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Ezra Einy, Ori Haimanko,
Diego Moreno, Aner Sela and
Benyamin Shitovitz

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Monaster Center for
Economic Research
Ben-Gurion University of the Negev
P.O. Box 653
Beer Sheva, Israel

Fax: 972-8-6472941
Tel: 972-8-6472286

Tullock Contests with Asymmetric Information

E. Einy,* O. Haimanko,* D. Moreno,† A. Sela,* and B. Shitovitz‡

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Abstract

Under standard assumptions about players' cost functions, we show that a Tullock contest with asymmetric information has a pure strategy equilibrium. Next we study Tullock contests in which players have a common value and a common state-independent linear cost function. A two-player contest in which one player has an information advantage has a unique equilibrium. In equilibrium both players exert the same expected effort, and although the player with an information advantage wins the prize with probability less than one-half, his payoff is greater or equal to that of his opponent. When there are more than two players in the contest, having information advantage leads to higher payoffs, but the other properties of equilibrium no longer hold.

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*Department of Economics, Ben-Gurion University of the Negev.

†Departamento de Economía, Universidad Carlos III de Madrid.

‡Department of Economics, University of Haifa.

1 Introduction

In a *Tullock contest* – Tullock (1980) – a player's probability of winning the prize is the ratio of the effort he exerts and the total effort exerted by all the players. Baye and Hoppe (2003) have identified a variety of economic settings (rent-seeking, innovation tournaments, patent races) which are strategically equivalent to a Tullock contest. Tullock contests also arise by design, e.g., sport competition, internal labor markets. A number of studies have provided an axiomatic justification to such contests, see, e.g., Skaperdas (1996) and Clark and Riis (1998)).

There is an extensive literature studying Tullock contests and its variations under complete information about the players' value of the prize and their cost of effort. Perez-Castrillo and Verdier (1992), Baye Kovenock and de Vries (1994), Szidarovszky and Okuguchi (1997), Cornes and Hartley (2005), Yamazaki (2008) and Chowdhury and Sheremeta (2009) study existence and uniqueness of equilibrium. Skaperdas and Gan (1995), Glazer and Konrad (1999), Konrad (2002), Cohen and Sela (2005) and Franke *et al.* (2011), study the effect on the players' behavior of changes in the payoff structure, and Schweinzer and Segev (2012) and Fu and Lu (2013) study optimal prize structures. See Konrad (2008) for a general survey.

In this paper we study Tullock contests under asymmetric information (i.e., when player's value for the prize and/or their cost of effort is private information), a topic seldom investigated in the literature. Fey (2008) and Wasser (2013) have recently provided an analysis of rent-seeking games under incomplete information. More closely related to our work is Warneryd (2003), which we discuss below.

In our setting, each player's value for the prize as well as his cost of effort depend on the state of nature. The set of states of nature is finite. Players have a common prior belief, but upon realization of the state of nature, and prior to taking action, each player observes some event that contains the realized state of nature. The information of each player at the moment of taking action is therefore described by a partition of the set of states of nature. (Jackson (1993) and Vohra (1999) have shown that this representation is equivalent to Harsanyi model of a Bayesian game using players' types.) A contest is therefore described by a set of players, a probability space describing players' prior uncertainty and their belief, a collection of partitions of the state space describing the players' information, a collection of state-dependent functions describing the players' values and costs, and a success function specifying the probability distribution used to allocate the prize for each profile of efforts. We assume throughout that the players cost functions are continuously differentiable, strictly increasing and convex with respect to effort, and that the cost of exerting no effort is zero in every state. (In a similar framework, Einy et al. (2001, 2002), Forges and Orzach (2011), and Malueg and Orzach (2009, 2012) study common-value first- and second-price auctions.)

We show that a Tullock contest has a pure strategy Bayesian equilibrium. The proof involves constructing a sequence of equilibria of contests obtained from the original Tullock contest by truncating the action space so that it is a closed and bounded interval whose lower bound approaches zero from above. We show that any limit point of a sequence of equilibria of these contests (which have an equilibrium by Nash's Theorem[1]) is an equilibrium of the original Tullock contest. A key step in the proof is to show that in any such limit point the total effort exerted by players is positive in every state of nature.

Our existence result applies regardless of whether players have private or common values, or whether their costs of effort is the same or different, and we make no presumption about the players' private information. Moreover, our result extends to a general class of Tullock *like* contests which success function is formed as the ratio between the score given to a player's effort and the total scores given to all players, provided each player's score function is strictly increasing and concave. (Warneryd (2012) establishes existence of equilibrium for common value Tullock contests when there are two types of players, those that have complete information and those who only have the prior information, and investigates which players are active, i.e., make a positive effort, in equilibrium.)

Next we study Tullock contests in which players have a common value for the prize and a common state independent linear cost function, to which we refer simply as *common-value Tullock contests*. We consider first two-player common-value Tullock contests in which one of the players has an information advantage over his opponent (i.e., the partition of one player is finer than that of his opponent). In our framework,

[1] The payoff functions in the truncated contests are continuous, and concave in players' own strategies, which allows the use of the Nash's theorem. However, it cannot be applied to the original, untruncated, contest, since the payoffs in it have discontinuity when all efforts are equal to zero.

when one player has an information advantage it can be assumed, without loss of generality, that one player observes the value while the other player has only the prior information about the value. Two-player common-value Tullock with this extreme information asymmetry have been studied by Warneryd (2003) in a setting were the players common value is a continuous random variable. We reproduce in our framework some of Warneryd (2003)'s results: We show that such contests have a unique (pure strategy) Bayesian equilibrium, which we characterize. In equilibrium both players exert the same expected effort and have a positive expected payoff, although the payoff of the player with an information advantage is greater or equal to that of his opponent. Moreover, the player with an information advantage wins the prize less frequently (i.e., with a smaller ex-ante probability) than the uninformed player. We also examine how players information affects the effort they exert and their payoffs. Assuming that the distribution of the players' value for the prize is not too disperse, we show that when one player is better informed than the other the total effort exerted by the players is smaller, and thus the share of the total surplus they capture is larger, than when both players have the same information.

We proceed to study whether these results for two-players common-value Tullock contests extend to contests with more than two players. We show that information advantage in rewarded in equilibrium in contests with any number of players: in any equilibrium of a common-value Tullock contests, if a player has an information advantage over another player then the payoff of the former is greater or equal to that of the later. This result is obtained by observing the formal equivalence between a common-value Tullock contest and a oligopoly with asymmetric information, and using the theorem of Einy, Moreno and Shitovitz (2002) that shows that in any Cournot Bayesian equilibrium of an oligopolistic industry a firm's information advantage is rewarded. The other properties of equilibrium of two-player contests, however, do not extend to contests with more than two players. Specifically, we show a three-player example in which two of the players have symmetric information which is superior to that of the third player, where the expected efforts exerted by players differ. We also provide an example of a contest in which all but one player have the same information and the remaining player has an information advantage, in which the ex-ante probability that the player with information advantage wins the prize is greater than that of any of the other players.

We also study the relative effectiveness of Tullock contests and all-pay auctions to induce players to exert effort. In the same framework we work at, and under the same assumptions, Einy *et al.* (2013) characterize the unique equilibrium of a two-player common-value all-pay auctions, which is in mixed strategies, and show that the expected payoff of the player with an information advantage is positive while the expected payoff of his opponent is zero, and that both the expected effort and the ex-ante probability of winning the prize are the same for both players. Using the results in Einy *et al.* (2013) and our results we show that the sign of the difference in the total effort exerted by players in a Tullock contest and an all-pay auction is undetermined, and may be either positive or negative depending on the distribution of the players' value for the prize – see Example 1. (Fang (2002) and Epstein, Mealem and Nitzan (2011) study the outcomes of Tullock contests and all-pay auction under complete information.)

