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PARTICIPATION CONSTRAINTS**

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# Allocation of Prizes in Contests with Participation Constraints

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## Abstract

We study all-pay contests with an exogenous minimal effort constraint where a player can participate in a contest only if his effort (output) is equal to or higher than the minimal effort constraint. Contestants are privately informed about a parameter (ability) that affects their cost of effort. The designer decides about the size and the number of prizes. We analyze the optimal prize allocation for the contest designer who wishes to maximize either the total effort or the highest effort. It is shown that if the minimal effort constraint is relatively high, the *winner-take-all* contest in which the contestant with the highest effort wins the entire prize sum does not maximize the expected total effort nor the expected highest effort. In that case, the *random* contest in which the entire prize sum is equally allocated to all the participants yields a higher expected total effort as well as a higher expected highest effort than the *winner-take-all* contest.

*Keywords:* Winner-take-all contests, all-pay auctions, participation constraints.

*JEL classification:* D44, O31, O32

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# 1 Introduction

In real-life contests, contestants often face participation constraints. For example, students who compete for grades in exams are required to achieve a minimal grade, or otherwise fail. Likewise, entry in professional sport competitions is often restricted, and only contestants who have achieved a certain predefined minimal requirement are allowed to compete. Researchers at universities too are required to achieve a minimal quality and quantity of output in order to be promoted. Indeed, initial research in contest design has found that limiting the number of contestants can be advantageous (see Baye et al., 1993, Taylor, 1995, and Fullerton & McAfee, 1999). Therefore it is clear that a contest designer who wishes to maximize the contestants' total effort should use endogenous participation constraints such as a reservation price or entry fees in order to exclude players with low valuations (abilities) from a contest. However, in many contests the participation constraints are exogenous (for example, the length (time) of R&D races or the minimum funds required for a candidate to participate in a political contest). In such contests, the contest designer will have limited control over the design since he cannot change the participation constraints.

In this paper, we assume all-pay contests with exogenous participation constraints and study the optimal allocation of prizes in these contests.<sup>1</sup> In an all-pay contest with a single prize the contestant with the highest effort (output) wins the entire prize, but all the contestants bear the cost of their effort.<sup>2</sup> It is well known that the all-pay contest under incomplete information with the optimal participation constraint is the optimal mechanism that maximizes the contestants' expected total effort (see Myerson (1981)). However, if the participation constraints are exogenous and are relatively high, neither is the efficiency of the standard all-pay contest with a single prize nor the optimal allocation of prizes clear. In contests without participation constraints, the *winner-take-all* structure in which the contestant with the highest effort (output) wins the

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<sup>1</sup>There might be other mechanisms which perform better than the all-pay contest in an environment with participation constraints, but we assume here that the general form of the contest is exogenous (all-pay) while the prize structure is endogenous.

<sup>2</sup>A different model emphasizes the use of contests to extract effort under "moral hazard" conditions (see Lazear and Rosen (1981), Green and Stokey (1983), Nalebuff and Stiglitz (1983), and Rosen (1986)). There output is a stochastic function of the unobservable effort, and the identity of the most productive agent is determined by an external shock.

entire prize sum is usually the optimal architecture that maximizes the contestants' expected total effort. For example, Barut and Kovenock (1998) studied a multi-prize all-pay contest under complete information, and showed that the revenue maximizing prize structure allows any combination of  $k - 1$  prizes, where  $k$  is the number of contestants. In particular, they showed that allocating the entire prize sum to a unique first prize is optimal. Moldovanu and Sela (2001, 2006) studied a one-stage all-pay contest and a two-stage all-pay contest with multiple prizes under incomplete information. In both cases, they showed that for a contest designer who maximizes the expected total effort, if the cost functions are linear in effort, it is optimal to allocate the entire prize sum to a single first prize. Schweinzer and Segev (2009) demonstrated that the optimal prize structure of symmetric  $n$ -player Tullock tournaments assigns the entire prize sum to the winner, provided that a symmetric pure strategy equilibrium exists. Fu and Lu (2009) studied a multi-stage sequential elimination Tullock contest, and showed that the optimal contest eliminates one contestant at each stage until the final, and the winner of the final takes the entire prize sum.

In our model of the all-pay contest with a minimal effort constraint, a contestant can participate only if his effort is equal to or higher than the minimal effort constraint. Contestants are privately informed about a parameter (ability) that affects their cost of effort. The designer decides about the size and number of prizes. He can allocate prizes such that the contestant with the highest effort wins a prize which is higher or equal to the prize of the contestant with the second highest effort, which is higher or equal to the prize of the contestant with the third highest effort, and so on. The designer maximizes either the expected total effort or the expected highest effort. According to our analysis, in contests with a relatively high minimal effort constraint, each contestant must have a chance to win the entire prize sum; otherwise the contestants will not participate in the contest. This result, however, eliminates the option of a contest with a finite number of prizes in which the entire prize is distributed among several contestants according to their efforts, and leaves us without many alternatives to the *winner-take-all* contest. The alternative we consider in this paper is the *random* contest in which all the participants, namely, the contestants who exert efforts which are higher or equal to the minimal effort constraint, have the same probability to win the entire prize sum. In the *random* contest, since the prize is randomly allocated among all the participants, it is obvious that

contestants do not have an incentive to exert efforts higher than the minimal effort constraint. Thus, the participants' effort in the *winner-take-all* contest is clearly higher than in the *random* contest. On the other hand, as we show, for any value of the minimal effort constraint, the number of participants in the *random* contest is higher than in the *winner-take-all* contest. Consequently, it is not clear that the *winner-take-all* contest yields a higher expected total effort than the *random* contest. Indeed, we demonstrate that if the minimal effort constraint is relatively high, the *winner-take-all* contest, in which the contestant with the highest effort wins the entire prize sum, does not maximize the expected total effort nor the expected highest effort. Furthermore, it is shown that, independent of the contestants' distribution of abilities, for high values of the minimum effort constraint, the expected total effort is higher when the entire prize is equally allocated to all the participants than when it is allocated to the contestant with the highest effort only. These results hold also in the case where the designer wishes to maximize the expected highest effort. In other words, even if the designer maximizes the expected highest effort, he will prefer equal allocation of the entire prize among all the participants in the contest over an allocation of the entire prize sum to the contestant with the highest effort.

