EFFICIENT AGGLOMERATION
OF SPATIAL CLUBS

Oded Hochman

Discussion Paper No. 09-03

March 2009

Monaster Center for
Economic Research
Ben-Gurion University of the Negev
P.O. Box 653
Beer Sheva, Israel

Fax: 972-8-6472941
Tel: 972-8-6472286

Efficient Agglomeration of Spatial Clubs

(or: The Agglomeration of Agglomerations)

Oded Hochman[1]

Revised February 2009

Abstract

The literature on agglomeration has focused largely on primary agglomeration caused by direct attraction effects. Here we focus on secondary and tertiary agglomerations caused by a primary agglomeration. Initially, scale economies in the provision of club goods (CGs) lead each CG to agglomerate in facilities of a club. This primary agglomeration causes a secondary concentration of population around these facilities, which in turn brings about a tertiary agglomeration of facilities of different clubs into centers. The agglomeration of facilities occurs only if a secondary concentration of population takes place. We analyze in detail two specific patterns of agglomeration. One is the central location pattern in which the facilities of all clubs agglomerate perfectly in the middle of their joint market area. The second is a triple-centered complex in which the center in the middle of the complex consists of perfectly agglomerated facilities of different clubs, each with a single facility per complex. The other two sub-centers consist of facilities of different clubs, each with two facilities per complex. These sub-centers are closer to the middle of the complex than to the boundaries and their facilities form condensed clusters of facilities that may contain residential land in between the facilities.

Keywords: agglomeration, clubs, complex, collective goods, local public goods, indirect attraction.

JEL Classification: R1, H4.

[1]Department of Economics, Ben Gurion University of the Negev, Beer Sheva, 84105, Israel.
Email: oded@bgu.ac.il

1. Introduction

The purpose of this paper is threefold: the first is to introduce an optimization model of an economy with spatial clubs, the second is to identify those forces in the economy that lead to the agglomeration of facilities of various clubs into multi-club centers and the last is to characterize these centers.

To facilitate the exposition we first introduce some terminology related to the theory of spatial clubs. A spatial club consists of facilities spread throughout the economy, each of which contains a concentration of the good provided by the club. A club-good (CG) is a good or service provided by each of the club facilities to their patrons. The provision of a CG by its club's facility is subject to scale economies. The patrons of a facility are a group of households who jointly consume the CG provided by the facility and are distinct from patrons of other facilities of the same club. In order to consume a particular CG a household has to commute to one of the facilities of the spatial club that provides this good. The market area of a facility is the area of residency of the facility's patrons.

Many local public goods are CGs as are many private consumption goods and services whose provision is subject to scale economies and therefore are provided collectively by spatial clubs. Real-life clubs such as country clubs, parks, museums, churches, etc. are also relevant to our model. In addition other institutions, not necessarily known as clubs, satisfy our specifications, for example, schools, police stations, theater and movie halls, restaurants, government offices, courthouses, shops and stores, and many more.. Notable among these various clubs is the 'production club' whose facilities include industrial areas and employment centers.

Three main reasons are typically offered to explain why both residential and non-residential activities agglomerate. One is reciprocal informational exchange, the second is increasing returns to scale and the last is spatial competition (see Fujita and Thisse (1996) for a comprehensive survey and Fujita and Thisse (2002) for recent theories on agglomeration). Most of these explanations are based on direct attraction forces such as the mutual attraction of units of an industry because their activity is enhanced when located close to each other.

In this paper, the primary agglomeration of CGs into facilities, is, similarly to other studies, a result of a direct attraction between units of a CG whose provision is subject to scale economies. Each CG agglomerates into its own facilities in order to provide the CG to households throughout the economy. We focus here, however, mainly on the secondary agglomerations of population around facilities and on the tertiary agglomerations of facilities of different clubs in centers in the midst of population concentrations.

The primary agglomeration of a CG in facilities attracts households to locate close to a facility in order to save commuting costs. The desire to save commuting costs is offset by congestion costs due to the limited supply of land in the proximity of the facility. The indirect attraction and the subsequent congestion cause secondary concentration of population around facilities, where the density of population decreases with its distance from the facility. In turn, the concentration of population around a facility causes facilities of different clubs to locate in the same vicinity in order to increase accessibility even further, thus creating tertiary agglomerations of facilities into centers in the midst of densely populated areas. All three stages of agglomeration, namely the primary agglomeration of CGs, the secondary concentration of population and the tertiary agglomeration of facilities into centers, occur simultaneously and the stages indicate the order of causality rather than the timing. Indeed, we show that tertiary agglomeration does not occur without a secondary concentration of population and that secondary agglomeration of population does not occur without the primary agglomeration of CGs into facilities.

In the 1960's urban economics models have dealt mainly with the secondary agglomeration of households in a residential ring surrounding a predetermined central business district (CBD), where all employment takes place. The concentration of industry in the CBD was exogenously assumed, the rationale being that the industry must be located in proximity to a sea port, train depot or other shipping facility through which the city's basic products can be exported to the rest of the world (e.g., Muth (1969)). Mills (1967) argued that the agglomeration of industry in a CBD is the result of the industry being subject to scale economies but he still assumed exogenous agglomeration. Instead of focusing on an endogenous CBD, Mills and his contemporaries concentrated on the residential ring. Henderson (1974) was the first to introduce a model in which an industry agglomerates endogenously into a CBD, however he still imposed on the model a single employment location surrounded by a residential ring. In the 1980's, Ogawa and Fujita (1980), Fujita and Ogawa (1982), and Fujita (1989) constructed simulation models of the agglomeration of an industry based on direct attraction effects. These simulations resulted in a variety of primary agglomerations. However, no secondary agglomeration of population and hence no tertiary agglomeration were possible, since a uniform density of population was everywhere assumed.

Recently, Lucas and Rosi-Hansberg (2002) incorporated both direct and indirect agglomeration engines into a single simulation model of an agglomerating industry and population/workers. But contrary to our model, in which facilities of different clubs agglomerate into centers, in their model only one type of facility exists and therefore no tertiary agglomeration can occur. Actually, none of the above models address the tertiary agglomeration of different primary agglomerations into centers in the midst of population concentrations as described in this paper.

Some studies in the literature (e.g., Fujita and Thisse (1986), Thisse and Wildasin (1992), Papageorgiou and Pines (1998) and papers surveyed by Berliant and ten Raa (1994)) investigate the agglomeration of facilities while imposing a uniform distribution of population. In this paper we show that effective agglomeration of facilities cannot occur without a secondary concentration of population and the agglomerations of facilities in the above studies are due to either the 'edge-of-economy effect', to indivisibility problems and/or to random technological effects. Therefore, to avoid confounding our own results we assume herein an economy without edges, i.e., our economy's territory is ring-shaped and fully occupied. In addition we investigate here only cases of full divisibility.

On this ring-shaped area of homogeneous land, we construct a model of an economy with spatial clubs using the conceptual framework of Hochman, Pines and Thisse (1995) (HPT hereafter).[2] In this economy there are many types of essential collective goods that require a wide variety of spatial clubs that a household must visit in order to consume the goods. The agglomeration of each CG into a separate facility results from scale economies in the provision of the good. Without such scale economies, each household would consume the CG privately in its own premises in order to avoid commuting costs. Since the direct attraction forces between units of a CG caused by scale economies are assumed to be internal to the facility, they are reflected only in the size of the facilities and not in their number. Thus, at any given site no more than one facility per club exists. We demonstrate that the population density is never uniform in a first-best allocation and that there are always areas in the economy in which population and facilities agglomerate.

Our model's results specify that in an optimal allocation the economy's territory is partitioned into identical *complexes*, where a complex is the smallest autonomous area in the economy, i.e., the smallest area in which all residents, and they alone, consume all the types of CGs in facilities

[2] While HPT focused on the finance of services rendered by the facilities, they disregarded spatial aspects and questions of agglomeration of facilities on which the present paper focuses.

located inside the complex. Thus, nobody commutes in or out of a complex. In a sense, this fact makes the complex the ideal municipality. In this paper we characterize an allocation by characterizing its representative complex.

A *complex configuration* is a vector of integers without a common multiplier that specifies the number of facilities of each club in the complex. Thus, the first entry in the vector is the number of facilities of club one in a complex, the second entry is the number of facilities of club two and so forth. Each model with a given set of functions which consists of feasible transportation cost functions and feasible provision cost functions, both for each club type, as well as of a utility function and a given complex configuration, have an optimal solution with identical and symmetric complexes. We refer to such a solution as a local optimum. In a global optimum the complex configuration is also chosen optimally.

Next, we characterize the spatial pattern of two local optimum solutions with two specific complex configurations.[3] In the first configuration, each club has a single facility per complex. With this configuration, the model results in monocentric complexes (cities) in which facilities of all clubs agglomerate perfectly in the center of the complex and share the whole complex as a common market area.[4] The population density and the housing price function in each of the complexes of this configuration increase with proximity to the complex's center, where both functions reach their peak. In addition, we provide specifications of a functions domain in which this solution is the unique global (over all possible configurations) optimum.

The second configuration that we investigate has two groups of clubs. Each club in the first group has a single facility per complex and each club in the second group has two facilities per complex. In the optimal allocation all the facilities of clubs of the first group agglomerate perfectly in the middle of each complex and the whole complex is their market area. The facilities of clubs of the second group are divided into two clusters each of which contains one facility of each club of the second group. The complex area is divided in the middle into two equal market areas, one for each cluster of facilities of the clubs of the second group. One cluster is located in the second quarter of the complex's area and the other in the third quarter. Thus, the clusters of the second group (DF clubs hereafter) are closer to the middle of the complex than to its boundaries. In other words, these clusters gravitate towards the center of the complex. The facilities in a cluster are close to each other but residential areas may exist between the facilities in the cluster, depending on whether or not the transportation cost functions of the different DF clubs are proportional to each other. Facilities with proportional transportation costs share the same facility location. Thus, while clubs of the second group do not necessarily agglomerate perfectly, they are drawn to each other and the cluster as a whole is drawn towards the facilities located in the middle of the complex. The complex is symmetric around its middle with a higher density of population between the clusters of DF clubs and the center of the complex than between the clusters and the boundaries.

Contrary to non-spatial clubs (e.g., Berglas (1976), Scotchmer and Wooders (1987); see also the survey by Scotchmer (2002) of spatial and non-spatial clubs), our optimal solution cannot be attained by a laissez faire allocation and sometimes not even by decentralization. In a laissez faire situation club owners are free to operate without restrictions, so they engage in spatial monopolistic competition, which, in general, does not yield an optimal allocation. We also show that for an economy with price taking agents there sometimes is a limited number of decentralization methods, each of which may fit under different conditions. Most decentralization methods involve subsidizing households and taxing facilities. However, such a decentralized

[3] These complex configurations are: (1,. . . ,1) and (1,. . . ,1,2,. . . ,2).

[4] By perfect agglomeration we mean that facilities are adjacent to each other without having any residential area between them.

solution may entail different subsidies to identical households that are located in different places and is, therefore, difficult to implement.

Five additional sections follow this introduction. Section 2 describes the setup of the model. The necessary conditions for Pareto optimum are described in section 3 and the decentralization of the optimal allocation is depicted in section 4. Section 5 contains our main results. First, in subsection 5.1, we present general characteristics of the solution. Then we proceed to describe a perfect agglomeration in subsection 5.2 and an imperfect agglomeration in 5.3. We conclude with a short summary and a few pointers for future research of global optimum solutions.

2. The Model Setup

The country's geography is designated by a ring of unit width, with a circle running through the middle of the ring being the axis (see Figure 1).

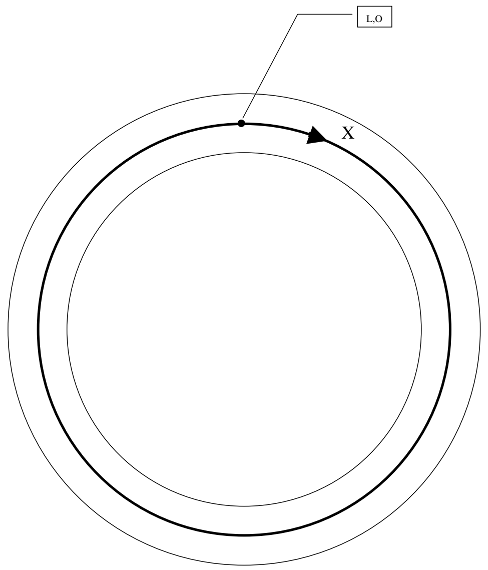

Figure 1: A Ring-Shaped Economy

We assume the circle's circumference is $\mathcal{L}$. Note that the total area of the ring in this case is also $\mathcal{L}$. An arbitrary point on the ring's axis is referred to as the origin. The location of any point on the axis of the ring is uniquely defined by its distance x from the origin in a clockwise direction (henceforth also the positive or the right direction). All points on the line segment perpendicular to the axis are designated as the same location because travel between these points involves no costs. The country accommodates $\mathcal{N}$ *households* (each time we introduce a concept it is italicized) which are identical to each other in all respects. We assume that these households are free to choose their residential location in the economy. Hence, all households must have the same utility level everywhere; otherwise they will migrate to the location with the higher utility. Each individual household derives utility from the consumption of a *composite good*, Z, and from *housing*, H, both of which the household consumes at its location of residency.

The household also derives utility from I types of *collective goods* (CGs hereafter), where G_i, is the quantity of the i^{th} CG the household consumes, $i = 1, ..., I$, according to a well-behaved utility function, $u(Z, H, G_1, \ldots, G_I)$. All goods are *essential*, and each CG is consumed at a special facility to which the household has to travel. Each individual is endowed with Y units of the composite good which can be used for private consumption and for the production of housing, CGs and transportation.

The economy contains I different clubs, one for each type of CG . A *club* of type i supplies units of the i-th CG through its $\tilde{m}_i$ *facilities* which are located throughout the economy. Each facility is identified by i, j, where $j \in (1, \ldots, \tilde{m}_i)$ is the index of the specific facility of club i, and $i \in (1, \ldots, I)$ refers to the club type. Facility i, j, whose location is designated by $x_{i,2j}$, provides G_{ij} units of the i-th CG to N_{ij} *patrons*, i.e., to individual households consuming the i-th CG in facility ij and residing within its *market area*, where a market area of a facility is a segment of the x-axis where all and only the facility's patrons live.[5] We also make the simplifying assumption that a facility does not occupy land and since, in practice, club facilities occupy only a small fraction of the total land available compared to residential land, the distortion caused by this assumption is negligible when considering the simplification involved. We represent facility ij's market area by the interval $[x_{i,2j-1}, x_{i,2j+1}]$. The union of the market areas of the $\tilde{m}_i$ facilities supplying the i-th CG coincides with the *residential area* $[0, \mathsf{L}]$ where L, the boundary of the residential area, fulfills the condition that $\mathsf{L} \leq \mathcal{L}$.[6] Accordingly, the spatial characteristics of each facility ij are fully specified by the following triplet of nodes (see Figure 2):

$x_{i,2j-1}$ = the left boundary of the ij-th facility's market area and the right boundary of the $i(j-1)$-th facility's market area,

$x_{i,2j}$ = the location of the ij-th facility, and

$x_{i,2j+1}$ = the right boundary of the ij-th facility's market area and the left boundary of the $i(j+1)$–th facility's market area.

Since each resident must consume all the types of club goods, the extreme boundaries must fulfill, $x_{i,2\tilde{m}_i+1} = \mathsf{L}$, and $x_{i,1} = 0$, for all i.[7]

We define the *clubs configuration* as the vector of integers $\{\tilde{m}_1, ..., \tilde{m}_I\}$, where $\tilde{m}_i$ is the number of facilities of type i in the economy. Thus, the clubs configuration is a vector of I integer variables.

To facilitate the analysis, we sort the clubs
configurations into classes, where each class is represented by a vector $(m_1, ..., m_I)$ ((m_i) for brevity) of I integers which have no common multiplier other than 1, i.e., for every $\lambda \geq 2$, at least one of the quotients m_i/λ, $i = 1, ..., I$, is not an integer. We term the configuration without a common multiplier *a basic configuration.* From here on we designate a club's configuration $(\tilde{m}_i)$ by $k(m_i)$, where (m_i) is the basic configuration designating the class, and the multiplier k is an additional integer-variable to be solved.

In an economy with population $\mathcal{N}$ and available land $\mathcal{L}$ there is a model with the clubs configuration $k(m_i)$. *A complex* in this economy is the optimal solution of a model whose population size is $\frac{\mathcal{N}}{k}$, $\frac{\mathsf{L}}{k}$ is its available land, its clubs configuration is the basic (m_i) and it has the same functions (costs, utility) as in the original model. In the solution of the complex, the common multiplier is 1, the configuration is the basic (m_i) and all its land, $\frac{\mathsf{L}}{k}$, is occupied by $\frac{\mathcal{N}}{k}$ households.

[5] By this we assume that a market area of a facility is a connected segment. In what follows we prove that, indeed, the market area of a facility of a club is a connected segment, provided $t_i(x)$, the club's commuting cost function, is linear in x (see Lemma 3). In the case of nonlinear transportation costs, connected market areas remain an assumption.

[6] By this, we make the assumption that the occupied area is continuous and the unoccupied area is concentrated at the end of the economy, L, and next to the origin, 0.

[7] In this model, the focus is on the case in which all available land is occupied, i.e., $\mathsf{L} = \mathcal{L}$, which implies that $0 \equiv x_{i,1} = x_{i,2\tilde{m}_i+1} = \mathsf{L} = \mathcal{L}, \forall i$. Therefore, calculations with the location variable x are modulo $\mathcal{L}$ (i.e. $\mathcal{L}+x = x$.). For example, for all i, $\tilde{m}_i$ and an arbitrary y, $0 < y < \mathcal{L}$, $x_{i,1} + y = x_{i,2\tilde{m}_i+1} + y = \mathcal{L}+y = y$.

The optimal solution of the model with the configuration $k(m_i)$ can now be described as k consecutive replications of the complex with the basic configuration (m_i). Each two consecutive complexes are adjacent and have a joint boundary. $\mathcal{L} - \mathsf{L} \geq 0$ is the vacant land at the edges. The common multiplier k, is now an integer variable measuring the number of complexes in the economy. Thus, by determining k and characterizing the complex, we characterize the solution of the general model.[8] In the rest of the paper we use the terms basic configuration and *complex configuration,* interchangeably.

Figure 2 depicts the layout of a complex with a basic configuration of $(1, 2, 3)$. For expositional purposes, we mark the nodes of each club on a different horizontal axis. Actually, they are all jointly located on the x axis.

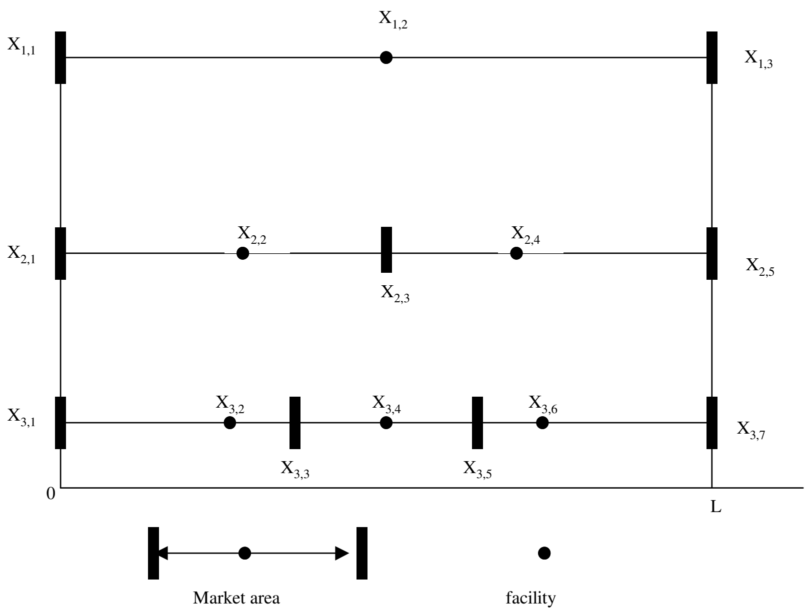

Figure 2: Facility Locations and Market Areas in a Complex with Basic Configuration (1,2,3).

We designate the length (also the area) of the complex by L and the population of the complex by N. Then $kL = \mathsf{L}$ and $kN = \mathcal{N}$. Accordingly, L is also the coordinate of the right boundary of the first complex (whose left boundary is the origin, 0) and the left boundary of the second complex, if it exists and so on. Since all complexes are identical, it is sufficient to solve only for one (the first) complex.

Since all goods are essential, the boundaries of each complex must coincide with the boundaries of the extreme facilities farthest from the center of each of the I CGs; hence

$$x_{i,1} = 0; \quad x_{i,2m_i+1} = L, \text{ for all } i \in \{1, \ldots, I\} \textit{ and } kL = \mathsf{L} \leq \mathcal{L}. \tag{1}$$

Equation (1) implies that, by assumption, the origin is a boundary of all clubs. Similarly, the relation between the complex and the overall population must be

$$N = \mathcal{N}/k. \tag{2}$$

[8] In HPT, a complex is defined as the smallest autonomous area in the economy, i.e., the smallest area in which its residents and only the residents of the area consume all the CGs in the area. It is clear from the discussion so far that our complex satisfies this definition.

In order to use a CG, the household incurs travel costs of a home-facility trip which is given by $t_i(\left|x - x_{i,2j_i(x)}\right|)$, where the argument of the function is the absolute value of the home-facility distance and $j^i(x)$ is the index j of the facility of club i whose residents at x use. The transportation cost function fulfills $\frac{\partial t_i(y)}{\partial y} \overset{def}{=} t_i' > 0$, $\frac{\partial^2 t_i(y)}{\partial y^2} \overset{def}{=} t_i'' \leq 0$, for all $y \geq 0$.

The *provision cost function,* $c^i(G_{ij}, N_{ij})$ (for brevity, hereafter $c^i(j)$) is the cost to facility i, j for providing its CG, G_{ij}, to N_{ij} households. The function $c^i(j)$ fulfills,

$$\begin{aligned}
&c_1^i(j) = \frac{\partial c^i(j)}{\partial G_{ij}} > 0,\ c_2^i(j) = \frac{\partial c^i(j)}{\partial N_{ij}} \geq 0,\ c_{11}^i(j) = \frac{\partial^2 c^i(j)}{\partial G_{ij}^2} > 0, \\
&\partial\left(\frac{c^i(j)}{N_{ij}}\right) \Big/ \partial N_{ij} \begin{cases} < 0\ if\ N_{ij} < \bar{N}_{ij}(G_{ij}), \\ \geq 0\ if\ N_{ij} \geq \bar{N}_{ij}(G_{ij}), \end{cases},\ \partial^2\left(\frac{c^i(j)}{N_{ij}}\right) \Big/ \partial\left(N_{ij}\right)^2 > 0 \\
&where\ 0 < \bar{N}_{ij}(G_{ij}) \leq \infty,\ and\quad G_{ij} \geq 0.
\end{aligned} \tag{3}$$

Thus, $\frac{c^i(j)}{N_{ij}}$ is either a U- or L-shaped function of N_{ij}.[9] The scale economies reflected in the second line of (3) are responsible for the concentration of club goods in facilities. Without these scale economies, a CG would be provided to a household at home, like z, and not in facilities where there is joint consumption of households. Each facility i, j is identified by its CG, G_{ij}, facility location, $x_{i,2j}$, market area, $(x_{i,2j-1}, x_{i,2j+1})$ and the population within its market area, N_{ij}.

A kind of club that requires special attention is the production club, which we designate by the index $i = 1$. Patrons N_{1j}, of facility $1, j$ of a production club work in the club's facility location $x_{1,2j}$, reside in the facility's market area and together with an input of G_{1j} units of composite good, produce a net positive output $\left(-c^1(G_{1j}, N_{1j}) > 0\right)$ of the composite good. Thus, $[G_{1j} - c^1(G_{1j}, N_{1j})']$ is the gross output of the j-th facility of club 1 and as such, is its production function. The general characteristics of a club's cost functions specified in (3), need some modification and interpretation in the case of production club. Thus, instead of (3) we assume,

$$\begin{aligned}
&\begin{matrix} c^1(G_{1j}, 0) = 0 \\ c_{11}^1(j) > 0, \end{matrix};\ c_1^1(j) \begin{cases} < 0,\ if\ G_{1j} < \bar{G}_{1j}(N_{1j}) \\ \geq 0,\ if\ G_{1j} \geq \bar{G}_{1j}(N_{1j}) \end{cases},\ \frac{\partial \bar{G}_{1j}(N_{1j})}{\partial N_{1j}} > 0 \\
&\begin{matrix} c_2^1(j) \leq 0, \\ c_{22}^1(j) \geq 0, \end{matrix},\ \partial\left(\frac{c^1(j)}{N_{ij}}\right) \Big/ \partial N_{ij} < 0;\ \partial^2\left(\frac{c^1(j)}{N_{ij}}\right) \Big/ \partial\left(N_{ij}\right)^2 > 0;
\end{aligned} \tag{4}$$

Accordingly, for $N_{1j} > 0$, the function $c^1(G_{1j}, N_{1,j})$ obtains negative values and is U- shaped as a function of G_{1j}, while the average, $\left(\frac{G_{1j} - c^1(G_{1j}, N_{1j})}{N_{1j}}\right)$ is increasing as a function of N_{1j} . This last property is a reflection of labor-oriented scale economies in production.[10] We also assume in the production club case that the marginal utility of G_{1j} is zero, i.e., $\partial u / \partial G_1 = 0$, which means that G_{1j} is a production factor that does not affect the household's well-being.

We adopt here the assumption accepted in urban economics literature of a non-atomic distribution of population over space. Thus, a household in our model is identified by its residence at x. In addition, we confine ourselves to allocations in which all households are identical in the sense that they all have the same utility function, skills, and initial endowment

[9]Note that $c_2^i(j) = 0$ implies that G_{ij} is a pure public good with an L-shaped average cost function. Then $c^i(G, N) = c^i(G, 1)$ for all values of N and G. When G is a private good distributed equally to each of the N residents, $c^i(G, N) = Nc^i(G, 1)$. Accordingly, as long as $c^i(G, N)$ fulfils, $c^i(G, 1) < c^i(G, N) < Nc^i(G, 1)$, G is a semi(congestable)-local public good.

[10]In what follows, results specific to the production club in subsequent sections will be given in footnotes.

and they all face the same transportation and provision cost structure. In that case, free choice of the location of residency implies an equal utility level for everyone everywhere, namely:

$$u(Z(x), H(x), G_{1,j^1(x)}, \ldots, G_{I,j^I(x)}) = U, \quad \text{for all } x \in [0, kL]\,, \tag{5}$$

where U is the common utility level for all households in the economy and $j^i(x)$ is the index of the facility providing the i-th CG to households living at x. We designate by $u_i(x)$ the derivative of $u(Z(x), H(x), G_{1,j^1(x)}, \ldots, G_{I,j^I(x)})$ with respect to the i-th variable of the utility function as specified in (5) , e.g., $u_2(x) = \frac{\partial u(Z(x),H(x),G_{1,j^1(x)},\ldots,G_{I,j^I(x)})}{\partial H(x)}$.