The rest of the paper is organized as follows: in Section 2 we describe the general setting. In Section 3 we establish that every Tullock contest has a pure strategy Bayesian equilibrium. Section 4 and 5 study common-value Tullock contests with two players, and with more players, respectively. Section 6 concludes. Long proofs are given in the Appendix.

2 Tullock Contests

A group of players $N = \{1, ..., n\}$, with $n \geq 2$, compete for a prize by choosing a level of *effort* in $\mathbb{R}_+$. Players' uncertainty about the state of nature is described by a probability space (Ω, p), where Ω is a finite set and p is a probability distribution over Ω describing the players' common prior belief about the realized state of nature. W.l.o.g. we assume that $p(\omega) > 0$ for every $\omega \in \Omega$. The private information about the state of nature of player $i \in N$ is described by a partition Π_i of Ω. The value for the prize for each player i is described by a random variable $V_i : \Omega \to \mathbb{R}_{++}$, i.e., if $\omega \in \Omega$ is realized then player i's ("private") value for the prize is $V_i(\omega)$. The cost of effort of each player $i \in N$ is described by a function $c_i : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$, which is continuously differentiable, strictly increasing and convex in effort x_i, and such that

$c_i(\cdot, 0) = 0$ on Ω.

A contest starts by a move of nature that selects a state ω from Ω according to the distribution p. Every player $i \in N$ observes the element $\pi_i(\omega)$ of Π_i which contains ω. Then players simultaneously choose their effort levels $(x_1, ..., x_n) \in \mathbb{R}^n_+$. The prize is awarded in a probabilistic fashion, according to a *success function* ρ, which attributed to each profile of effort levels $x \in \mathbb{R}^n_+$ a probability distribution $\rho(x)$ in the n-simplex according to which the prize recipient is chosen. Hence, the payoff of player $i \in N$, $u_i : \Omega \times \mathbb{R}^n_+ \to \mathbb{R}$, is given for every $\omega \in \Omega$ and $x \in \mathbb{R}^n_+$ by

$$u_i(\omega, x) = \rho_i (x) V_i(\omega) - c_i (\omega, x_i) . \tag{1}$$

Thus, a *contest* is described by a collection $(N, (\Omega, p), \{\Pi_i\}_{i \in N}, \{V_i\}_{i \in N}, \{c_i\}_{i \in N}, \rho)$.

In a contest, a pure strategy of player $i \in N$ is a Π_i-measurable function $X_i : \Omega \to \mathbb{R}_+$ (i.e., X_i is constant on every element of Π_i), that represents i's choice of effort in each state of nature following the observation of his private information. We denote by S_i the set of strategies of player i, and by $S = \prod_{i=1}^n S_i$ the set of strategy profiles. For any strategy $X_i \in S_i$ and $\pi_i \in \Pi_i$, $X_i (\pi_i)$ stands for the constant value that $X_i (\cdot)$ takes on π_i. Also, given a strategy profile $X = (X_1, ..., X_n) \in S$, we denote by X_{-i} the profile obtained from X by suppressing the strategy of player $i \in N$. Throughout the paper we restrict attention to pure strategies.

Let $X = (X_1, ..., X_n)$ be a strategy profile. We denote by $U_i(X)$ the expected payoff of player i, which is given by

$$U_i(X) \equiv E[u_i(\cdot, (X_1 (\cdot) , ..., X_n (\cdot))].$$

For $\pi_i \in \Pi_i$, we denote by $U_i(X \mid \pi_i)$ the expected payoff of player i conditional on π_i, i.e.,

$$U_i(X \mid \pi_i) \equiv E[u_i(\cdot, (X_1 (\cdot) , ..., X_n (\cdot)) \mid \pi_i].$$

An N-tuple of strategies $X^* = (X^*_1, ..., X^*_N)$ is a Bayesian equilibrium if for every player $i \in N$, and every strategy $X_i \in S_i$

$$U_i(X^*) \geq U_i(X^*_{-i}, X_i); \tag{2}$$

or equivalently,

$$U_i(X^* \mid \pi_i) \geq U_i(X^*_{-i}, X_i \mid \pi_i) \tag{3}$$

for every $\pi_i \in \Pi_i$.

3 Existence of Equilibrium in Tullock Contests

Tullock contests are identified by a class of success functions ρ^T such that for $x \in \mathbb{R}_+^n \backslash \{0\}$ the probability that player $i \in N$ wins the prize is

$$\rho_i^T(x) = \frac{x_i}{\bar{x}}, \tag{4}$$

where $\bar{x} \equiv \sum_{k=1}^N x_k$ is the total effort exerted by the players. Theorem 1 establishes that under our assumptions a Tullock contest has a pure strategy equilibrium.

Theorem 1. *Every Tullock contest has a (pure) strategy Bayesian equilibrium.*

Note that Theorem 1 makes no presumption about the players' private information, and applies regardless of whether players have private or common values, or whether their costs of effort is the same or different. A direct implication of Theorem 1 is the existence of equilibrium for a general class of success functions. For this class of success functions, Szidarovszky and Okuguchi (1997) have established existence of a unique equilibrium when players have complete information.

Corollary 1. *Every contest in which the success function ρ is given for $x \in \mathbb{R}_+^n \backslash \{0\}$ and $i \in N$ by*

$$\rho_i(x) = \frac{g_i(x_i)}{\sum_{j=1}^n g_j(x_j)},$$

where, for every $j \in N$, $g_j : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing and concave bijection[2]*, has a Bayesian equilibrium.*

Proof. Let $C = (N, (\Omega, p), \{\Pi_i\}_{i\in N}, \{V_i\}_{i\in N}, \{c_i\}_{i\in N}, \rho)$ be a contest satisfying the assumptions of Corollary 1 for $(g_1, ..., g_n)$. The Tullock contest $(N, (\Omega, p), \{\Pi_i\}_{i\in N}, \{V_i\}_{i\in N}, \{\bar{c}_i\}_{i\in N}, \rho^T)$ where $\bar{c}_i(\cdot, \cdot) = c_i\left(\cdot, g_i^{-1}(\cdot)\right)$ for every $i \in N$ and $\rho^T(0) = \rho(0)$ has a Bayesian equilibrium $X^* = (X_1^*, ..., X_n^*)$ by Theorem 1. It is easy to see that $Y^* = (g_1^{-1} \circ X_1^*, ..., g_n^{-1} \circ X_n^*)$ is a Bayesian equilibrium of C. ■

[2]Functions g_j do not, in fact, need to be bijections, for our claim to hold. This can be shown using the same argument as in the proof below, provided Theorem 1 is extended to hold for contests where the levels of effort are restricted to be in a set $[a, b)$ for $0 \le a \le b$ and $b \in \mathbb{R}_+ \cup \{\infty\}$ (under the additional assumption that $\lim_{x\to b} c_i(\cdot, x) = \infty$). This extension of Theorem 1 can be obtained by essentially the same proof as the one given in the Appendix.

4 Two-Player Common-Value Tullock Contests

Henceforth we study contests in which players have a common value for the prize and a common state-independent linear cost function, i.e., for all $i \in N$, $V_i = V$, and $c_i(\cdot, x) \equiv x$ on Ω. We refer to these contests as *common-value contests,* and they are described by a collection $(N, (\Omega, p), (\Pi_i)_{i \in N}, V, \rho)$. Let us index the set of states of nature as $\Omega = \{\omega_1, ..., \omega_m\}$, write $p(\omega_k) = p_k$ and $V(\omega_k) = v_k$ for $k \in \{1, ..., m\}$, and assume, w.l.o.g., that $0 < v_1 \leq v_2 < ... \leq v_m$.