We also study the case where there are prize caps such that the value of a prize is smaller than the entire prize sum. Therefore, at least two contestants win prizes given that there are more than one participant. It is shown that even with prize caps, independent of the contestants' distribution of abilities, the expected total effort in the *random* contest is higher than in the *winner-take-all* contest for high levels of the minimal effort constraint. Therefore in contests with prize caps and a minimal effort constraint the *random* contest is still a legitimate contest form.

In the literature on contest theory several researchers have studied the effect of bid caps in all-pay auctions. Che and Gale (1998) calculated the bidding equilibrium of a complete information all-pay auction with two bidders having different valuations for a prize and linear cost functions, and demonstrated that a bid cap can increase the players' total bid. Gaviious, Moldovanu and Sela (2003) studied symmetric all-pay auctions under incomplete information and showed that, regardless of the number of bidders, if agents have linear or concave cost functions then setting a bid cap is not profitable for a designer who wishes to maximize

the average bid. They also showed that if agents have convex cost functions, then effectively capping the bids is profitable for a designer facing a sufficiently large number of bidders.<sup>3</sup> In the all-pay auction with a bid cap, similarly to our random contest with a minimal effort constraint, the prize is randomly allocated among the contestants who exert an effort equal to the effort (bid) cap. However, the effect of the minimal effort constraint on the contestants' effort is completely different than the effect of the effort cap especially when their levels are high. While for a contest designer who wishes to maximize the expected total effort random allocation of the prize sum by using a high bid cap is not effective, we show that if the minimal effort constraint is sufficiently high, the prize sum should be randomly allocated among the contestants who exert an effort equal to the minimal effort constraint.

The rest of the paper is organized as follows: In Section 2 we present the model. In Section 3 we analyze the symmetric equilibrium effort functions and the expected total effort in the *winner-take-all* contest in which the contestant with the highest effort wins the entire prize. In Section 4 we analyze the expected total effort in the *random* contest in which the entire prize sum is equally allocated among all the participants. In Section 5 we compare the total effort in both forms of the contest, and in Section 6 we compare the expected highest effort between these two contests. Section 7 analyzes the equilibrium and the total effort in both contest forms when there are prize caps and Section 8 concludes.

## 2 The model

Consider an all-pay contest with  $n \geq 2$  contestants. Each contestant  $i$  makes an effort  $x_i$ . These efforts are submitted simultaneously. An effort  $x_i$  causes a cost  $\frac{x_i}{c_i}$  where  $c_i \geq 0$  is the ability (or type) of contestant  $i$  which is private information to  $i$ . Abilities are drawn independently of each other from an interval  $[0, 1]$  according to a distribution function  $F$  which is common knowledge. We assume that  $F$  has a continuous density  $dF > 0$ . There also exists an exogenous minimal effort  $1 \geq d \geq 0$  such that a contestant can participate in the contest only if his effort is higher or equal to  $d$ .

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<sup>3</sup>See also Kaplan, Luski and Wettstein (2003) about contests with bid caps.

The designer decides about the size and number of prizes for which he has a fixed total prize sum equal to 1. He can allocate prizes such that the contestant with the highest effort wins a prize which is higher or equal to the prize of the contestant with the second highest effort which is higher or equal to the prize of the contestant with the third highest effort, and so on. We assume that the designer maximizes either the expected value of total effort of the contestants or the expected value of their highest effort. Each contestant  $i$  chooses his effort in order to maximize his expected utility given the other competitors' actions and the values of the prizes.

### 3 The *winner-take-all* contest

Assume first a *winner-take-all* contest in which the designer allocates the entire prize sum to the contestant with the highest effort given that his effort is higher or equal to the minimal effort constraint  $d$ . We focus on a symmetric equilibrium where all participants use the same, strictly monotonic equilibrium effort function  $b(c)$ . Applying the revelation principle, player  $i$  with ability  $c$  chooses to behave as an agent with ability  $s$  to solve the following optimization problem:

$$\max_s F(s)^{n-1} - \frac{b(s)}{c}$$

In equilibrium, the above maximization problem must be solved by  $s = c$ . Then, the calculation of the equilibrium effort yields

$$b(c) = cF(c)^{n-1} - \int_0^c F(y)^{n-1} dy + k$$

where  $k$  is a constant. Given that there is a minimal effort constraint  $d$ , there exists a cutoff  $\tilde{c}$  such that all the contestants with lower types (abilities) than  $\tilde{c}$  decide to stay out of the contest and all the contestants with higher types than  $\tilde{c}$  (participants) decide to participate in the contest. The effort of the contestant with type  $\tilde{c}$  (cutoff) is equal to  $d$  and his expected payoff is equal to zero. Thus, we have

$$\tilde{c}F(\tilde{c})^{n-1} - b(\tilde{c}) = \tilde{c}F(\tilde{c})^{n-1} - d = 0 \tag{1}$$

By (1) we obtain that  $k = \int_0^{\tilde{c}} F(y)^{n-1} dy$ , so the equilibrium effort for every  $c \geq \tilde{c}$  is given by

$$b(c) = cF(c)^{n-1} - \int_{\tilde{c}}^c F(y)^{n-1} dy \quad (2)$$

Thus, the contestants' expected total effort in the *winner-take-all* contest is given by

$$TE_W = n \int_{\tilde{c}}^1 b(c)F'(c)dc = n \int_{\tilde{c}}^1 (cF(c)^{n-1} - \int_{\tilde{c}}^c F(y)^{n-1} dy)F'(c)dc \quad (3)$$

From equation (1) it can be verified that  $\tilde{c} \geq d$  and  $\tilde{c} = d$  only when both of these parameters are equal to 1. Thus by (3) we can see that  $TE_W(d = 0) > 0$  and  $TE_W(d = 1) = 0$ . The relation between the expected total effort  $TE_W$  and the value of the minimal effort constraint  $d$  is as follows:

**Proposition 1** *In the winner-take-all contest, the expected total effort increases in (sufficiently) small values of the minimal effort constraint, and decreases in (sufficiently) large values of the minimal effort constraint.*

**Proof.** See Appendix. ■

By Proposition 1 we can see that if the minimal effort constraint  $d$  is relatively high, the contest designer does not have any incentive to manipulate the level of the participation constraint. The reason is that he cannot impose a minimal effort constraint smaller than  $d$ , and if he imposes an endogenous minimal effort constraint larger than  $d$ , the utility of the contest designer who maximizes the total effort will be reduced.