We now turn to housing construction. Let $H^s(x)$ be the amount of housing constructed per unit land at x. $H^s(x)$ is produced by land and the composite good. The amount of composite good used in the production per unit of land at x is $c_h(H^s(x))$, with $\frac{dc_h(H^s)}{dH^s} \overset{def}{=} c_h^{'}(H^s) > 0$ and $\frac{d^2c_h(H^s)}{d(H^s)^2} \overset{def}{=} c_h^{"}(H^s) > 0$. We term $c_h(H^s)$ as the *housing cost function.* The material balance for housing implies

$$n(x)H(x) = H^s(x), \tag{6}$$

where $n(x)$ is the *population density function*.

The club membership constraint can be written as:

$$\int\limits_{x_{i,2j-1}}^{x_{i,2j+1}} n(x)dx - N_{ij} = 0 \quad \forall i \in \{1, \ldots, I\}, \text{ and } \ j \in \{1, \ldots, m_i\}, \tag{7}$$

and

$$N - \sum_{j=1}^{m_i} N_{i,j} = 0 \quad \forall \ i \in \{1, \ldots, I\}. \tag{8}$$

The *housing price function,* $p_h(x)$, is defined as:

$$p_h(x) \overset{def}{=} u_2(x) \,/ u_1(x), \tag{9}$$

where the composite good Z is the numeraire. From (9) and (5) we substitute out $H(x)$ and $Z(x)$ to obtain the *compensated demand function for housing*, namely

$$H(x) = h[p_h(x), G_{1,j^1(x)}, \ldots, G_{I,j^i(x)}, U] \tag{10}$$

and the *compensated demand function for the composite good*, which is

$$Z(x) = z[p_h(x), G_{1,j^1(x)}, \ldots, G_{I,j^{IJ}(x)}, U], \tag{11}$$

where $p_h(x)$ together with the different CGs and the utility level, U, are arguments in both of the above functions. Let the *aggregate expenditure function* for the (representative) complex be given by $E(N, U)$ where

$$\begin{aligned} E(N,U) &= \int\limits_0^L [n(x)z(\cdot) + c_h(H^s)]dx + \sum_{i=1}^{I}\sum_{j=1}^{m_i} c^i(j) \\ &+ \sum_{i=1}^{I}\sum_{j=1}^{m_i} \int\limits_{x_{i,2j-1}}^{x_{i,2j+1}} n(x)t_i(|x - x_{i,2j}|)dx. \end{aligned} \tag{12}$$

The three terms of the complex's aggregate expenditure function in (12) are the expenditures on consumption and housing production (the first term), the provision cost of all CGs (the second term), and the total transportation costs (the third term). Accordingly, $kE(N,U)$ is the economy's aggregate expenditure function.

Recalling that each individual is endowed with Y units of the composite good, the complex's material balance of the composite good requires that

$$E(N,U) - NY = 0. \qquad (13)$$

In other words, the complex's aggregate expenditure must equal the complex's aggregate supply of the composite good.

The above set of equations (1)-(13) defines the constraints of a feasible spatial resource allocation for the whole economy. Necessary conditions for a Pareto optimal allocation are given in the next section.

3. The Optimal Solution

The necessary conditions for a Pareto optimal allocation in the economy are obtained by maximizing the common utility level, U, subject to the constraints (1)-(13). The Lagrangian and the formal derivation of the first order conditions are specified in Appendix 7.1. In solving the model, we assume for simplicity that the variable k, the number of complexes in the economy, is a real variable and not an integer. By making this assumption, we disregard the factual indivisibility of complexes and allow a fraction of a complex in the solution.[11] The necessary conditions in this section are given for a single complex. In our economy there are k such identical complexes. Another assumption we make is that the complex configuration, $(m_1, ..., m_I)$, is a given vector of I integers. Therefore, the necessary conditions below are for a local optimum. Additional conditions for the global optimum, in which the optimal complex configuration is determined as well, follow in a subsequent section.

The equations in this section are calculated from the necessary conditions derived in Appendix 7.1. The equations here are easier to interpret than the original ones but still constitute a full set of necessary conditions for a Pareto optimal complex, equivalent in every way to the original conditions derived in the Appendix.

3.1. Households and Housing

3.1.1. Housing Construction

In (9) $P_h(x)$ is defined as the quotient $u_2(x)/u_1(x)$. A necessary condition for the efficient allocation given in (14) below, states that *the marginal cost of housing construction*, $c_h'(H^s(x))$, equals $P_h(x)$, i.e.,

$$P_h(x) \left(\equiv u_2(x)/u_1(x)\right) = c_h'(H^s(x)), \ \textit{for all } x \qquad (14)$$

where $H^s(x)$ is the amount of housing constructed per unit land at x and $c_h'(H^s(x))$ is an abbreviation of $\frac{\partial c_h(H^s(x))}{\partial H^s(x)}$. It follows from (14) that $P_h(x)$ is, indeed, the housing price function. Observe that we can solve equation (14) to obtain $H^s(P_h(x))$.

[11] If k is not an integer, there must be a fraction of a complex in the solution. Obviously, an actual allocation contains only complete complexes, which is the case for an integer k. Thus, in the optimal solution, with an integer k, each complex is either smaller or larger than the optimal complex of the solution with a real k, and the utility level is lower as well. The distortion is negligible for a real but relatively large k. The problem of indivisibilities of economic entities is quite common in the economic literature (e.g. the indivisibility of the firm). In our case the problem might be more severe since k is likely to be small. Thus, we can see that the subject of indivisibility of optimal complexes deserves a separate study.

3.1.2. Rent Function

The rent at x, $R(x)$, is defined in (15) below as the difference between the revenue and the cost of construction per unit of land at x. Thus

$$R(x) \overset{def}{=} P_h(x) H^s(P_h(x)) - c_h(H^s(P_h(x))), \ for \ all \ x. \tag{15}$$

The properties of the rent function are given in Appendix 7.2. Note that even though in general housing price functions and rent functions are competitive equilibrium tools, they are well defined in this optimization model and have the same properties as in an equilibrium since housing and land have no external effects associated with them.

Taking the integral of the rent function over the entire country yields ALR, the *aggregate land rent in the economy*, i.e.,

$$ALR = k\int_0^L R(x)dx. \tag{16}$$

Note that the right hand side of the ALR equation above consists of the aggregate land rents *in a complex* multiplied by the number of complexes in the economy.

3.1.3. The Optimal 'Budget Constraint'

Let $j^i(x)$ be the index of the facility of club i to which a household residing at x travels. We define $Tr(x)$ as the *travel and recreation expenditure of a household residing at* x, commuting to facilities $j^i(x)$ located at $x_{i,2j^i(x)}$, paying commuting costs $t_i\left(\left|x - x_{i,2j^i(x)}\right|\right)$ and *congestion tolls*, $c_2^i\left(j^i(x)\right)$, for $i = 1, ..., I$.Thus,

$$Tr(x) \overset{def}{=} \sum_{i=1}^{I}\left[c_2^i\left(j^i(x)\right) + t_i\left(\left|x - x_{i,2j^i(x)}\right|\right)\right]. \tag{17}$$

Note that $Tr(x)$ is a continuous and differentiable function of x everywhere except at facility locations, $x_{i,2j^i(x)}$, where $Tr(x)$ is continuous but not differentiable.

The following equation (18), the *household's optimal 'budget constraint'* at x, is a necessary condition for Pareto optimum.[12] The congestion tolls included in $Tr(x)$ are what distinguish the necessary condition below from an equilibrium budget constraint. We also define in (18) the *household's optimal expenditure function at* x, $e(p_h(x), G_{1,j^1(x)}, \ldots, G_{I,j^I(x)}, T_r(x), U)$, *in short* $e(x)$.

$$\begin{aligned} Y + \nu &= z(p_h(x), G_{1,j^1(x)}, \ldots, G_{I,j^I(x)}, U) + p_h(x)h(p_h(x), G_{1,j^1(x)}, \ldots, G_{I,j^i(x)}, U) + \\ +T_r(x) &\overset{def}{=} e(p_h(x), G_{1,j^1(x)}, \ldots, G_{I,j^I(x)}, T_r(x), U), \ for \ all \ x. \end{aligned} \tag{18}$$

We see that in (18), $p_h(x)$ indeed serves as the housing price and the household's income $Y + v$ is independent of location and consists of the initial endowment of an individual household, Y, plus ν − *an equal share of total alternative -shadow-land- rents* in the economy.[13] Thus, a

[12] Note that if $i = 1$ is a production club, then the expression $\left(-c_2^1\right)$ is the marginal product of labor, which attains positive values and appears as income in the household's optimal budget constraint. In this case the model has a non-zero solution even if Y vanishes.

[13] Namely, $\nu = \frac{LR_A}{N}$, where $R_A \geq 0$ and if $kL < \mathcal{L}$ than $R_A = 0$. See also (26), (25) and the discussion that follows at the end of section 3.

household behaves in the optimum as a utility maximizer who considers as given: his income; the location $x_{i,2j}$ of all facilities (i,j); the quantities of CGs, G_{ij}, in these facilities; and the congestion tolls $c_2^i(j) \overset{def}{=} \frac{\partial c^i(G_{ij},N_{ij})}{\partial N_{ij}}$ the household is required to pay when it uses facility i,j. Each club $i \in (1,...,I)$ has m_i facilities spread throughout the complex and a household at x visits one facility of each club i.

3.2. Clubs

The external effects in the model are concentrated in clubs and therefore most of the equations in this section are not equilibrium relations.

3.2.1. Samuelson's Rule

The necessary condition in (19) below, determines G_{ij}, the optimal amount of CG for facility j in club i. The equation below is a version of Samuelson's well known rule about public goods.

$$\int_{x_{i,2j-1}}^{x_{i,2j+1}} \left[\frac{u_{i+2}}{u_1} n\right] dx = c_1^j(j), \forall i,j, \tag{19}$$

where $c_1^i(j) = \frac{\partial c^i(j)}{\partial G_{ij}}$. On the right-hand side of (19) is the marginal rate of substitution in production between the CG and the composite good and on the left-hand side of (19) is the sum of the marginal rates of substitution in consumption of the users of facility i,j, where the marginal rate of substitution is between the CG and the composite good.[14]

3.2.2. Optimal Facility Location

The optimal facility location, $x_{i,2j}$, should satisfy the necessary condition in (20) below, which is also a necessary condition for the facility location to minimize aggregate transportation costs of patrons to facility (i,j).

$$\int_{x_{i,2j-1}}^{x_{i,2j}} n(x)\, t_i'(x_{i,2j}-x)\, dx = \int_{x_{i,2j}}^{x_{i,2j+1}} n(x)\, t_i'(x-x_{i,2j})\, dx, \forall i,j. \tag{20}$$

In (20) the aggregate marginal transportation costs of patrons on one side of a facility equal the aggregate marginal transportation costs on the other side, so that a marginal shift in the facility location does not change aggregate transportation costs to the facility. It should be noted that linear t_i in (20) implies that on each side of the facility reside an equal number of patrons. The following lemma can now be proved;

Lemma 1 *A club's facility location is an interior point of the club's market area, and therefore of the complex. The market area of a facility is in a bounded segment of the complex.*

The proof of the first part of the lemma follows directly from (20) which requires that patrons should reside on both sides of the facility location. The proof of the second part of the lemma follows from the finiteness of the household's income which allows it to travel only a bounded distance.

[14] For club 1, the production club, after substituting $u_3 = 0$ in (19) reads, $0 = c_1^1(j)$. Indeed, the left-hand side of (19) vanishes when $i = 1$ since $u_3 = 0$. To understand the meaning of (19) when $i = 1$, consider the production function $G_{1,j} - c^1(j)$ in perfect competition. The equality between the value of the marginal product of $G_{1,j}$ and the price of $G_{1,j}$, which is 1, i.e., $\frac{\partial}{\partial G_{1,j}}\left(G_{1,j} - c^1(j)\right) = 1$, results in $0 = c_1^1(j)$, which is, as we just showed, (19) for i=1. So in the case of the production club this condition is fulfilled in perfect competition.

3.3. Bid Price Functions and Nodes

Bid price functions of housing and land are essentially tools of competitive equilibrium analysis. They can be employed in our optimization model since the land and housing markets are free from external effects. The bid price functions below are defined for given facility locations and the CGs in them, and for a given optimal utility level. The crucial assumption which allows bid functions analysis is the assumption of households' freedom to choose their location of residency, which implies equal utility level to identical households everywhere. This assumption is, indeed, part of this model as well as part of other urban competitive models. For a proof that bid housing price functions analysis is compatible with the necessary conditions of this optimization model, see sections 7.1.1 and 7.1.2 in the Appendix.

3.3.1. Bid Housing Price Functions

Let $Tr\left(x, j^1, ..., j^I\right)$ be the sum of the home-facility commuting costs plus the congestion tolls $c_2^i\left(j^i\right)$ a household residing at x pays when traveling to each of the I facilities, $j^1, ..., j^I$, as specified in (18), where j^i is the index of facility j of club i, i.e., $j^i \in (1, ..., m_i)$. The facility j^i is located at $x_{i,2j^i}$,with a given quantity of CG, G_{i,j^i}, i.e.,

$$Tr\left(x, j^1, ..., j^I\right) \overset{def}{=} \sum_{i=1}^{I}\left[c_2^i\left(j^i\right) + t_i\left(\left|x - x_{i,2j^i}\right|\right)\right], \tag{21}$$

$$for\ all\ x,\ j\ and\ i\ s.t.,\ 0 \le x \le L,\ j^i \in (1, ..., m_i),\ i = 1, ..., I.$$

For the household to reside at x and travel to the given I facilities (j^i), the household's optimal budget constraint must fulfill,

$$Y + \nu = z(p_h(x), \left(G_{i,j^i}\right), U) + p_h(x)\, h(p_h(x), \left(G_{i,j^i}\right), U) + T_r\left(x, \left(j^i\right)\right). \tag{22}$$

where $\left(G_{i,j^i}\right)$=$G_{1,j^1}, ..., G_{I,j^i}$; $\left(j^i\right) = j^1, ..., j^I$; $z(p_h(x), \left(G_{i,j^i}\right), U)$ is the compensated demand function for the composite good Z, defined in (11) and $h(p_h(x), \left(G_{i,j^i}\right), U)$ is the compensated demand function for housing H, defined in (10).

The vector $\left(\left(G_{i,j^i}\right), \left(j^i\right), U\right)$ is fixed and given and so is the household's income $Y + v$. The only variable remaining to be determined at a given location x is the price of housing, $p_h(x)$. By substituting out $p_h(x)$ from (22) we obtain the *bid housing-pice of a household residing at x and traveling to facilities at* $\left(x_{i,2j^i}\right)$ where the household uses the CGs, $\left(G_{i,j^i}\right)$. We designate this bid housing-price function by $p_h^b\left(x, j^1, ..., j^I\right)$ .What distinguishes one bid housing price function from another is the set of facilities to which the household travels. Income and utility levels are the same for everybody everywhere and are known parameters as are the CGs and facility locations. Therefore, once the facilities' indices of a bid housing price function are known, all information is revealed. Each bid housing-price function has a different set of I facilities. In each of two different sets of indices there is at least one facility that the other lacks. For some vectors (J^i), there may be locations x for which p_h, substituted out of (22), is negative. In such cases we set the bid housing price equal to zero. We can now prove the following lemma.

Lemma 2 *The bid housing price function is a continuous function of the distance x and twice differentiable, with a positive second derivative everywhere except at the I facility locations $(x_{i,2ji})$ where it is continuous but not differentiable.*[15]

A household at location x, by choosing to travel to facilities that yield the highest bid housing price is actually choosing to attain the utility level at location x by spending the least

[15]For proof of Lemma 2 see Appendix 7.2

of all possible costs other than the cost of housing. Such behavior by all households leads to an efficient allocation. In competitive markets, a household at x travels to the facilities that yield the highest bid housing price at x, because he then can outbid others competing for housing at x. Accordingly, $p_h(x)$, the housing price function at x, fulfills

$$P_h(x) = \max_{j^1,\ldots,j^i} p_h^b\left(x, j^1, \ldots, j^I\right) = p_h^b\left(x, j^1(x), \ldots, j^I(x)\right), \forall x \ where \ j^i \in (1, \ldots, m_i), \ i = 1, \ldots, I. \tag{23}$$

The vector of indexes of facilities $\left(j^1(x), \ldots, j^I(x)\right)$ to which a household residing in x travels to, is merely the vector $\left(j^1, \ldots, j^I\right)$ that maximizes $p_h^b(x, (j^i))$ in (23). Thus, the upper boundary curve of all bid housing price functions as defined in (23), besides being the housing price function also determines the facility locations to which a household at x travels.

3.3.2. Bid Rent Functions

We define the bid rent functions as

$$R^b\left(x, j^1, \ldots, j^I\right) = p_h^b\left(x, j^1, \ldots, j^I\right) H^s\left(p_h^b\left(x, j^1, \ldots, j^I\right)\right) - C_h\left(H^s\left(p_h^b(x, j^1, \ldots, j^I)\right)\right)$$

The bid rent is a monotonic increasing function of $p_h^b\left(x, j^1, \ldots, j^I\right)$ and fulfills $R^b\left(p_h^b = 0\right) = 0$. Therefore, in most cases we can use either the bid rent function or the bid price function.

3.3.3. Boundaries and Facility Locations

In the optimal allocation a *node* x_b on the x-axis is *a boundary point between club-i market areas,* if there are points x_l and x_r, $x_l < x_b < x_r$, such that all residents living in (x_l, x_b) consume the i-th CG in a facility to the left of x_b, and all residents in (x_b, x_r) consume the i-th CG in a facility to the right of x_b.

Let x_b be a boundary point of clubs $i_1, \ldots, i_K$, $1 \leq K \leq I$ and of them only (when $K = I$, x_b is the boundary of the complex). For brevity of notation we also designate by K the set $(i_k, k = 1, \ldots, K)$ and by $I - K$, the set $((i_k \notin K) \ and \ (i_k \in (1, \ldots, I)))$. There is a point x_l, $x_l < x_b$, (x_l can be any point between x_b and the next boundary point to the left of x_b) that residents at every point x, $x_l < x < x_b$ use the I CGs at the same facilities. We designate these facilities by $j_o^1, \ldots, j_o^I$, i.e.,$j_o^i = j^i(x)$, $x_l < x < x_b$. In the same way, there is a point x_r, $x_r > x_b$, where all residents in the segment $x_b < x < x_r$ use the I CGs at the same facilities. In this segment, if $i \in K$, then $j_o^i + 1$ is the facility in which residents consume the i-th CG and if $i \in I - K$, j_o^i is still the facility in which residents of x consume the i-th CG. The necessary condition associated with the boundary x_b now follows,

$$p_h(x) = \max_{j^1,\ldots,j^I} p_h^b\left(x, j^1, \ldots, j^I\right) \begin{cases} = p_h^b\left(x, j_o^1, \ldots, j_o^I\right), & for \ x, \ s.t. \quad x_l < x \leq x_b, \\ = p_h^b\left(x, \left(j_o^i + 1, \forall i \in K\right) \ and \ \left(j_o^i, \forall i \in (I-K)\right)\right), & \\ & for \ x, \ s.t. \quad x_b < x \leq x_r, \end{cases}$$

$$and \ \ p_h(x_b) = p_h^b\left(x_b, j_o^1, \ldots, j_o^I\right) = p_h^b\left(x_b; \left(j_o^i + 1, \forall i \in K\right) \ and \ \left(j_o^i, \forall i \in (I-K)\right)\right). \tag{24}$$

Equation (24) states that the bid function $p_h^b\left(x;, j_o^i, \ldots, j_o^I\right)$ and the bid function $p_h^b\left(x; \left[j_o^i + 1, \forall \ i \in K\right] \cup \left[j_o^i, \ \forall i\right.\right.$ intersect at x_b and are equal to the housing price there. Hence, the two bid functions must coincide with the housing price function in a neighborhood of x_b as well. Note that the lowest line in (24) is the actual necessary condition. Below, x_b is indexed according to the rules set up in Section 2.

$$x_b = x_{i_1, 2j_o{}^{i_1}+1} =, .., = x_{i_K, 2j_o{}^{i_K}+1}.$$

For the proof that (24) is compatible with the necessary conditions, see Appendix 7.1.2.

The location of facility j of club i in our model is a node located at $x_{i,2j}$. The transportation cost function, $t_i(|x - x_{i,2j}|)$, is a continuous and differentiable function of x everywhere except at $x = x_{i,2j}$ where it is not differentiable. Since $Tr(x)$ in (21) contains transportation cost functions, it is continuous and twice differentiable everywhere except at facility locations where it is continuous but not differentiable. This property is passed on to p_h^b solved from (22) and (23) (see lemma 2). Since $p_h(x)$, the housing price function itself, consists of segments of bid housing price functions that intersect at boundaries, it must be continuous and twice differentiable too except at facility locations and boundaries where it is continuous but not differentiable. To sum up the analysis we write it in the form of a corollary,

Corollary 1 *The housing price function, $P_h(x)$, is a continuous and twice differentiable function of x with a positive second derivative everywhere, except in nodes where it is continuous but not differentiable.*

Consecutive facilities of the same club may hold different quantities of the CG. Hence, households residing on different sides of a clubs' boundary may consume different quantities of one or more CGs (depending on whether the boundary is of one or more clubs and whether consecutive clubs have different quantities of their

CG). With discontinuous changes in quantities of CGs consumed in consecutive facilities, discontinuous changes in households' consumption of housing and the composite good may be observed as well when crossing a clubs' boundary. In Appendix 7.1.1 we show that where housing is concerned, the quantity of housing consumed and produced as well as the population density, are continuous functions at a boundary, as stated in the following Proposition.

Proposition 2 *The household's housing consumption, $H(x)$, is continuous everywhere, including in boundary and facility locations. Also continuous everywhere are the density of population, $n(x)$, and the supply of housing, $H^s(x)$.*

It should be noted that unlike the continuity of the supply and demand of housing, the household consumption of composite good may be discontinuous in boundaries. For details and proof of the proposition, see Appendix 7.1.1 and 7.1.2.

3.3.4. <u>Market Areas</u>

In section 2 we assumed that a market area served by a facility is a connected segment of the x-axis. Thus far we used this assumption only for simplifying the notation. Now we prove this assumption endogenously in Lemma 3 for clubs with linear transportation cost functions.

Lemma 3 *The market area of a club with linear transportation cost function is a connected segment of the x-axis.*

For a proof see Appendix 7.3.1. Lemma 3 and Lemma 1 yield the next Proposition:

Proposition 3 *The market area of a club's facility is a bounded area and the facility is located in its interior. Market areas of clubs with a linear transportation cost function are connected.*

Recall that in this study we investigate only allocations in which market areas are connected.

3.4. The Henry George Rule

The alternative land rent, R_A, is the land rent at the boundaries of a complex, i.e., $R_A = R(L)$. R_A is the lowest land rent anywhere in the complex. A necessary condition for Pareto optimum of an economy with identical households is the following relation:

$$\nu = \frac{R_A \mathcal{L}}{\mathcal{N}} \tag{25}$$

where v is the household's income from its share of alternative land rents (see also (18)). The Kunn-Tucker conditions imply that If $\mathsf{L}<\mathcal{L} \Rightarrow R_A = v = 0$ and when $\mathsf{L}=\mathcal{L} \Rightarrow v, R_A \geq 0$.

The last necessary condition for an optimum is the Henry George rule,

$$DLR \equiv \int_0^L \left(R(x) - R_A\right) dx. = \sum_{i=1}^{I} \sum_{j=1}^{m_i} \left(c^i(j) - N_{ij} c_2^i(j)\right), \tag{26}$$

The term $\int_0^L \left(R(x) - R_A\right) dx > 0$, is the *differential land rents* (DLR). Since the DLR on the left hand side of (26) is positive, so is the term on the right hand side of the equation, i.e., the aggregate provision cost, $\sum_{i=1}^{I} \sum_{j=1}^{m_i} c^i(j)$, minus the aggregate congestion tolls, $\sum_{i=1}^{I} \sum_{j=1}^{m_i} N_{ij} c_2^i(j)$ (See also (17) and (18)). This means that congestion tolls alone cannot be the sole source of financing the clubs' operations. In (26) the DLR exactly equals the remaining deficit of the clubs after congestion tolls are paid to the clubs.[16] Therefore, the only net profits in the economy are the alternative land rents. It follows from (25) that in the optimum the overall profits in the economy, if any (i.e., if $R_A > 0$), are distributed among the general population.

4. Decentralization

In this section we deviate from the analysis of agglomeration to discuss briefly the issue of decentralizing the optimal allocation described in the previous section. A laissez faire allocation would not be efficient because of the lack of incentive of club owners to provide the optimal amount of CGs, to impose optimal user charges and to optimally locate the facilities. In actual fact, each facility owner does possess market power and if left to his own devices, will engage in monopolistic competition. To achieve the optimum, a local government (of a complex) has to intervene in the economic operations that take place in its jurisdiction. The government may intervene either directly by providing by itself the optimal CGs in facilities located optimally and by taxing land rents which, together with congestion tolls collected from users, can finance its operations and ensure the fulfillment of the necessary conditions. This type of direct intervention, however, is problematic since, besides there being a lack of information about optimal quantities of CGs, locations of facilities and exact corrective taxes, it requires constant management of facilities. Throughout the ages, governments, especially local ones, have proved themselves to be highly inefficient in managing economic activities, club facilities being no exception.

Conversely, decentralization of CG provision requires of a local government only the determination of prices and income transfers between sectors and their imposition by taxation and subsidization. The Second Fundamental Theorem of Welfare Theory (e.g., see Mas-Colell et al., (1995) Ch. 16, Proposition 16.D.1) proves that, in general, it is possible to decentralize a

[16] In the case of the industrial club, the term $\left(-N_{1,j} c_2^1(j) > 0\right)$ is the wages paid to the workers in the facility and $\left(-c^1(j)\right)>0$ is the value added over the value of the input of the composite good, G_{1j}. Therefore, $c^1(j) - N_{1j} c_2^1(j) > 0$ is the deficit of the production club's facility. Therefore, each facility has to recieve a subsidy from the local government that can be financed by an optimal taxation of land rents. This result is well-known in the literature.

Pareto optimal allocation with specifications fitting our model's assumptions. In Mas-Colell et. al., it is shown that every Pareto optimal allocation (x^*, y^*) (his notation) has a price vector $p = (p_1, ..., p_L) \neq 0$, such that (x^*, y^*, p) is a price quasi-equilibrium with transfers. In other words, in a sufficiently well-behaved economy if agents are price takers, there exist prices and income transfers that yield the optimal solution as a market allocation. In practice, however, the actual determination of these transfers and prices is still an open question that we refrain from investigating at this time.

In the case of non-spatial clubs, an efficient equilibrium exists that does not require any government intervention (e.g., see the outset in HPT). However, in the case of spatial clubs, government intervention is needed to instigate the provision of optimal quantities of CGs at the optimal nodes, since club operators possess market power and do not have any incentive to behave competitively.