In this section we study two-player common-value Tullock contests in which player 2 has an information advantage over player 1 (i.e., $n = 2$ and Π_2 is finer than Π_1). Thus, we may assume w.l.o.g. that the only information player 1 has about the state is the common prior belief, i.e., $\Pi_1 = \{\Omega\}$, whereas player 2 has perfect information about the state of nature, i.e., $\Pi_2 = \{\{\omega_1\}, ..., \{\omega_m\}\}$. In such contests a strategy profile is a pair (X, Y), where X can be identified with $x \in \mathbb{R}_+$ that specifies player 1's unconditional effort, and Y can be identified with $(y_1, ..., y_m) \in \mathbb{R}^m_+$ that specifies the effort of player 2 in each of the m states of nature. Thus, abusing notation, we shall write $X = x$ and $Y = (y_1, ..., y_m)$ whenever appropriate.

The following notation will be useful in characterizing the pure strategy Bayesian equilibria of a Tullock contest. For $k \in \{1, ..., m\}$ write

$$A_k = \left(\sum_{s=k}^{m} p_s \sqrt{v_s} \right) \left(1 + \sum_{s=k}^{m} p_s \right)^{-1}. \tag{5}$$

Note that

$$A_1 = \frac{E(\sqrt{V})}{2}.$$

Lemma 1 establishes a key property of the sequence $\{A_k\}_{k=1}^{m}$.

Lemma 1. *If* $\sqrt{v_{\bar{k}}} > A_{\bar{k}}$ *for some* $\bar{k} < m$, *then* $\sqrt{v_k} > A_k$ *and* $A_{\bar{k}} > A_k$ *for all* $k > \bar{k}$.

Proposition 1 shows that a two-player common-value Tullock contest in which player 2 has an information advantage has a unique pure strategy equilibrium with the following explicit description. Let $k^* \in \{1, ..., m\}$ be the smallest index such that $\sqrt{v_k} > A_k$. Since

$$\sqrt{v_m} > \frac{p_m}{(1 + p_m)} \sqrt{v_m} = A_m,$$

k^* is well defined.

Proposition 1. *A two-player common-value Tullock contest in which player 2 has an information advantage has a unique Bayesian equilibrium* (X^*, Y^*) *given by*

$$x^* = A_{k^*}^2,$$

$$y_k^* = 0$$

for all $k < k^*$*, and*

$$y_k^* = A_{k^*}\left(\sqrt{v_k} - A_{k^*}\right)$$

for all $k \geq k^*$.

Proposition 1 in particular implies uniqueness and symmetry of equilibrium in the complete information case, i.e., when $m = 1$. (Note that in this case $k^* = 1$, and therefore $y_1^* = A_1(\sqrt{v_1} - A_1) = v_1/2 - v_1/4 = A_1^2 = x^*$. This result is well known in the literature.) When $m > 1$, we have $\sqrt{v_1} > A_1 = E(\sqrt{V})/2$ (and hence $k^* = 1$) whenever the distribution of values is not too disperse; e.g., this inequality holds when $v_m < 4v_1$. When this is the case, the unique equilibrium is interior. For future references we state this observation in Remark 2.

Remark 2. *Consider a two-player common-value Tullock contest in which player 2 has an information advantage. The unique Bayesian equilibrium is interior if and only if* $\sqrt{v_1} > E(\sqrt{V})/2$*, i.e., the distribution of values is not too disperse.*

Interestingly, when one player has superior information the expected effort exerted by players in the equilibrium of the contest is the same.

Proposition 2. *In a two-player common-value Tullock contest in which player 2 has an information advantage both players exert the same (expected) effort, i.e.,*

$$E(Y^*) = A_{k^*}^2 = x^* = X^*. \tag{6}$$

Hence the expected total effort is

$$TE = X^* + E(Y^*) = 2A_{k^*}^2.$$

Proof. By Proposition 1,

$$\begin{aligned}
E(Y^*) &= \sum_{s=1}^{m} p_s y_s^* \\
&= \sum_{s=k^*}^{m} p_s A_{k^*}\left(\sqrt{v_k} - A_{k^*}\right) \\
&= A_{k^*} \sum_{s=k^*}^{m} p_s \sqrt{v_k} - A_{k^*}^2 \sum_{s=k^*}^{m} p_s \\
&= A_{k^*}^2 \left(1 + \sum_{s=k^*}^{m} p_s\right) - A_{k^*}^2 \sum_{s=k^*}^{m} p_s \\
&= A_{k^*}^2. \blacksquare
\end{aligned}$$

In a two-player common-value Tullock contest in which player 2 has an information advantage the equilibrium probabilities that player 1 wins the prize when the state is ω_k is

$$\rho_{1k}^* := \rho_1^T(x^*, y_k^*) = \frac{A_{k^*}^2}{A_{k^*}^2 + A_{k^*}\left(\sqrt{v_k} - A_{k^*}\right)} = \frac{A_{k^*}}{\sqrt{v_k}}$$

when $k \geq k^*$, whereas the probability that player 2 wins the prize is $\rho_{2k}^* = 1 - \rho_{1k}^*$. Thus, the larger is the realized value of the prize, the smaller (larger) is the probability that player 1 (player 2) wins the prize, i.e., $\rho_{1k'}^* \leq \rho_{1k}^*$ and $\rho_{2k'}^* \geq \rho_{2k}^*$ for $k' > k \geq k^*$, with a strict inequality if $v_{k'} > v_k$. Of course, the larger is the realized value of the prize, the larger is the effort of player 2, i.e.,

$$y_{k'}^* = A_{k^*}\left(\sqrt{v_{k'}} - A_{k^*}\right) \geq A_{k^*}\left(\sqrt{v_k} - A_{k^*}\right) = y_k^*.$$

for $k' > k \geq k^*$ (with a strict inequality if $v_{k'} > v_k$). Additionally, for $k' > k \geq k^*$,

$$\rho_{1k'}^* v_{k'} = A_{k^*}\sqrt{v_{k'}} \geq A_{k^*}\sqrt{v_k} = \rho_{1k}^* v_k$$

(with a strict inequality if $v_{k'} > v_k$), i.e., the larger is the realized value of the prize, the larger is the conditional expected payoff of player 1; also,

$$\rho_{2k'}^* v_{k'} \geq \rho_{2k}^* v_{k'} \geq \rho_{2k}^* v_k$$

(with a strict inequality if $v_{k'} > v_k$), i.e., the larger is the realized value of the prize, the larger is the conditional expected payoff of player 2. Write $\bar{\rho}_i^* = E(\rho_i^*)$ for the ex-ante probability that player i wins the prize. Proposition 3 establishes another interesting property of equilibrium.

Proposition 3. *Consider a two-player common-value Tullock contest in which player 2 has an information advantage. If $v_1 < v_2 < ... < v_m$, then the ex-ante probability that player 1 wins the prize is greater than that of player 2, i.e., $\bar{\rho}_1^* > \bar{\rho}_2^*$.*

Remark 3 states that under symmetric information each player exerts an expected effort equal to $E(V)/4$. The proof of this result is straightforward, and is therefore omitted.

Remark 3. *A two-player common-value Tullock contest in which players have symmetric information has a unique pure strategy equilibrium, which is symmetric and involves each player exerting an expected effort equal to $E(V)/4$.*

The surplus captured by the players in a contest is the difference between the expected (total) surplus $E(V)$ and the expected total effort they exert. In Proposition 4 below we show that when player 2 has an information advantage, in an interior equilibrium players exert less effort, and therefore capture a greater surplus, than when they are symmetrically informed.