Suppose now that the contest designer allocates the entire-prize sum to the two contestants with the highest efforts given that their efforts are higher or equal to  $d$ . In this case, the prize for the contestant with the highest effort is  $a \geq 0.5$ , and the prize for the contestant with the second highest effort is  $1 - a$ . Then, it can be verified that the expected highest effort is smaller or equal to  $a < 1$ . If the value of the minimal effort is larger than the value of the first prize,  $d > a$ , the contest is not efficient since no contestant will participate in it. Hence, if the minimal effort constraint  $d$  is relatively high, an allocation of a finite number of prizes is not a relevant option for a designer who wishes to maximize the total effort. As such, we can conclude that in contests with a relatively high minimal effort constraint, each contestant must have a chance to win the entire prize sum or else the contestants will not participate in the contest. Obviously this



argument leaves us with few alternatives to the *winner-take-all* contest where the minimal effort constraint has relatively high values. In the next section we discuss the alternative we find the most plausible.

## 4 The *random* contest

Consider the *random* contest in which the contest designer equally allocates the entire prize sum to all the contestants who exert efforts higher or equal to the minimal effort constraint  $d$ ; that is, all the participants in the contest have the same probability to win the entire prize sum. Denote by  $\hat{c}$  the cutoff, such that all the contestants with lower types (abilities) than  $\hat{c}$  decide to stay out of the contest, and all the contestants with higher types than  $\hat{c}$  decide to participate in the contest. Since the allocation of the prize sum does not depend on the effort level, given that it is higher or equal to the minimal effort constraint  $d$ , the participants do not have any incentive to exert efforts higher than the minimal effort constraint  $d$ . The probability of winning with an effort of  $d$  is

$$\begin{aligned}
Pr(win) &= F(\hat{c})^{n-1} + \frac{1}{2} \binom{n-1}{1} F(\hat{c})^{n-2} (1 - F(\hat{c})) + \frac{1}{3} \binom{n-1}{2} F(\hat{c})^{n-3} (1 - F(\hat{c}))^2 \\
&+ \frac{1}{4} \binom{n-1}{3} F(\hat{c})^{n-4} (1 - F(\hat{c}))^3 + \dots + \frac{1}{n} \binom{n-1}{n-1} (1 - F(\hat{c}))^{n-1} \\
&= \sum_{j=1}^n \frac{1}{j} \binom{n-1}{j-1} F(\hat{c})^{n-j} (1 - F(\hat{c}))^{j-1} \\
&= \frac{1}{n(1 - F(\hat{c}))} \sum_{j=1}^n \binom{n}{j} F(\hat{c})^{n-j} (1 - F(\hat{c}))^j \\
&= \frac{1 - F(\hat{c})^n}{n(1 - F(\hat{c}))} .
\end{aligned}$$

Thus, the expected payoff of contestant  $i$  with ability of  $c$  that submits an effort of  $d$  is

$$\frac{1 - F(\hat{c})^n}{n(1 - F(\hat{c}))} - \frac{d}{c}$$

The expected total effort is

$$TE_R = dn(1 - F(\hat{c})) \quad (4)$$

where the cutoff  $\hat{c}$  satisfies

$$\begin{aligned} \frac{1 - F(\hat{c})^n}{n(1 - F(\hat{c}))} - \frac{d}{\hat{c}} &= \frac{1}{n(1 - F(\hat{c}))}(1 - F(\hat{c}))(1 + F(\hat{c}) + \dots + F(\hat{c})^{n-1}) - \frac{d}{\hat{c}} \\ &= \frac{1 + F(\hat{c}) + \dots + F(\hat{c})^{n-1}}{n} - \frac{d}{\hat{c}} = 0 \end{aligned} \quad (5)$$

By (5) we obtain that  $\hat{c} \geq d$  and  $\hat{c} = d$  only when both of these parameters are equal to 1. We can see that  $TE_R(d = 0) = TE_R(d = 1) = 0$  and, similarly to the *winner-take-all* contest, we obtain that

**Proposition 2** *In the random contest the total effort increases in (sufficiently) small values of the minimal effort constraint, and decreases in (sufficiently) large values of the minimal effort constraint.*

**Proof.** See Appendix. ■

Thus, by Proposition 2, like in the *winner-take-all* contest, the contest designer in the *random* contest does not have any incentive to manipulate the level of the minimal effort constraint, given that it is relatively high.

**Proposition 3** *For any value of the minimal effort constraint, the number of participants in the random contest is equal to or larger than in the winner-take-all contest.*

**Proof.** See Appendix. ■

By Proposition 3, the number of participants in the *random* contest is larger than in the *winner-take-all* contest. On the other hand, the participants' efforts in the *winner-take-all* contest are higher than in the *random* contest. In the next section we deal with the problem of which contest form, the *random* contest or the *winner-take-all* contest, yields the higher total effort.

## 5 Total effort

In this section we assume that the contest designer wishes to maximize the expected total effort. Without a minimal effort constraint ( $d = 0$ ), Moldovanu and Sela (2001) have shown that the *winner-take-all* contest is the optimal design that maximizes the contestants' expected total effort, given that their cost functions are linear. Below we show that Moldovanu-Sela's result does not necessarily hold in contests with minimal effort constraints.