We first investigate the case in which club operators can locate facilities only in predetermined sites matching optimal facility locations. We will partially relax this restriction later on. There is no unique way to decentralize our optimum and for different clubs, different methods may be more suitable. A natural way to decentralize our optimum is to allow each facility operator to charge each user the congestion toll $c_2^i(j)$, which ensures the fulfillment of (18). The facility's income from user charges is then $N_{ij}c_2^i(j)$ and, in general, this toll is not sufficient to cover the full cost of running an optimal facility, i.e., the facility's loss is $c^i(j) - N_{ij}c_2^i(j) > 0$ and the local government has to provide the missing funds to cover facilities losses.[17] The General Henry George Rule (26) ensures that the differential land rents, taxable by the local government, are sufficient to cover the total deficit.[18]

The above decentralization method, in which facility operators charge patrons with congestion tolls and are subsidized by the local government, suffers from lack of incentive to behave efficiently by facility operators. By doing nothing and acquiring the government's subsidy, a facility operator obtains the subsidy as positive profits, while by behaving optimally he only ends up without losses (see HPT). Another problem with this method is the lack of government knowledge of how to divide taxed differential land rents into subsidies between different facilities.

Despite the drawbacks of the decentralization method discussed above, there are circumstances in which it is the appropriate one. Consider, for example, the case in which the provision costs are divided into costs of constructing a facility (fixed costs) and marginal costs of operations increasing with the number of users. In such a case, the government can construct the facility, thus paying the fixed costs, and then lease the facility to a private operator who is allowed to charge users the marginal cost while maintaining current operations and paying the variable costs. Knight (1924) showed that there are circumstances under which user charges that maximize profits are exactly equal to optimal congestion tolls in a road system. Indeed, if the facility operator incurs positive profits, the government can obtain these profits as lease payment and redistribute it to households of users.

Another decentralization method is applicable to cases in which division to fixed and increasing marginal costs are not relevant. We let an asterisk designate optimal values of variables

[17]Not all facilities must suffer losses and some may even have profits, however, when pulling together all the clubs there are losses. To see that consider the following Henry George (HG) rule (see also (26)), $0 < DLR \equiv \int_0^L (R(x) - R_A)\, dx = \sum_{i=1}^{I} \sum_{j=1}^{m_i} \left(c^i(j) - N_{ij}c_2^i(j)\right)$. It follows that the double summation in HG rule above is positive, however, it may contain some individual negative terms, each of which belonging to a club. Such clubs need to be taxed instead of subsidized.

[18]In the case of the industrial club, $\left(-c_2^1 > 0\right)$, is the marginal productivity of labor that equals the wage rate. Our assumptions imply that $\left(-N_{1j}c_2^1\right)$, the total wages paid by the industry, are larger than the net production, $\left(-c^i\right)$. Therefore, the industrial club has to be subsidized by the local government. This result is well known in the literature.

and $p^d_{G_{ij}}$, where $p^d_{G_{ij}} \overset{def}{\equiv} \frac{c^i_1\left(G^*_{ij},N^*_{ij}\right)}{N^*_{ij}}$, be the price a household *pays* per unit of G_{ij} it consumes at facility (i,j).[19] Let the price, $p_{G_{ij}}$, be the price the facility ij operator *receives* per unit of CG he provides, which is $p_{G_{ij}} \overset{def}{\equiv} N^*_{ij}p^d_{G_{ij}} = c^i_1\left(G^*_{ij},N^*_{ij}\right)$. Note that in this case the club has positive profits since, $p_{G_{ij}}G^*_{ij} = c^i_1\left(G^*_{ij},N^*_{ij}\right)G^*_{ij} > c^i\left(G^*_{ij},N^*_{ij}\right)$, where both c^i_1 and c^i_{11} are positive (see Section 2). Finally let $S(x)$ be a government subsidy to a household located at x, $S(x) \overset{def}{\equiv} \sum_i \left[\frac{c^i_1\left(G^*_{i,j^i(x)},N^*_{i,j^i(x)}\right)}{N^*_{i,j^i(x)}}G^*_{i,j^i(x)} - c^i_2\left(G^*_{i,j^i(x)},N^*_{i,j^i(x)}\right)\right]$ where the summation is over all the clubs for which this method of decentralization is used. This subsidy compensates households for those charges which are higher than the congestion tolls. The government can finance this subsidy by taxing the facilities' profits. We can now prove the following Proposition,

Proposition 4 *The price vector* $\left(p^d_{G_{ij}}, p_{G_{ij}}, p^*_h(x)\right)$*, the household's subsidy function* $S(x)$*, the model setup in Section 2 for a given basic configuration and the optimal facilities locations constitute a price quasi-equilibrium with transfers that yield the model's Pareto optimal allocation.*

For the proof see section 7.4.1 Note that in this decentralization method, facility operators are price takers and customers pursue the least expensive facility which fulfills their need.

If all facilities of a club are the same, i.e., they all have the same number of patrons and the same amount of CG, the subsidies to a household are identical everywhere. However, if there are clubs with three or more facilities in a complex, some of them may have different amounts of patronage than others. In this case, when facilities of the same club are not identical, the required subsidies to households become location-dependent and may differ between neighborhoods. In practice, local governments do not bother to return the income they tax from clubs to the particular users and instead add this income to the general municipal income by which they provide the general population with goods and services.[20]

So far we have assumed that club managers face predetermined facility locations in a complex, which to the most extent resembles real life. Club sizes and locations are detailed in city master plans, their number is regulated and each club requires a permit. As such, no decentralization of the choice of club locations is really required. The fact that in real life decentralization of the choice of facility locations does not take place is a clear indication of the complexity of such a process.

In what follows we investigate the decentralization of locating facilities purely for academic interest. The optimal facility location is the one that minimizes overall commuting costs

[19]The price $p^d_{G_{ij}}$ defined here does not equate with the household's marginal rate of substitution between the club good and the composite good consumed by the individual. It is, therefore, not really a (Lindhal) price but more a lumpsum tax. However, a price-taking individual will consume the correct optimal CG since this is the quantity of the CG provided by the closest facility and it is the better option of the CG consumption compared to other facilities of the same type. This lumpsum is preferred over Lindhal pricing since all users of a facility pay the same.

[20]Retail stores are facilities of a club with, yet, another method of financing its operations. Stores provide the service of distributing consumption goods to the general public. They buy goods from producers at gross prices and sell them at higher prices. Stores differ from each other in the type of goods they sell, their diversity and prices, accessibility to the store, etc. In practice, although these stores are very competitive, they are not price-takers and the method of payment for their services is not as in Proposition 4. Yet their allocation could be optimal if the government would tax the profits of the stores and refund buyers for excess payment. In practice, taxes on stores are high but the refunding of buyers is practically impossible and, as before, the tax income becomes part of the government's general budget.

from the market area, i.e., (20) has to be fulfilled. If households are left to pay for their own commuting, facility operators will choose facility locations that maximize their patronage and profits and disregard the effect the facility location has on commuting costs. This may lead operators to locate their facilities inefficiently. For example, if two facilities of the same club are in a complex, both of them will locate in the center of the complex, each trying to add to its market area the more densely populated areas in the center of the complex while giving up sparsely populated areas closer to the complex boundary. To induce facility managers to locate efficiently, their goal function should include the minimization of their patrons' total commuting costs, so that (20) is satisfied. To achieve this goal, each facility operator should transport his patrons by himself, in return for a predetermined lumpsum payment. The lumpsum should be the same to all residents living on the same side of the facility and equal to the commuting costs of an individual living at the boundary of the market area. With this method of payment, a facility manager has an incentive to choose a facility location that minimizes overall transportation costs, since he will be maximizing his profits from transportation. Indeed, a first-order condition for such a minimization is (20). At the same time, the local government should tax the additional profits of the club owner and redistribute them among the club's patrons so that the lumpsum transportation payment of a household minus the transport subsidy it receives equals the household's actual transportation costs. In this case, the redistributed amounts vary from one location to another even within the market area of the same facility and even if all clubs are the same. Beside all the above drawbacks which render this decentralization unpractical, this decentralized method of choosing a facility location, suffers from the inherent problem that the facility operator can only acquire monetary travel costs. Costs involving the value of travel time must be borne by the individuals themselves. Thus, the facility operator may only minimize partial commuting costs and does not locate the facility optimally. In view of these drawbacks, we would conclude that the determination of potential facility locations should be left to city planners.

5. Agglomeration of Spatial Clubs and Households

In this section we investigate general agglomeration trends of spatial clubs and households in optimal allocations and elaborate on allocations of two simple basic configurations, each of which characterizes a particular type of club's agglomeration: the first deals with perfect agglomeration of facilities of different clubs and the second involves imperfect agglomeration of facilities of different clubs. We give an example in which perfect agglomeration of facilities in the center of a complex is a unique global optimum. In addition, we show that a local optimum solution of a basic configurations may have a domain in the functions space, in which it is a global optimum. As a reminder, a complex configuration is a vector with I integer components m_i, which do not have a common multiplier. Each m_i designates the number of facilities of club i in a complex. The variable k measures the number of complexes in the economy.

5.1. General Characteristics

In a ring-shaped economy that is partially unoccupied, even if we assume that the occupied land constitutes a single connected segment $(0, \mathsf{L})$, $0 < \mathsf{L}\,(=kL) < \mathcal{L}$, and all the unoccupied land is the segment $(\mathsf{L}, \mathcal{L})$, there are two edges to the occupied land: $\mathsf{L}(\equiv kL)$ and $O(\equiv \mathcal{L})$.[21] Since all CGs are essential, these two edges must be boundary points to all clubs, i.e., the origin, O, is the left boundary of the first facility of each club and L is the right boundary point of the

[21] If the occupied land is not connected, there are more than just two edges to the economy, a fact that strengthens our arguments.

last facility of each club.

To see that an agglomeration of clubs in an economy with edges and a uniform population distribution is ineffective, consider the following example of an allocation of I clubs in such an economy. Each of the clubs has the same number of identical market areas and facilities that are located in their midst, i.e., $m_i = 1, \forall i$, and the number of complexes, k, is also the total number of facilities of each club. Since the extreme two boundaries of every club coincide, it follows that all market areas are common to all clubs and the facilities of all I clubs are jointly located in the center of each market area. In other words, facilities of all clubs agglomerate in a single location at the center of each complex.

We also assume in this example that the population is uniformly distributed over space and that the quantity of CG in each of the facilities of a club is the same. Another assumption is that all households have the same resources and utility level, hence, all households must also consume the same amount of composite good. In short, in the economy just constructed, all households have the same utility level and consume identical bundles of housing, composite good and CGs. Market areas are common to all I clubs and in the center of each market area facilities of all I clubs are agglomerated.

Now, suppose the economy no longer has edges and $kL\,(\equiv \mathsf{L}) = O(\equiv \mathcal{L})$. Then the previous allocation of clubs with common market areas and agglomeration of facilities of the I clubs exists in the edgeless economy as well, but in this case the last boundary of the last market area of each club coincide with each other as well as with the first boundary of the first market area of each club. However, in this edgeless economy, unlike the economy with edges, there are no points that *must* be a boundary to all types of clubs. Actually, a club in the edgeless economy is free to have its boundaries anywhere as long as the distance between two consecutive boundaries of the same club are constant and equal to L. Therefore, all clubs can be arbitrarily arranged in a consecutive order and the location of boundaries and hence of facilities of different clubs, can be arranged so that the distance of a facility of one club from the next consecutive club's facility is L/I. The sizes of a club market areas remain unchanged as in the allocation with agglomeration, the location of each facility remains in the middle of its market area and the quantity of CG in each facility remains as is. The result of such an allocation is first of all that there is no agglomeration of facilities; in fact, the facilities are distributed evenly throughout the ring. Secondly, since the market areas are the same in the two allocations and the distribution of population is uniform, the number of patrons and travel distances in each market area remain the same as in the economy with edges. Consequently, total commuting costs in each facility are unchanged as well as total provision costs. It follows that each household consumes the same basket as before and therefore has the same utility level, but this time there is no agglomeration of facilities in the allocation.

The above example implies that an allocation with an agglomeration of facilities of different clubs in a second best, edgeless economy, constrained to a uniform population distribution, is just one of an infinite number of equivalent allocations, all with the same consumption bundle and utility level but without an agglomeration of clubs. *This, in turn, implies that the agglomeration of facilities of different clubs does not contribute to welfare in an economy with a uniform population distribution and is therefore, an ineffective agglomeration.* The fact that in the above example of an economy with edges, there is a single allocation and in this allocation facilities agglomerate, is entirely due to the economy's edges and to the technical coincidence that all clubs have market areas of the same size. Therefore, in order to avoid confounding the main issues of this paper and to concentrate on essentials, from here on we restrict our analysis to solutions of the model that satisfy the following Condition A:

<u>Condition A</u> *In an optimal allocation investigated here:*

(i) *The number of complexes, k, is an integer.*

(ii) *There is no vacant land in the economy, i.e., $\mathsf{L}\,(=kL)=\mathcal{L}$ and $R_A>0$,*

where R_A is the shadow rent at a complex boundary.

Part (i) of Condition A is intended to avoid the problem of indivisibility of optimal complexes by dealing only with population sizes that are integer multiplications of an optimal complex size. In that we follow HPT. Part (ii) is intended to achieve an edgeless economy to avoid the 'edge-of-the-economy' effect. Under Condition A, for every area $\mathcal{L}$ of the economy we have a lower bound of $\mathcal{N}$, $\underline{\mathcal{N}}(\mathcal{L})$, such that every $\mathcal{N}$ fulfilling Condition A, also fulfills $\mathcal{N}>\underline{\mathcal{N}}(\mathcal{L})$. Then, $\mathsf{L}\,(\equiv kL)=\mathcal{L}$.

We attribute the term *central location pattern (CLP)* to a club's location pattern in which every market area is common to all clubs and facilities of all clubs are located in the center of these joint market areas. Thus, in the above example, the initial location pattern with agglomeration of facilities of all I clubs is a CLP.

We now introduce a new tool to aid in the proof of the next proposition, a rotation of a club system while keeping the population and all the rest of the clubs unmoved. This tool is useful in an edgeless, ring shaped, uniform density economy.

<u>Definition:</u> *Let a rotation of club i be a shift to the right by a given distance of all the nodes of club i, while the rest of the economy remains unchanged.*

Club i nodes are the boundaries and facility locations of club i of all the facilities in the economy and in a rotation of club i, the locations of nodes of clubs other than i remain constant as do the quantities of CGs in the facilities of all clubs, including club i. This rotation maintains constant distances between club i nodes and keeps the locations of households unchanged.

We now return to the first best allocation to prove the following Proposition:

Proposition 5 *In a first-best allocation of a club economy the population density is <u>never</u> uniform, i.e., there are segments of the economy in which $n\,(x)\neq\mathcal{N}/\mathcal{L}$.*[22]

<u>Proof:</u> The proof is by contradiction and it applies to cases satisfying Condition A. Suppose there is an optimal allocation in which $n\,(x)=\mathcal{N}/\mathcal{L}$ *for all* x. We show below that this assumption leads to a contradiction.

We first argue that in an economy with a uniformly distributed population, symmetry considerations alone imply that all the facilities of a club are identical and each household in the economy has the same consumption basket. The optimality of a club is determined in this economy, by the choice of three parameters: the size (length), l_{ij}, of the market area, which in turn determines N_{ij}, the patronage of the facility, where $N_{ij}=nl_{ij}$, the second parameter is the facility location, $x_{i,2j}$, which is always in the midst of the facility and the third parameter is the quantity of CG in the facility, $G_{i,j}$. These three parameters depend only on the density and homogeneity of the population that are the same everywhere, which implies that in all the facilities of a club these three parameters have the same values. Condition A implies that there

[22]When there is at least one transportation cost function whose second derivative is strictly negative, i.e., there is at least one i_o s.t., $t''_{i_0}<0$, we can strengthen the proposition's result. Actually, if $t''_{i_0}<0$, there is no segment in the economy in which the density of population is constant. To see this, consider (B2) in Appendix 7.2, in which we see that if $\dot{p}_h$ vanishes at a point x, $\ddot{p}_h$ must be positive there. Since $\dot{p}_h=0$ if and only if the gradient of the density function is zero as well, i.e., $\dot{n}\,(x)=0$, the assertion follows. ■

are no divisibility problems in the economy. The same symmetry considerations also imply that housing consumption, $H(x)$, is the same everywhere, since the demand for housing and construction conditions are the same everywhere. The same amount of CGs everywhere together with the same utility level to all, u, the consumption of the composite good, $z(x)$, and with it the whole consumption basket are identical everywhere. Thus, in such an optimal allocation, $H(x) = H,\ Z(x) = Z$, and $G_{ij} = G_i$ for all i, j and x, where H, Z, G_i and u are constants.

Consider now a rotation of a club; the consumption of any household in the economy does not change by the rotation: some households may have their relative location changed within the same market area of a facility and others may have the facility changed as well. However, since all facilities of a club contain the same amount of CG and each location in the economy is contained in one of the facilities of the club, all households consume the same amount of CG before and after the rotation.

Since the rotation keeps the length of the market area unchanged and the density of population constant, the patronage of a facility remains N_i. The overall commuting costs in a facility of club i, $\int_0^l nt_i(|y - l/2|)\,dy$, is the same both, before and after a rotation. Commuting costs of a particular household to the nearest facility may increase or decrease, but since the number of facilities does not change and neither do commuting costs in a facility, overall commuting costs in a club do not change.

When rotating a club, we are free to determine the location of one of its nodes, all other nodes are than determined by this choice since the order of and distances between nodes must be maintained. Accordingly, we rotate each club so that they all share one facility location, say, x_2, i.e., $x_{i2} = x_2$, for all $i = 1, ..., I$. Thus, after the rotations, there is a neighborhood of x_2 the size of the smallest market area, in which residents commute only to x_2 to consume all types of CGs.

We now construct two equations describing the optimal 'price of housing function' after the rotations in the neighborhood of x_2. From equations (17) and (18) we obtain (i) below and from the definition of $p_h(x)$ in (9) we obtain (ii) below.

$$p_h^{uo}(x) = \frac{1}{H}\left[Y + \nu - Z - \sum_{i=1}^{I} c_2^i(G_i, N_i) - \sum_{i=1}^{I} t_i(|x - x_2|)\right]. \qquad (i)$$

$$p_h^{uo}(x) = \frac{u_2(x)}{u_1(x)}. \qquad (ii)$$

$\forall x,\ x_2 - \varepsilon \le x \le x_2 + \varepsilon,$ *where* ε *is the length of the smallest market area.*

If we will show that the two expressions of p_h^{uo} in (i) and (ii) obtain different values in some locations we could reach the contradiction that the optimal allocation with uniform population distribution is inconsistent and therefore does not exist. From the discussion above it follows that the values of H, Z, G_i and N_i are the same constants everywhere and do not vary by rotations. Transportation costs to all facilities in the neighborhood of x_2, however, increase with the distance from x_2, as seen in (i) above. Thus the housing price function after the rotations and in the neighborhood of x_2, i.e., $p_h^{uo}(x)$, on the one hand in (i) declines when the distance from x_2 increases and on the other hand in (ii) is a constant, since everywhere, including the neighborhood of x_2, H, Z, and G_i are constants and do not vary with distance. Since $p_h^{uo}(x)$ must be the same in (i) and (ii), this inequality is a contradiction. Therefore our initial assumption that $n(x) = \mathcal{N}/\mathcal{L}$ everywhere, is not correct and there are locations in which the density of population, $n(x)$, is different from $\mathcal{N}/\mathcal{L}$. ■

Another property of an optimal solution in a spatial club economy is that the allocation is symmetric with respect to the center of the complex. The spatial *symmetric structure* is as follows: a club i with an odd number of facilities in a complex, m_i, has facility $j = \frac{m_i+1}{2}$ located in the middle of the complex with its market area spread symmetrically around the facility. The remaining $m_i - 1$ facilities, an even number, are arranged consecutively and located symmetrically with respect to the center of the complex, so that each facility has its mirror image facility on the other side of the center. Thus, facilities j and j' are two symmetric facilities if $j + j' - 1 = m_i$. If m_i is an even number there is no facility in the center and instead a boundary is located there. In this case of an even m_i, all the facilities are symmetrically located around the center so that a facility j is the mirror image of its symmetric facility j' on the other side of the center where $j + j' - 1 = m_i$. The population density is also symmetric around the center of the complex. It should be reminded that in each complex configuration there is at least one club with an odd m_i otherwise the configuration would have a common multiplier and would not be basic. Therefore, there is at least one facility in the center of each complex. We can now prove the existence of the symmetry in an optimal allocation in Proposition 6 below.

Proposition 6 *The optimal complex in a solution of the model that satisfies Condition A is located in a symmetric structure (as described above).*

Proof: To prove the Proposition we need to show that the necessary conditions are consistent with a symmetric structure. We do so by assuming that the first-order conditions are fulfilled in a symmetric structure and show that it does not lead to any contradiction. When checking the consistency of the necessary conditions we need to concentrate mainly on their spatial aspects. Due to Condition A it is sufficient to analyze a representative complex only.

We designate by L the right boundary and length of the representative complex and by o its left boundary. Accordingly, the center of the representative complex is $L/2$, it turns out that $L/2 = x_{i,m_i+1}$ for all i and if m_i is odd, x_{i,m_i+1} is the location of the median facility $\frac{m_i+1}{2}$ and if m_i is even, x_{i,m_i+1} is the boundary between the two middle facilities, $\frac{m_i}{2}$ and $\frac{m_i}{2} + 1$ in the complex. For each x, $o \leq x \leq L/2$ there is a point x', symmetric to x with respect to $L/2$, such that $x' = L - x$, then $L/2 \leq x' \leq L$.

We start the consistency check with equation (18) by showing that the (bid) housing price function(s) calculated from the equation is symmetric. We first show that the function $Tr\left(x\right)$ in (17) is symmetric with respect to $L/2$. If x, $o \leq x \leq L/2$, is in the market area of facility $j = \frac{m_i+1}{2}$ of club i, where m_i is odd, then so is x' and $t_i\left(|x - L/2|\right) = t_i\left(|x' - L/2|\right)$. If x, $o \leq x \leq L/2$ is in a facility of club i with an even m_i or in facility j, $j \leq m_i$ *and* $j \neq \frac{m_i+1}{2}$ of club i that has an odd m_i, then$\left(x_{i,2j} - x\right) = \left(x' - x_{i,2j'}\right)$ and $t_i\left(|x_{i,2j} - x|\right) = t_i\left(\left|x' - x_{i,2j'}\right|\right)$, where $j + j' - 1 = m_i$, $x_{i,2j'} = L - x_{i,2j}$ and $x = L - x'$. So far we have shown that commuting costs of a household at x to facilities of all I clubs are the same as they are to a household at x', where $x' = L - x$.

Symmetry also implies equality of patronage in symmetric facilities, i.e., $N_{i,j} = N_{i,j'}$, as well as equality between the CGs. Hence $G_{i,j} = G_{i,j'}$, where $j + j' - 1 = m_i$. The equalities of the patronage and CGs between symmetric facilities imply that so are the congestion tolls, i.e., $c_2^i\left(G_{i,j}, N_{i,j}\right) = c_2^i\left(G_{i,j'}, N_{i,j'}\right)$. If x is a point in the market area of facility $j = \frac{m_i+1}{2}$ of club i that has an odd m_i, then so is its symmetric point $x' = L - x$, and households in the two locations pay the same congestion toll in facility $j = \frac{m_i+1}{2}$ of club i.

The implications from the arguments above are that two households, one residing at x and the other at x', travel to symmetric facilities, consume the same amounts of CGs and pay the same commuting costs and congestion tolls. This implies that $Tr\left(\cdot\right)$ in (17) fulfills

$$Tr\left(x\right) = Tr\left(x'\right), \ \forall x, x', \ s.t. \ x + x' = L \tag{27}$$

We are now able to show that (18) is consistent with a symmetric structure. From (27) it follows that $Y + v$ -$Tr(x) = Y + v$ -$Tr(x')$. Therefore, $p_h(x)h(p_h(x)) + z(p_h(x))$, the term equal to $Y + v$ -$Tr(x)$ in (18) after moving $Tr(x)$ to the other side of the equation, must also be equal to $p_h(x')h(p_h(x')) + z(p_h(x'))$. The term $p_h(\cdot)h(\cdot) + z(\cdot)$ is a monotonic increasing function of $p_h(\cdot)$, its only independent argument once all the CGs are given.[23] It follows that the bid housing price function solved from $p_h(\cdot)h(\cdot)+z(\cdot)$ are symmetric, i.e., $p_h^b(x, j^1, .., j^I) = p_h^b(x', j^{1\prime}, .., j^{I\prime})$, where $j^i + j^{i\prime} - 1 = m_i$; and $x + x' = L$, $j^{i\prime}, j^i \in (1, ..., m_i)$, $0 \le x \le L/2$,$L/2 \le x' \le L$. This in turn implies that the housing price function itself is symmetric, i.e., $p_h(x) = p_h(x')$, where $x + x' = L$. The symmetry of $p_h(\cdot)$ implies that $n(x) (= n(p_h(x)))$ is also symmetric around $L/2$, since it too is a monotonic increasing function of $p_h(\cdot)$ when the facility locations are given (see Appendix 7.2). For given facility locations we can obtain from (24) the boundaries of symmetric facilities by using bid housing price functions. The symmetry of these functions together with the symmetry of facility locations imply the symmetry of the boundaries. When the boundaries are given, (20) implies the symmetry of facility locations. The consistency of a symmetric complex with the rest of the necessary conditions and constraints follows immediately. ■

Propositions 5 and 6 state that a first-best allocation is symmetric and that its population density is never uniform. The question is whether agglomerations of facilities of various clubs actually take place in an optimal allocation. To answer this question we characterize two optimal allocations, each with a simple yet different complex configuration, where we show that the concentration of households and agglomeration of clubs' facilities do occur in an edgeless economy.

5.2. Perfect Agglomeration

The term *perfect agglomeration* refers to an agglomeration of facilities of different clubs located at the same place.[24] An allocation with a central location pattern (CLP) in which all the facilities in a complex are located in the center of the complex is an example of perfect agglomeration. In what follows we show that our model with the configuration $\overbrace{(1, ..., 1)}^{I}$ has an optimal solution with a CLP that satisfies the necessary conditions specified in Section 3. From Proposition 6 we know that all the facilities in a CLP allocation are located in the center of a symmetric complex. By the superscript c we designate the optimal values of variables of the model with the configuration $\overbrace{(1, ..., 1)}^{I}$. In the Proposition below we investigate properties of the model's solution.

Proposition 7 *The optimal allocation of a spatial clubs economy with the configuration* $\left(\overbrace{1, ..., 1}^{I}\right)$ *that satisfies Condition A consists of k complexes, each with a CLP, in which facilities of all the clubs are located in the center of the complex and the population is distributed symmetrically around the complex center. The population density function and the price of housing function are both symmetrical around the complex's center, continuous and differentiable everywhere except at the nodes where both functions are continuous but not differential. Both functions are declining with distance from the center and the housing price function has a positive second*

[23]To see this, note that since h and z are compensated demand functions, $p_h \frac{\partial h}{\partial p_h} + \frac{\partial z}{\partial p_h} = 0$. When we substitute the above expression into the differential of the term $p_h h + z$ with respect to p_h we obtain $\frac{\partial(p_h h+z)}{\partial p_h} = h > 0$.