Proposition 4. *Consider a two-player common-value Tullock contest in which player 2 has an information advantage. If $v_1 < v_m$ and the distribution of values is not too disperse, i.e., $\sqrt{v_1} > E(\sqrt{V})/2$, then the players' exert less effort and hence capture a greater share of the surplus than when both players have symmetric information.*

Proof. When player 2 has an information advantage, then $\sqrt{v_1} > E(\sqrt{V})/2$ implies that the equilibrium is interior by Remark 2, and therefore the expected total effort is $TE = 2A_1^2 = \left(E(\sqrt{V})\right)^2/2$ by Proposition 2. When players have symmetric information the expected total effort $\overline{TE}$ is $\overline{TE} = E(V)/2$ by Remark 3. Then $v_1 < v_m$ together with Jensen's inequality imply

$$\overline{TE} - TE = \frac{E(V)}{2} - \frac{\left(E(\sqrt{V})\right)^2}{2} > 0. \blacksquare$$

Warneryd (2003) establishes counterparts to Propositions 1 to 4 when the players' common-value V is a continuous random variable, and shows that the fully informed player obtains a greater payoff than the uninformed player. This latter result also holds when V is a discrete random variable, and, as it turns out, even outside the two-player case. Indeed, we will show in Theorem 2 in the next section that in a common-value Tullock contest with two or more players, when a player has an information advantage over another player (not necessarily an extreme one), then in any Bayesian equilibrium the payoff of the former is greater or equal to that of the latter.

We conclude this section studying what can be said about the players' expected total effort in all pay auctions and Tullock contests. The contests arising in many economic and political applications are effectively all pay auctions either by design (e.g., sports or political competition) or by the nature of the problem (e.g., a patent races).

A *common-value all-pay auction* is a common-value contest in which the success function is given for $x \in \mathbb{R}_+^n$ by $\rho^{APA}(x) = 1/m(x)$ if $x_i = \max\{x_j\}_{j \in N}$, and $\rho^{APA}(x) = 0$ otherwise, where $m(x) = |k \in N : x_k = \max\{x_j\}_{j \in N}|$. Einy *et. al.* (2013) show that in unique equilibrium of a two-player common-value all-pay auction in which $v_1 < ... < v_m$ and player 2 observes the value while player 1 does not, the players' total expected effort is

$$TE^{APA} = 2\sum_{s=1}^{m} p_s \left(\sum_{k=1}^{s-1} p_k v_k + \frac{1}{2} p_s v_s\right) = 2\sum_{s=1}^{m} p_s \sum_{k=1}^{s-1} p_k v_k + \sum_{s=1}^{m} p_s^2 v_s.$$

Hence the difference between total efforts in an all-pay auction and a Tullock contest is

$$\Delta := TE^{APA} - TE = 2\sum_{s=1}^{m} p_s \sum_{k=1}^{s-1} p_k v_k + \sum_{s=1}^{m} p_s^2 v_s - 2A_{k^*}^2.$$

For simplicity, consider the case where there are only two states of nature, i.e., $m = 2$. If the equilibrium of the Tullock contest is interior, then

$$\begin{aligned}
\Delta &= 2p_1 p_2 v_1 + \left(p_1^2 v_1 + p_2^2 v_2\right) - 2A_1^2 \\
&= 2p_1 p_2 v_1 + p_1^2 v_1 + p_2^2 v_2 - 2\frac{\left(p_1\sqrt{v_1} + p_2\sqrt{v_2}\right)^2}{4} \\
&= 2p_1 p_2 v_1 + \frac{1}{2}\left(p_1\sqrt{v_1} - p_2\sqrt{v_2}\right)^2 \\
&> 0.
\end{aligned}$$

Hence an all-pay auction generates more effort that a Tullock contest. However, if the Tullock contest has a corner equilibrium, then

$$\begin{aligned}
\Delta &= 2p_1p_2v_1 + \left(p_1^2v_1 + p_2^2v_2\right) - 2A_2^2 \\
&= 2p_1p_2v_1 + p_1^2v_1 + p_2^2v_2 - 2\frac{\left(p_2\sqrt{v_2}\right)^2}{(1+p_2)^2} \\
&= p_1v_1(1+p_2) - p_2^2v_2\left(\frac{2}{(1+p_2)^2} - 1\right).
\end{aligned}$$

Thus, Δ may be either positive or negative depending on the distribution of the players' common value – see Example 1 below. Hence the level of effort generated by these two contests cannot be ranked in general.

The following example illustrates our findings.

Example 1. Let $m = 2$, $p_1 = 1 - p$, $v_1 = 1$, and $v_2 = v$, where $p \in (0,1)$ and $v \in (1,\infty)$. Then $E(V) = 1 - p(1-v)$, $E(\sqrt{V}) = 1 - p(1-\sqrt{v})$, $A_1 = E(\sqrt{V})/2$, and $A_2 = p\sqrt{v}/(1+p)$. If $v < (1+p)^2/p^2$, then $\sqrt{v_1} = 1 > A_1$ and $k^* = 1$; otherwise $k^* = 2$.

In a Tullock contest in which player 2 observes the value but player 1 does not, the unique equilibrium is

$$X^* = A_1^2,\ Y^* = \left(A_1\left(1 - A_1\right), A_1\left(\sqrt{v} - A_1\right)\right),$$

and the total effort is $TE = 2A_1^2 = [1 - p(1-\sqrt{v})]^2/2$ when $v < (1+p)^2/p^2$. Otherwise, the unique equilibrium is

$$X^* = A_2^2,\ Y^* = \left(0, A_2\left(\sqrt{v} - A_2\right)\right),$$

and the total effort is $TE = 2A_2^2 = 2p^2v/(1+p)^2$. If $v < (1+p)^2/p^2$, then the ex-ante probability that player 1 wins the prize is

$$\bar{\rho}_1^* = (1-p)A_1 + p\frac{A_1}{\sqrt{v}} = \frac{1}{2}\left(p + (1-p)\sqrt{v}\right)\frac{1 - p + p\sqrt{v}}{\sqrt{v}} \geq \frac{1}{1+p} > \frac{1}{2}.$$

Otherwise, this probability is

$$\bar{\rho}_1^* = (1-p) + p\frac{A_2}{\sqrt{v}} = (1-p) + \frac{p^2}{1+p} = \frac{1}{1+p} > \frac{1}{2}.$$

Hence, consistently with Proposition 3 the uninformed player wins the prize more frequently than the informed player. Further, if $v < (1+p)^2/p^2$, then

$$\begin{aligned}
2\left[U_2(X^*,Y^*) - U_1(X^*,Y^*)\right] &= (1-p)\frac{A_1(1-A_1)-A_1^2}{A_1^2+A_1(1-A_1)} + pv\frac{A_1(\sqrt{v}-A_1)-A_1^2}{A_1^2+A_1(\sqrt{v}-A_1)} \\
&= (1-p)\,p\left(1-\sqrt{v}\right)^2 \\
&> 0.
\end{aligned}$$

And if $v \geq (1+p)^2/p^2$, then

$$\begin{aligned}
2\left[U_2(X^*,Y^*) - U_1(X^*,Y^*)\right] &= -(1-p) + pv\frac{A_2(\sqrt{v}-A_2)-A_2^2}{A_2^2+A_2(\sqrt{v}-A_2)} \\
&= \frac{1-p}{p+1}\left(p(v-1)-1\right) \\
&> \frac{1-p}{p} \\
&> 0.
\end{aligned}$$

That is, the payoff of the informed player is greater or equal to that of the uninformed player. (We show in Theorem below the information advantage is always rewarded in a common-value Tullock contest, regardless of the number of players and the number of states of nature.)

Under symmetric information the equilibrium total effort in a Tullock contest is $E(V)/2 > \max\{2A_1^2, 2A_2^2\}$, i.e., the total effort when player 2 has an information advantage is less than when both players have the same information.