Denote by  $d_{R-opt}$  the optimal minimal effort constraint that maximizes the contestants' expected total effort in the *random* contest. The *winner-take-all* contest obviously yields a higher expected total effort than the *random* contest for  $d = 0$  and also for sufficiently small values of the minimal effort constraint. The following result gives the highest values of minimal effort constraint,  $d = d_{R-opt}$ , for which the *random* contest most certainly will not be efficient for a contest designer who maximizes the total effort.

**Proposition 4** *For any value of the minimal effort constraint which is smaller than or equal to the optimal minimal effort constraint in the random contest,  $d \leq d_{R-opt}$ , a mixed structure of the winner-take-all contest and the random contest yields a higher total effort than the random contest.*<sup>4</sup>

**Proof.** See Appendix. ■

So far we have shown that the *random* contest is not a relevant option for a designer who maximizes the total effort if the minimal effort constraint has relatively low values,  $d \leq d_{R-opt}$ . The following example shows that for relatively high values of the minimal effort constraint,  $d > d_{R-opt}$ , the *random* contest might be a better option than the *winner-take-all* contest for a designer who wishes to maximize the expected total effort.

**Example 1** *Suppose that the number of contestants is  $n = 4$  and the contestants' abilities are distributed*

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<sup>4</sup>In a mixed structure of the *winner-take-all* contest and the *random* contest, the contest designer allocates the prize sum by having the contestant with the highest effort win a prize equal to  $(1-a)$ , while a prize equal to  $a$  is shared by all the contestants with efforts higher than or equal to the minimal effort constraint  $d$ .

according to

$$F(c) = c$$

Then, by (1) +(3) and (4)+(5) we obtain the values of the total effort as a function of the minimal effort constraint  $d$  in the winner-take-all contest and in the random contest. These values are displayed in the

Table below.

$d$	Total effort - winner-take-all contest	Total effort - random contest
0.2	0.5860	0.4361
0.3	0.5447	0.5116
0.4	<b>0.4910</b>	<b>0.5349</b>
0.5	<b>0.4272</b>	<b>0.5178</b>
0.6	<b>0.3550</b>	<b>0.4661</b>
0.7	<b>0.2755</b>	<b>0.3874</b>
0.8	<b>0.1894</b>	<b>0.2811</b>
0.9	<b>0.0974</b>	<b>0.1540</b>
1	<b>0</b>	<b>0</b>

We can see that for all  $0.4 \leq d < 1$  the total effort in the *random* contest is larger than the total effort in the *winner-take-all* contest.

The following result generalizes the findings of Example 1.

**Theorem 1** *For every distribution of the contestants' abilities  $F$ , there exists a number  $0 < d_t(F) < 1$  such that for every minimal effort constraint  $d \geq d_t(F)$ , the expected total effort in the random contest is higher than in the winner-take-all.*

**Proof.** See Appendix. ■

Theorem 1 demonstrates that the *random* contest is a legitimate option for a designer who maximizes the expected total effort if the values of the minimal effort constraint are relatively high.

## 6 Highest effort

The contest designer's goal is not necessarily to maximize the contestants' expected total effort. Rather he may wish to maximize the expected highest effort. The expected highest effort in the *winner-take-all* contest is

$$\begin{aligned} HE_W &= \int_{\tilde{c}}^1 b(c) n F(c)^{n-1} F'(c) dc \\ &= \int_{\tilde{c}}^1 \left( c F(c)^{n-1} - \int_{\tilde{c}}^c F(y)^{n-1} dy \right) n F(c)^{n-1} F'(c) dc \end{aligned} \quad (6)$$

where  $b(c)$  is the equilibrium effort function given by (2) and the cutoff  $\tilde{c}$  is given by (1).

The expected highest effort in the *random* contest is

$$HE_R = d(1 - F(\hat{c})^n) \quad (7)$$

where the cutoff  $\hat{c}$  is given by (5).

Note that the highest effort in the *random* contest is smaller or equal to  $d$ , while in the *winner-take-all* contest it is larger or equal to  $d$ . However, the probability that all the contestants will choose to stay out of the *winner-take-all* contest is higher than in the *random* contest. Thus, it is not clear that even the expected highest effort in the *winner-take-all* is higher than in the *random* contest. Indeed, we have

**Theorem 2** *For every distribution of the contestants' abilities  $F$ , there exists a number  $0 < d_h(F) < 1$ , such that for every minimal effort constraint  $d \geq d_h(F)$ , the expected highest effort in the random contest is higher than in the winner-take-all.*

**Proof.** See Appendix. ■

Theorem 2 shows that even if the designer maximizes the expected highest effort, he will prefer to equally allocate the entire prize among all the participants in the contest rather than to allocate the entire prize sum to the contestant with the highest effort. The intuitive explanation to this result is that when the minimal effort constraint  $d$  is very high, the expected number of participants is relatively low such that there is no meaningful difference between the expected total effort and the expected highest effort. However, it could be

easily verified that in the case where a designer maximizes the expected highest effort, the advantage of the *random* contest over the *winner-take-all* contest is valid for a smaller range of the minimal effort constraint values than in the case where a designer maximizes the expected total effort.

## 7 Prize caps

Suppose now that the designer has a fixed total prize sum equal to 2, and assume also that there exists a prize cap such that the maximal value of a single prize is 1. In such a case, it can be shown that for the designer who wishes to maximize the total effort in the *winner-take-all* contest it is optimal to allocate two prizes, each equal to the prize cap 1, for the contestants with the two highest efforts given that their efforts are higher than or equal to the minimal effort constraint  $d$ . In the *random* contest, if only one contestant decides to participate with an effort equal to  $d$ , he wins a prize equal to 1, and in any other case with  $k \geq 2$  participants, each participant has the same probability to win a prize equal to 1.