[24]In a model where facilities occupy space, perfect agglomeration means that the areas occupied by the facilities are adjacent to each other with no households in between.

derivative.

Proof: In a CLP $x_{i,2j} = x^c_{2j}, \forall i$, since all facilities are located in the center of the j-th complex, for all $j = 1, ..., k$. Substituting x^c_{2j} into equation $(B1)$ in Appendix 7.2 yields,

$$\dot{p}_h(x) = -\frac{1}{h(x)} \sum_{i=1}^{I} t'_i(|x - x^c_{2j}|) sign(x - x_{2j});$$
$$\forall x,\ x^c_{2j-1} < x < x^c_{2j+1}, and\ \forall j,\ j \in (1, ..., k), \tag{28}$$

where a dot above a function designates differentiation with respect to distance and x^c_{2j-1} and x^c_{2j+1} are the boundaries of complex j and of all the facilities in the complex. Let $y_j \overset{def}{=} |x - x^c_{2j}|$; s.t., $x^c_{2j-1} \leq x \leq x^c_{2j+1}, \forall j,\ j \in (1, ..., k)$, be the distance of a point x in complex j from the center of the complex, x^c_{2j}. Then $0 \leq y_j \leq L/2, \forall j$. If we designate points in complex j by y_j then $y_j = 0$ is the center of complex j, and $y_j = L/2$ are the boundaries of the complex. Note that each y_j, $0 < y_j \leq L/2$, stands for two symmetric points in the complex. In what follows we avoid using the index j or superscript c, unless there is a possibility of confusion. From (28), after substituting into it y for $|x - x^c_{2j}|$, we obtain

$$\frac{\partial p_h(y)}{\partial y} = -\frac{1}{h(y)} \sum_{i=1}^{I} t'_i(y). \tag{29}$$

Equation (29) implies that the first derivative of $p_h(y)$ is negative. This means that $p_h(y)$ is monotonically decreasing, its highest value is at the center where $y = 0$ and its lowest value at the boundaries where $y = L/2$. The second derivative of $p_h(y)$ is always positive (see $(B2)$ in Appendix 7.2), which means that the rate of decline decreases with distance from the center. From $(B4)$ in Appendix 7.2, we learn that $n(p_h)$ for a given set of CGs, (G_i), is a monotonic increasing function of p_h. This implies that n^c, like p^c_h, is monotonically decreasing with y, and attains its highest value in the center of the complex where $y = 0$, and its lowest value at the boundaries where $y = L/2$. ■

Corollary 8 *In an optimal allocation with a CLP, the agglomeration of facilities of different clubs in the center of each complex is accompanied by a concentration of households around the center.*

The proof follows directly from Proposition 7.

Definition: An optimal allocation of the model with a given set of functions and the basic configuration M, is a global optimum if any optimal allocation of the model with the same functions but with a basic configuration other than M has a lower value of the goal function.

In the example below we present additional specifications to the model's general functions introduced in Section 2. The allocation with the CLP of Proposition 7 together with the complex configuration $\left(\overbrace{1, ..., 1}^{I}\right)$ is the global optimum solution of the model whose functions fulfill the specifications in the example below.

EXAMPLE: Functions Specifications For A Global Perfect Agglomeration

Consider a model of an economy with spatial clubs which in addition to the conditions on the functions set in Section 2, satisfies the following more specific conditions;

1. The utility function is of the form $u = U(H, Z, \psi(G_1, \ldots, G_I))$, *where* $\psi(G_1, \ldots, G_I)$ *is invariant for permutations of the set* $(G_1, \ldots, G_I)$, *e.g.,* $\psi(G_1, \ldots, G_I) = \prod_{i=1}^{I} G_i$.

2. All clubs share the same transportation cost function, i.e., $t_i(y) = t(y), \forall i$ *an the same provision cost function, i.e.,* $c^i(G_{ij}, N_{ij}) = c(G_{ij}, N_{ij}), \forall i, j.$

The housing price function of a CLP in an optimal complex is depicted in Figure 3.

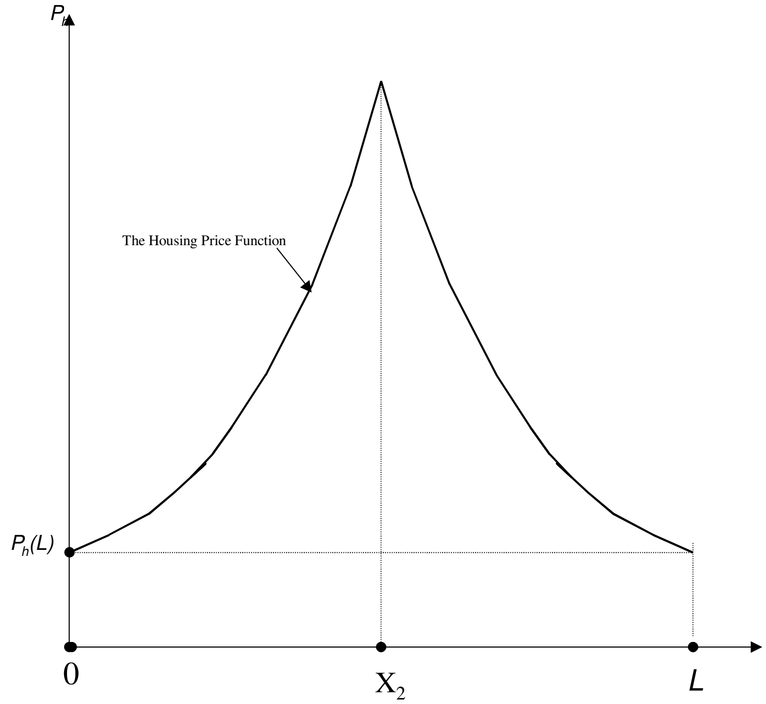

Figure 3: The Housing Price Function of an Optimal Complex with the Configuration (1,...,1).

Lemma 4 *An optimal CLP allocation as described in Proposition 7 is a global optimum solution to the model with functions from the above Example.*

Proof: To show that the global optimum with functions from the Example has a CLP, we show that such a global optimum has the configuration $\left(\overbrace{1,...,1}^{I}\right)$. If in a global optimum solution to the model with functions in the Example above, the sizes of a particular club's market areas, the quantity of CGs in each of the club's facilities and the club's facility locations and boundaries satisfy the necessary conditions specified in section 3 and are therefore optimal for this one club. These same values also satisfy the necessary conditions of all other clubs and the same values are optimal for all clubs. The reason for this is that all clubs have the same cost functions and utility function. This implies that every market area is common to all clubs, which in turn, implies that each market area is a separate complex. The configuration of such a complex is $\left(\overbrace{1,...,1}^{I}\right)$ and an allocation with a CLP is, according to 7, its optimal solution.■

It should be noted that a marginal change in the number of facilities in a complex is impossible and the smallest change is of one more (or less) facility. Therefore, sufficiently small variations in the specifications of the functions would leave the basic configuration of the global optimum unchanged. For example, if instead of using the utility function $u = H \cdot Z \cdot \prod_{i=1}^{I} G_i$

in the Example, the utility function used is $u = H \cdot Z \cdot \prod_{i=1}^{I} G_i^{1+\alpha_i}$, where $|\alpha_i|$ are sufficiently small yet different from each other, the global optimum allocation would still be a CLP. The same is true for small variations in the transportation cost functions of the different clubs or small differences in their provision cost functions. However, while the basic configuration of the global optimum may not change due to small variations, all other variables change continuously.

5.3. Imperfect Agglomeration

In this section we characterize the optimal allocation of the model with the basic configuration given below,

$$(m_i = 1,\ \forall i = 1, ..., I_1\ \ \&\ \ m_i = 2\,, \forall i = I_1 + 1, ..., I)\,;\ 1 \leq I_1 < I < \infty. \tag{30}$$

While in perfect agglomeration we investigate the agglomeration of facilities of all clubs in a single location at the center of the complex, in this section we investigate an example of *an imperfect agglomeration.* Such an agglomeration takes place when facilities of different clubs agglomerate in clusters around the center of the complex but steer clear of it. By saying that the agglomeration is imperfect we mean that the clusters may contain dwellings between the facilities.

We first introduce the symmetric structure of the allocation with the configuration given in (30) as specified in Proposition 6. The symmetric structure possesses the following properties: (1) Each of the clubs $i \in 1, ..., I_1$, (henceforth SF clubs) have one facility located in the middle of the complex and its market area is the whole complex. (2) The two facilities of each of the clubs $i \in (I_1 + 1, ..., I)$ (henceforth DF clubs), are symmetrically located on each side of the center of the complex and each of their market areas is extended between a complex boundary and the center. Altogether, the complex has I_1 facilities of SF clubs and $2\,(I - I_1)$ facilities of DF clubs, $(I - I_1)$ facilities of DF clubs on each side of the complex center. The properties of the allocation with the configuration (30) are depicted in the series of lemmas presented in the rest of this section.

Lemma 5 *In the optimal allocation of the model with the configuration (30) discussed above, all the facility locations of the DF clubs are in the second and third quarters of the complex area.*[25] *The average density of the population residing between the two facilities of the DF club located farthest from the center (one facility to the left and one to the right of the center), is higher than it is between these two DF facilities and the boundaries.*

Proof: Without loss of generality, we consider only the first half of the representative complex, namely the segment $(0, L/2)$ (an equivalent analysis would apply to the other half $(L/2, L)$). Let $\underline{x}_2$ in $(0, L/2)$ be the facility location of the DF club closest to the origin, 0, and farthest from the center $L/2$ (if there is more than one such club, one of them is chosen arbitrarily). This means that all facility locations of the DF clubs in $(0, L/2)$ are located between $\underline{x}_2$ and the center, $L/2$. We show below that $\underline{x}_2$ is in the second quarter of the complex area and therefore so are the rest of the DF locations.

To see that $\underline{x}_2$ is in the second quarter, consider a location x that is to the left of $\underline{x}_2$, i.e., $0 \leq x < \underline{x}_2$. The point x has a point x', $\underline{x}_2 \leq x' \leq L/2$, symmetric to x with respect to $\underline{x}_2$, so that $\underline{x}_2 - x = x' - \underline{x}_2$. Note that all the individuals residing in the segment $(0, L/2)$ travel to

[25] By the term 'quarter' we refer to a segment which results from a division of the complex's length into four equal consecutive segments. The first quarter is the segment farthest to the left and the other three quarters are numbered consecutively in the clockwise direction.

the same facilities. An individual residing in x' travels to the same facilities as the individual at x, however, the distances from x' to all the facilities that are not located in $\underline{x}_2$ are shorter than from x, because all the facilities are located to the right of $\underline{x}_2$. Even if all the DF clubs are located in $\underline{x}_2$, the distance to travel from $x' = 2\underline{x}_2 - x$ to the SF-clubs in the center is still shorter. Accordingly, commuting costs from x' to all facilities are lower than from x, while all other arguments of $p_h^{ia}(x)$ are the same. In $(B1)$ in Appendix 7.2 it is shown that an increase in transportation costs, while all arguments of $p_h(x)$ are kept constant, causes $p_h(x)$ to decline. Therefore, $p_h^{ia}(x)$, the housing price at x, $0 \leq x < \underline{x}_2$, must be lower than $p_h^{ia}(x')$, the housing price at x'. Consequently, because $n(p_h)$ is monotonic increasing in p_h while the rest of the variables remain constant it follows that $n^{ia}(x) < n^{ia}(x')$. Consider now (20) with respect to facility ($\underline{i}$,1) located in $\underline{x}_2$. On the left-hand side of the equation, for every x which has a symmetric x', $t'_{\underline{i}}(\underline{x}_2 - x)$ is equal to $t'_{\underline{i}}(x' - \underline{x}_2)$ on the right-hand side. However, $t'_{\underline{i}}(\underline{x}_2 - x)$ on the left-hand side of the equation is weighted by $n^{ia}(x)$, which is lower than $n^{ia}(x')$, the weight of $t'_{\underline{i}}\left(x' - \underline{x}_2\right)$ on the right-hand side. This implies that for the equality in (20) to hold, the interval $(0, \underline{x}_2)$, in which the weights are lower, must be longer than the interval $(\underline{x}_2, L/2)$ in which the weights are higher. This, in turn, implies that $\underline{x}_2 - 0 > L/2 - \underline{x}_2$, which means that $\underline{x}_2 > L/4$ (see Figure 4). This is the first item we have to prove in the lemma.

We just proved that $n^{ia}(x') > n^{ia}(x)$ for all x located to the left of $\underline{x}_2$ and having a symmetric point x' with respect to $\underline{x}_2$. Additionally, there are points between the origin and the point $(2\underline{x}_2 - L/2)$, the point symmetric to $L/2$ with respect to $\underline{x}_2$, which is located to the left of $\underline{x}_2$. These points left of $(2\underline{x}_2 - L/2)$ have no matching symmetric points and since the transportation costs of residents in these locations are higher than in any point x that has a matching symmetric point, the value of $p_h^{ia}(x)$ for $x < (2\underline{x}_2 - L/2)$ must be lower than it is for any $x>(2\underline{x}_2 - L/2)$. Furthermore, as $x < (2\underline{x}_2 - L/2)$ approaches the origin, $p_h^{ia}(x)$ continues to decline. To see this, consider equation $(B1)$ in Appendix 7.2. In this equation, $\dot{Tr}(x)$ is positive (negative) when x increases (declines) since $sign(x - x_{i,2ji})$ is negative for all i. Therefore, $\dot{p}_h^{ia} = -\frac{\dot{Tr}}{h(x)}$ is positive, i.e., $p_h^{ia}(x)$ declines when x approaches zero. In turn, it follows from (B4) in Appendix 7.2 that the density of population at the unmatched points to the left of $(2\underline{x}_{i2} - L/2)$ is lower than at any other point to the right of $(2\underline{x}_{i2} - L/2)$. Thus, the average density of population between $\underline{x}_2$ and the origin must be lower than between $\underline{x}_2$ and $L/2$.■

In what follows we show that all the DF clubs agglomerate in two clusters, one in the second quarter of the complex and the other in the third. Let $\overline{x}_2$ designate the facility location of the DF club in $(0, L/2)$, which is located closest to $L/2$. At the DF club closest to the center of the complex (20) is satisfied with the point $\overline{x}_2$ as the location of the facility. This means that there is a positive distance between $\overline{x}_2$ and the boundary at $L/2$, equal to the short side of the market area of the facility closest to the center. It follows that all the facilities of the DF clubs in the first half of the complex are located between $\overline{x}_2$ $(< L/2)$ and $\underline{x}_2$ $(> L/4)$ and are clustered together close to each other in the second quarter of the complex (and consequently, the DF clubs in the second half of the complex are clustered in the third quarter of the complex). We term such close grouping of facility locations, a *cluster of DF facilities.* In the lemma above we showed that such clusters of DF clubs are located closer to the center of the complex than to the boundaries. In such cases we say that the DF clusters *gravitate towards the center of the complex.*

To clarify the role of transportation costs in an imperfect agglomeration of DF clubs, consider the following Lemma;

Lemma 6 *In an allocation with the basic configuration specified in (30), different DF clubs with* *proportional transportation cost functions share the same facility locations.*[26]

Proof: Suppose that i and i' are two clubs with proportional transportation costs, i.e., $t_i(x)/t_{i'}(x) = \alpha_{ii'}, \forall x$, where $\alpha_{ii'}$, the factor of proportionality, is constant. Then the proportionality is retained by the derivatives as well as the functions and $t_i'(x) = \alpha_{ii'} t_{i'}'(x)$ .Thus, if (20) holds for club i it holds for its proportional club i' at the same facility location as well. To see this consider (20) for club i, in which we substitute $\alpha_{ii'} t_{i'}'(x)$ for $t_i'(x)$ and then we eliminate the proportionality factor $\alpha_{ii'}$ from the equation to obtain (20) for club i' at the same facility location as club i. ■

Note that all linear transportation cost functions are proportional and therefore DF clubs with linear transportation cost functions agglomerate perfectly at a single location. Examples can be constructed of non-proportional transportation cost functions of DF clubs that yield different facility locations for each club.

Figure 4 depicts the housing price function in an optimal complex with the configuration (1,2).

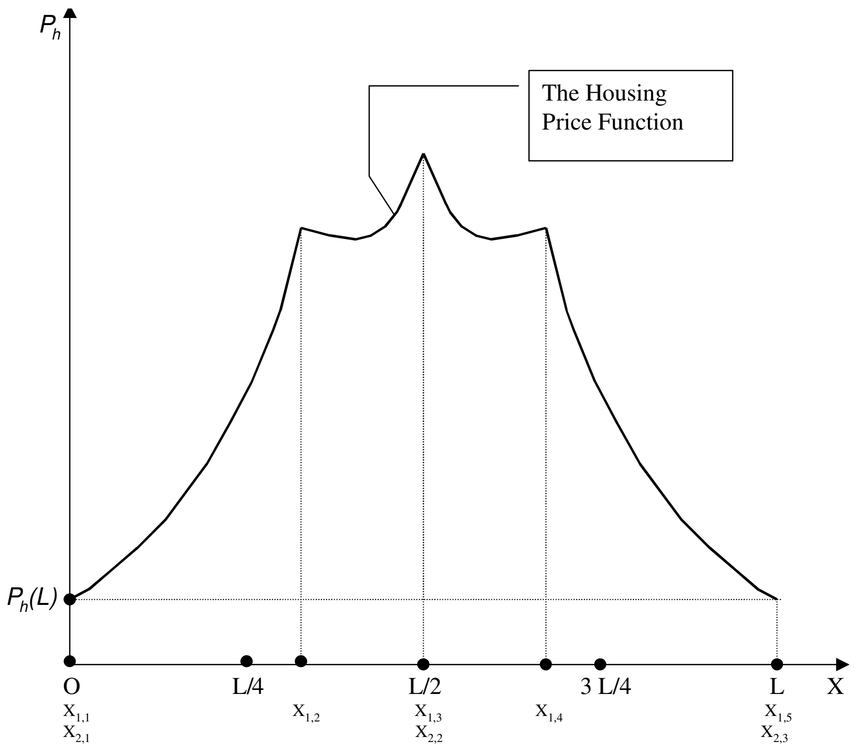

Figure 4: The Housing Price Function in a Complex with the Configuration (1,2)

We can now summarize the analysis of imperfect agglomeration performed in this section by the following Proposition;

Proposition 9 *In an optimal allocation with the basic configuration specified in* (30)

(*i*) *The facilities of SF clubs in the complex, i.e., of clubs* $i \in (1, ..., I_1)$, *are all perfectly agglomerated in the center of the complex;*

(*ii*) *The facilities of DF clubs, i.e., of clubs* $i \in (I_1 + 1, ..., I)$, *agglomerate imperfectly in clusters that gravitate towards the center of the complex, i.e., the clusters agglomerate in the second and third quarters of the complex.*

[26] Recall that in our model, if two facilities of different club types share the same location, it means that they are adjacent to each other with no residential area between them.

(*iii*) *The average density of population between the clusters of the DF clubs and the center of the complex is higher than the average density between these clusters and the boundaries of the complex.*

(*iv*) *If, in a cluster, two DF clubs have proportional transportation cost functions, they share the same facility location.*

6. Summary and Concluding Remarks

The purpose of this paper was to characterize optimal allocations of an economy with spatial clubs and to investigate agglomeration trends of households and club facilities in it. Our results showed that each local optimum could be decentralized, sometimes in more than one way, although most were difficult, if not impossible, to implement. In an optimal allocation of clubs, the primary agglomeration was of club goods into facilities due to scale economies in their provision to the population. The primary agglomeration led to a secondary agglomeration of population which, in turn, led to the tertiary agglomeration of facilities of different clubs in centers. The three types of agglomerations occurred simultaneously and their ordering is due to causality not timing: Without the primary agglomeration there would not be a secondary one, and without the secondary one there would not be a tertiary agglomeration. Furthermore, an optimal allocation would never have a uniform population distribution and neither would an allocation with a uniform distribution of population have an efficient agglomeration of facilities. We then showed that the price of housing as well as the supply and demand for housing functions were continuous functions of the distance, x, as was the density of population function. We also showed that the optimum complex was symmetric with respect to its center.

We characterized two allocations, each with a specific complex configuration: in the first allocation, each complex contained one facility of each club and in the second allocation each complex contained both: clubs that had one facility per complex and clubs that had two facilities per complex. We identified two distinct types of agglomerations of club facilities: the perfect agglomeration and the imperfect one. In the perfect agglomeration, facilities of different clubs agglomerated perfectly in the center of the complex, where they are adjacent to each other without residential activity between them. In the second allocation, besides the perfect agglomeration of facilities of some of the clubs in the center of the complex, facilities of the rest of the clubs agglomerated imperfectly in clusters, but households still may were residing between them. While the clusters as a whole were away from the center of the complex, but drawn towards it.

In respect to the issue of global optimum solutions of the model, the solutions specified were mostly to the model with a predetermined complex configuration. We termed such an allocation a local optimum since the global optimum included a solution to the configuration variables as well. The only global optimum was the solution to the model with functions specified in the Example in section 5.2 together with the configuration $(1, ..., 1)$.

The purposes of this paper were completely satisfied so far. One avenue for future research may focus on the relation between certain costs and utility functions and their global optimal configuration. Such a research may shade light on questions like what functions would result in a hierarchy of clubs in a global optimum or what causes certain types of clubs to be imbedded in other club facilities.

REFERENCES

1. Berglas, E. (1976) "On the Theory of Clubs," *American Economic Review,* May (Paper and Proceedings), 66(2), 116-121.

2. Berliant, M. and T. ten Raa (1994) "Regional Science: the State of the Art," *Regional Science and Urban Economics*, 24, 631-647.
3. Fujita, M. (1989) *Urban Economic Theory*, Cambridge: Cambridge University Press.
4. Fujita, M. and H. Ogawa (1982) "Multiple Equilibria and Structural Transition of Non-monocentric Urban Configurations," *Regional Science and Urban Economics*, 12, 161-196.
5. Fujita, M. and J. F. Thisse (1986) "Spatial Competition with Land Market: Hotelling and Von Thunen United," Review of Economic Studies53,819-41.
6. Fujita, M. and J. F. Thisse (1996) "Economics of Agglomeration," *Journal of the Japanese and International Economies*, 10, 339-378.
7. Fujita, M. and J. F. Thisse (2002) *Economics of Agglomeration,* Cambridge, Cambridge University Press.
8. Henderson, J. V. (1974) "The Sizes and Types of Cities," *American Economic Review*, 64, 640-56.
9. Hochman, O., D. Pines and J.-F. Thisse (1995) "On the Optimal Structure of Local Governments," *American Economic Review*, 85, 1224-1240.
10. Knight, F. H. (1924) "Some Fallacies in the Interpretation of Social Cost," The Quarterly Journal of Economics, 38, 4, 582-606.
11. Mills, E. S. (1967) "An Aggregative Model of Resource Allocation in a Metropolitan Area," *American Economic Review,* 57, 197-210.
12. Mills, E. S. and B. W. Hamilton (1989) *Urban Economics*, Scott, Foresman, 4th ed.
13. Ogawa, H. and M. Fujita, (1980) "Equilibrium Land Use Patterns in a Nonmonocentric City," *Journal of Regional Science*, 20:4, 455-475.
14. Muth R. F. (1969) *Cities and Housing,* Chicago, University of Chicago Press.
15. Lucas, R. E. Jr. and E. Rosi-Hansberg (2002) "On The Internal Structure of Cities," *Econometrica,* 70,4,1445-1476.
16. Papageorgiou, Y. and D. Pines (1999) *An Essay on Urban Economic Theory*, Kluwer Academic Press, Ch. 10.
17. Scotchmer, S. (2002) "Local Public Goods and Clubs" in *Handbook of Public Economics,* Vol. IV, (A. Anerbach and M. Feldstein, Eds), North-Holland.
18. Scotchmer, S. and M. H. Wooders (1987) "Competitive Equilibrium and the Core in Club Economies with Anonymous Crowding," *Journal of Public Economics,* 34(2), 159-174.
19. Solow, R. W. and W. S. Vickrey (1971) "Land Use in a Long Narrow City," *Journal of Economic Theory*, 3, 430-447.
20. Thisse, J.-F. and D. E. Wildasin (1992) "Public Facility Location and Urban Spatial Structure: Equilibrium and Welfare Analysis," *Journal of Public Economics*, 48, 83-118.