In an all pay auction in which player 2 observes the value but player 1 does not, the equilibrium total effort is

$$TE^{APA} = 2(1-p)\,p + (1-p)^2 + p^2v = (1-p)(1+p) + p^2v.$$

As we have shown above, if $v < (1+p)^2/p^2$, then the expected total effort in the unique equilibrium of the Tullock contest, which is interior, is $TE = 2A_2^2 = [1-p(1-\sqrt{v})]^2/2$. Hence

$$TE^{APA} - TE = (1-p)(1+p) + p^2v - \frac{\left(1-p(1-\sqrt{v})\right)^2}{2} > 2p > 0.$$

However, if $v \geq (1+p)^2/p^2$, then the expected total effort in the unique equilibrium of the Tullock contest, which is a corner equilibrium, is $TE = 2p^2v/(1+p)^2$. Assume that $p = 1/4$. Then

$$TE^{APA} - TE = \frac{15}{16} - \frac{7}{400}v.$$

Hence $TE^{APA} < TE$ for $v > 375/7$.

5 n-Player Common-Value Tullock Contests

In this section we study whether the properties of two-player common-value Tullock contests extend to contests with more than two players. We begin by establishing in Theorem 2 a general property of common-value Tullock contests: these contests reward information advantage. Theorem 2 is a direct implication of the theorem of Einy, Moreno and Shitovitz (2002).

Theorem 2. *Let $X^* = (X_1^*, ..., X_n^*)$ be any equilibrium of an n-player common-value Tullock contest. If player i has an information advantage over player j, then $U_i(X^*) \geq U_j(X^*)$.*

Proof. An n-player common-value Tullock contest $(N, (\Omega, p), (\Pi_i)_{i\in N}, V)$ is formally identical to what Einy, Moreno and Shitovitz (2002) refer to as an oligopolist industry $(N, (\Omega, p), P, c, (\Pi_i)_{i\in N})$, where the demand and cost functions are defined for $(\omega, x) \in \Omega \times \mathbb{R}_{++}$ as

$$P(\omega, x) = \frac{V(\omega)}{x},$$

and

$$c(\omega, x) = x,$$

respectively. With this convention, the state-dependent profit of firm $i \in N$ in the industry coincides with the payoff of player $i \in N$ in the contest, i.e., for $\omega \in \Omega$ and $X \in S$,

$$\begin{aligned} u_i(\omega, X) &= \frac{V(\omega)}{\sum_{s=1}^n X_s} X_i(\omega) - X_i(\omega) \\ &= P(\omega, \sum_{s=1}^n X_s(\omega)) X_i(\omega) - c(\omega, X_i(\omega)). \end{aligned}$$

Theorem 2 then follows from the theorem of Einy, Moreno and Shitovitz (2002).[3] ■

[3] The demand function $P(\omega, x)$ is not differentiable at $x = 0$ – it is not even defined – and therefore

The following example shows that Proposition 2 does not extend to common-value Tullock contests with more than two players. In the example, player 1 has only prior information whereas players 2 and 3 have complete information. In equilibrium the expected effort of the uninformed player is below that of each of the informed players.

Example 2. Consider a 3-player common-value Tullock contest in which $m = 2$, $p_1 = p_2 = 1/2$, $v_1 = 1$ and $v_2 = 2$. Player 1 has no information, i.e., his information partition is $\Pi_1 = \{\omega_1, \omega_2\}$, and players 2 and 3 have complete information, i.e., their information partitions are $\Pi_2 = \Pi_3 = \{\{\omega_1\}, \{\omega_2\}\}$. In the interior equilibrium of this contest, which is readily calculated by solving the system of equations formed by the players' reaction functions, the effort of player 1 is $X_1^* = 0.30899$ while the efforts of players 2 and 3 are $X_2^* = X_3^* = (0.20342, 0.46933)$. Note that

$$X_1^* = 0.30899 < \frac{1}{2}(0.20342 + 0.46933) = E(X_2^*) = E(X_3^*),$$

i.e., the effort of player 1 is less than the expected effort of players 2 and 3.

The next example shows that Proposition 3 does not extend to contests with more than two players. In the example there is an informed player and a number of uninformed players. Contrary to the spirit of Proposition 3, the ex-ante probability that the informed player wins the prize is above that of the uninformed players.

Example 3. Consider an eight player common-value Tullock contest in which $m = 2$, $p_1 = p_2 = 1/2$, $v_1 = 1$ and $v_2 = 2$. Players 1 to 7 have no information, i.e., their information partition is $\Pi_i = \{\omega_1, \omega_2\}$ for $i \in \{1, ..., 7\}$, and player 8 is completely informed, i.e., his information partitions is $\Pi_8 = \{\{\omega_1\}, \{\omega_2\}\}$. This contest has a (corner) equilibrium given by

$$X_1^* = ... = X_7^* = 0.15551, \ X_8^* = (0, 0.38694).$$

In equilibrium, the ex-ante probability that player $i \in \{1, 2, ..., 7\}$ wins the prize is

$$\bar{\rho}_i^* = \frac{1}{2}(\frac{1}{7} + \frac{0.155\,51}{7(0.155\,51) + 0.386\,94}) = 0.12413,$$

does not formally satisfy the assumptions of Einy, Moreno and Shitovitz (2002). However, it is easy to see that in any equilibrium X of a common-value Tullock contest the total effort is positive in all states of nature, i.e., $\bar{X}(\cdot) > 0$. Thus the non-differentiability at 0 is irrelevant, and the proof of the theorem in Einy, Moreno and Shitovitz (2002) applies in this case with no change.

whereas the ex-ante probability that player 8 win the prize is

$$\bar{\rho}_8^* = 1 - 7(0.12413) = 0.13109.$$

Thus, the informed player wins the prize more frequently than an uninformed player.

6 Concluding remarks

Under broad conditions, Tullock contests have pure strategy equilibria. Two-player common-value Tullock contests in which one player has an information advantage exhibit interesting properties: an equilibrium is unique, although it may not be interior. And regardless of whether the equilibrium is interior or not, both players exert the same expected effort, although the player with an information advantage obtains a payoff greater or equal to his opponent, and wins the object less frequently than him. When the equilibrium is interior, which occurs when the distribution of the players' common value is not too disperse, the players exert less effort than when they are symmetrically informed. (It is an open question whether this property holds when the distribution of values is sufficiently disperse and the unique equilibrium is a corner equilibrium.) While the information advantage is rewarded in common-value Tullock contests regardless of whether there are two or more players, the other properties of equilibrium obtained for two-player contests may not hold in contests with more than two players. Interestingly, a Tullock contest may generate more effort than an all-pay auction.

7 Appendix

Proof of Theorem 1. Let $C = (N, (\Omega, p), \{\Pi_i\}_{i\in N}, \{V_i\}_{i\in N}, \{c_i\}_{i\in N}, \rho^T)$ be a Tullock contest. Since the cost function of each player is strictly increasing and convex in the player's effort, it follows from (1) that there exists $Q > 0$ such that $u_i(\cdot, x) < 0$ for every $i \in N$ and every $x \in \mathbb{R}^n_+$, provided $x_i > Q$. For any $0 < \varepsilon < Q$ consider a variant of the contest, denoted by C_ε, in which the effort set of each player i is restricted to be the bounded interval $[\varepsilon, Q]$. In C_ε, the set of strategies of player i, $S_{i,\varepsilon}$, is identifiable with the compact set $[\varepsilon, Q]^{\Pi_i}$ via the the bijection $\mathbf{x}_i \longleftrightarrow (\mathbf{x}_i(\pi_i))_{\pi_i\in\Pi_i}$. Player i's expected payoff function U_i is continuous on $S_\varepsilon = \times_{i=1}^n S_{i,\varepsilon}$ (since the success function ρ in (4) is continuous if efforts are restricted to $[\varepsilon, Q]$), and it is concave in i's own strategy (as the state-dependent payoff function $u_i(\cdot, x)$ is concave in the variable x_i if efforts are restricted to $[\varepsilon, Q]$). Nash's Theorem thus guarantees existence of a Bayesian equilibrium in C_ε; pick one such equilibrium and denote it by $X^*_\varepsilon = (X^*_{1,\varepsilon}, ..., X^*_{n,\varepsilon})$.