In the *winner-take-all* contest with two identical prizes, each of them equal to 1, contestant  $i$  with ability  $c$  chooses to behave as an agent with ability  $s$  to solve the following optimization problem:

$$\max_s F(s)^{n-1} + (n-1)F(s)^{n-2}(1-F(s)) - \frac{\beta(s)}{c}$$

where  $\beta(c)$  is the symmetric equilibrium effort function given by

$$\beta(c) = cF(c)^{n-1} + c(n-1)F(c)^{n-2}(1-F(c)) - \int_{\vec{c}}^c (F(y)^{n-1} + (n-1)F(y)^{n-2}(1-F(y)))dy$$

where the cutoff  $\vec{c}$  is given by

$$\vec{c} F(\vec{c})^{n-1} + \vec{c}(n-1)F(\vec{c})^{n-2}(1-F(\vec{c})) = d \quad (8)$$

The contestants' expected total effort in the *winner-take-all* contest is given by

$$\begin{aligned} TE_{WC} &= n \int_{\vec{c}}^1 \beta(c) F'(c) dc = \\ & n \int_{\vec{c}}^1 (cF(c)^{n-1} + c(n-1)F(c)^{n-2}(1-F(c)) - \int_{\vec{c}}^c (F(y)^{n-1} + (n-1)F(y)^{n-2}(1-F(y)))dy) F'(c) dc \end{aligned} \quad (9)$$

In the *random* contest, the probability of winning one of the two prizes with an effort of  $d$  is

$$\begin{aligned}
Pr_c(win) &= F(\overleftarrow{c})^{n-1} + \binom{n-1}{1} F(\overleftarrow{c})^{n-2} (1 - F(\overleftarrow{c})) + \frac{2}{3} \binom{n-1}{2} F(\overleftarrow{c})^{n-3} (1 - F(\overleftarrow{c}))^2 \\
&+ \frac{2}{4} \binom{n-1}{3} F(\overleftarrow{c})^{n-4} (1 - F(\overleftarrow{c}))^3 + \dots + \frac{2}{n} \binom{n-1}{n-1} (1 - F(\overleftarrow{c}))^{n-1} \\
&= \sum_{j=1}^n \frac{2}{j} \binom{n-1}{j-1} F(\overleftarrow{c})^{n-j} (1 - F(\overleftarrow{c}))^{j-1} - F(\overleftarrow{c})^{n-1} \\
&= \frac{2}{n(1 - F(\overleftarrow{c}))} \sum_{j=1}^n \binom{n}{j} F(\overleftarrow{c})^{n-j} (1 - F(\overleftarrow{c}))^j - F(\overleftarrow{c})^{n-1} \\
&= \frac{2(1 - F(\overleftarrow{c})^n)}{n(1 - F(\overleftarrow{c}))} - F(\overleftarrow{c})^{n-1}
\end{aligned}$$

The expected total effort is

$$TE_{RC} = dn(1 - F(\overleftarrow{c})) \quad (10)$$

where the cutoff  $\overleftarrow{c}$  satisfies

$$\frac{2(1 - F(\overleftarrow{c})^n)}{n(1 - F(\overleftarrow{c}))} - F(\overleftarrow{c})^{n-1} - \frac{d}{\overleftarrow{c}} = \frac{2(1 + F(\overleftarrow{c}) + \dots + F(\overleftarrow{c})^{n-1})}{n} - F(\overleftarrow{c})^{n-1} - \frac{d}{\overleftarrow{c}} = 0 \quad (11)$$

**Theorem 3** *For every distribution of the contestants' abilities  $F$ , there exists a number  $0 < d_c(F) < 1$  such that for every minimal effort constraint  $d \geq d_c(F)$ , the expected total effort in the random contest with a prize cap is higher than in the winner-take-all contest with a prize cap.*

**Proof.** See Appendix. ■

Theorem 3 demonstrates that if the levels of the prizes are limited, the *random* contest is still a legitimate option for a designer who maximizes the expected total effort if the values of the minimal effort constraint are relatively high.

## 8 Concluding remarks

We studied all-pay contests with a minimal effort constraint and the following two forms of prize allocation:

1. A *winner-take-all contest* where the contestant with the highest effort (output) wins the entire prize sum.
2. A *random contest* where all the participants have the same probability to win the entire prize sum. We

showed that independent of the distribution of the contestants' abilities, if the minimal effort constraint is relatively high, the *random contest* yields a higher expected total effort than the *winner-take-all contest*.

Our results were shown to hold for high values of the minimal effort constraint, but we also demonstrated by an example that the results hold even for minimal effort constraint values that are not relatively high.

The main implication of this paper is that the existence of participation constraints in contests is a possible explanation for why multiple-prize contests exist in the real world. Another implication of our results relates to the system of grading in many universities worldwide given that a designer (lecturer) has a constraint on the average of the students' grades in each class. In various universities, grades fall in the range of 0-100.

In Israel, for example, BA students whose grades are smaller than 56 and MA students whose grades are smaller than 65 fail. In that case, the minimal effort constraints fall in the range in which the *random contest* might yield a higher expected total effort as well as a higher expected highest effort than the *winner-take-all contest*. Indeed, contests among students at universities are composed of a mixture of the *random contest* and the *winner-take-all contest*. However, according to our results, if the minimal output (grade) constraint is sufficiently high, the *random contest* is a better option than the *winner-take-all contest* and then binary grades of fail/pass should be applied.