7. Appendices

7.1. Deriving the Necessary Conditions for a Pareto Optimum

The Lagrangian, $\mathbb{L}$, of the problem set up in Section 2 is given below in $(A1)$. The Lagrange multipliers are: $[\lambda(x), \alpha(x), \delta_{ij}, \gamma_i, \rho, \omega, \eta, \ell_i, \theta_i]$. The constraints multiplied by ω, ℓ_i, θ_i in the Lagrangian are effective equalities and their multipliers can obtain any sign. The rest of the constraints are inequalities and their multipliers must be non-negative. When these multipliers are positive the constraint they multiply is effective and when a constraint is not effective, its multiplier vanishes. Thus,

$$
\begin{aligned}
\mathbb{L} &= U - \int_0^L \lambda(x)\left(U - u(Z(x), H(x), G_{1,j_1(x)}, \ldots, G_{I,j_I(x)})\right) dx - \sum_i \gamma_i \left(N - \sum_{j=1}^{m_i} N_{i,j}\right) \\
&- \int_0^{\mathsf{L}} \alpha(x)\left(n(x)H(x) - H^s(x)\right) dx - \sum_i \sum_j \delta_{ij}\left(N_{ij} - \int\limits_{x_{i,2j-1}}^{x_{i,2j+1}} n(x)dx\right) \\
&-\eta\left[\int\limits_0^L [n(x)Z(x) + c_h(H^s)]dx + \sum_{i=1}^I \sum_{j=1}^{m_i} \int\limits_{x_{i,2j-1}}^{x_{i,2j+1}} n(x)t_i(|x - x_{i,2j}|)dx + \sum_{i=1}^I \sum_{j=1}^{m_i} c^i(i,j) - NY\right] \\
&- \sum_i \ell_i\left(x_{i,2m_i+1} - L\right) - \sum_i \theta_i x_{i,1} - \rho\left(L - \mathcal{L}/k\right) - \omega\left(\mathcal{N}/k - N\right). \qquad (A1)
\end{aligned}
$$

The following equations are first-order conditions for the maximization of the model. To obtain a first-order condition we differentiate the Lagrangian with respect to a variable of the model and equate the result to zero. In each of the first-order conditions, the particular derivative is written at the beginning of each equation and to the left of the double slashes.

$$
\frac{\partial \mathbb{L}}{\partial x_{i,2m_i+1}}\Bigg\| \; n(L)\frac{\delta_{i,m_i}}{\eta} - n(L)\left[t_i(|L - x_{i,2m_i}|)\right] = \frac{\ell_i}{\eta}. \qquad (A2)
$$

$$
\frac{\partial \mathbb{L}}{\partial L}\Bigg\| \; \sum_i \frac{\ell_i}{\eta} - \left[n(L)Z(L) + c_h(H^s(L))\right] = \frac{\rho}{\eta}. \qquad (A3)
$$

$$
\frac{\partial \mathbb{L}}{\partial N_{ij}}\Bigg\| \; \frac{\gamma_i}{\eta} - c_2^i(G_{ij}, N_{ij}) = \frac{\delta_{ij}}{\eta} \qquad (A4)
$$

Substituting $(A4)$ for $j = m_i$ into $(A2)$ and the result into $(A3)$, yields

$$
n(L)\sum_i \frac{\gamma_i}{\eta} - \frac{\rho}{\eta} = n(L)\sum_{i=1}^I \left[t_i(|L - x_{i,2m_i}|) + c_2^i(i, m_i)\right] + n(L)z(L) + c_h(H^s(L)). \qquad (A5)
$$

$$
\frac{\partial \mathbb{L}}{\partial n(x)}\Bigg\| \; \sum_i \frac{\delta_{ij_i(x)}}{\eta} - \frac{\alpha(x)}{\eta}H(x) - z(x) - \sum_{i=1}^I t_i(|x - x_{i,2j_i(x)}|) = 0. \qquad (A6)
$$

Substituting $(A4)$ into $(A6)$ yields

$$
\sum_i \frac{\gamma_i}{\eta} - \left[z(x) + \frac{\alpha(x)}{\eta}H(x) + \sum_i c_2^i(i, j_i(x)) + \sum_{i=1}^I t_i(|x - x_{i,2j_i(x)}|)\right] = 0, \; \forall x, \; 0 \le x \le L. \quad (A7)
$$

$$
\frac{\partial \mathbb{L}}{\partial Z(x)}\Bigg\| \; \lambda(x) = \frac{\eta n(x)}{u_1(x)}. \qquad (A8)
$$

$$\left.\frac{\partial \mathbb{L}}{\partial H(x)}\right\| \begin{array}{l}\lambda\left(x\right)u_2(x)-\alpha\left(x\right)n(x)=0\\ \Longrightarrow n(x)\left[\frac{\alpha(x)}{\eta}-\frac{u_2(x)}{u_1(x)}\right]=0.\end{array} \quad (A9)$$

$$\left.\frac{\partial \mathbb{L}}{\partial H^s\left(x\right)}\right\| \frac{\alpha\left(x\right)}{\eta}-c_h'(H^s)=0. \quad (A10)$$

We multiply $(A7)$ for $x=L$ by $n\left(L\right)$ and then substitute the result together with $(A10)$ and $n\left(L\right)H\left(L\right)=H^s\left(L\right)$ into $(A5)$, to obtain;

$$H^s\left(L\right)c_h'(H^s\left(L\right))-c_h(H^s\left(L\right))=\frac{\rho}{\eta}. \quad (A11)$$

Equation $(A7)$ implies that $\frac{\rho}{\eta}$, the shadow price of the occupied land constraint, is also the land rent at L, the boundary of the complex. By applying the Kunn-Tucker rule to (A11) we obtain: $\frac{\rho}{\eta}>0\Longrightarrow L=\mathcal{L}/k$ and $L<\mathcal{L}/k\Longrightarrow\frac{\rho}{\eta}=0$. This, in turn, implies that $\frac{\rho}{\eta}$ is the alternative land rent (R_A in the text proper). We continue now with the derivation of the rest of the first-order conditions

$$\left.\frac{\partial \mathbb{L}}{\partial G_{i,j}}\right\| \int_{x_{i,2j-1}}^{x_{i,2j+1}} n(x)\left[u_{i+2}(x)/\,u_1(x)\right]dx-c_1^i(i,j)\ =0. \quad (A12)$$

$$\left.\frac{\partial \mathbb{L}}{\partial x_{i,2j}}\right\| \int_{x_{i,2j-1}}^{x_{i,2j}} n(x)t_i'(x_{i,2j}-x)dx=\int_{x_{i,2j}}^{x_{i,2j+1}} n(x)t_i'(x-x_{i,2j})dx. \quad (A13)$$

$$\left.\frac{\partial \mathbb{L}}{\partial N}\right\| Y+\frac{\omega}{\eta}-\sum_i\frac{\gamma_i}{\eta}=0. \quad (A14)$$

$$\left.\frac{\partial \mathbb{L}}{\partial k}\right\| \rho\mathcal{L}=\omega\mathcal{N}. \quad (A15)$$

Since ρ is non-negative according to $(A15)$, so must ω be non-negative. We now substitute $(A14)$ into $(A7)$ to obtain:

$$Y+\frac{\omega}{\eta}-\left[z(x)+\frac{\alpha\left(x\right)}{\eta}H(x)+\sum_i c_2^i(i,j_i\left(x\right))+\sum_{i=1}^{I}t_i(|x-x_{i,2j_i(x)}|)\right]=0. \quad (A16)$$

Note that $(A9)$ and $(A10)$ imply that $\frac{\alpha(x)}{\eta}$ is equal to $p_h\left(x\right)$, the housing price function in the text proper, and equation $(A16)$ is the so-called 'household's optimal budget constraint', where $\sum_i c_2^i(i,j_i\left(x\right))$ is the sum of all 'congestion tolls' to be paid by the household at x to each of the I facilities it patronizes. The term $\frac{\omega}{\eta}$ is the household's share in the overall alternative land rents, as can be verified from $(A11)$ and $(A15)$.

Now we substitute $(A11)$ into $(A15)$ to replace ρ with ω, and substitute the result into $(A16)$ to eliminate ω. We multiply the result by $n\left(x\right)$, integrate between $[0,L]$ and into the result we add the resource constraint (12) with (13) substituted into it. We then substitute $(A10)$ and $(A11)$ into the result to obtain

$$\int_0^L\left[c_h'(H^s(x))H^s(x)-c_h(H^s(x))-\frac{\rho}{\eta}\right]dx+\sum_i\sum_j\left[N_{i,j}c_2^i(G_{i,j},N_{i,j})-c^i(G_{i,j},N_{i,j})\right]=0. \quad (A17)$$

This is the Henry George rule for the complex.

7.1.1. Continuity of Functions at Boundaries

The derivations below concern variables at the boundaries of clubs. In the text we use bid rent functions to derive these relations. In this Appendix we use direct differentiation and show that the technique of bid functions used in the text satisfies the necessary conditions derived in this Appendix.

First we note that households residing to the left of a boundary travel to the left in order to consume the CG, while residents to the right of a boundary travel to the right. This implies residents in a neighborhood of a boundary consume each CG whose market areas are separated by the boundary, at one of two different facilities. These two facilities provide the same CG but they may be located at different distances from the boundary and contain different quantities of the CG. Therefore the quantity of housing and composite good consumed by households at the vicinity of a boundary may be discontinuous and when approaching a boundary from the left the consumption basket may differ from the one when approaching the boundary from the right. Consider such a boundary point x_o between facilities j_k& $j_k+1 \in (1, ..., m_{i_k})$, of clubs $i_k \in (1, ..., I)$, $k = 1, ..., K$; $1 \leq K < I$, i.e., $x_o = x_{i_1,2j_1+1} = ... = x_{i_K,2j_K+1}$ and x_o is an interior point to all clubs i, s.t., $i \neq i_k$&$i \in (1, ..., I)$. In the Lagrangian $(A1)$, every integral which contains x_o in its domain can be split at x_o into two integrals: in one integral x_o is the upper limit and in the other integral x_o is the lower limit, i.e., $\int_0^L f(x)\,dx = \int_0^{x_o} f(x)\,dx + \int_{x_o}^L f(x)\,dx$. When the integrand is continuous at x_o, splitting the integral does not affect the outcome. However, there are control variables that may be discontinuous at such a boundary point, x_o. Households residing at x_0 may commute either to the left of x_0 or to its right to consume the CGs for which x_0 is a boundary. These two types of households, in addition to possibly using different quantities of CGs, may differ in commuting costs and congestion tolls as well as in the amounts of housing and composite good they consume. Thus, the variables that may be discontinuous at x_o besides G_{i_k}, are $H(x_o)$, $H^s(x_o)$, $n(x_o)$ and $Z(x_o)$. We designate by the superscript $+$ the limit at $x_o$of these variables when approaching x_o from the left and by the superscript $-$ the limit of the variables when approaching x_o from the right. Accordingly, at x_o the variables $H^+(x_o)$, $Z^+(x_o)$, $n^+(x_o)$ and $H^{s+}(x_o)$ are each left-continuous and $H^-(x_0)$, $Z^-(x_0)$, $n^-(x_o)$ and $H^{s-}(x_0)$ are each right-continuous. Note that Lagrange multipliers that are functions of x, such as $\lambda(x)$ and $\alpha(x)$, may also split at the boundaries.

We now introduce these split variables into the Lagrangian at boundary points and then derive the necessary conditions associated with them. First, note that the differentiation with respect to the split variables themselves yields the same equations as the derivation with respect to the same continuous variables but with the split variables replacing the continuous ones. New necessary conditions are obtained only when differentiating with respect to the location of the boundary, x_o. In the Lagrangian, besides the integrals with limits in boundaries such as $x_{i,2j\pm1}$, there are three additional places where these split variables may appear: one is in the integrals of the resource constraint that are multiplied by η; the second is in the utility constraint that is multiplied by $\lambda(x)$; and the third is in the equality of demand and supply of housing at each location that is multiplied by $\alpha(x)$.

When differentiating with respect to x_o either the utility constraint or the equality between the demand and supply of housing, we obtain the constraint multiplied by the Lagrange multiplier at the boundary. This expression vanishes and therefore can be ignored. It should be noted that households at a boundary $x_o = x_{i_k,j_k+1}$, may either commute to their left or to their right. If a household commutes to his left to facility i_k, j_k at $x_{i_k,2j_k}$ he bears commuting costs of $t_{i_k}(x_o - x_{i_k,2j_k})$, congestion tolls $c_2^{i_k}(i_k, j_k)$ and the variables associated with it at x_o are right-continuous (with superscript $+$). If, however, a household commutes to his right to facility i_k, j_k+1 at $x_{i_k,2(j_k+1)}$, his commuting costs are $t_{i_k}(x_{i_k,2j_k+1} - x_o)$, congestion tolls $c_2^{i_k}(i_k, j_k+1)$

and the variables associated with the household at x_o are left-continuous (with superscript -). Accordingly, the differentiation of the Lagrangian with respect to x_o $(= x_{i_k,2j_k+1})$ is given below,

$$-n^+(x_o)\frac{\sum_k \delta_{i_k,j_k}}{\eta} + n^+(x_o)Z^+(x_o) + c_h(H^{s+}(x_o)) + n^+(x_o)\sum_k t_{i_k}(x_o - x_{i_k,2j_k}) =$$
$$-n^-(x_o)\frac{\sum_k \delta_{i_k,j_k+1}}{\eta} + n^-(x_o)Z^-(x_o) + c_h(H^{s-}(x_o) + n^-(x_0)\sum_k t_{i_k}(|x_{i_k,2(j_k+1)} - x_o).$$

After substituting $(A4)$ into the above equation and rearranging terms, we obtain the following necessary condition for efficiency,

$$n^+(x_o)\left[\sum_k \left(c_2^{i_k}(i_k,j_k) + t_{i_k}(x_o - x_{i_k,2j_k}) - \frac{\gamma_{i_k}}{\eta}\right) + Z^+(x_o)\right] + c_h(H^{s+}(x_o)) =$$
$$n^-(x_o)\left[\sum_k \left(c_2^{i_k}(i_k,j_k+1) + t_{i_k}(x_{i_k,2(j_k+1)} - x_o) - \frac{\gamma_{i_k}}{\eta}\right) + Z^-(x_o)\right] + c_h(H^{s-}(x_o)). \quad (A18)$$

$P_h(x)$ in the text is a continuous function of x (see Corollary 1) and in this Appendix it is equal to $\frac{\alpha(x)}{\eta}$ (see $(A9)$), hence $\frac{\alpha(x)}{\eta}$ is continuous as well. Condition $(A10)$ here, implies that $H^s(x_o)$ is also continuous, i.e., $H^s(x_o) = (c_h')^{-1}\left(\frac{\alpha(x_o)}{\eta}\right)$. This, in turn, implies that $i)$ $c_h(H^{s\pm}(x_o)) = c_h(H^s(x_o))$. and $ii)$ $n^{\pm}(x_o)\,H^{\pm}(x_o) = H^s(x_o)$.

We now subtract from the square brackets in the left-hand side of $(A18)$ the null-valued left-hand side of $(A7)$ for the variables at x_o with the superscript $+$. With these right-continuous variables the commuting is to the left to clubs $i_k, j_k, k = 1, ..., K$. Next, we subtract from the square brackets on the right-hand side of $(A18)$, the null-valued left-hand side of $(A7)$ for the variables at x_o with the superscript $-$, where commuting to clubs $i_k, j_k + 1$ is to the right of x_o to $x_{i_k,2(j_k+1)}$. Since x_o is an interior point of all clubs other than i_k, commuting costs to and congestion tolls at any club $i, (i \in 1, ..., I \cap i \neq i_k, k = 1, ..., K)$, are the same for all households at x_o. In addition, we substitute $i)$ and $ii)$ above into the result to obtain, after rearranging terms

$$n^-(x_o)\left[\sum_{i\neq i_o}\frac{\gamma_i}{\eta} - \sum_{i\neq i_o} c_2^i(i, j^i(x_o)) - \sum_{i\neq i_o}^{I} t_i(|x_o - x_{i,2j_i(x_o)}|)\right] + c_h(H^s(x_o)) - \frac{\alpha(x_o)}{\eta}H^s(x_o) =$$
$$n^+(x_o)\left[\sum_{i\neq i_k}\frac{\gamma_i}{\eta} - \sum_{i\neq i_k} c_2^i(i, j^i(x_o)) - \sum_{i\neq i_k}^{I} t_i(|x_o - x_{i,2j_i(x_o)}|)\right] + c_h(H^s(x_o)) - \frac{\alpha(x_o)}{\eta}H^s(x_o).$$

By reducing equal terms from both sides of the above equation we obtain $n^+(x_o) = n^-(x_o)$, i.e., the density of the population is a continuous function of x in the boundary x_o. This implies that $H(x_o) = \frac{H^s(x_o)}{n(x_o)}$ is continuous as well at x_o.

To summarize, in this section we proved that the continuity of the housing price function at a boundary implies the continuity of the population density function, the housing supply function and the housing demand function at the boundary. ■

7.1.2. Necessary Conditions for Boundaries and Bid Housing Price Functions

We now substitute the equality between the variables $n(x_0)$, $H^s(x_0)$ and $H(x_0)$ with superscript $+$ to their counterparts with superscript $-$ into $(A18)$ and after reducing equal terms we obtain,

$$\sum_k \left[c_2^{i_k}(i_k, j_k) + t_{i_k}(x_o - x_{i_k,2j_k})\right] + Z^+(x_o) = \sum_k \left[c_2^{i_k}(i_k, j_k+1) + t_{i_k}(x_{i_k,2(j_k+1)} - x_o)\right] + Z^-(x_o). \tag{A19}$$

Condition $(A19)$ is the necessary condition for x_o to be a boundary point between facilities j_k and j_k+1 of club i_k, for all $k = 1, ..., K$, i.e., $x_o = x_{i_1,2j_1+1} = ... = x_{i_K,2j_K+1}$ and an interior point in all other clubs. In the Lemma below we prove that this condition is equivalent to the determination of boundary points by the intersection of bid rent functions (see section 3.3.3).

Lemma 7 *Equation $(A19)$ holds for x_o if and only if $(A20)$ below holds.*

$$P_h^b\left(x_o; \left[j_k^{i_k}, \forall k \in (1, ..., K)\right] \cup \left[\ \left(j^i(x_o), \forall i \in (1, ..., I)\right) \cap (i \neq i_k, \forall k)\right]\right) =$$
$$P_h^b\left(x_o; \left[j_k^{i_k}+1, \forall k \in (1, ..., K)\right] \cup \left[\ \left(j^i(x_o), \forall i \in (1, ..., I)\right) \cap (i \neq i_k, \forall k)\right]\right). \qquad ((A20))$$

It should be noted that the indexes $j_k^{i_k}$ above are identical to the pairs i_k, j_k.

Proof: First we show that $(A20)$ implies $(A19)$. From (22) we obtain at x_o for facilities $j_k^{i_k}$:

$$H(x_o)\, P_h^b\left(x_o; \left[j_k^{i_k}, \forall k \in (1, ..., K)\right] \cup \left[\ \left(j^i(x_o), \forall i \in (1, ..., I)\right) \cap (i \neq i_k, \forall k)\right]\right) = Y + v -$$
$$\left\{Z^+(x_o) + \sum_k \left[c_2^{i_k}(j_k^{i_k}) + t_{i_k}(x_o - x_{i_k,2j_k})\right] + \sum_{i \neq i_k}\left[c_2^i\left(j^i(x_o)\right) + t_i\left(\left|x_o - x_{i,2j(x_o)}\right|\right)\right]\right\}, \quad (i)$$

whereas for facilities $j_k^{i_k}+1$ we obtain,

$$H(x_o)\, P_h^b\left(x_o; \left[j_k^{i_k}+1, \forall k \in (1, ..., K)\right] \cup \left[\ \left(j^i(x_o), \forall i \in (1, ..., I)\right) \cap (i \neq i_k, \forall k)\right]\right) = Y + v -$$
$$\left\{Z^-(x_o) + \sum_k \left[c_2^{i_k}(j_k+1) + t_{i_k}(x_{i_k,2(j_k+1)} - x_o)\right] + \sum_{i \neq i_k}\left[c_2^i\left(j^i(x_o)\right) + t_i\left(\left|x_o - x_{i,2j(x_o)}\right|\right)\right]\right\} (ii)$$

We now multiply $(A20)$ by $H(x_o)$ and then substitute into its left-hand side the right-hand side of (i) above, and into the right-hand side of the extended $(A20)$, we substitute the right-hand side of (ii) above. Then by reducing identical terms from both sides of the equation we obtain $(A19)$. Thus we showed that $(A20)$ implies $(A19p)$.

To show that $(A19)$ implies $(A20)$ we claim the following : from the right-hand side of (22) for facilities $j_k^{i_k}$ at the boundary x_o, we subtract the left-hand side of $(A19)$ for i_k and $j_k^{i_k}$. We then equate the result to the right-hand side of (22) for facilities $j_k^{i_k}+1$ at the boundary x_o, from which we subtracted the right-hand side of $(A19)$ for i_k and $j_k^{i_k}$. By reducing identical terms from both sides of the resulting equality, we obtain $(A20)$.■

7.1.3. Prices and Shadow Prices

The prices we used in all sections of the paper and their shadow counterparts in this Appendix are presented in the following table.

Variable in text	Variable in Appendix	Description of variable
R_A	$\frac{\varrho}{\eta}$	The alternative land rent
$p_h(x)$	$\frac{\alpha(x)}{\eta}$	The housing price function
υ	$\frac{\omega}{\eta}$	The share of a household in alternative land rents

7.2. Characterizing the Bid Housing Price and Other Related Functions

The following differentiation of the (bid) housing price function proves Lemma 2. Differentiating (22) with respect to distance, bearing in mind that no facility is located in x, yields the Muthian spatial equilibrium condition,[27]

$$h(x,p_h^b)\dot{p}_h^b(x,j^1,...,j^I)+\dot{T}r\,(x,j^1,...,j^I)\equiv 0 \; where \; \dot{T}r\,(x,j^1,...,j^I)=$$
$$=\sum_{i=1}^{I} t_i'(|x-x_{i,2ji}|)sign(x-x_{i,2ji}). \tag{(B1)}$$

A dot above a function designates differentiation with respect to x. The reader should bear in mind that according to our assumptions $t_i'(y)=\frac{dt_i(y)}{dy}>0$ and $t_i''(y)=\frac{d^2t_i(y)}{dy^2}\leq 0$.

Equation $(B1)$ implies that a marginal displacement at a given location causes a marginal change in the bid-housing-price function proportional to the sum of all marginal changes in the home-facility commuting costs to the facilities of clubs $j_1,...,j_I$. The factor of proportionality is $-1/h(x|\,p_h^b(x))$, i.e., minus the reciprocal of the amount of housing consumed by a household at x, provided $p_h^b(x)$ is the price of housing. Note that since $t_i(|y|)$ is not differentiable at $y=0$, at the facility locations, $x_{i,2j_i}$, $p_h^b\left(x/\left(j^i\right)\right)$ is continuous but not differentiable. For an x that is not a facility location, the second derivative of the bid housing price is obtained by differentiating $(B1)$ with respect to distance, thus

$$\ddot{p}_h^b=-\frac{\frac{\partial h}{\partial p_h^b}\left(\dot{p}_h^b\right)^2+\sum\limits_{i=1}^{I}t_i''(|x-x_{i,2j_i}|)}{h(\cdot)}\geq 0. \tag{B2}$$

Thus, $(B2)$ implies that $p_h^b(x)$ is a concave function of x.

Since the housing price function, $p_h(x)$, at a location x that is not a node coincides with one of the bid rent functions, it has all the properties of a bid housing price function, except at boundaries and facility locations where it is continuous but not differentiable. We now turn to other continuous functions that depend on $p_{h(x)}$ (see Appendix 7.1.1). By differentiating (14) we obtain

$$\frac{dH^s}{dp_h}=\frac{1}{c_h''(H^s)}>0\Rightarrow \dot{H}^s=\frac{dH^s}{dp_h}\dot{p}_h=sign\left(\dot{p}_h\right)\frac{dH^s}{dp_h}\left|\dot{p}_h\right| \tag{B3}$$

Equation $(B3)$ implies that the supply of housing at a given location is an increasing function of its product's price there, and that $\dot{H}^s$ has the same sign as $\dot{p}_h$.

---

[27]The function $sign(x)$ is given by $sign(x)=\begin{cases}1\; for\; x>0\\ 0\; for\; x=0\\ -1\; for\; x<0\end{cases}$.

The function $sign(x)$ is differentiable everywhere except at $x=0$. Furthermore, $|x|=x\cdot sign(x)$ and $\partial|x|/\partial x=sign(x)$, except atr $x=0$, where it is not defined.

The density function, $n(p_h) = H^s(p_h(x))/h(p_h(x))$ (defined as the number of households per unit of land) increases with the price of housing. To see this, we make the following differentiation:

$$\frac{\partial n(x)}{dp_h} = \frac{d(H^s/h)}{dp_h} = \frac{h\,\partial H^s/\partial p_h - H^s \partial h/\partial p_h}{h^2} > 0. \qquad (B4)$$

The sign of $(B4)$ follows from $(B3)$ and from the substitution effect which implies that $\partial h(\cdot)/\partial p_h(\cdot) < 0$ in (10). It follows from $(B4)$ that the density $n(x) = H^s(x)/h(x)$ increases with distance the same way that $p_h(x)$ does.

By differentiating the land rent function in (15) and using $(B1)$ as well as (14), we obtain

$$\dot{R}(x) = H^s(x)\dot{p}_h(x), \qquad (B5)$$

which implies that $R(x)$ varies with distance in the same way that $p_h(x)$ does. By differentiating $\dot{R}(x)$, we obtain

$$\ddot{R}(x) = \dot{H}^s\dot{p}_h + H^s\ddot{p}_h \geq 0 \qquad (B6)$$

Together, equations $(B6)$ and $(B2)$ imply that, in the general case, R, like p_h, is a concave function of x.

The functions $p_h^b(x)$ and $p_h(x)$, are also functions of the parameters U, Y and G_{ij}. By differentiation of (18) as well as (5), with respect to Y, taking into account that only variables controlled by the consumer may be indirectly affected, namely $H(x)$ and $Z(x)$, we obtain

$$\frac{\partial p_h(x)}{\partial Y} = \frac{1}{h(x)} \geq 0 \qquad (B7)$$

In the same way we obtain for G_{ij}

$$\frac{\partial p_h(x)}{\partial G_{ij}} = \frac{1}{h(x)}\frac{U_{i+2}}{U_1} > 0, x_{i,2j-1} \leq x \leq x_{i,2j+1} \qquad (B8)$$

7.3. Proof for Section 3

7.3.1. Proof of Lemma 3

The proof is by contradiction. We assume that the market area is not connected and show that this assumption leads to a contradiction. Without loss of generality, let the disconnected market area be of club 1(not necessarily the industrial club).

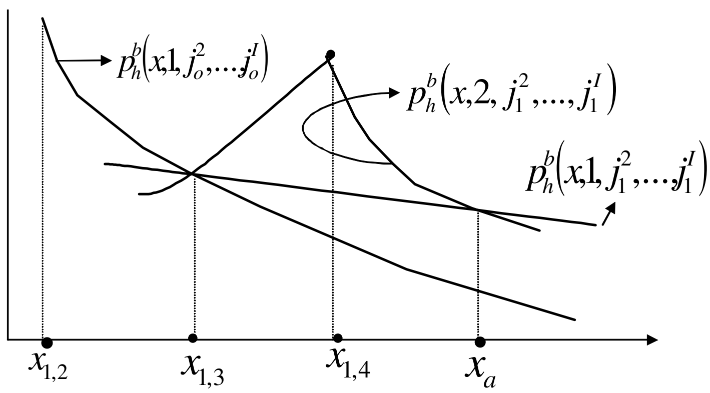

Figure 5: Bid housing price functions in a disconnected market area.

In Figure 5, facility $1,1$ (facility 1 of club 1) providing $G_{1,1}$ of the CG of club 1 is located in $x_{1,2}$, and facility $1,2$ providing $G_{1,2}$ is located in $x_{1,4}$. The locations $x_{1,3}$ and x_a are boundaries between the market areas of facility $1,1$ and facility $1,2$. There are two parts of the market area of facility $1,1$: the first lies to the left of $x_{1,3}$ and includes $x_{1,2}$ and the second is spread to the right of x_a. The market area of facility $1,2$ is between $x_{1,3}$ and x_a. Thus, the market area of facility $1,1$ is disconnected and we show here that such a layout leads to a contradiction when transportation costs are linear. To avoid the question of where people residing in a boundary use the CGs, n what follows we assume that market areas consist of half-closed segments, e.g., $(x_{i,2j-1}, x_{i,2j+1}]$.

The proof is divided into two parts. In the first part, the housing price function in the connected segment of the market area $\left(x_{1,3}, x_{\alpha}\right)$ coincides with a single bid housing price function. In the second part, we extend the proof to the more general case.

