 Concluding remarks

In models with asymmetric information, differences in players' information usually result in different equilibrium strategies, probabilities of winning, and expected payoffs. In this model we show that when the players' information endowments can be ranked, with one player having an IA over his opponent, the ex-ante distributions of players efforts, as well as their ex-ante probabilities of winning the contest, are the same in equilibrium. The difference in information only manifested itself in the different expected payoffs.

In the model without an IA, we show that equilibrium efforts still have the same distribution for all players, and thus the expected efforts of all players are equal, but their probabilities of winning may differ. We also show that the conditional expected payoffs of the players increase in the expected value of winning (as conveyed by the revealed information).

In the model with an IA we also observe that the player with better information has a positive expected payoff while his opponent's expected payoff is zero, and that the highest expected total effort is obtained when the difference in the players' information is as small as possible. Thus, a contest designer who wishes to maximize the expected total effort has an incentive to reduce the difference in information between the players.

Our results are established under the postulate that the information set of each player is connected with respect to the value of winning the contest. This, and the assumption that different information endowments can be ranked (the IA case), are found to be sufficient for the existence of a unique equilibrium. When the different information endowments cannot be ranked (the no IA case) we present a sufficient condition for the existence of the equilibrium.

Our results yield the conclusion that in common-value all-pay auctions the players' information does not affect the ratio of the players' expected efforts. Moreover, in the IA model, the players' information does not affect their probabilities of winning the contest. It is interesting to note that this property does not hold for other forms of contests such as Tullock contests under the IA assumption (see Einy et al. 2013 and Warneryd (2003, 2012)).

8 Appendix

Proof of Proposition 1

Fix an equilibrium (F_1, F_2) in the auction G. We will prove that $(F_1, F_2) = (F_1^*, F_2^*)$.

In what follows, for $k = 1, 2$ and $\omega \in \Omega$, $F_k(\omega, \cdot)$ will be treated either as a *probability distribution* on $\mathbb{R}_+$, or as the corresponding *c.d.f.*, depending on the context. Also, as F_1 is state-independent, $F_1(\omega, \cdot)$ will be shortened to $F_1(\cdot)$, whenever convenient.

Notice that $F_k(\cdot, \{c\}) \equiv 0$ for any effort $c > 0$ and $k = 1, 2$. Indeed, if $F_k(\omega, \{c\}) > 0$ for some k and ω, then $F_m(\omega', (c-\varepsilon, c]) = 0$ for the other player m and every $\omega' \in \Omega$, and some sufficiently small $\varepsilon > 0$. But then k would be strictly better off by shifting the probability from c to $c - \frac{\varepsilon}{2}$, a contradiction to F_k being an equilibrium strategy. Thus, $F_1(\cdot), F_2(\omega, \cdot)$ are non-atomic on $(0, \infty)$ for every $\omega \in \Omega$. Notice also that there is no interval $(a, b) \subset (0, \infty)$ on which in some state of nature *only one* player places positive probability according to his equilibrium strategy. Indeed, otherwise there would exist $a' > a$ such that only one player places positive probability on (a', b), and it would then be profitable for that player to deviate (in at least one state of nature, if this is the informed player 2) by shifting positive probability from (a', b) to a'.

Suppose now that there is a bounded interval $(a, b) \subset (0, \infty)$ such that $F_1((a, b)) = 0$ (and thus $F_2(\omega, (a, b)) = 0$ for every $\omega \in \Omega$, by the previous paragraph), but $F_1([0, a]) > 0$ and $F_1([b, \infty)) > 0$. By extending this interval if necessary, it can also be assumed that (a, b) is maximal with respect to this property, i.e., that $F_1([\max(a-\varepsilon, 0), a]) > 0$ and $F_1([b, b+\varepsilon]) > 0$ for every small enough $\varepsilon > 0$. However, using the fact that $F_2(\omega, \cdot)$ is non-atomic on $(0, \infty)$ for every $\omega \in \Omega$, the expected payoff of player 1 at $\frac{a+b}{2}$ is strictly bigger than his payoff for any effort in $[b, b+\varepsilon]$, if $\varepsilon > 0$ is small enough. This contradicts the assumption that $F_1([b, b+\varepsilon]) > 0$. This contradiction shows that there exists no interval (a, b) as above, meaning that $F_1(\cdot)$ must have *full support* on some closed interval. Denote this interval[13] by $[c, d]$. Notice also that, for every $\omega \in \Omega$, $F_2(\omega, \cdot)$ must be supported on the interval $[c, d]$ (though there need not be full support), since otherwise there would be an interval where only player 2 places positive probability, and this was ruled out.