## 9 Appendix

### 9.1 Proof of Proposition 1

By (3) we have

$$\begin{aligned}\frac{dTE_W(d, \tilde{c})}{d\tilde{c}} &= n \int_{\tilde{c}}^1 F(\tilde{c})^{n-1} F'(c) dc - n\tilde{c}F(\tilde{c})^{n-1} F'(\tilde{c}) \\ &= nF(\tilde{c})^{n-1}(1 - F(\tilde{c}) - \tilde{c}F'(\tilde{c}))\end{aligned}$$

By (1) and the *implicit functions theorem* we obtain

$$\frac{d\tilde{c}}{dd} = \frac{1}{F(\tilde{c})^{n-1} + (n-1)\tilde{c}F(\tilde{c})^{n-2}F'(\tilde{c})}$$

Thus,

$$\frac{dTE_W(d, \tilde{c})}{dd} = \frac{dTE_W}{d\tilde{c}} \frac{d\tilde{c}}{dd} = \frac{nF(\tilde{c})^{n-1}(1 - F(\tilde{c}) - \tilde{c}F'(\tilde{c}))}{F(\tilde{c})^{n-1} + (n-1)\tilde{c}F(\tilde{c})^{n-2}F'(\tilde{c})} \quad (12)$$

Since the cutoff  $\tilde{c}$  approaches zero when  $d$  approaches zero, and  $1 - F(\tilde{c}) - \tilde{c}F'(\tilde{c}) > 0$  when  $\tilde{c}$  approaches zero, we obtain that  $\frac{dTE_W}{dd} > 0$  for sufficiently small values of  $d$ . Since the cutoff  $\tilde{c}$  approaches 1 when  $d$  approaches 1, we have

$$\lim_{d \rightarrow 1} \frac{dTE_W(d, \tilde{c})}{dd} = -\frac{nF'(1)}{1 + (n-1)F'(1)} < 0 \quad (13)$$

*Q.E.D.*

### 9.2 Proof of Proposition 2

By (5) and the *implicit functions theorem* we have

$$\frac{d\hat{c}}{dd} = \frac{1}{\frac{1}{n}(1 + F(\hat{c}) + \dots + F(\hat{c})^{n-1}) + \frac{\hat{c}}{n}(F'(\hat{c}) + \dots + (n-1)F(\hat{c})^{n-2}F'(\hat{c}))} \quad (14)$$

By (14) and (4) we have

$$\begin{aligned}\frac{dTE_R(d, \hat{c})}{dd} &= \frac{dTE_R}{dd} + \frac{dTE_R}{d\hat{c}} \frac{d\hat{c}}{dd} \\ &= n(1 - F(\hat{c})) + \frac{-dnF'(\hat{c})}{\frac{1}{n}(1 + F(\hat{c}) + \dots + F(\hat{c})^{n-1}) + \frac{\hat{c}}{n}(F'(\hat{c}) + \dots + (n-1)F(\hat{c})^{n-2}F'(\hat{c}))}\end{aligned} \quad (15)$$

Since  $\hat{c}$  approaches zero when  $d$  approaches zero, we have

$$\lim_{d \rightarrow 0} \frac{dTE_R(d, \hat{c})}{dd} = n > 0$$

Since  $\hat{c}$  approaches 1 when  $d$  approaches 1, we have

$$\lim_{d \rightarrow 1} \frac{dTE_R(d, \hat{c})}{dd} = \frac{-nF'(1)}{1 + \frac{n-1}{2}F'(1)} < 0 \quad (16)$$

*Q.E.D.*

### 9.3 Proof of Proposition 3

By (1) the cutoff in the *winner-take-all* contest satisfies

$$\tilde{c}F(\tilde{c})^{n-1} = d$$

By (5) the cutoff in the *random* contest satisfies

$$\frac{\hat{c}(1 - F(\hat{c})^n)}{n(1 - F(\hat{c}))} = d$$

Let  $w(c) = cF(c)^{n-1}$  and  $r(c) = \frac{c(1-F(c)^n)}{n(1-F(c))}$ . Note that  $w(c)$  and  $r(c)$  are increasing functions that satisfy  $w(0) = r(0) = 0$ . Thus, in order to show that  $\hat{c} \leq \tilde{c}$ , it is sufficient to show that  $r(c) \geq w(c)$  for all  $0 \leq c \leq 1$ .

Indeed,

$$r(c) = \frac{c(1 - F(c)^n)}{n(1 - F(c))} = \frac{c(1 + F(c) + F(c)^2 + \dots + F(c)^{n-1})}{n} \geq \frac{ncF(c)^{n-1}}{n} = w(c)$$

*Q.E.D.*

### 9.4 Proof of Proposition 4

Suppose that the contest designer allocates the prize sum by having the contestant with the highest effort win a prize equal to  $(1 - a)$ , while a prize equal to  $a$  is shared by all the contestants with efforts which are higher or equal to the minimal effort constraint  $d$ . The symmetric equilibrium in this case is given by

$$\gamma(c) = (1-a)cF(c)^{n-1} - \int_{\bar{c}}^c (1-a)F(y)^{n-1}dy + s$$

where  $\bar{c}$  is the cutoff and  $s$  is a constant. Since the expected payoff of type  $\bar{c}$  is zero and his effort is equal to the minimal effort level  $d$ , we obtain that  $s$  and  $\bar{c}$  are given by

$$\begin{aligned} d &= \frac{a\bar{c}(1-F(\bar{c})^n)}{n(1-F(\bar{c}))} + (1-a)\bar{c}F(\bar{c})^{n-1} \\ s &= \frac{a\bar{c}(1-F(\bar{c})^n)}{n(1-F(\bar{c}))} \end{aligned} \quad (17)$$

The expected total effort is

$$\begin{aligned} TE(a) &= n \int_{\bar{c}}^1 ((1-a)cF(c)^{n-1} - \int_{\bar{c}}^c (1-a)F(y)^{n-1}dy + \frac{a\bar{c}(1-F(\bar{c})^n)}{n(1-F(\bar{c}))})F'(c)dc \\ &= n \int_{\bar{c}}^1 ((1-a)cF(c)^{n-1} - \int_{\bar{c}}^c (1-a)F(y)^{n-1}dy)F'(c)dc + a\bar{c}(1-F(\bar{c})^n) \end{aligned}$$