The function $p_h^b\left(x,1,j_o^2,...,j_o^I\right)$ (see Figure 5) is the bid housing price of residents who travel to $x_{1,2}$ to consume $G_{1,1}$ and to $x_{i,2j_o^i}$, $i=2,...,I$ (the points are not depicted in Figure 5) to consume the rest of the CGs. $p_h^b\left(x,1,j_o^2,...,j_o^I\right)$ coincide with the housing price function in the segment $(x_{1,2}, x_{1,3}]$. The function $p_h^b\left(x,2,j_1^2,...,j_1^I\right)$ is the bid of residents at x who travel to $x_{1,4}$, as well as to $x_{i,2j_1^i}$, $i=2,...,I$ (these points are also not depicted in Figure 5) and this bid function coincides with the housing price function in the segment $(x_{1,3}, x_a]$. In addition to the above two bid housing price functions, we consider the bid housing price function, $p_h^b\left(x,1,j_1^2,...,j_1^I\right)$. In the segment $(x_{1,3}, x_a]$ this function is the highest bid for housing that residents are willing to pay for housing while patronizing facility $(1,1)$. The reason is that the optimal vector of facilities of all clubs other than 1 at $(x_{1,3}, x_a]$ is $j_1^2,...,j_1^I$, the same vector as is in the housing price function in the segment $(x_{1,3}, x_a]$. In other words, among all bid housing price functions with households that patronize facility $(1,1)$, the bid function $p_h^b\left(x,1,j_1^2,...,j_1^I\right)$ is the highest in $(x_{1,3}, x_a]$. In this case the market area of facility $(1,1)$ would exist to the right of x_a only if the functions $p_h^b\left(x,1,j_1^2,...,j_1^I\right)$ and $p_h^b\left(x,2,j_1^2,...,j_1^I\right)$ intersect at x_a.

From (21) we learn that the only difference between $Tr\left(x,1,j_1^2,...,j_1^I\right)$ and $Tr\left(x,2,j_1^2,...,j_1^I\right)$, is the cost terms associated with facilities of club 1. At the point $x_{1,4}$, the function $Tr\left(x_{1,4},1,j_1^2,...,j_1^I\right)$ must be higher than $Tr\left(x_{1,4},2,j_1^2,...,j_1^I\right)$ otherwise no one would travel to facility $(1,2)$ and the market area of facility $(1,1)$ would be connected. From equation $(B1)$ in Appendix 7.2, it follows that in locations to the right of $x_{1,4}$, the equality $\dot{Tr}\left(x,2,j_1^2,...,j_1^I\right) = \dot{Tr}\left(x,1,j_1^2,...,j_1^I\right)$ must hold. The reason for the equality of the two $\dot{Tr}$-s is that since from all locations to the right of $x_{1,4}$ households commute in both cases to the same facilities of clubs i, $t_i'(x)$ for $i>1$ is the same in both the above Tr functions. In addition, $sign\left(x-x_{1,2}\right) = sign\left(x-x_{1,4}\right), \forall x > x_{1,4}$, and by assumption, $t_1'(y)$ =Constant. Hence, the two bid functions, $p_h^b\left(x,2,j_1^2,...,j_1^I\right)$ and $p_h^b\left(x,1,j_1^2,...,j_1^I\right)$ at x_a have the same slopes as is shown below in $(D1)$ and calculated from equation $(B1)$ in Appendix 7.2.

$$\dot{p}_h^b\left(x_a,1,j_1^2,...,j_1^I\right) = -\frac{\dot{Tr}\left(x_a,1,j_1^2,...,j_1^I\right)}{H(x_a)} = -\frac{\dot{Tr}\left(x_a,2,j_1^2,...,j_1^I\right)}{H(x_a)} = \dot{p}_h^b\left(x_a,2,j_1^2,...,j_1^I\right).. \tag{(D1)}$$

In the middle of $(D1)$, the two $\dot{Tr}$-functions in the numerators above are equal, due to the linearity of the transportation cost functions and so is $H(x_a)$ (the two p_h^b functions in the compensated demand for housing in the denominator are the same at their intersection point). Thus, at x_a both p_h^b functions, as well as their derivatives, are the same. This means that the two bid functions do not intersect at x_a but are tangent to each other. This is a contradiction, which means that facility 1,1 is connected. This completes the first part of the proof.

In the second stage we prove the lemma for the case in which the housing price function $p_h(x)$, $x_{1,3} < x < x_a$ consists of segments of different bid functions, each with a different vector $\left(J^2, ..., j^I\right)$. All of these bid functions of which p_h consists, have $j^1 = 2$, i.e., in all of them residents travel to facility 2 of club 1. In this case, let $\bar{p}_h^b\left(x, 2, j_1^2, ..., j_1^I\right)$ be the bid function that in $(x_{1,4}, x_a)$ coincides with the last segment of the price function, namely, the segment that ends in x_a. By $\bar{p}_h^b\left(x, 1, j_1^2, ..., j_1^I\right)$ we designate the bid function of a household that patronizes $\left(1, j_1^2, ..., j_1^I\right)$. This function is the highest bid of patrons of $(1,1)$ in a sufficiently small segment to the left of x_a. The two bid functions, $\bar{p}_h^b\left(x, 2, j_1^2, ..., j_1^I\right)$ and $\bar{p}_h^b\left(x, 1, j_1^2, ..., j_1^I\right)$ intersect at x_a. From here on the proof proceeds exactly as the proof in part one. This completes the proof of the Lemma.■

7.4. Proofs for Section 4

7.4.1. Proof of Proposition 4

The profit function of a contractor at x facing the given price of housing $p_h^*(x)$ is $\pi(x) = p_h^*(x) H^s(x) - c_h(H^s(x))$. Hence, contractors maximizing their profits at x by choosing $H^s(x)$, lead to the fulfillment of $p_h^*(x) = c_h'(H^s(x))$. Thus causes $H^s(x)$ to equal $H^{s*}(x)$. The subsidy $S(x)$ ensures that a household at x can purchase the optimal consumption basket at the given prices. The price of housing being optimal implies that so is $H(x)$. Together $H^*(x)$ and $H^{s*}(x)$ imply in turn that $n(x) = n^*(x) = \frac{H^{s*}(x)}{H^*(x)}$ and upon integration that indeed, $N_{ij} = N_{ij}^*$.

Let $\pi_{ij} = p_{G_{ij}} G_{ij} - c^i\left(G_{ij}, N_{ij}^*\right)$ be the profit function that an operator of facility ij maximizes by choosing G_{ij} for given $p_{G_{ij}}$ and N_{ij}^*. The necessary condition for this maximization is $= c_1^i\left(G_{ij}, N_{ij}^*\right)$ and since $p_{G_{ij}} \stackrel{def}{=} c_1^i(G_{ij}^*, N_{ij}^*)$, this condition yields $G_{ij} = G_{ij}^*$. Each individual pays $p_{G_{ij}}^d = \frac{p_{G_{ij}}}{N_{ij}^*}$. This ensures that the overall payments paid for $G_{i,j}$ by residents of the market area of facility (i,j), i.e., $N_{ij}^* p_{G_{ij}}^d = p_{G_{ij}} \left(= c_1^i(G_{ij}^*, N_{ij}^*)\right)$, are sufficient to induce the facility operator to provide the optimal CG.

The Henry George rule ensures that aggregate land rents, in addition to aggregate clubs' profits, that are all within the complex's jurisdiction, exactly match the funds needed to finance the required transfers to residents. ■

Efficient Agglomeration of Spatial Clubs

(or: The Agglomeration of Agglomerations)

Oded Hochman[1]

Revised February 2009

Abstract

The literature on agglomeration has focused largely on primary agglomeration caused by direct attraction effects. Here we focus on secondary and tertiary agglomerations caused by a primary agglomeration. Initially, scale economies in the provision of club goods (CGs) lead each CG to agglomerate in facilities of a club. This primary agglomeration causes a secondary concentration of population around these facilities, which in turn brings about a tertiary agglomeration of facilities of different clubs into centers. The agglomeration of facilities occurs only if a secondary concentration of population takes place. We analyze in detail two specific patterns of agglomeration. One is the central location pattern in which the facilities of all clubs agglomerate perfectly in the middle of their joint market area. The second is a triple-centered complex in which the center in the middle of the complex consists of perfectly agglomerated facilities of different clubs, each with a single facility per complex. The other two sub-centers consist of facilities of different clubs, each with two facilities per complex. These sub-centers are closer to the middle of the complex than to the boundaries and their facilities form condensed clusters of facilities that may contain residential land in between the facilities.

Keywords: agglomeration, clubs, complex, collective goods, local public goods, indirect attraction.

JEL Classification: R1, H4.

[1]Department of Economics, Ben Gurion University of the Negev, Beer Sheva, 84105, Israel.
Email: oded@bgu.ac.il

1. Introduction

The purpose of this paper is threefold: the first is to introduce an optimization model of an economy with spatial clubs, the second is to identify those forces in the economy that lead to the agglomeration of facilities of various clubs into multi-club centers and the last is to characterize these centers.

To facilitate the exposition we first introduce some terminology related to the theory of spatial clubs. A spatial club consists of facilities spread throughout the economy, each of which contains a concentration of the good provided by the club. A club-good (CG) is a good or service provided by each of the club facilities to their patrons. The provision of a CG by its club's facility is subject to scale economies. The patrons of a facility are a group of households who jointly consume the CG provided by the facility and are distinct from patrons of other facilities of the same club. In order to consume a particular CG a household has to commute to one of the facilities of the spatial club that provides this good. The market area of a facility is the area of residency of the facility's patrons.

Many local public goods are CGs as are many private consumption goods and services whose provision is subject to scale economies and therefore are provided collectively by spatial clubs. Real-life clubs such as country clubs, parks, museums, churches, etc. are also relevant to our model. In addition other institutions, not necessarily known as clubs, satisfy our specifications, for example, schools, police stations, theater and movie halls, restaurants, government offices, courthouses, shops and stores, and many more.. Notable among these various clubs is the 'production club' whose facilities include industrial areas and employment centers.

Three main reasons are typically offered to explain why both residential and non-residential activities agglomerate. One is reciprocal informational exchange, the second is increasing returns to scale and the last is spatial competition (see Fujita and Thisse (1996) for a comprehensive survey and Fujita and Thisse (2002) for recent theories on agglomeration). Most of these explanations are based on direct attraction forces such as the mutual attraction of units of an industry because their activity is enhanced when located close to each other.

In this paper, the primary agglomeration of CGs into facilities, is, similarly to other studies, a result of a direct attraction between units of a CG whose provision is subject to scale economies. Each CG agglomerates into its own facilities in order to provide the CG to households throughout the economy. We focus here, however, mainly on the secondary agglomerations of population around facilities and on the tertiary agglomerations of facilities of different clubs in centers in the midst of population concentrations.

The primary agglomeration of a CG in facilities attracts households to locate close to a facility in order to save commuting costs. The desire to save commuting costs is offset by congestion costs due to the limited supply of land in the proximity of the facility. The indirect attraction and the subsequent congestion cause secondary concentration of population around facilities, where the density of population decreases with its distance from the facility. In turn, the concentration of population around a facility causes facilities of different clubs to locate in the same vicinity in order to increase accessibility even further, thus creating tertiary agglomerations of facilities into centers in the midst of densely populated areas. All three stages of agglomeration, namely the primary agglomeration of CGs, the secondary concentration of population and the tertiary agglomeration of facilities into centers, occur simultaneously and the stages indicate the order of causality rather than the timing. Indeed, we show that tertiary agglomeration does not occur without a secondary concentration of population and that secondary agglomeration of population does not occur without the primary agglomeration of CGs into facilities.

In the 1960's urban economics models have dealt mainly with the secondary agglomeration of households in a residential ring surrounding a predetermined central business district (CBD), where all employment takes place. The concentration of industry in the CBD was exogenously assumed, the rationale being that the industry must be located in proximity to a sea port, train depot or other shipping facility through which the city's basic products can be exported to the rest of the world (e.g., Muth (1969)). Mills (1967) argued that the agglomeration of industry in a CBD is the result of the industry being subject to scale economies but he still assumed exogenous agglomeration. Instead of focusing on an endogenous CBD, Mills and his contemporaries concentrated on the residential ring. Henderson (1974) was the first to introduce a model in which an industry agglomerates endogenously into a CBD, however he still imposed on the model a single employment location surrounded by a residential ring. In the 1980's, Ogawa and Fujita (1980), Fujita and Ogawa (1982), and Fujita (1989) constructed simulation models of the agglomeration of an industry based on direct attraction effects. These simulations resulted in a variety of primary agglomerations. However, no secondary agglomeration of population and hence no tertiary agglomeration were possible, since a uniform density of population was everywhere assumed.

Recently, Lucas and Rosi-Hansberg (2002) incorporated both direct and indirect agglomeration engines into a single simulation model of an agglomerating industry and population/workers. But contrary to our model, in which facilities of different clubs agglomerate into centers, in their model only one type of facility exists and therefore no tertiary agglomeration can occur. Actually, none of the above models address the tertiary agglomeration of different primary agglomerations into centers in the midst of population concentrations as described in this paper.

Some studies in the literature (e.g., Fujita and Thisse (1986), Thisse and Wildasin (1992), Papageorgiou and Pines (1998) and papers surveyed by Berliant and ten Raa (1994)) investigate the agglomeration of facilities while imposing a uniform distribution of population. In this paper we show that effective agglomeration of facilities cannot occur without a secondary concentration of population and the agglomerations of facilities in the above studies are due to either the 'edge-of-economy effect', to indivisibility problems and/or to random technological effects. Therefore, to avoid confounding our own results we assume herein an economy without edges, i.e., our economy's territory is ring-shaped and fully occupied. In addition we investigate here only cases of full divisibility.

On this ring-shaped area of homogeneous land, we construct a model of an economy with spatial clubs using the conceptual framework of Hochman, Pines and Thisse (1995) (HPT hereafter).[2] In this economy there are many types of essential collective goods that require a wide variety of spatial clubs that a household must visit in order to consume the goods. The agglomeration of each CG into a separate facility results from scale economies in the provision of the good. Without such scale economies, each household would consume the CG privately in its own premises in order to avoid commuting costs. Since the direct attraction forces between units of a CG caused by scale economies are assumed to be internal to the facility, they are reflected only in the size of the facilities and not in their number. Thus, at any given site no more than one facility per club exists. We demonstrate that the population density is never uniform in a first-best allocation and that there are always areas in the economy in which population and facilities agglomerate.

Our model's results specify that in an optimal allocation the economy's territory is partitioned into identical *complexes*, where a complex is the smallest autonomous area in the economy, i.e., the smallest area in which all residents, and they alone, consume all the types of CGs in facilities

[2]While HPT focused on the finance of services rendered by the facilities, they disregarded spatial aspects and questions of agglomeration of facilities on which the present paper focuses.

located inside the complex. Thus, nobody commutes in or out of a complex. In a sense, this fact makes the complex the ideal municipality. In this paper we characterize an allocation by characterizing its representative complex.

A *complex configuration* is a vector of integers without a common multiplier that specifies the number of facilities of each club in the complex. Thus, the first entry in the vector is the number of facilities of club one in a complex, the second entry is the number of facilities of club two and so forth. Each model with a given set of functions which consists of feasible transportation cost functions and feasible provision cost functions, both for each club type, as well as of a utility function and a given complex configuration, have an optimal solution with identical and symmetric complexes. We refer to such a solution as a local optimum. In a global optimum the complex configuration is also chosen optimally.

Next, we characterize the spatial pattern of two local optimum solutions with two specific complex configurations.[3] In the first configuration, each club has a single facility per complex. With this configuration, the model results in monocentric complexes (cities) in which facilities of all clubs agglomerate perfectly in the center of the complex and share the whole complex as a common market area.[4] The population density and the housing price function in each of the complexes of this configuration increase with proximity to the complex's center, where both functions reach their peak. In addition, we provide specifications of a functions domain in which this solution is the unique global (over all possible configurations) optimum.

The second configuration that we investigate has two groups of clubs. Each club in the first group has a single facility per complex and each club in the second group has two facilities per complex. In the optimal allocation all the facilities of clubs of the first group agglomerate perfectly in the middle of each complex and the whole complex is their market area. The facilities of clubs of the second group are divided into two clusters each of which contains one facility of each club of the second group. The complex area is divided in the middle into two equal market areas, one for each cluster of facilities of the clubs of the second group. One cluster is located in the second quarter of the complex's area and the other in the third quarter. Thus, the clusters of the second group (DF clubs hereafter) are closer to the middle of the complex than to its boundaries. In other words, these clusters gravitate towards the center of the complex. The facilities in a cluster are close to each other but residential areas may exist between the facilities in the cluster, depending on whether or not the transportation cost functions of the different DF clubs are proportional to each other. Facilities with proportional transportation costs share the same facility location. Thus, while clubs of the second group do not necessarily agglomerate perfectly, they are drawn to each other and the cluster as a whole is drawn towards the facilities located in the middle of the complex. The complex is symmetric around its middle with a higher density of population between the clusters of DF clubs and the center of the complex than between the clusters and the boundaries.

Contrary to non-spatial clubs (e.g., Berglas (1976), Scotchmer and Wooders (1987); see also the survey by Scotchmer (2002) of spatial and non-spatial clubs), our optimal solution cannot be attained by a laissez faire allocation and sometimes not even by decentralization. In a laissez faire situation club owners are free to operate without restrictions, so they engage in spatial monopolistic competition, which, in general, does not yield an optimal allocation. We also show that for an economy with price taking agents there sometimes is a limited number of decentralization methods, each of which may fit under different conditions. Most decentralization methods involve subsidizing households and taxing facilities. However, such a decentralized

[3]These complex configurations are: (1,...,1) and (1,...,1,2,...,2).

[4]By perfect agglomeration we mean that facilities are adjacent to each other without having any residential area between them.

solution may entail different subsidies to identical households that are located in different places and is, therefore, difficult to implement.

Five additional sections follow this introduction. Section 2 describes the setup of the model. The necessary conditions for Pareto optimum are described in section 3 and the decentralization of the optimal allocation is depicted in section 4. Section 5 contains our main results. First, in subsection 5.1, we present general characteristics of the solution. Then we proceed to describe a perfect agglomeration in subsection 5.2 and an imperfect agglomeration in 5.3. We conclude with a short summary and a few pointers for future research of global optimum solutions.

2. The Model Setup

The country's geography is designated by a ring of unit width, with a circle running through the middle of the ring being the axis (see Figure 1).

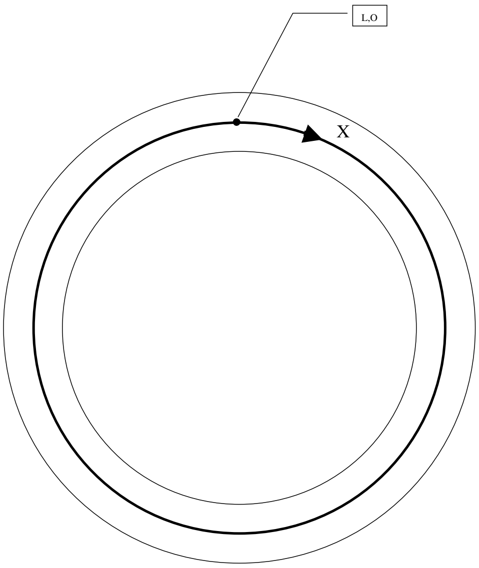

Figure 1: A Ring-Shaped Economy

We assume the circle's circumference is $\mathcal{L}$. Note that the total area of the ring in this case is also $\mathcal{L}$. An arbitrary point on the ring's axis is referred to as the origin. The location of any point on the axis of the ring is uniquely defined by its distance x from the origin in a clockwise direction (henceforth also the positive or the right direction). All points on the line segment perpendicular to the axis are designated as the same location because travel between these points involves no costs. The country accommodates $\mathcal{N}$ *households* (each time we introduce a concept it is italicized) which are identical to each other in all respects. We assume that these households are free to choose their residential location in the economy. Hence, all households must have the same utility level everywhere; otherwise they will migrate to the location with the higher utility. Each individual household derives utility from the consumption of a *composite good*, Z, and from *housing*, H, both of which the household consumes at its location of residency.

The household also derives utility from I types of *collective goods* (CGs hereafter), where G_i, is the quantity of the i^{th} CG the household consumes, $i = 1, ..., I$, according to a well-behaved utility function, $u(Z, H, G_1, \ldots, G_I)$. All goods are *essential*, and each CG is consumed at a special facility to which the household has to travel. Each individual is endowed with Y units of the composite good which can be used for private consumption and for the production of housing, CGs and transportation.

The economy contains I different clubs, one for each type of CG . A *club* of type i supplies units of the i-th CG through its $\tilde{m}_i$ *facilities* which are located throughout the economy. Each facility is identified by i, j, where $j \in (1, \ldots, \tilde{m}_i)$ is the index of the specific facility of club i, and $i \in (1, \ldots, I)$ refers to the club type. Facility i, j, whose location is designated by $x_{i,2j}$, provides G_{ij} units of the i-th CG to N_{ij} *patrons*, i.e., to individual households consuming the i-th CG in facility ij and residing within its *market area*, where a market area of a facility is a segment of the x-axis where all and only the facility's patrons live.[5] We also make the simplifying assumption that a facility does not occupy land and since, in practice, club facilities occupy only a small fraction of the total land available compared to residential land, the distortion caused by this assumption is negligible when considering the simplification involved. We represent facility ij's market area by the interval $[x_{i,2j-1}, x_{i,2j+1}]$. The union of the market areas of the $\tilde{m}_i$ facilities supplying the i-th CG coincides with the *residential area* $[0, \mathsf{L}]$ where L, the boundary of the residential area, fulfills the condition that $\mathsf{L} \leq \mathcal{L}$.[6] Accordingly, the spatial characteristics of each facility ij are fully specified by the following triplet of nodes (see Figure 2):

$x_{i,2j-1}$ = the left boundary of the ij-th facility's market area and the right boundary of the $i(j-1)$-th facility's market area,

$x_{i,2j}$ = the location of the ij-th facility, and

$x_{i,2j+1}$ = the right boundary of the ij-th facility's market area and the left boundary of the $i(j+1)$–th facility's market area.

Since each resident must consume all the types of club goods, the extreme boundaries must fulfill, $x_{i,2\tilde{m}_i+1} = \mathsf{L}$, and $x_{i,1} = 0$, for all i.[7]

We define the *clubs configuration* as the vector of integers $\{\tilde{m}_1, ..., \tilde{m}_I\}$, where $\tilde{m}_i$ is the number of facilities of type i in the economy**.** Thus, the clubs configuration is a vector of I integer variables.

To facilitate the analysis, we sort the clubs
configurations into classes, where each class is represented by a vector $(m_1, ..., m_I)$ ((m_i) for brevity) of I integers which have no common multiplier other than 1, i.e., for every $\lambda \geq 2$, at least one of the quotients m_i/λ, $i = 1, ..., I$, is <u>*not*</u> an integer. We term the configuration without a common multiplier *a basic configuration.* From here on we designate a club's configuration $(\tilde{m}_i)$ by $k\,(m_i)$, where (m_i) is the basic configuration designating the class, and the multiplier k is an additional integer-variable to be solved.

In an economy with population $\mathcal{N}$ and available land $\mathcal{L}$ there is a model with the clubs configuration $k\,(m_i)$. *A complex* in this economy is the optimal solution of a model whose population size is $\frac{\mathcal{N}}{k}$, $\frac{\mathsf{L}}{k}$ is its available land, its clubs configuration is the basic (m_i) and it has the same functions (costs, utility) as in the original model. In the solution of the complex, the common multiplier is 1, the configuration is the basic (m_i) and all its land, $\frac{\mathsf{L}}{k}$, is occupied by $\frac{\mathcal{N}}{k}$ households.

[5]By this we assume that a market area of a facility is a connected segment. In what follows we prove that, indeed, the market area of a facility of a club is a connected segment, provided $t_i\,(x)$, the club's commuting cost function, is linear in x (see Lemma 3). In the case of nonlinear transportation costs, connected market areas remain an assumption.

[6]By this, we make the assumption that the occupied area is continuous and the unoccupied area is concentrated at the end of the economy, L, and next to the origin, 0.

[7]In this model, the focus is on the case in which all available land is occupied, i.e., $\mathsf{L} = \mathcal{L}$, which implies that $0 \equiv x_{i,1} = x_{i,2\tilde{m}_i+1} = \mathsf{L} = \mathcal{L}$, $\forall i$. Therefore, calculations with the location variable x are modulo $\mathcal{L}$ (i.e. $\mathcal{L}+x = x$.). For example, for all i, $\tilde{m}_i$ and an arbitrary y, $0 < y < \mathcal{L}$, $x_{i,1} + y = x_{i,2\tilde{m}_i+1} + y = \mathcal{L}+y = y$.

The optimal solution of the model with the configuration $k(m_i)$ can now be described as k consecutive replications of the complex with the basic configuration (m_i). Each two consecutive complexes are adjacent and have a joint boundary. $\mathcal{L} - \mathsf{L} \geq 0$ is the vacant land at the edges. The common multiplier k, is now an integer variable measuring the number of complexes in the economy. Thus, by determining k and characterizing the complex, we characterize the solution of the general model.[8] In the rest of the paper we use the terms basic configuration and *complex configuration,* interchangeably.

Figure 2 depicts the layout of a complex with a basic configuration of $(1, 2, 3)$. For expositional purposes, we mark the nodes of each club on a different horizontal axis. Actually, they are all jointly located on the x axis.

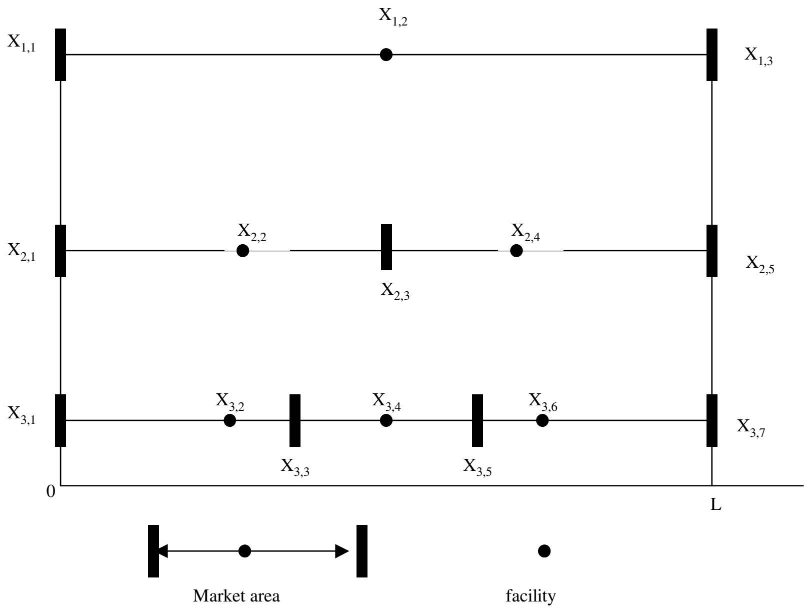

Figure 2: Facility Locations and Market Areas in a Complex with Basic Configuration (1,2,3).

We designate the length (also the area) of the complex by L and the population of the complex by N. Then $kL = \mathsf{L}$ and $kN = \mathcal{N}$. Accordingly, L is also the coordinate of the right boundary of the first complex (whose left boundary is the origin, 0) and the left boundary of the second complex, if it exists and so on. Since all complexes are identical, it is sufficient to solve only for one (the first) complex.