Note next that $c = 0$. Indeed, if $c > 0$ then $F_2(\cdot, \{c\}) \equiv 0$, and thus player 1 has a negative expected payoff for efforts in $[c, c+\varepsilon]$ for all small enough $\varepsilon > 0$ (because with efforts in $[c, c+\varepsilon]$ he loses the contest almost for sure while expending positive effort of at least c). He would then profitably deviate from F_1 by shifting the probability from $[c, c+\varepsilon]$ to effort 0. Thus, indeed, $c = 0$. Note also that the interval $[0, d]$ is non-degenerate, i.e., $0 < d$, since otherwise the equilibrium strategies would prescribe the constant effort 0, and it is clear that each player would have a profitable unilateral deviation to some $\varepsilon > 0$.

Given i, $i = 1, ..., n$, we will now show that $F_2(\omega_i, \cdot)$ has full support on a (possibly degenerate) *subinterval*

[13] The interval must be bounded as no efforts above v_n will be made in equilibrium, due to the associated negative payoff.

of $[0,d]$. Indeed, if not, there would exist an open subinterval $(a,b)\subset[0,d]$ such that $F_2(\omega_i,(a,b))=0$, but $F_2(\omega_i,[0,a])>0$ and $F_2(\omega_i,[b,d])>0$. Since $F_1((a,b))>0$, there must be $j\neq i$ such that $F_2(\omega_j,(a,b))=0>0$. Assume that $i<j$ (the opposite case is treated similarly). Then there are $x\in[b,d]$ and $y\in(a,b)$ such that

$$\begin{aligned} v_iF^1(x)-x &= E_2\left(\{\omega_i\},F^1,x\right) && (19)\\ &\geq E_2\left(\{\omega_i\},F^1,y\right)=v_iF^1(y)-y && (20)\end{aligned}$$

and

$$\begin{aligned} v_jF^1(x)-x &= E_2\left(\{\omega_j\},F^1,x\right) && (21)\\ &\leq E_2\left(\{\omega_j\},F^1,y\right)=v_jF^1(y)-y. && (22)\end{aligned}$$

But $x>y$, and therefore

$$(v_j-v_i)F^1(x)>(v_j-v_i)F^1(y) \tag{23}$$

since $v_i<v_j$ and the c.d.f. F^1 is strictly increasing on $[0,d]$. Adding (23) to the inequality in (19)-(20) contradicts the inequality obtained in (21)-(22), and therefore no such (a,b) exists. Consequently, each $F_2(\omega_i,\cdot)$ has full support on some subinterval[14] $[a_i,b_i]$ of $[0,d]$. Moreover, if $i<j$ then $[a_i,b_i]$ lies below $[a_j,b_j]$ (barring boundary points), since otherwise it would have been possible to find $x>y$, where $x\in[a_i,b_i]$ and $y\in[a_j,b_j]$, such that inequalities (19)-(20) and (21)-(22) hold. As above, this would lead to a contradiction via (23).