We show that

$$\lim_{\varepsilon \to 0+} \inf \bar{X}^*_\varepsilon (\cdot) > 0.$$

Indeed, suppose to the contrary that there is a vanishing positive sequence $\{\varepsilon_k\}_{k=1}^\infty$ such that

$$\lim_{k \to \infty} \min_{\omega \in \Omega} \bar{X}^*_{\varepsilon_k} (\omega) = 0, \tag{7}$$

and fix $\omega^* \in \Omega$ such that

$$\bar{X}^*_{\varepsilon_k} (\omega^*) = \min_{\omega \in \Omega} \bar{X}^*_{\varepsilon_k} (\omega) \tag{8}$$

for infinitely many k (and thus, w.l.o.g., for every k). Since the expected payoff of player i is negative in every state of nature when $x_i = Q$, for any sufficiently small ε_k the equilibrium strategy X^*_{i,ε_k} satisfies $X^*_{i,\varepsilon_k} (\cdot) < Q$. Thus, for a given $\pi_i \in \Pi_i$, $X^*_i (\pi_i) \in [\varepsilon_k, Q)$. Additionally, X_i and $X_i (\pi_i)$ can both be viewed as the argument of the function $U_i(X^*_{-i,\varepsilon_k}, X_i \mid \pi_i)$, since $X_i (\pi_i)$ is the only numerical input needed to determine the conditional expected payoff of player i given π_i, when the equilibrium strategies of players other than i are X^*_{-i,ε_k}. Since the equilibrium strategy X^*_{i,ε_k} is a (local) maximizer of $U_i(X^*_{-i,\varepsilon_k}, X_i \mid \pi_i)$ by (3),

$$\left. \frac{dU_i(X^*_{-i,\varepsilon_k}, X_i, \mid \pi_i)}{dX_i (\pi_i)} \right|_{X_i(\pi_i) = X^*_{i,\varepsilon_k}(\pi_i)} \leq 0.$$

That is,

$$\left. \frac{dE[u_i(\cdot, X^*_{-i,\varepsilon_k} (\cdot), X_i (\pi_i) \mid \pi_i]}{dX_i (\pi_i)} \right|_{X_i(\pi_i) = X^*_{i,\varepsilon_k}(\pi_i)} \leq 0,$$

or, equivalently,

$$E \left[\frac{du_i(\cdot, X^*_{-i,\varepsilon_k} (\cdot), X^*_{i,\varepsilon_k} (\pi_i))}{dx_i} \mid \pi_i \right] \leq 0.$$

Using (4) and (1) we calculate the derivative explicitly,

$$E \left[\frac{V_i(\cdot)}{\bar{X}^*_{\varepsilon_k} (\cdot)} - \frac{X^*_{i,\varepsilon_k} (\pi_i) V_i(\cdot)}{\bar{X}^*_{\varepsilon_k} (\cdot)^2} - \frac{d}{dx_i} c_i \left(\cdot, X^*_{i,\varepsilon_k} (\pi_i)\right) \mid \pi_i \right] \leq 0.$$

Thus

$$E\left[\frac{V_i(\cdot)}{\bar{X}^*_{\varepsilon_k}(\cdot)}-\frac{d}{dx_i}c_i\left(\cdot,X^*_{i,\varepsilon_k}(\pi_i)\right)\mid\pi_i\right]-X^*_{i,\varepsilon_k}(\pi_i)\,E\left[\frac{V_i(\cdot)}{\bar{X}^*_{\varepsilon_k}(\cdot)^2}\mid\pi_i\right]\le 0,$$

which leads to

$$X^*_{i,\varepsilon_k}(\pi_i)\ge\frac{E\left[\frac{V_i(\cdot)}{\bar{X}^*_{\varepsilon_k}(\cdot)}-\frac{d}{dx_i}c_i\left(\cdot,X^*_{i,\varepsilon_k}(\pi_i)\right)\mid\pi_i\right]}{E\left[\frac{V_i(\cdot)}{\bar{X}^*_{\varepsilon_k}(\cdot)^2}\mid\pi_i\right]}.\tag{9}$$

Inequality (9) holds, in particular, for $\pi_i=\pi_i(\omega^*)$. Since $X^*_{i,\varepsilon_k}(\omega^*)=X^*_{i,\varepsilon_k}(\pi_i(\omega^*))$ (as, by definition, $\omega^*\in\pi_i(\omega^*)$), (9) yields

$$X^*_{i,\varepsilon_k}(\omega^*)\ge\frac{E\left[\frac{V_i(\cdot)}{\bar{X}^*_{\varepsilon_k}(\cdot)}-\frac{d}{dx_i}c_i\left(\cdot,X^*_{i,\varepsilon_k}(\omega^*)\right)\mid\pi_i(\omega^*)\right]}{E\left[\frac{V_i(\cdot)}{\bar{X}^*_{\varepsilon_k}(\cdot)^2}\mid\pi_i(\omega^*)\right]}.\tag{10}$$

Summing over $i\in N$ we obtain

$$\bar{X}^*_{\varepsilon_k}(\omega^*)\ge\sum_{i=1}^{n}\frac{E\left[\frac{V_i(\cdot)}{\bar{X}^*_{\varepsilon_k}(\cdot)}-\frac{d}{dx_i}c_i\left(\cdot,X^*_{i,\varepsilon_k}(\omega^*)\right)\mid\pi_i(\omega^*)\right]}{E\left[\frac{V_i(\cdot)}{\bar{X}^*_{\varepsilon_k}(\cdot)^2}\mid\pi_i(\omega^*)\right]},$$

or (since $\bar{X}^*_{\varepsilon_k}(\omega^*)\ge n\varepsilon>0$)

$$1\ge\sum_{i=1}^{n}\frac{E\left[\frac{\bar{X}^*_{\varepsilon_k}(\omega^*)}{\bar{X}^*_{\varepsilon_k}(\cdot)}V_i(\cdot)-\bar{X}^*_{\varepsilon_k}(\omega^*)\,\frac{d}{dx_i}c_i\left(\cdot,X^*_{i,\varepsilon_k}(\omega^*)\right)\mid\pi_i(\omega^*)\right]}{E\left[\frac{\bar{X}^*_{\varepsilon_k}(\omega^*)^2}{\bar{X}^*_{\varepsilon_k}(\cdot)^2}V_i(\cdot)\mid\pi_i(\omega^*)\right]}.\tag{11}$$

By the definition of $\bar{X}^*_{\varepsilon_k}(\omega^*)$ (see (8)),

$$0\le\frac{\bar{X}^*_{\varepsilon_k}(\omega^*)}{\bar{X}^*_{\varepsilon_k}(\omega)}\le 1$$

for every $\omega\in\Omega$. Hence we assume w.l.o.g. (by moving to a subsequence if necessary) that the limit

$$a(\omega)=\lim_{k\to\infty}\frac{\bar{X}^*_{\varepsilon_k}(\omega^*)}{\bar{X}^*_{\varepsilon_k}(\omega)}$$

exists for every $\omega \in \Omega$. Note also that $a(\omega) = 1$ for $\omega = \omega^*$, which occurs with positive probability by our assumption on p, and thus

$$\lim_{k\to\infty} E\left[\frac{\bar{X}^*_{\varepsilon_k}(\omega^*)^2}{\bar{X}^*_{\varepsilon_k}(\cdot)^2} V_i(\cdot) \mid \pi_i(\omega^*)\right] = E\left[a(\cdot)^2 V_i(\cdot) \mid \pi_i(\omega^*)\right] > 0. \tag{12}$$