Since all goods are essential, the boundaries of each complex must coincide with the boundaries of the extreme facilities farthest from the center of each of the I CGs; hence

$$x_{i,1} = 0; \quad x_{i,2m_i+1} = L, \text{ for all } i \in \{1, \ldots, I\} \textit{ and } kL = \mathsf{L} \leq \mathcal{L}. \tag{1}$$

Equation (1) implies that, by assumption, the origin is a boundary of all clubs. Similarly, the relation between the complex and the overall population must be

$$N = \mathcal{N}/k. \tag{2}$$

[8] In HPT, a complex is defined as the smallest autonomous area in the economy, i.e., the smallest area in which its residents and only the residents of the area consume all the CGs in the area. It is clear from the discussion so far that our complex satisfies this definition.

In order to use a CG, the household incurs travel costs of a home-facility trip which is given by $t_i(|x - x_{i,2j_i(x)}|)$, where the argument of the function is the absolute value of the home-facility distance and $j^i(x)$ is the index j of the facility of club i whose residents at x use. The transportation cost function fulfills $\frac{\partial t_i(y)}{\partial y} \overset{def}{=} t_i' > 0$, $\frac{\partial^2 t_i(y)}{\partial y^2} \overset{def}{=} t_i'' \leq 0$, for all $y \geq 0$.

The *provision cost function,* $c^i(G_{ij}, N_{ij})$ (for brevity, hereafter $c^i(j)$) is the cost to facility i,j for providing its CG, G_{ij}, to N_{ij} households. The function $c^i(j)$ fulfills,

$$\begin{aligned}
&c_1^i(j) = \frac{\partial c^i(j)}{\partial G_{ij}} > 0,\ c_2^i(j) = \frac{\partial c^i(j)}{\partial N_{ij}} \geq 0,\ c_{11}^i(j) = \frac{\partial^2 c^i(j)}{\partial G_{ij}^2} > 0, \\
&\partial\left(\frac{c^i(j)}{N_{ij}}\right) \Big/ \partial N_{ij} \begin{cases} < 0 \; if \; N_{ij} < \bar{N}_{ij}(G_{ij}), \\ \geq 0 \; if \; N_{ij} \geq \bar{N}_{ij}(G_{ij}), \end{cases},\ \partial^2\left(\frac{c^i(j)}{N_{ij}}\right) \Big/ \partial (N_{ij})^2 > 0 \\
&where \; 0 < \bar{N}_{ij}(G_{ij}) \leq \infty, \; and \quad G_{ij} \geq 0.
\end{aligned} \tag{3}$$

Thus, $\frac{c^i(j)}{N_{ij}}$ is either a U- or L-shaped function of N_{ij}.[9] The scale economies reflected in the second line of (3) are responsible for the concentration of club goods in facilities. Without these scale economies, a CG would be provided to a household at home, like z, and not in facilities where there is joint consumption of households. Each facility i,j is identified by its CG, G_{ij}, facility location, $x_{i,2j}$, market area, $(x_{i,2j-1}, x_{i,2j+1})$ and the population within its market area, N_{ij}.

A kind of club that requires special attention is the production club, which we designate by the index $i = 1$. Patrons N_{1j}, of facility $1,j$ of a production club work in the club's facility location $x_{1,2j}$, reside in the facility's market area and together with an input of G_{1j} units of composite good, produce a net positive output $\left(-c^1(G_{1j}, N_{1j}) > 0\right)$ of the composite good. Thus, $[G_{1j} - c^1(G_{1j}, N_{1j})']$ is the gross output of the j-th facility of club 1 and as such, is its production function. The general characteristics of a club's cost functions specified in (3), need some modification and interpretation in the case of production club. Thus, instead of (3) we assume,

$$\begin{aligned}
&\begin{matrix} c^1(G_{1j}, 0) = 0 \\ c_{11}^1(j) > 0, \end{matrix};\ c_1^1(j) \begin{cases} < 0, \; if \; G_{1j} < \bar{G}_{1j}(N_{1j}) \\ \geq 0, \; if \; G_{1j} \geq \bar{G}_{1j}(N_{1j}) \end{cases},\ \frac{\partial \bar{G}_{1j}(N_{1j})}{\partial N_{1j}} > 0 \\
&\begin{matrix} c_2^1(j) \leq 0, \\ c_{22}^1(j) \geq 0, \end{matrix},\ \partial\left(\frac{c^1(j)}{N_{ij}}\right) \Big/ \partial N_{ij} < 0;\ \partial^2\left(\frac{c^1(j)}{N_{ij}}\right) \Big/ \partial (N_{ij})^2 > 0;
\end{aligned} \tag{4}$$

Accordingly, for $N_{1j} > 0$, the function $c^1(G_{1j}, N_{1,j})$ obtains negative values and is U- shaped as a function of G_{1j}, while the average, $\left(\frac{G_{1j} - c^1(G_{1j}, N_{1j})}{N_{1j}}\right)$ is increasing as a function of N_{1j} . This last property is a reflection of labor-oriented scale economies in production.[10] We also assume in the production club case that the marginal utility of G_{1j} is zero, i.e., $\partial u / \partial G_1 = 0$, which means that G_{1j} is a production factor that does not affect the household's well-being.

We adopt here the assumption accepted in urban economics literature of a non-atomic distribution of population over space. Thus, a household in our model is identified by its residence at x. In addition, we confine ourselves to allocations in which all households are identical in the sense that they all have the same utility function, skills, and initial endowment

[9] Note that $c_2^i(j) = 0$ implies that G_{ij} is a pure public good with an L-shaped average cost function. Then $c^i(G, N) = c^i(G, 1)$ for all values of N and G. When G is a private good distributed equally to each of the N residents, $c^i(G, N) = Nc^i(G, 1)$. Accordingly, as long as $c^i(G, N)$ fulfils, $c^i(G, 1) < c^i(G, N) < Nc^i(G, 1)$, G is a semi(congestable)-local public good.

[10] In what follows, results specific to the production club in subsequent sections will be given in footnotes.

and they all face the same transportation and provision cost structure. In that case, free choice of the location of residency implies an equal utility level for everyone everywhere, namely:

$$u(Z(x), H(x), G_{1,j^1(x)}, \ldots, G_{I,j^I(x)}) = U, \quad \text{for all } x \in [0, kL], \tag{5}$$

where U is the common utility level for all households in the economy and $j^i(x)$ is the index of the facility providing the i-th CG to households living at x. We designate by $u_i(x)$ the derivative of $u(Z(x), H(x), G_{1,j^1(x)}, \ldots, G_{I,j^I(x)})$ with respect to the i-th variable of the utility function as specified in (5), e.g., $u_2(x) = \frac{\partial u(Z(x),H(x),G_{1,j^1(x)},\ldots,G_{I,j^I(x)})}{\partial H(x)}$.

We now turn to housing construction. Let $H^s(x)$ be the amount of housing constructed per unit land at x. $H^s(x)$ is produced by land and the composite good. The amount of composite good used in the production per unit of land at x is $c_h(H^s(x))$, with $\frac{dc_h(H^s)}{dH^s} \overset{def}{=} c_h^{'}(H^s) > 0$ and $\frac{d^2 c_h(H^s)}{d(H^s)^2} \overset{def}{=} c_h^{''}(H^s) > 0$. We term $c_h(H^s)$ as the *housing cost function.* The material balance for housing implies

$$n(x)H(x) = H^s(x), \tag{6}$$

where $n(x)$ is the *population density function*.

The club membership constraint can be written as:

$$\int_{x_{i,2j-1}}^{x_{i,2j+1}} n(x)dx - N_{ij} = 0 \quad \forall i \in \{1, \ldots, I\}, \text{ and } j \in \{1, \ldots, m_i\}, \tag{7}$$

and

$$N - \sum_{j=1}^{m_i} N_{i,j} = 0 \quad \forall\ i \in \{1, \ldots, I\}. \tag{8}$$

The *housing price function*, $p_h(x)$, is defined as:

$$p_h(x) \overset{def}{=} u_2(x)\,/u_1(x), \tag{9}$$

where the composite good Z is the numeraire. From (9) and (5) we substitute out $H(x)$ and $Z(x)$ to obtain the *compensated demand function for housing*, namely

$$H(x) = h[p_h(x), G_{1,j^1(x)}, \ldots, G_{I,j^i(x)}, U] \tag{10}$$

and the *compensated demand function for the composite good*, which is

$$Z(x) = z[p_h(x), G_{1,j^1(x)}, \ldots, G_{I,j^{IJ}(x)}, U], \tag{11}$$

where $p_h(x)$ together with the different CGs and the utility level, U, are arguments in both of the above functions. Let the *aggregate expenditure function* for the (representative) complex be given by $E(N, U)$ where

$$\begin{aligned} E(N,U) \quad = \quad & \int_0^L [n(x)z(\cdot) + c_h(H^s)]dx + \sum_{i=1}^{I}\sum_{j=1}^{m_i} c^i(j) \\ & + \sum_{i=1}^{I}\sum_{j=1}^{m_i} \int_{x_{i,2j-1}}^{x_{i,2j+1}} n(x)t_i(|x - x_{i,2j}|)dx. \end{aligned} \tag{12}$$

The three terms of the complex's aggregate expenditure function in (12) are the expenditures on consumption and housing production (the first term), the provision cost of all CGs (the second term), and the total transportation costs (the third term). Accordingly, $kE(N,U)$ is the economy's aggregate expenditure function.

Recalling that each individual is endowed with Y units of the composite good, the complex's material balance of the composite good requires that

$$E(N,U) - NY = 0. \qquad (13)$$

In other words, the complex's aggregate expenditure must equal the complex's aggregate supply of the composite good.

The above set of equations (1)-(13) defines the constraints of a feasible spatial resource allocation for the whole economy. Necessary conditions for a Pareto optimal allocation are given in the next section.

3. The Optimal Solution

The necessary conditions for a Pareto optimal allocation in the economy are obtained by maximizing the common utility level, U, subject to the constraints (1)-(13). The Lagrangian and the formal derivation of the first order conditions are specified in Appendix 7.1. In solving the model, we assume for simplicity that the variable k, the number of complexes in the economy, is a real variable and not an integer. By making this assumption, we disregard the factual indivisibility of complexes and allow a fraction of a complex in the solution.[11] The necessary conditions in this section are given for a single complex. In our economy there are k such identical complexes. Another assumption we make is that the complex configuration, $(m_1, ..., m_I)$, is a given vector of I integers. Therefore, the necessary conditions below are for a local optimum. Additional conditions for the global optimum, in which the optimal complex configuration is determined as well, follow in a subsequent section.

The equations in this section are calculated from the necessary conditions derived in Appendix 7.1. The equations here are easier to interpret than the original ones but still constitute a full set of necessary conditions for a Pareto optimal complex, equivalent in every way to the original conditions derived in the Appendix.

3.1. Households and Housing

3.1.1. Housing Construction

In (9) $P_h(x)$ is defined as the quotient $u_2(x)/u_1(x)$. A necessary condition for the efficient allocation given in (14) below, states that *the marginal cost of housing construction*, $c_h'(H^s(x))$, equals $P_h(x)$, i.e.,

$$P_h(x) \left(\equiv u_2(x)/u_1(x)\right) = c_h'(H^s(x)), \ \textit{for all } x \qquad (14)$$

where $H^s(x)$ is the amount of housing constructed per unit land at x and $c_h'(H^s(x))$ is an abbreviation of $\frac{\partial c_h(H^s(x))}{\partial H^s(x)}$. It follows from (14) that $P_h(x)$ is, indeed, the housing price function. Observe that we can solve equation (14) to obtain $H^s(P_h(x))$.

[11] If k is not an integer, there must be a fraction of a complex in the solution. Obviously, an actual allocation contains only complete complexes, which is the case for an integer k. Thus, in the optimal solution, with an integer k, each complex is either smaller or larger than the optimal complex of the solution with a real k, and the utility level is lower as well. The distortion is negligible for a real but relatively large k. The problem of indivisibilities of economic entities is quite common in the economic literature (e.g. the indivisibility of the firm). In our case the problem might be more severe since k is likely to be small. Thus, we can see that the subject of indivisibility of optimal complexes deserves a separate study.

3.1.2. Rent Function

The rent at x, $R(x)$, is defined in (15) below as the difference between the revenue and the cost of construction per unit of land at x. Thus

$$R(x) \overset{def}{=} P_h(x) H^s(P_h(x)) - c_h(H^s(P_h(x))), \ for \ all \ x. \tag{15}$$

The properties of the rent function are given in Appendix 7.2. Note that even though in general housing price functions and rent functions are competitive equilibrium tools, they are well defined in this optimization model and have the same properties as in an equilibrium since housing and land have no external effects associated with them.

Taking the integral of the rent function over the entire country yields ALR, the *aggregate land rent in the economy*, i.e.,

$$ALR = k \int_0^L R(x)dx. \tag{16}$$

Note that the right hand side of the ALR equation above consists of the aggregate land rents *in a complex* multiplied by the number of complexes in the economy.

3.1.3. The Optimal 'Budget Constraint'

Let $j^i(x)$ be the index of the facility of club i to which a household residing at x travels. We define $Tr(x)$ as the *travel and recreation expenditure of a household residing at* x, commuting to facilities $j^i(x)$ located at $x_{i,2j^i(x)}$, paying commuting costs $t_i\left(\left|x - x_{i,2j^i(x)}\right|\right)$ and *congestion tolls*, $c_2^i\left(j^i(x)\right)$, for $i = 1, ..., I$.Thus,

$$Tr(x) \overset{def}{=} \sum_{i=1}^{I} \left[c_2^i\left(j^i(x)\right) + t_i\left(\left|x - x_{i,2j^i(x)}\right|\right)\right]. \tag{17}$$

Note that $Tr(x)$ is a continuous and differentiable function of x everywhere except at facility locations, $x_{i,2j^i(x)}$, where $Tr(x)$ is continuous but not differentiable.

The following equation (18), the *household's optimal 'budget constraint'* at x, is a necessary condition for Pareto optimum.[12] The congestion tolls included in $Tr(x)$ are what distinguish the necessary condition below from an equilibrium budget constraint. We also define in (18) the *household's optimal expenditure function at* x, $e(p_h(x), G_{1,j^1(x)}, \ldots, G_{I,j^I(x)}, T_r(x), U)$, *in short* $e(x)$.

$$\begin{aligned} Y + \nu &= z(p_h(x), G_{1,j^1(x)}, \ldots, G_{I,j^I(x)}, U) + p_h(x)h(p_h(x), G_{1,j^1(x)}, \ldots, G_{I,j^i(x)}, U) + \\ +T_r(x) &\overset{def}{=} e(p_h(x), G_{1,j^1(x)}, \ldots, G_{I,j^I(x)}, T_r(x), U), \ for \ all \ x. \end{aligned} \tag{18}$$

We see that in (18), $p_h(x)$ indeed serves as the housing price and the household's income $Y + v$ is independent of location and consists of the initial endowment of an individual household, Y, plus ν – *an equal share of total alternative -shadow-land- rents* in the economy.[13] Thus, a

[12]Note that if $i = 1$ is a production club, then the expression $\left(-c_2^1\right)$ is the marginal product of labor, which attains positive values and appears as income in the household's optimal budget constraint. In this case the model has a non-zero solution even if Y vanishes.

[13]Namely, $\nu = \frac{LR_A}{N}$, where $R_A \geq 0$ and if $kL < \mathcal{L}$ than $R_A = 0$. See also (26), (25) and the discussion that follows at the end of section 3.

household behaves in the optimum as a utility maximizer who considers as given: his income; the location $x_{i,2j}$ of all facilities (i,j); the quantities of CGs, G_{ij}, in these facilities; and the congestion tolls $c_2^i(j) \stackrel{def}{=} \frac{\partial c^i(G_{ij},N_{ij})}{\partial N_{ij}}$ the household is required to pay when it uses facility i,j. Each club $i \in (1,...,I)$ has m_i facilities spread throughout the complex and a household at x visits one facility of each club i.

3.2. Clubs

The external effects in the model are concentrated in clubs and therefore most of the equations in this section are not equilibrium relations.

3.2.1. Samuelson's Rule

The necessary condition in (19) below, determines G_{ij}, the optimal amount of CG for facility j in club i. The equation below is a version of Samuelson's well known rule about public goods.

$$\int_{x_{i,2j-1}}^{x_{i,2j+1}} \left[\frac{u_{i+2}}{u_1} n\right] dx = c_1^j(j), \forall i,j, \tag{19}$$

where $c_1^i(j) = \frac{\partial c^i(j)}{\partial G_{ij}}$. On the right-hand side of (19) is the marginal rate of substitution in production between the CG and the composite good and on the left-hand side of (19) is the sum of the marginal rates of substitution in consumption of the users of facility i,j, where the marginal rate of substitution is between the CG and the composite good.[14]

3.2.2. Optimal Facility Location

The optimal facility location, $x_{i,2j}$, should satisfy the necessary condition in (20) below, which is also a necessary condition for the facility location to minimize aggregate transportation costs of patrons to facility (i,j).

$$\int_{x_{i,2j-1}}^{x_{i,2j}} n(x)\, t_i'(x_{i,2j} - x)\, dx = \int_{x_{i,2j}}^{x_{i,2j+1}} n(x)\, t_i'(x - x_{i,2j})\, dx, \forall i,j. \tag{20}$$

In (20) the aggregate marginal transportation costs of patrons on one side of a facility equal the aggregate marginal transportation costs on the other side, so that a marginal shift in the facility location does not change aggregate transportation costs to the facility. It should be noted that linear t_i in (20) implies that on each side of the facility reside an equal number of patrons. The following lemma can now be proved;

Lemma 1 *A club's facility location is an interior point of the club's market area, and therefore of the complex. The market area of a facility is in a bounded segment of the complex.*

The proof of the first part of the lemma follows directly from (20) which requires that patrons should reside on both sides of the facility location. The proof of the second part of the lemma follows from the finiteness of the household's income which allows it to travel only a bounded distance.

[14] For club 1, the production club, after substituting $u_3 = 0$ in (19) reads, $0 = c_1^1(j)$. Indeed, the left-hand side of (19) vanishes when $i = 1$ since $u_3 = 0$. To understand the meaning of (19) when $i = 1$, consider the production function $G_{1,j} - c^1(j)$ in perfect competition. The equality between the value of the marginal product of $G_{1,j}$ and the price of $G_{1,j}$, which is 1, i.e., $\frac{\partial}{\partial G_{1,j}}\left(G_{1,j} - c^1(j)\right) = 1$, results in $0 = c_1^1(j)$, which is, as we just showed, (19) for i=1. So in the case of the production club this condition is fulfilled in perfect competition.

3.3. Bid Price Functions and Nodes

Bid price functions of housing and land are essentially tools of competitive equilibrium analysis. They can be employed in our optimization model since the land and housing markets are free from external effects. The bid price functions below are defined for given facility locations and the CGs in them, and for a given optimal utility level. The crucial assumption which allows bid functions analysis is the assumption of households' freedom to choose their location of residency, which implies equal utility level to identical households everywhere. This assumption is, indeed, part of this model as well as part of other urban competitive models. For a proof that bid housing price functions analysis is compatible with the necessary conditions of this optimization model, see sections 7.1.1 and 7.1.2 in the Appendix.

3.3.1. Bid Housing Price Functions

Let $Tr\left(x, j^1, ..., j^I\right)$ be the sum of the home-facility commuting costs plus the congestion tolls $c_2^i\left(j^i\right)$ a household residing at x pays when traveling to each of the I facilities, $j^1, ..., j^I,$ as specified in (18), where j^i is the index of facility j of club i, i.e., $j^i \in (1, ..., m_i)$. The facility j^i is located at $x_{i,2j^i}$,with a given quantity of CG, G_{i,j^i}, i.e.,

$$Tr\left(x, j^1, ..., j^I\right) \overset{def}{=} \sum_{i=1}^{I} \left[c_2^i\left(j^i\right) + t_i\left(\left|x - x_{i,2j^i}\right|\right)\right], \tag{21}$$

$$for\ all\ x,\ j\ and\ i\ s.t.,\ 0 \leq x \leq L,\ j^i \in (1, ..., m_i),\ i = 1, ..., I.$$

For the household to reside at x and travel to the given I facilities (j^i), the household's optimal budget constraint must fulfill,

$$Y + \nu = z(p_h(x), \left(G_{i,j^i}\right), U) + p_h(x)\, h(p_h(x), \left(G_{i,j^i}\right), U) + T_r\left(x, \left(j^i\right)\right). \tag{22}$$

where $\left(G_{i,j^i}\right)$=$G_{1,j^1}, ..., G_{I,j^i}$; $\left(j^i\right) = j^1, ...,\ j^I$; $z(p_h(x), \left(G_{i,j^i}\right), U)$ is the compensated demand function for the composite good Z, defined in (11) and $h(p_h(x), \left(G_{i,j^i}\right), U)$ is the compensated demand function for housing H, defined in (10).

The vector $\left(\left(G_{i,j^i}\right), \left(j^i\right), U\right)$ is fixed and given and so is the household's income $Y + v$. The only variable remaining to be determined at a given location x is the price of housing, $p_h(x)$. By substituting out $p_h(x)$ from (22) we obtain the *bid housing-pice of a household residing at x and traveling to facilities at* $\left(x_{i,2j^i}\right)$ where the household uses the CGs, $\left(G_{i,j^i}\right)$. We designate this bid housing-price function by $p_h^b\left(x, j^1, ..., j^I\right)$ .What distinguishes one bid housing price function from another is the set of facilities to which the household travels. Income and utility levels are the same for everybody everywhere and are known parameters as are the CGs and facility locations. Therefore, once the facilities' indices of a bid housing price function are known, all information is revealed. Each bid housing-price function has a different set of I facilities. In each of two different sets of indices there is at least one facility that the other lacks. For some vectors (J^i), there may be locations x for which p_h, substituted out of (22), is negative. In such cases we set the bid housing price equal to zero. We can now prove the following lemma.

Lemma 2 *The bid housing price function is a continuous function of the distance x and twice differentiable, with a positive second derivative everywhere except at the I facility locations $(x_{i,2ji})$ where it is continuous but not differentiable.*[15]

A household at location x, by choosing to travel to facilities that yield the highest bid housing price is actually choosing to attain the utility level at location x by spending the least

[15]For proof of Lemma 2 see Appendix 7.2

of all possible costs other than the cost of housing. Such behavior by all households leads to an efficient allocation. In competitive markets, a household at x travels to the facilities that yield the highest bid housing price at x, because he then can outbid others competing for housing at x. Accordingly, $p_h(x)$, the housing price function at x, fulfills

$$P_h(x) = \max_{j^1,...,j^i} p_h^b\left(x, j^1, ..., j^I\right) = p_h^b\left(x, j^1(x), ..., j^I(x)\right), \forall x \ \mathit{where} \ j^i \in (1, ..., m_i), \ i = 1, ..., I. \tag{23}$$

The vector of indexes of facilities $\left(j^1(x), ..., j^I(x)\right)$ to which a household residing in x travels to, is merely the vector $\left(j^1, ..., j^I\right)$ that maximizes $p_h^b(x, (j^i))$ in (23). Thus, the upper boundary curve of all bid housing price functions as defined in (23), besides being the housing price function also determines the facility locations to which a household at x travels.

3.3.2. Bid Rent Functions

We define the bid rent functions as

$$R^b\left(x, j^1, ..., j^I\right) = p_h^b\left(x, j^1, ..., j^I\right) H^s\left(p_h^b\left(x, j^1, ..., j^I\right)\right) - C_h\left(H^s\left(p_h^b(x, j^1, ..., j^I)\right)\right)$$

The bid rent is a monotonic increasing function of $p_h^b\left(x, j^1, ..., j^I\right)$ and fulfills $R^b\left(p_h^b = 0\right) = 0$. Therefore, in most cases we can use either the bid rent function or the bid price function.

3.3.3. Boundaries and Facility Locations

In the optimal allocation a *node* x_b on the x-axis is *a boundary point between club-i market areas,* if there are points x_l and x_r, $x_l < x_b < x_r$, such that all residents living in (x_l, x_b) consume the i-th CG in a facility to the left of x_b, and all residents in (x_b, x_r) consume the i-th CG in a facility to the right of x_b.

Let x_b be a boundary point of clubs $i_1, ..., i_K$, $1 \leq K \leq I$ and of them only (when $K = I$, x_b is the boundary of the complex). For brevity of notation we also designate by K the set $(i_k, k = 1, ..., K)$ and by $I - K$, the set $((i_k \notin K) \ \mathit{and} \ (i_k \in (1, ..., I)))$. There is a point x_l, $x_l < x_b$, (x_l can be any point between x_b and the next boundary point to the left of x_b) that residents at every point x, $x_l < x < x_b$ use the I CGs at the same facilities. We designate these facilities by $j_o^1, ..., j_o^I$, i.e.,$j_o^i = j^i(x)$, $x_l < x < x_b$. In the same way, there is a point x_r, $x_r > x_b$, where all residents in the segment $x_b < x < x_r$ use the I CGs at the same facilities. In this segment, if $i \in K$, then $j_o^i + 1$ is the facility in which residents consume the i-th CG and if $i \in I - K$, j_o^i is still the facility in which residents of x consume the i-th CG. The necessary condition associated with the boundary x_b now follows,

$$p_h(x) = \max_{j^1,...,j^I} p_h^b\left(x, j^1, ..., j^I\right) \begin{cases} = p_h^b\left(x, j_o^1, ..., j_o^I\right), & \mathit{for} \ x, \ s.t. \quad x_l < x \leq x_b, \\ = p_h^b\left(x, \left(j_o^i + 1, \forall i \in K\right) \ \mathit{and} \ \left(j_o^i, \forall i \in (I-K)\right)\right), & \\ \qquad \mathit{for} \ x, \ s.t. \quad x_b < x \leq x_r, & \end{cases}$$

$$\mathit{and} \ \ p_h(x_b) = p_h^b\left(x_b, j_o^1, ..., j_o^I\right) = p_h^b\left(x_b; \left(j_o^i + 1, \forall i \in K\right) \ \mathit{and} \ \left(j_o^i, \forall i \in (I-K)\right)\right). \tag{24}$$

Equation (24) states that the bid function $p_h^b\left(x;, j_o^i, ..., j_o^I\right)$ and the bid function $p_h^b\left(x; \left[j_o^i + 1, \forall \ i \in K\right] \cup \left[j_o^i, \ \forall i\right.\right.$ intersect at x_b and are equal to the housing price there. Hence, the two bid functions must coincide with the housing price function in a neighborhood of x_b as well. Note that the lowest line in (24) is the actual necessary condition. Below, x_b is indexed according to the rules set up in Section 2.

$$x_b = x_{i_1, 2j_o{}^{i_1}+1} =, .., = x_{i_K, 2j_o{}^{i_K}+1}.$$

For the proof that (24) is compatible with the necessary conditions, see Appendix 7.1.2.