Thus, the intervals $\{[a_i,b_i]\}_{i=1}^n$ are disjoint (barring boundary points), and "ordered" according to the index i on the interval $[0,d]$. Moreover, $\cup_{i=1}^n[a_i,b_i]=[0,d]$, since otherwise there would be a "gap" (a,b) on which only player 1 places positive probability, which is impossible as we have seen earlier. It follows that there are points $0=x_0\leq x_1\leq ...<x_n\equiv d$ such that $[a_i,b_i]=[x_{i-1},x_i]$ for every $i=1,2,...,n$, i.e., $F^1(\cdot)$ has full support on $[0,x_n]$, and, for $i=1,...,n$, $F_2(\omega_i,\cdot)$ has full support on $[x_{i-1},x_i]$. Denote by i_0 the smallest integer with $x_{i_0}>0$.[15]

Since $F^1(\cdot)$ has full support on $[0,x_n]$ and $F_2(\omega,\cdot)$ has no atoms (except possibly at 0), player 1 is indifferent between any two efforts in $(0,x_n]$. Thus, the following equality must hold for every $i=i_0,...,n$

[14] All these subintervals are either non-degenerate (of positive length), or $\{0\}$, as only the latter can be an atom of $F_2(\omega_i,\cdot)$.

[15] Since each interval $[x_{i-1},x_i]$ is either non-degenerate or $\{0\}$, $0=x_0=...=x_{i_0-1}<x_{i_0}<...<x_n$.

and every positive $x \in [x_{i-1}, x_i]$:

$$\sum_{j=1}^{i-1} p_j v_j + p_i v_i F_2(\omega_i, x) - x = E_1(x, F_2) = \lim_{y \searrow 0} E_1(y, F_2) \equiv e_1 \geq 0.$$

In particular,

$$F_2(\omega_i, x) = \frac{x - \sum_{j=1}^{i-1} p_j v_j + e_1}{p_i v_i} \tag{24}$$

for every $i = i_0, ..., n$ and every positive $x \in [x_{i-1}, x_i]$. Since $F_2(\omega_i, \cdot)$ is supported on $[x_{i-1}, x_i]$, we have $F_2(\omega_i, x_i) = 1$, and thus

$$x_i = \sum_{j=1}^{i} p_j v_j - e_1 \tag{25}$$

for every $i = i_0, ..., n$.

Since, for $i = i_0, ..., n$, $F_2(\omega_i, \cdot)$ has full support on $[x_{i-1}, x_i]$ and $F_1(\cdot)$ has no atoms (except, possibly, at 0), player 2 is indifferent between all positive efforts in $[x_{i-1}, x_i]$. Thus, the following equality must hold for every positive $x \in [x_{i-1}, x_i]$:

$$\begin{aligned} v_i F_1(x) - x &= E_2(\{\omega_i\}, F_1, x) \\ &= E_2(\{\omega_i\}, F_1, x_i) = v_i F_1(x_i) - x_i. \end{aligned}$$

In particular,

$$F_1(x) = \frac{x}{v_i} + F_1(x_i) - \frac{x_i}{v_i},$$

and using the fact that $F_1(x_n) = 1$ and (25), we obtain

$$F_1(x) = \frac{x + e_1}{v_i} + \sum_{j=1}^{i-1} p_j \left[1 - \frac{v_j}{v_i}\right] \tag{26}$$

for every $i = i_0, ..., n$, and every positive $x \in [x_{i-1}, x_i]$.

If $e_1 > 0$, it follows from (26) that $F_1(\cdot)$ has an atom at effort 0. Then, obviously $F_2(\omega_i, \cdot)$ cannot have an atom at 0, for any i, since otherwise each player would have a profitable unilateral deviation that shifts the probability from zero to an effort slightly above zero. In particular, all intervals $\{[x_{i-1}, x_i]\}_{i=1}^{n}$ are non-degenerate, i.e., $i_0 = 1$. But then, by (24), $F_2(\omega_1, \cdot)$ has an atom at 0, a contradiction. We conclude that $e_1 = 0$.

If $i_0 > 1$, $x_{i_0-1} = 0$, and thus (24) should hold for $i = i_0$ and any sufficiently small x. But then, if $x < p_1 v_1$

$$F_2(\omega_{i_0}, x) = \frac{x - \sum_{j=1}^{i_0-1} p_j v_j}{p_{i_0} v_{i_0}} \leq \frac{x - p_1 v_1}{p_{i_0} v_{i_0}} < 0,$$

and thus $F_2(\omega_{i_0}, x)$ is not a c.d.f., a contradiction. Consequently, $i_0 = 1$.