Also, (7) and (8) imply

$$\lim_{k\to\infty} E\left[\bar{X}^*_{\varepsilon_k}(\omega^*) \frac{dc_i\left(\cdot, X^*_{i,\varepsilon_k}(\omega^*)\right)}{dx_i} \mid \pi_i(\omega^*)\right] = 0. \tag{13}$$

Taking limit of the right-hand side of (11), which exists by (12) and (13), we get

$$1 \geq \sum_{i=1}^{n} \frac{E[a(\cdot) V_i(\cdot) \mid \pi_i(\omega^*)]}{E[a(\cdot)^2 V_i(\cdot) \mid \pi_i(\omega^*)]}.$$

Furthermore, as $0 \leq a(\cdot)^2 \leq a(\cdot) \leq 1$, we obtain

$$1 \geq \sum_{i=1}^{n} \frac{E[a(\cdot) V_i(\cdot) \mid \pi_i(\omega^*)]}{E[a(\cdot)^2 V_i(\cdot) \mid \pi_i(\omega^*)]} \geq n.$$

Since by assumption $n \geq 2$, we have reached a contradiction. This proves that, indeed,

$$\liminf_{\varepsilon\to 0+} \bar{X}^*_{\varepsilon}(\cdot) > 0. \tag{14}$$

Now let $\{\varepsilon^*_k\}_{k=1}^{\infty}$ be a vanishing positive sequence such that the limit

$$X^*_i(\omega) \equiv \lim_{k\to\infty} X^*_{i,\varepsilon_k}(\omega)$$

exists for every $i \in N$ and $\omega \in \Omega$. (Such a sequence exists since all $X^*_{i,\varepsilon}(\omega)$ belong to the compact interval $[0, Q]$.) Obviously, $X^* = (X^*_1, ..., X^*_n)$ constitutes a strategy profile in the contest C, and it follows from (14) that

$$\bar{X}^*(\cdot) > 0. \tag{15}$$

We show that X^* is a Bayesian equilibrium of C.

Since the state-dependent payoff function $u_i(\cdot, x)$ is continuous at any point x with $\bar{x} > 0$, for every $i \in N$, every $\pi_i \in \Pi_i$, and every sequence $\{Y_k\}_{k=0}^{\infty}$ of strategy profiles such that $Y_0(\cdot) > 0$ and $Y_{i,0}(\omega) = \lim_{k\to\infty} Y_{i,k}(\omega)$ for every i and ω, we have

$$\lim_{k\to\infty} U_i(Y_{1,k}, ..., Y_{n,k} \mid \pi_i) = U_i(Y_{1,0}, ..., Y_{n,0} \mid \pi_i). \tag{16}$$

Since every X_ε^* is a Bayesian equilibrium in C_ε, for every sufficiently large k and every strategy X_i of player i satisfying $0 < X_i(\cdot) \leq Q$ we have

$$U_i(X^*_{\varepsilon_k^*} \mid \pi_i) \geq U_i(X^*_{-i,\varepsilon_k^*}, X_i \mid \pi_i). \tag{17}$$

Applying the limit as $k \to \infty$ to both sides of inequality (17), it follows from (16) (and the fact (15)) that

$$U_i(X^* \mid \pi_i) \geq U_i(X^*_{-i}, X_i \mid \pi_i) \tag{18}$$

for every strategy X_i of player i satisfying $0 < X_i(\cdot) \leq Q$ and every $\pi_i \in \Pi_i$.

It is easy to see that

$$\lim \inf_{x_i \to 0+} U_i(X^*_{-i}, x_i \mid \pi_i) \geq U_i(X^*_{-i}, 0 \mid \pi_i),$$

where $x_i > 0$ (respectively, $x_i = 0$) is identified with a strategy of i for which $X_i(\pi_i) = x_i$ (respectively, $X_i(\pi_i) = 0$). Thus (18) in fact holds for every strategy X_i satisfying $0 \leq X_i(\cdot) \leq Q$ (i.e., the deviations of i may be zero at some states of nature).

Finally, note that player i can improve upon any strategy X_i for which $X_i(\omega) > Q$ at some ω by lowering the effort on $\pi_i(\omega)$ to zero and thus receiving non-negative expected payoff conditional on $\pi_i(\omega)$. Thus, in contemplating a unilateral deviation from X_i^*, player i is never worse off by limiting himself to strategies X_i satisfying $0 \leq X_i(\cdot) \leq Q$. But this implies that (18) holds for every strategy $X_i \in S_i$. Since this is the case for every $i \in N$, we have shown that X^* is a Bayesian equilibrium of C. ■

Proof of Lemma 1. Assume that $\sqrt{v_{\bar{k}}} > A_{\bar{k}}$ for some $\bar{k} < m$.

We show that $\sqrt{v_k} > A_k$ for all $k > \bar{k}$. Suppose not; let $\hat{k} > \bar{k}$ be the first index $k > \bar{k}$ such that for $\sqrt{v_k} \leq A_k$. Note that $v_{\hat{k}} \geq v_{\hat{k}-1}$ and $\sqrt{v_{\hat{k}-1}} > A_{\hat{k}-1}$ imply

$$\begin{aligned}
\left(1 + \sum_{s=\hat{k}}^{m} p_s\right)\sqrt{v_{\hat{k}}} &\geq \left(1 + \sum_{s=\hat{k}-1}^{m} p_s\right)\sqrt{v_{\hat{k}-1}} - p_{\hat{k}-1}\sqrt{v_{\hat{k}-1}} \\
&> \left(1 + \sum_{s=\hat{k}-1}^{m} p_s\right) A_{\hat{k}-1} - p_{\hat{k}-1}\sqrt{v_{\hat{k}-1}} \\
&= \sum_{s=\hat{k}-1}^{m} p_s\sqrt{v_s} - p_{\hat{k}-1}\sqrt{v_{\hat{k}-1}} \\
&= \left(1 + \sum_{s=\hat{k}}^{m} p_s\right) A_{\hat{k}},
\end{aligned}$$

which contradicts the assumption that $\sqrt{v_{\hat{k}}} \leq A_{\hat{k}}$.

Now we show that $A_{\bar{k}} > A_k$ for all $k > \bar{k}$. Suppose not; let $\tilde{k} > \bar{k}$ be the first index $k > \bar{k}$ such that $A_{\bar{k}} \leq A_k$. Since $\sqrt{v_{\tilde{k}-1}} > A_{\tilde{k}-1}$ (as we have just shown), then

$$\begin{aligned}\left(1 + \sum_{s=\tilde{k}-1}^{m} p_s\right) A_{\tilde{k}-1} &= \sum_{s=\tilde{k}-1}^{m} p_s \sqrt{v_s} \\ &= p_{\tilde{k}-1}\sqrt{v_{\tilde{k}-1}} + \sum_{s=\tilde{k}}^{m} p_s \sqrt{v_s} \\ &> p_{\tilde{k}-1} A_{\tilde{k}-1} + \left(1 + \sum_{s=\tilde{k}}^{m} p_s\right) A_{\tilde{k}}.\end{aligned}$$

Hence

$$\left(1 + \sum_{s=\tilde{k}-1}^{m} p_s\right) A_{\tilde{k}-1} - p_{\tilde{k}-1} A_{\tilde{k}-1} > \left(1 + \sum_{s=\tilde{k}}^{m} p_s\right) A_{\tilde{k}},$$

i.e.,

$$\left(1 + \sum_{s=\tilde{k}}^{m} p_s\right) A_{\tilde{k}-1} > \left(1 + \sum_{s=\tilde{k}}^{m} p_s\right) A_{\tilde{k}}.$$

Thus, $A_{\bar{k}} \geq A_{\tilde{k}-1} > A_{\tilde{k}}$, which contradicts the choice of $\tilde{k}$. ■

Proof of Proposition 1. Let (X, Y), where $X = x$ and $Y = (y_1, ..., y_m)$, be a Bayesian equilibrium, whose existence is guaranteed by Theorem 1. We show that $x > 0$. If $x = 0$, then $\rho_2^T(0) = 1$, since otherwise player 2 does not have a best response against $x = 0$. But then $y_1 = y_2 = ... = y_m = 0$, and therefore player 1 can profitably deviate by exerting an arbitrarily small effort $\varepsilon > 0$. Hence $x > 0$. Moreover, $y_k > 0$ for some $k \in \{1, ..., m\}$ since otherwise $x > 0$ is not a best response of player 1.