Then, we have

$$\begin{aligned} \lim_{a \rightarrow 1} \frac{dTE}{da} &= \lim_{a \rightarrow 1} \left( \frac{dTE}{da} + \frac{dTE}{d\bar{c}} \frac{d\bar{c}}{da} \right) = n \int_{\bar{c}}^1 (-cF(c)^{n-1} + \int_{\bar{c}}^c F(y)^{n-1}dy)F'(c)dc + \bar{c}(1-F(\bar{c})^n) \\ &\quad + \frac{d\bar{c}}{da}(1-F(\bar{c})^n - \bar{c}nF(\bar{c})^{n-1}F'(\bar{c})) \end{aligned}$$

By (17)

$$\lim_{a \rightarrow 1} \bar{c}(a) = \frac{dn(1-F(\bar{c}))}{1-F(\bar{c})^n}$$

Thus,

$$\begin{aligned} \lim_{a \rightarrow 1} \frac{dTE}{da} &= n \int_{\bar{c}}^1 (-cF(c)^{n-1} + \int_{\bar{c}}^c F(y)^{n-1}dy)F'(c)dc + nd(1-F(\bar{c})) + \frac{d\bar{c}}{da}(1-F(\bar{c})^n - \bar{c}nF(\bar{c})^{n-1}F'(\bar{c})) \\ &< -nd(1-F(\bar{c})) + nd(1-F(\bar{c})) + \frac{d\bar{c}}{da}(1-F(\bar{c})^n - \bar{c}nF(\bar{c})^{n-1}F'(\bar{c})) \\ &= \frac{d\bar{c}}{da}(1-F(\bar{c})^n - \bar{c}nF(\bar{c})^{n-1}F'(\bar{c})) \end{aligned}$$

It can be shown that  $\frac{d\bar{c}}{da} \leq 0$ ; that is, the higher the prize shared by all the participants is, the higher is the number of participants in the contest. Note that in the *random* contest we have

$$\begin{aligned} \frac{dTE_R(d, \hat{c})}{dd} &= \frac{dTE_R}{dd} + \frac{dTE_R}{d\hat{c}} \frac{d\hat{c}}{dd} = n(1-F(\hat{c})) - dnF'(\hat{c}) \frac{n(1-F(\hat{c}))}{1-F(\hat{c})^n - \hat{c}nF(\hat{c})^{n-1}F'(\hat{c}) + dnF'(\hat{c})} \\ &= \frac{n(1-F(\hat{c}))(1-F(\hat{c})^n - \hat{c}nF(\hat{c})^{n-1}F'(\hat{c}))}{1-F(\hat{c})^n - \hat{c}nF(\hat{c})^{n-1}F'(\hat{c}) + dnF'(\hat{c})} \end{aligned}$$

Thus, the cutoff  $\hat{c}_m$  that maximizes the expected total effort in the *random* contest satisfies  $1 - F(\hat{c}_m)^n - \hat{c}_m n F(\hat{c}_m)^{n-1} F'(\hat{c}_m) = 0$  and for every  $\hat{c} \leq \hat{c}_m$  there exists  $1 - F(\hat{c})^n - \hat{c} n F(\hat{c})^{n-1} F'(\hat{c}) > 0$ . That is, we obtain that for every  $d \leq d_{R-opt}$

$$\lim_{a \rightarrow 1} (1 - F(\bar{c})^n - \bar{c} n F(\bar{c})^{n-1} F'(\bar{c})) = 1 - F(\hat{c})^n - \hat{c} n F(\hat{c})^{n-1} F'(\hat{c}) > 0$$

Hence,  $\lim_{a \rightarrow 1} \frac{dTE}{da} < 0$ , that is, decreasing the total prize allocated to all the participants or, alternatively, increasing the prize for the contestant with the highest effort, necessarily increases the contestants' expected total effort.

*Q.E.D.*

## 9.5 Proof of Theorem 1

By (3) and (4) we have

$$\Delta T(d = 1) = TE_R(d = 1) - TE_W(d = 1) = 0$$

By (13) and (16) we obtain

$$\lim_{d \rightarrow 1} \frac{d\Delta T}{dd} = \lim_{d \rightarrow 1} \frac{dTE_R(d, \hat{c})}{dd} - \frac{dTE_W(d, \hat{c})}{dd} = -\frac{nF'(1)}{1 + \frac{n-1}{2}F'(1)} + \frac{nF'(1)}{1 + (n-1)F'(1)} < 0$$

Thus, there exists  $0 < d_t < 1$  such that for all  $d \geq d_t$ ,  $\Delta T(d) = TE_R(d) - TE_W(d) \geq 0$ . *Q.E.D.*

## 9.6 Proof of Theorem 2

By (6) and (7) we have

$$\Delta H(d = 1) = HE_R(d = 1) - HE_W(d = 1) = 0$$

The first derivatives are

$$\begin{aligned} \frac{dHE_R(d, \hat{c})}{dd} &= \frac{dHE_R}{dd} + \frac{dHE_R}{d\hat{c}} \frac{d\hat{c}}{dd} = \\ &= 1 - F(\hat{c})^n - \frac{dnF(\hat{c})^{n-1}F'(\hat{c})}{\frac{1}{n}(1 + F(\hat{c}) + \dots + F(\hat{c})^{n-1}) + \frac{\hat{c}}{n}(F'(\hat{c}) + \dots + (n-1)F(\hat{c})^{n-2}F'(\hat{c}))} \end{aligned}$$

and

$$\frac{dHE_W(d, \tilde{c})}{dd} = \frac{dHE_W}{d\tilde{c}} \frac{d\tilde{c}}{dd} = \frac{F(\tilde{c})^{n-1}(1 - F(\tilde{c})^n - n\tilde{c}F(\tilde{c})^{n-1}F'(\tilde{c}))}{F(\tilde{c})^{n-1} + (n-1)\tilde{c}F(\tilde{c})^{n-2}F'(\tilde{c})}$$

If  $d$  approaches 1 we have

$$\lim_{d \rightarrow 1} \frac{d\Delta H}{dd} = \lim_{d \rightarrow 1} \frac{dHE_R(d, \tilde{c})}{dd} - \lim_{d \rightarrow 1} \frac{dHE_W(d, \tilde{c})}{dd} = \frac{-nF'(1)}{1 + \frac{n-1}{2}F'(1)} + \frac{nF'(1)}{1 + (n-1)F'(1)} < 0$$