It now follows from (25), (24), and (26) that

$$x_i = \sum_{j=1}^{i} p_j v_j$$

for every $i = 1, ..., n$, that

$$F_1(x) = \frac{x}{v_i} + \sum_{j=1}^{i-1} p_j \left[1 - \frac{v_j}{v_i}\right]$$

for every $i = 1, ..., n$ and every $x \in [x_{i-1}, x_i]$, and that

$$F_2(\omega_i, x) = \frac{x - \sum_{j=1}^{i-1} p_j v_j}{p_i v_i}$$

for every $i = 1, ..., n$ and positive $x \in [x_{i-1}, x_i]$. Thus, (F_1, F_2) coincides with (F_1^*, F_2^*) as described in (4) and (7). Q.E.D.

Proof of Proposition 5

For any $i = 0, ..., n$ consider the player j for whom $\pi^i \in \Pi_j$, and assume that j's rival (denoted $-j$) uses the strategy F_{-j}^{**}. The expected payoff of player j conditional on the event π^i is given as follows. If $1 \leq i \leq n$, and j exerts effort $x \in [x_{i-1}, x_i]$, then

$$\begin{aligned} E_j(\pi^i, x, F_{-j}^{**}) &= \\ &= p^{i,i+1} v_i F_{-j}^{**}\left(\pi^{i-1}, x\right) - x \\ &= p^{i,i+1} v_i \frac{x + p^{i,i+1} v_i - x_i}{p^{i,i+1} v_i} - x \\ &= p^{i,i+1} v_i - x_i. \end{aligned} \tag{27}$$

If $1 \le i \le n-1$ and j exerts effort $x \in [x_i, x_{i+1}]$, then

$$\begin{aligned} E_j(\pi^i, x, F^{**}_{-j}) &= \\ &= p^{i,i+1}v_i + p^{i+1,i}v_{i+1}F^{**}_{-j}\left(\pi^{i+1}, x\right) - x \\ &= p^{i,i+1}v_i + p^{i+1,i}v_{i+1}\frac{x - x_i}{p^{i+1,i}v_{i+1}} - x \\ &= p^{i,i+1}v_i - x_i. \end{aligned} \tag{28}$$

Now set $v_0 = 0$. Then (28) applies also when $i = 0$ (in which case $j = 1$), i.e. (28) holds for every $0 \le i \le n-1$.

Equalities (27) and (28) establish the following fact:

Fact 1. When player j's opponent uses F^{**}_{-j}, player j is: (i) indifferent between all efforts in $[x_{i-1}, x_{i+1}]$ given the event $\pi^i \in \Pi_j$ for $1 \le i \le n-1$; (ii) indifferent between all efforts in $[x_0, x_1]$ given π^0 (if j is player 1); (iii) indifferent between all efforts in $[x_{n-1}, x_n]$ given π^n (if j is player 2).

It can be shown by induction on i that, for $i = 2, 3, ..., n$,

$$p^{i,i+1}v_i - x_i = \sum_{k=1}^{i-1} p^{k,k+1}(p^{k+1,k+2}v_{k+1} - v_k) \ge 0. \tag{29}$$

The expression in (29) is non-negative as every summand in $\sum_{k=1}^{i-1} p^{k,k+1}(p^{k+1,k+2}v_{k+1} - v_k)$ is non-negative by assumption (15). When $i = 0$ or $i = 1$, equality (29) remains meaningful if the sum is defined as 0. It then follows from (29) and (27), (28) that:

Fact 2. The conditional expected payoffs of player j considered in (27) and (28) are non-negative for the corresponding efforts.