Since $x > 0$ maximizes player 1's payoff given Y, then

$$\frac{\partial}{\partial x}\left(\sum_{s=1}^{m} p_s \left(v_s \frac{x}{x + y_s} - x\right)\right) = \sum_{s=1}^{m} p_s v_s \frac{y_s}{(x + y_s)^2} - 1 = 0. \tag{19}$$

And since y_s maximizes player 2's payoff in state ω_s given x, then

$$\frac{\partial}{\partial y_s}\left(v_s \frac{y_s}{x + y_s} - y_s\right) = v_s \frac{x}{(x + y_s)^2} - 1 \leq 0, \tag{20}$$

(with equality if $y_s > 0$) for each $s = 1, ..., m$.

Notice next that if $y_k > 0$ for some $k < m$, then $y_{k'} > 0$ for all $k' > k$. Since $x > 0$, if $y_k > 0$ then $y_k = \sqrt{x}\left(\sqrt{v_k} - \sqrt{x}\right)$ by (20), and since $v_{k'} \geq v_k$ for all $k' > k$, $\sqrt{x}\left(\sqrt{v_{k'}} - \sqrt{x}\right) > 0$, i.e.,

$$v_{k'}\frac{x}{x^2} - 1 > 0,$$

for all $k' > k$. Then $y_{k'} = 0$ would violate inequality (20) for $s = k'$. Hence $y_{k'} > 0$.

Let k° be the smallest index such that $y_k > 0$. Thus, $x > 0$ and (19) imply

$$\sum_{s=1}^{m} p_s v_s \frac{y_s}{(x + y_s)^2} = \sum_{s=k^\circ}^{m} p_s v_s \frac{y_s}{(x + y_s)^2} = 1,$$

and (20) implies $y_{k'} = \sqrt{x}\left(\sqrt{v_{k'}} - \sqrt{x}\right) > 0$ for all $k' \geq k^\circ$. Hence $x = A_{k^\circ}^2$, $y_k = A_{k^\circ}\left(\sqrt{v_k} - A_{k^\circ}\right)$ for all $k \geq k^\circ$, and $y_k = 0$ for all $k < k^\circ$.

We now show that $k^\circ = k^*$, which establishes that the profile $(x^*, y_1^*, ..., y_m^*)$ identified in Proposition 1 is the unique equilibrium. Assume first that $k^\circ < k^*$. Then $\sqrt{v_{k^\circ}} \leq A_{k^\circ}$ since k^* is the smallest index such that $\sqrt{v_k} > A_k$, and hence $y_{k^\circ} = \sqrt{x}\left(\sqrt{v_{k^\circ}} - \sqrt{x}\right) = A_{k^\circ}\left(\sqrt{v_{k^\circ}} - A_{k^\circ}\right) \leq 0$, a contradiction as $y_{k^\circ} > 0$ by the definition of k°. Assume next that $k^\circ > k^*$. In this case, $y_{k^*} = 0$. Since $\sqrt{v_{k^*}} > A_{k^*}$, by Lemma 1

$$A_{k^*}^2 > A_{k^\circ}^2 = x, \tag{21}$$

and therefore

$$v_{k^*}\frac{x}{x^2} - 1 = \frac{A_{k^\circ}^2}{A_{k^\circ}^4}\left(v_{k^*} - A_{k^\circ}^2\right) > 0.$$

This stands in contradiction to (20), as $y_{k^*} = 0$ by the definition of $k^\circ (> k^*)$. We conclude that indeed $k^\circ = k^*$. ■

Proof of Proposition 3. Let us be given a two-player common-value Tullock contest in which player 2 has an information advantage over player 1. Given $(y_{k^*}, ..., y_m) \in \mathbb{R}_+^{k^*}$ define the function

$$\bar{p}_2\left(y_{k^*}, ..., y_m\right) := \sum_{k=k^*}^{m} \frac{p_k y_k}{y_k + \sum_{s=k^*}^{m} p_s y_s}.$$

Hence, recalling (6), $\bar{\rho}_2 = \bar{p}_2\left(y_{k^*}^*, ..., y_m^*\right)$. We show that a maximum point $\overline{y}$ of $\bar{p}_2$ on $K = \left\{(y_{k^*}, ..., y_m) \in \mathbb{R}_+^{k^*} \mid y_{k*} \leq y_{k*+1} ... \leq y_m\right\}$ must satisfy $\overline{y}_{k*} = ... = \overline{y}_m$. Hence

$$\max_K \bar{p}_2 = \frac{\sum_{s=k^*}^{m} p_s}{1 + \sum_{s=k^*}^{m} p_s} \leq \frac{1}{2}. \tag{22}$$

Since $y^*_{k^*} < ... < y^*_m$ (the inequalities are strict, which follows from our assumption that $v_1 < v_2 < ... < v_m$ and the expressions for $(y^*_k)^m_{k=k^*}$ given in Proposition 1), (22) implies

$$\bar{\rho}_2 = \bar{p}_2 \left(y^*_{k^*}, ..., y^*_m\right) < \max_K \bar{p}_2 \leq 1/2,$$

which establishes Proposition 3.

Differentiating $\bar{p}_2$ with respect to y_k for $k \in \{k^*, ..., m\}$ we get

$$\frac{\partial \bar{p}_2}{\partial y_k} = p_k \left(\sum_{t=k^*, t\neq k}^{m} \frac{p_t y_t}{\left(y_k + \sum_{s=k^*}^m p_s y_s\right)^2} - \sum_{t=k^*, t\neq k}^{m} \frac{p_t y_t}{\left(y_t + \sum_{s=k^*}^m p_s y_s\right)^2} \right). \tag{23}$$

For every $(y_{k^*}, ..., y_m) \in K$ such that $y_{k^*} < y_{k^*+1} \leq ... \leq y_m$, $\partial \bar{p}_2 / \partial y_{k^*} (y) > 0$, and therefore necessarily $\overline{y}_{k^*} = \overline{y}_{k^*+1}$. Suppose now that it has already been shown that $\overline{y}_{k^*} = \overline{y}_{k^*+1} = ... = \overline{y}_k$, $m-1 \geq k > 1$. We show that $\overline{y}_{k+1} = \overline{y}_k$ as well. Indeed, if $\overline{y}_{k^*} = \overline{y}_{k^*+1} = ... = \overline{y}_k < \overline{y}_{k+1} \leq ... \leq \overline{y}_m$, then by (23) we obtain that $\partial \bar{p}_2 / \partial y_k (\overline{y}) > 0$, a contradiction. Thus $\overline{y}_{k^*} = ... = \overline{y}_m$. ■

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