Thus, there exists  $0 < d_h < 1$  such that for all  $d \geq d_h$ ,  $\Delta H(d) = HE_R(d) - HE_W(d) \geq 0$ . *Q.E.D.*

## 9.7 Proof of Theorem 3

Consider first the *random* contest. By (11) and the *implicit functions theorem* we have

$$\frac{d\overleftarrow{c}}{dd} = \frac{1}{h(\overleftarrow{c})} \quad (18)$$

where

$$\begin{aligned} h(\overleftarrow{c}) &= \frac{2}{n}(1 + F(\overleftarrow{c}) + \dots + F(\overleftarrow{c})^{n-1}) + \frac{2\overleftarrow{c}}{n}(F'(\overleftarrow{c}) + \dots + (n-1)F(\overleftarrow{c})^{n-2}F'(\overleftarrow{c})) \\ &\quad - F^{n-1}(\overleftarrow{c}) - \overleftarrow{c}(n-1)F(\overleftarrow{c})^{n-2}F'(\overleftarrow{c}) \end{aligned}$$

By (10) and (18) we have,

$$\begin{aligned} \frac{dTE_{RC}(d, \overleftarrow{c})}{dd} &= \frac{dTE_{RC}}{dd} + \frac{dTE_{RC}}{d\overleftarrow{c}} \frac{d\overleftarrow{c}}{dd} \\ &= n(1 - F(\overleftarrow{c})) - dnF'(\overleftarrow{c}) \frac{1}{h(\overleftarrow{c})} \end{aligned}$$

Since  $\overleftarrow{c}$  approaches 1 when  $d$  approaches 1, we have

$$\lim_{d \rightarrow 1} \frac{dTE_{RC}(d, \overleftarrow{c})}{dd} = \frac{-nF'(1)}{2 + (n-1)F'(1) - 1 - (n-1)F'(1)} = -nF'(1) < 0 \quad (19)$$

Consider now the *winner-take-all* contest. By (9) we have

$$\begin{aligned} \frac{dTE_{WC}(d, \overrightarrow{c})}{d\overrightarrow{c}} &= n \int_{\overrightarrow{c}}^1 (F(\overrightarrow{c})^{n-1} + (n-1)F(\overrightarrow{c})^{n-2}(1 - F(\overrightarrow{c})))F'(c)dc \\ &\quad - n\overrightarrow{c}F'(\overrightarrow{c})(F(\overrightarrow{c})^{n-1} + (n-1)F(\overrightarrow{c})^{n-2}(1 - F(\overrightarrow{c}))) \\ &= (nF(\overrightarrow{c})^{n-1} + n(n-1)F(\overrightarrow{c})^{n-2}(1 - F(\overrightarrow{c})))(1 - F(\overrightarrow{c}) - \overrightarrow{c}F'(\overrightarrow{c})) \end{aligned}$$

By (8) and the *implicit functions theorem* we obtain

$$\frac{d\vec{c}}{dd} = \frac{1}{g(\vec{c})}$$

where

$$\begin{aligned} g(\vec{c}) &= F(\vec{c})^{n-1} + (n-1)\vec{c}F(\vec{c})^{n-2}F'(\vec{c}) + (n-1)F(\vec{c})^{n-2}(1-F(\vec{c})) \\ &\quad + (n-1)(n-2)\vec{c}F(\vec{c})^{n-3}(1-F(\vec{c}))F'(\vec{c}) - (n-1)\vec{c}F(\vec{c})^{n-2}F'(\vec{c}) \end{aligned}$$

Thus,

$$\frac{dTE_{WC}(d, \vec{c})}{dd} = \frac{dTE_{WC}}{d\vec{c}} \frac{d\vec{c}}{dd} = \frac{(nF(\vec{c})^{n-1} + n(n-1)F(\vec{c})^{n-2}(1-F(\vec{c}))) (1-F(\vec{c}) - \vec{c}F'(\vec{c}))}{g(\vec{c})}$$

Since the cutoff  $\vec{c}$  approaches 1 when  $d$  approaches 1, we have

$$\lim_{d \rightarrow 1} \frac{dTE_{WC}(d, \vec{c})}{dd} = -nF'(1) < 0 \quad (20)$$

By (20) and (19) we obtain

$$\lim_{d \rightarrow 1} \frac{dTE_{RC}(d, \overleftarrow{c})}{dd} - \frac{dTE_{WC}(d, \vec{c})}{dd} = -nF'(1) + nF'(1) = 0$$

and

$$\begin{aligned} \lim_{d \rightarrow 1} \frac{d^2 \Delta T_C}{dd^2} &= \lim_{d \rightarrow 1} \frac{d^2 TE_{RC}(d, \overleftarrow{c})}{dd^2} - \lim_{d \rightarrow 1} \frac{d^2 TE_{WC}(d, \vec{c})}{dd^2} \\ &= -2nF'(1) - nF''(1) - n(n-1)(n-2)(F'(1))^3 + 2(2+6+\dots+(n-1)(n-2))(F'(1))^3 \\ &\quad - (-2nF'(1) - nF''(1) - n(n-1)(n-2)(F'(1))^3) \\ &= 2(2+6+\dots+(n-1)(n-2))(F'(1))^3 \end{aligned}$$

Thus,  $\lim_{d \rightarrow 1} \frac{d^2 \Delta T_C}{dd^2} \geq 0$ . We obtained that the point  $d = 1$  is a minimum point of  $TE_{RC}(d, \overleftarrow{c}) - TE_{WC}(d, \vec{c})$

which implies that there exists  $0 < d_c < 1$  such that for all  $d \geq d_c$ ,  $TE_{RC}(d, \overleftarrow{c}) - TE_{WC}(d, \vec{c}) \geq 0$ . *Q.E.D*

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