Next consider $\pi^i \in \Pi_j$, for some $0 \le i \le n-2$ and player j. Notice that, given the event π^i, if $y \in [x_{i+1}, x_{i+2}]$ then

$$\begin{aligned} E_j(\pi^i, y, F^{**}_{-j}) &= p^{i,i+1}v_i + p^{i+1,i}v_{i+1}F^{**}_{-j}\left(\pi^{i+1}, y\right) - y \\ &= p^{i,i+1}v_i + p^{i+1,i}v_{i+1}\frac{y + p^{i+2,i+3}v_{i+2} - x_{i+2}}{p^{i+2,i+3}v_{i+2}} - y \\ &\le p^{i,i+1}v_i + p^{i+1,i}v_{i+1}\frac{x_{i+1} + p^{i+2,i+3}v_{i+2} - x_{i+2}}{p^{i+2,i+3}v_{i+2}} - x_{i+1} \\ &= p^{i,i+1}v_i + p^{i+1,i}v_{i+1}\frac{x_{i+1} - x_i}{p^{i+1,i}v_{i+1}} - x_{i+1} \\ &= p^{i,i+1}v_i - x_i = E_j(\pi^i, x_i, F^{**}_{-j}). \end{aligned} \tag{30}$$

The inequality in (30) holds since, by (15), $p^{i+2,i+3}v_{i+2} > p^{i+1,i}v_{i+1}$, and the last equality in (30) holds by (28). Since obviously, if $y > x_{i+2}$ and $i \leq n-2$,

$$E_j(\pi^i, y, F_{-j}^{**}) \leq E_j(\pi^i, x_{i+2}, F_{-j}^{**}) \tag{31}$$

and, if $y > x_n$,

$$\begin{aligned} E_1(\pi^{n-1}, y, F_2^{**}) &\leq E_1(\pi^{n-1}, x_n, F_2^{**}) = E_1(\pi^{n-1}, x_{n-1}, F_2^{**}), \\ E_2(\pi^n, y, F_1^{**}) &\leq E_2(\pi^n, x_n, F_1^{**}) \end{aligned} \tag{32}$$

Then (31), (32) and (30) establish the following:

Fact 3. When player j's rival uses F_{-j}^{**}, player j (weakly) prefers effort x_i to any effort above $x_{\min(i+1,n)}$, given the event $\pi^i \in \Pi_j$ for $0 \leq i \leq n$.

Now consider $\pi^i \in \Pi_j$, for some $2 \leq i \leq n$ and player j. Given the event π^i, if $y \in [x_{i-2}, x_{i-1}]$ then

$$\begin{aligned} E_j(\pi^i, y, F_{-j}^{**}) &= \\ &= p^{i,i+1}v_i F_{-j}^{**}\left(\pi^{i-1}, y\right) - y \\ &= p^{i,i+1}v_i \frac{y - x_{i-2}}{p^{i-1,i-2}v_{i-1}} - y \\ &\leq p^{i,i+1}v_i \frac{x_{i-1} - x_{i-2}}{p^{i-1,i-2}v_{i-1}} - x_{i-1} \\ &= p^{i,i+1}v_i - x_i = E_j(\pi^i, x_i, F_{-j}^{**}). \end{aligned} \tag{33}$$

The inequality in (33) holds since by (15) $p^{i,i+1}v_i \geq p^{i-1,i-2}v_{i-1}$, and the last equality in (33) holds by (27). Note also that when $i \geq 2$ and $0 \leq y \leq x_{i-2}$,

$$E_j(\pi^i, y, F_{-j}^{**}) \leq 0. \tag{34}$$

Then (33), (34), and Fact 2 lead to the following:

Fact 4. When player j's rival uses F_{-j}^{**}, player j (weakly) prefers effort x_i to any effort below x_{i-1}, given the event $\pi^i \in \Pi_j$ for $2 \leq i \leq n$.

Facts 1, 3, and 4 show that for any $0 \leq i \leq n$, conditional on the event $\pi^i \in \Pi_j$, any effort in the support of $F_j^{**}\left(\pi^i, \cdot\right)$ is a best response of player j against the mixed strategy F_{-j}^{**} of his rival. Thus F_j^{**} is also an unconditional best response of player j, which means that F^{**} is indeed an equilibrium of G. Q.E.D.

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