The location of facility j of club i in our model is a node located at $x_{i,2j}$. The transportation cost function, $t_i\left(|x - x_{i,2j}|\right)$, is a continuous and differentiable function of x everywhere except at $x = x_{i,2j}$ where it is not differentiable. Since $Tr\left(x\right)$ in (21) contains transportation cost functions, it is continuous and twice differentiable everywhere except at facility locations where it is continuous but not differentiable. This property is passed on to p_h^b solved from (22) and (23) (see lemma 2). Since $p_h\left(x\right)$, the housing price function itself, consists of segments of bid housing price functions that intersect at boundaries, it must be continuous and twice differentiable too except at facility locations and boundaries where it is continuous but not differentiable. To sum up the analysis we write it in the form of a corollary,

Corollary 1 *The housing price function, $P_h\left(x\right)$, is a continuous and twice differentiable function of x with a positive second derivative everywhere, except in nodes where it is continuous but not differentiable.*

Consecutive facilities of the same club may hold different quantities of the CG. Hence, households residing on different sides of a clubs' boundary may consume different quantities of one or more CGs (depending on whether the boundary is of one or more clubs and whether consecutive clubs have different quantities of their

CG). With discontinuous changes in quantities of CGs consumed in consecutive facilities, discontinuous changes in households' consumption of housing and the composite good may be observed as well when crossing a clubs' boundary. In Appendix 7.1.1 we show that where housing is concerned, the quantity of housing consumed and produced as well as the population density, are continuous functions at a boundary, as stated in the following Proposition.

Proposition 2 *The household's housing consumption, $H\left(x\right)$, is continuous everywhere, including in boundary and facility locations. Also continuous everywhere are the density of population, $n\left(x\right)$, and the supply of housing, $H^s\left(x\right)$.*

It should be noted that unlike the continuity of the supply and demand of housing, the household consumption of composite good may be discontinuous in boundaries. For details and proof of the proposition, see Appendix 7.1.1 and 7.1.2.

3.3.4. Market Areas

In section 2 we assumed that a market area served by a facility is a connected segment of the x-axis. Thus far we used this assumption only for simplifying the notation. Now we prove this assumption endogenously in Lemma 3 for clubs with linear transportation cost functions.

Lemma 3 *The market area of a club with linear transportation cost function is a connected segment of the x-axis.*

For a proof see Appendix 7.3.1. Lemma 3 and Lemma 1 yield the next Proposition:

Proposition 3 *The market area of a club's facility is a bounded area and the facility is located in its interior. Market areas of clubs with a linear transportation cost function are connected.*

Recall that in this study we investigate only allocations in which market areas are connected.

3.4. The Henry George Rule

The alternative land rent, R_A, is the land rent at the boundaries of a complex, i.e., $R_A = R(L)$. R_A is the lowest land rent anywhere in the complex. A necessary condition for Pareto optimum of an economy with identical households is the following relation:

$$\nu = \frac{R_A \mathcal{L}}{\mathcal{N}} \tag{25}$$

where v is the household's income from its share of alternative land rents (see also (18)). The Kunn-Tucker conditions imply that If $\mathsf{L}<\mathcal{L} \Rightarrow R_A = v = 0$ and when $\mathsf{L}=\mathcal{L} \Rightarrow v, R_A \geq 0$.

The last necessary condition for an optimum is the Henry George rule,

$$DLR \equiv \int_0^L (R(x) - R_A)\, dx. = \sum_{i=1}^{I} \sum_{j=1}^{m_i} \left(c^i(j) - N_{ij} c_2^i(j) \right), \tag{26}$$

The term $\int_0^L (R(x) - R_A)\, dx > 0$, is the *differential land rents* (DLR). Since the DLR on the left hand side of (26) is positive, so is the term on the right hand side of the equation, i.e., the aggregate provision cost, $\sum_{i=1}^{I} \sum_{j=1}^{m_i} c^i(j)$, minus the aggregate congestion tolls, $\sum_{i=1}^{I} \sum_{j=1}^{m_i} N_{ij} c_2^i(j)$ (See also (17) and (18)). This means that congestion tolls alone cannot be the sole source of financing the clubs' operations. In (26) the DLR exactly equals the remaining deficit of the clubs after congestion tolls are paid to the clubs.[16] Therefore, the only net profits in the economy are the alternative land rents. It follows from (25) that in the optimum the overall profits in the economy, if any (i.e., if $R_A > 0$), are distributed among the general population.

4. Decentralization

In this section we deviate from the analysis of agglomeration to discuss briefly the issue of decentralizing the optimal allocation described in the previous section. A laissez faire allocation would not be efficient because of the lack of incentive of club owners to provide the optimal amount of CGs, to impose optimal user charges and to optimally locate the facilities. In actual fact, each facility owner does possess market power and if left to his own devices, will engage in monopolistic competition. To achieve the optimum, a local government (of a complex) has to intervene in the economic operations that take place in its jurisdiction. The government may intervene either directly by providing by itself the optimal CGs in facilities located optimally and by taxing land rents which, together with congestion tolls collected from users, can finance its operations and ensure the fulfillment of the necessary conditions. This type of direct intervention, however, is problematic since, besides there being a lack of information about optimal quantities of CGs, locations of facilities and exact corrective taxes, it requires constant management of facilities. Throughout the ages, governments, especially local ones, have proved themselves to be highly inefficient in managing economic activities, club facilities being no exception.

Conversely, decentralization of CG provision requires of a local government only the determination of prices and income transfers between sectors and their imposition by taxation and subsidization. The Second Fundamental Theorem of Welfare Theory (e.g., see Mas-Colell et al., (1995) Ch. 16, Proposition 16.D.1) proves that, in general, it is possible to decentralize a

[16] In the case of the industrial club, the term $\left(-N_{1,j} c_2^1(j) > 0\right)$ is the wages paid to the workers in the facility and $\left(-c^1(j)\right)>0$ is the value added over the value of the input of the composite good, G_{1j}. Therefore, $c^1(j) - N_{1j} c_2^1(j) > 0$ is the deficit of the production club's facility. Therefore, each facility has to recieve a subsidy from the local government that can be financed by an optimal taxation of land rents. This result is well-known in the literature.

Pareto optimal allocation with specifications fitting our model's assumptions. In Mas-Colell et. al., it is shown that every Pareto optimal allocation (x^*, y^*) (his notation) has a price vector $p = (p_1, ..., p_L) \neq 0$, such that (x^*, y^*, p) is a price quasi-equilibrium with transfers. In other words, in a sufficiently well-behaved economy if agents are price takers, there exist prices and income transfers that yield the optimal solution as a market allocation. In practice, however, the actual determination of these transfers and prices is still an open question that we refrain from investigating at this time.

In the case of non-spatial clubs, an efficient equilibrium exists that does not require any government intervention (e.g., see the outset in HPT). However, in the case of spatial clubs, government intervention is needed to instigate the provision of optimal quantities of CGs at the optimal nodes, since club operators possess market power and do not have any incentive to behave competitively.

We first investigate the case in which club operators can locate facilities only in predetermined sites matching optimal facility locations. We will partially relax this restriction later on. There is no unique way to decentralize our optimum and for different clubs, different methods may be more suitable. A natural way to decentralize our optimum is to allow each facility operator to charge each user the congestion toll $c_2^i(j)$, which ensures the fulfillment of (18). The facility's income from user charges is then $N_{ij}c_2^i(j)$ and, in general, this toll is not sufficient to cover the full cost of running an optimal facility, i.e., the facility's loss is $c^i(j) - N_{ij}c_2^i(j) > 0$ and the local government has to provide the missing funds to cover facilities losses.[17] The General Henry George Rule (26) ensures that the differential land rents, taxable by the local government, are sufficient to cover the total deficit.[18]

The above decentralization method, in which facility operators charge patrons with congestion tolls and are subsidized by the local government, suffers from lack of incentive to behave efficiently by facility operators. By doing nothing and acquiring the government's subsidy, a facility operator obtains the subsidy as positive profits, while by behaving optimally he only ends up without losses (see HPT). Another problem with this method is the lack of government knowledge of how to divide taxed differential land rents into subsidies between different facilities.

Despite the drawbacks of the decentralization method discussed above, there are circumstances in which it is the appropriate one. Consider, for example, the case in which the provision costs are divided into costs of constructing a facility (fixed costs) and marginal costs of operations increasing with the number of users. In such a case, the government can construct the facility, thus paying the fixed costs, and then lease the facility to a private operator who is allowed to charge users the marginal cost while maintaining current operations and paying the variable costs. Knight (1924) showed that there are circumstances under which user charges that maximize profits are exactly equal to optimal congestion tolls in a road system. Indeed, if the facility operator incurs positive profits, the government can obtain these profits as lease payment and redistribute it to households of users.

Another decentralization method is applicable to cases in which division to fixed and increasing marginal costs are not relevant. We let an asterisk designate optimal values of variables

[17] Not all facilities must suffer losses and some may even have profits, however, when pulling together all the clubs there are losses. To see that consider the following Henry George (HG) rule (see also (26)), $0 < DLR \equiv \int_0^L (R(x) - R_A)\, dx = \sum_{i=1}^{I} \sum_{j=1}^{m_i} \left(c^i(j) - N_{ij}c_2^i(j)\right)$. It follows that the double summation in HG rule above is positive, however, it may contain some individual negative terms, each of which belonging to a club. Such clubs need to be taxed instead of subsidized.

[18] In the case of the industrial club, $\left(-c_2^1 > 0\right)$, is the marginal productivity of labor that equals the wage rate. Our assumptions imply that $\left(-N_{1j}c_2^1\right)$, the total wages paid by the industry, are larger than the net production, $\left(-c^i\right)$. Therefore, the industrial club has to be subsidized by the local government. This result is well known in the literature.

and $p^d_{G_{ij}}$, where $p^d_{G_{ij}} \overset{def}{\equiv} \frac{c^i_1\left(G^*_{ij},N^*_{ij}\right)}{N^*_{ij}}$, be the price a household *pays* per unit of G_{ij} it consumes at facility (i,j).[19] Let the price, $p_{G_{ij}}$, be the price the facility ij operator *receives* per unit of CG he provides, which is $p_{G_{ij}} \overset{def}{\equiv} N^*_{ij} p^d_{G_{ij}} = c^i_1\left(G^*_{ij}, N^*_{ij}\right)$. Note that in this case the club has positive profits since, $p_{G_{ij}} G^*_{ij} = c^i_1\left(G^*_{ij}, N^*_{ij}\right) G^*_{ij} > c^i\left(G^*_{ij}, N^*_{ij}\right)$, where both c^i_1 and c^i_{11} are positive (see Section 2). Finally let $S(x)$ be a government subsidy to a household located at x, $S(x) \overset{def}{\equiv} \sum_i \left[\frac{c^i_1\left(G^*_{i,j^i(x)},N^*_{i,j^i(x)}\right)}{N^*_{i,j^i(x)}} G^*_{i,j^i(x)} - c^i_2\left(G^*_{i,j^i(x)}, N^*_{i,j^i(x)}\right)\right]$ where the summation is over all the clubs for which this method of decentralization is used. This subsidy compensates households for those charges which are higher than the congestion tolls. The government can finance this subsidy by taxing the facilities' profits. We can now prove the following Proposition,

Proposition 4 *The price vector* $\left(p^d_{G_{ij}}, p_{G_{ij}}, p^*_h(x)\right)$*, the household's subsidy function* $S(x)$*, the model setup in Section 2 for a given basic configuration and the optimal facilities locations constitute a price quasi-equilibrium with transfers that yield the model's Pareto optimal allocation.*

For the proof see section 7.4.1 Note that in this decentralization method, facility operators are price takers and customers pursue the least expensive facility which fulfills their need.

If all facilities of a club are the same, i.e., they all have the same number of patrons and the same amount of CG, the subsidies to a household are identical everywhere. However, if there are clubs with three or more facilities in a complex, some of them may have different amounts of patronage than others. In this case, when facilities of the same club are not identical, the required subsidies to households become location-dependent and may differ between neighborhoods. In practice, local governments do not bother to return the income they tax from clubs to the particular users and instead add this income to the general municipal income by which they provide the general population with goods and services.[20]

So far we have assumed that club managers face predetermined facility locations in a complex, which to the most extent resembles real life. Club sizes and locations are detailed in city master plans, their number is regulated and each club requires a permit. As such, no decentralization of the choice of club locations is really required. The fact that in real life decentralization of the choice of facility locations does not take place is a clear indication of the complexity of such a process.

In what follows we investigate the decentralization of locating facilities purely for academic interest. The optimal facility location is the one that minimizes overall commuting costs

[19]The price $p^d_{G_{ij}}$ defined here does not equate with the household's marginal rate of substitution between the club good and the composite good consumed by the individual. It is, therefore, not really a (Lindhal) price but more a lumpsum tax. However, a price-taking individual will consume the correct optimal CG since this is the quantity of the CG provided by the closest facility and it is the better option of the CG consumption compared to other facilities of the same type. This lumpsum is preferred over Lindhal pricing since all users of a facility pay the same.

[20]Retail stores are facilities of a club with, yet, another method of financing its operations. Stores provide the service of distributing consumption goods to the general public. They buy goods from producers at gross prices and sell them at higher prices. Stores differ from each other in the type of goods they sell, their diversity and prices, accessibility to the store, etc. In practice, although these stores are very competitive, they are not price-takers and the method of payment for their services is not as in Proposition 4. Yet their allocation could be optimal if the government would tax the profits of the stores and refund buyers for excess payment. In practice, taxes on stores are high but the refunding of buyers is practically impossible and, as before, the tax income becomes part of the government's general budget.

from the market area, i.e., (20) has to be fulfilled. If households are left to pay for their own commuting, facility operators will choose facility locations that maximize their patronage and profits and disregard the effect the facility location has on commuting costs. This may lead operators to locate their facilities inefficiently. For example, if two facilities of the same club are in a complex, both of them will locate in the center of the complex, each trying to add to its market area the more densely populated areas in the center of the complex while giving up sparsely populated areas closer to the complex boundary. To induce facility managers to locate efficiently, their goal function should include the minimization of their patrons' total commuting costs, so that (20) is satisfied. To achieve this goal, each facility operator should transport his patrons by himself, in return for a predetermined lumpsum payment. The lumpsum should be the same to all residents living on the same side of the facility and equal to the commuting costs of an individual living at the boundary of the market area. With this method of payment, a facility manager has an incentive to choose a facility location that minimizes overall transportation costs, since he will be maximizing his profits from transportation. Indeed, a first-order condition for such a minimization is (20). At the same time, the local government should tax the additional profits of the club owner and redistribute them among the club's patrons so that the lumpsum transportation payment of a household minus the transport subsidy it receives equals the household's actual transportation costs. In this case, the redistributed amounts vary from one location to another even within the market area of the same facility and even if all clubs are the same. Beside all the above drawbacks which render this decentralization unpractical, this decentralized method of choosing a facility location, suffers from the inherent problem that the facility operator can only acquire monetary travel costs. Costs involving the value of travel time must be borne by the individuals themselves. Thus, the facility operator may only minimize partial commuting costs and does not locate the facility optimally. In view of these drawbacks, we would conclude that the determination of potential facility locations should be left to city planners.

5. Agglomeration of Spatial Clubs and Households

In this section we investigate general agglomeration trends of spatial clubs and households in optimal allocations and elaborate on allocations of two simple basic configurations, each of which characterizes a particular type of club's agglomeration: the first deals with perfect agglomeration of facilities of different clubs and the second involves imperfect agglomeration of facilities of different clubs. We give an example in which perfect agglomeration of facilities in the center of a complex is a unique global optimum. In addition, we show that a local optimum solution of a basic configurations may have a domain in the functions space, in which it is a global optimum. As a reminder, a complex configuration is a vector with I integer components m_i, which do not have a common multiplier. Each m_i designates the number of facilities of club i in a complex. The variable k measures the number of complexes in the economy.

5.1. General Characteristics

In a ring-shaped economy that is partially unoccupied, even if we assume that the occupied land constitutes a single connected segment $(0, \mathsf{L})$, $0 < \mathsf{L}\,(=kL) < \mathcal{L}$, and all the unoccupied land is the segment $(\mathsf{L}, \mathcal{L})$, there are two edges to the occupied land: $\mathsf{L}(\equiv kL)$ and $O(\equiv \mathcal{L})$.[21] Since all CGs are essential, these two edges must be boundary points to all clubs, i.e., the origin, O, is the left boundary of the first facility of each club and L is the right boundary point of the

[21] If the occupied land is not connected, there are more than just two edges to the economy, a fact that strengthens our arguments.

last facility of each club.

To see that an agglomeration of clubs in an economy with edges and a uniform population distribution is ineffective, consider the following example of an allocation of I clubs in such an economy. Each of the clubs has the same number of identical market areas and facilities that are located in their midst, i.e., $m_i = 1, \forall i$, and the number of complexes, k, is also the total number of facilities of each club. Since the extreme two boundaries of every club coincide, it follows that all market areas are common to all clubs and the facilities of all I clubs are jointly located in the center of each market area. In other words, facilities of all clubs agglomerate in a single location at the center of each complex.

We also assume in this example that the population is uniformly distributed over space and that the quantity of CG in each of the facilities of a club is the same. Another assumption is that all households have the same resources and utility level, hence, all households must also consume the same amount of composite good. In short, in the economy just constructed, all households have the same utility level and consume identical bundles of housing, composite good and CGs. Market areas are common to all I clubs and in the center of each market area facilities of all I clubs are agglomerated.

Now, suppose the economy no longer has edges and $kL\,(\equiv \mathsf{L}) = O(\equiv \mathcal{L})$. Then the previous allocation of clubs with common market areas and agglomeration of facilities of the I clubs exists in the edgeless economy as well, but in this case the last boundary of the last market area of each club coincide with each other as well as with the first boundary of the first market area of each club. However, in this edgeless economy, unlike the economy with edges, there are no points that *must* be a boundary to all types of clubs. Actually, a club in the edgeless economy is free to have its boundaries anywhere as long as the distance between two consecutive boundaries of the same club are constant and equal to L. Therefore, all clubs can be arbitrarily arranged in a consecutive order and the location of boundaries and hence of facilities of different clubs, can be arranged so that the distance of a facility of one club from the next consecutive club's facility is L/I. The sizes of a club market areas remain unchanged as in the allocation with agglomeration, the location of each facility remains in the middle of its market area and the quantity of CG in each facility remains as is. The result of such an allocation is first of all that there is no agglomeration of facilities; in fact, the facilities are distributed evenly throughout the ring. Secondly, since the market areas are the same in the two allocations and the distribution of population is uniform, the number of patrons and travel distances in each market area remain the same as in the economy with edges. Consequently, total commuting costs in each facility are unchanged as well as total provision costs. It follows that each household consumes the same basket as before and therefore has the same utility level, but this time there is no agglomeration of facilities in the allocation.

The above example implies that an allocation with an agglomeration of facilities of different clubs in a second best, edgeless economy, constrained to a uniform population distribution, is just one of an infinite number of equivalent allocations, all with the same consumption bundle and utility level but without an agglomeration of clubs. *This, in turn, implies that the agglomeration of facilities of different clubs does not contribute to welfare in an economy with a uniform population distribution and is therefore, an ineffective agglomeration.* The fact that in the above example of an economy with edges, there is a single allocation and in this allocation facilities agglomerate, is entirely due to the economy's edges and to the technical coincidence that all clubs have market areas of the same size. Therefore, in order to avoid confounding the main issues of this paper and to concentrate on essentials, from here on we restrict our analysis to solutions of the model that satisfy the following Condition A:

<u>Condition A</u> *In an optimal allocation investigated here:*

(i) *The number of complexes, k, is an integer.*

(ii) *There is no vacant land in the economy, i.e.,* $\mathsf{L}\,(= kL) = \mathcal{L}$ *and* $R_A > 0$,

where R_A *is the shadow rent at a complex boundary.*

Part (i) of Condition A is intended to avoid the problem of indivisibility of optimal complexes by dealing only with population sizes that are integer multiplications of an optimal complex size. In that we follow HPT. Part (ii) is intended to achieve an edgeless economy to avoid the 'edge-of-the-economy' effect. Under Condition A, for every area $\mathcal{L}$ of the economy we have a lower bound of $\mathcal{N}$, $\underline{\mathcal{N}}(\mathcal{L})$, such that every $\mathcal{N}$ fulfilling Condition A, also fulfills $\mathcal{N} > \underline{\mathcal{N}}(\mathcal{L})$. Then, $\mathsf{L}\,(\equiv kL) = \mathcal{L}$.

We attribute the term *central location pattern (CLP)* to a club's location pattern in which every market area is common to all clubs and facilities of all clubs are located in the center of these joint market areas. Thus, in the above example, the initial location pattern with agglomeration of facilities of all I clubs is a CLP.

We now introduce a new tool to aid in the proof of the next proposition, a rotation of a club system while keeping the population and all the rest of the clubs unmoved. This tool is useful in an edgeless, ring shaped, uniform density economy.

<u>*Definition:*</u> *Let a rotation of club i be a shift to the right by a given distance of all the nodes of club i, while the rest of the economy remains unchanged.*

Club i nodes are the boundaries and facility locations of club i of all the facilities in the economy and in a rotation of club i, the locations of nodes of clubs other than i remain constant as do the quantities of CGs in the facilities of all clubs, including club i. This rotation maintains constant distances between club i nodes and keeps the locations of households unchanged.

We now return to the first best allocation to prove the following Proposition:

Proposition 5 *In a first-best allocation of a club economy the population density is <u>never</u> uniform, i.e., there are segments of the economy in which* $n(x) \neq \mathcal{N}/\mathcal{L}$.[22]

<u>Proof:</u> The proof is by contradiction and it applies to cases satisfying Condition A. Suppose there is an optimal allocation in which $n(x) = \mathcal{N}/\mathcal{L}$ *for all* x. We show below that this assumption leads to a contradiction.

We first argue that in an economy with a uniformly distributed population, symmetry considerations alone imply that all the facilities of a club are identical and each household in the economy has the same consumption basket. The optimality of a club is determined in this economy, by the choice of three parameters: the size (length), l_{ij}, of the market area, which in turn determines N_{ij}, the patronage of the facility, where $N_{ij} = nl_{ij}$, the second parameter is the facility location, $x_{i,2j}$, which is always in the midst of the facility and the third parameter is the quantity of CG in the facility, $G_{i,j}$. These three parameters depend only on the density and homogeneity of the population that are the same everywhere, which implies that in all the facilities of a club these three parameters have the same values. Condition A implies that there

[22]When there is at least one transportation cost function whose second derivative is strictly negative, i.e., there is at least one i_o s.t., $t''_{i_0}<0$, we can strengthen the proposition's result. Actually, if $t''_{i_0}<0$, there is no segment in the economy in which the density of population is constant. To see this, consider (B2) in Appendix 7.2, in which we see that if $\dot{p}_h$ vanishes at a point x, $\ddot{p}_h$ must be positive there. Since $\dot{p}_h = 0$ if and only if the gradient of the density function is zero as well, i.e., $\dot{n}(x) = 0$, the assertion follows. ■

are no divisibility problems in the economy. The same symmetry considerations also imply that housing consumption, $H(x)$, is the same everywhere, since the demand for housing and construction conditions are the same everywhere. The same amount of CGs everywhere together with the same utility level to all, u, the consumption of the composite good, $z(x)$, and with it the whole consumption basket are identical everywhere. Thus, in such an optimal allocation, $H(x) = H,\ Z(x) = Z$, and $G_{ij} = G_i$ for all i, j and x, where $H, Z,\ G_i$ and u are constants.

Consider now a rotation of a club; the consumption of any household in the economy does not change by the rotation: some households may have their relative location changed within the same market area of a facility and others may have the facility changed as well. However, since all facilities of a club contain the same amount of CG and each location in the economy is contained in one of the facilities of the club, all households consume the same amount of CG before and after the rotation.

Since the rotation keeps the length of the market area unchanged and the density of population constant, the patronage of a facility remains N_i. The overall commuting costs in a facility of club i, $\int_0^l nt_i(|y - l/2|)\,dy$, is the same both, before and after a rotation. Commuting costs of a particular household to the nearest facility may increase or decrease, but since the number of facilities does not change and neither do commuting costs in a facility, overall commuting costs in a club do not change.

When rotating a club, we are free to determine the location of one of its nodes, all other nodes are than determined by this choice since the order of and distances between nodes must be maintained. Accordingly, we rotate each club so that they all share one facility location, say, x_2, i.e., $x_{i2} = x_2$, for all $i = 1, ..., I$. Thus, after the rotations, there is a neighborhood of x_2 the size of the smallest market area, in which residents commute only to x_2 to consume all types of CGs.

We now construct two equations describing the optimal 'price of housing function' after the rotations in the neighborhood of x_2. From equations (17) and (18) we obtain (i) below and from the definition of $p_h(x)$ in (9) we obtain (ii) below.

$$p_h^{uo}(x) = \frac{1}{H}\left[Y + \nu - Z - \sum_{i=1}^{I} c_2^i(G_i, N_i) - \sum_{i=1}^{I} t_i(|x - x_2|)\right]. \qquad (i)$$

$$p_h^{uo}(x) = \frac{u_2(x)}{u_1(x)}. \qquad (ii)$$

$\forall x,\ x_2 - \varepsilon \le x \le x_2 + \varepsilon,$ *where* ε *is the length of the smallest market area.*

If we will show that the two expressions of p_h^{uo} in (i) and (ii) obtain different values in some locations we could reach the contradiction that the optimal allocation with uniform population distribution is inconsistent and therefore does not exist. From the discussion above it follows that the values of H, Z, G_i and N_i are the same constants everywhere and do not vary by rotations. Transportation costs to all facilities in the neighborhood of x_2, however, increase with the distance from x_2, as seen in (i) above. Thus the housing price function after the rotations and in the neighborhood of x_2, i.e., $p_h^{uo}(x)$, on the one hand in (i) declines when the distance from x_2 increases and on the other hand in (ii) is a constant, since everywhere, including the neighborhood of x_2, H, Z, and G_i are constants and do not vary with distance. Since $p_h^{uo}(x)$ must be the same in (i) and (ii), this inequality is a contradiction. Therefore our initial assumption that $n(x) = \mathcal{N}/\mathcal{L}$ everywhere, is not correct and there are locations in which the density of population, $n(x)$, is different from $\mathcal{N}/\mathcal{L}$. ■

Another property of an optimal solution in a spatial club economy is that the allocation is symmetric with respect to the center of the complex. The spatial *symmetric structure* is as follows: a club i with an odd number of facilities in a complex, m_i, has facility $j = \frac{m_i+1}{2}$ located in the middle of the complex with its market area spread symmetrically around the facility. The remaining $m_i - 1$ facilities, an even number, are arranged consecutively and located symmetrically with respect to the center of the complex, so that each facility has its mirror image facility on the other side of the center. Thus, facilities j and j' are two symmetric facilities if $j + j' - 1 = m_i$. If m_i is an even number there is no facility in the center and instead a boundary is located there. In this case of an even m_i, all the facilities are symmetrically located around the center so that a facility j is the mirror image of its symmetric facility j' on the other side of the center where $j + j' - 1 = m_i$. The population density is also symmetric around the center of the complex. It should be reminded that in each complex configuration there is at least one club with an odd m_i otherwise the configuration would have a common multiplier and would not be basic. Therefore, there is at least one facility in the center of each complex. We can now prove the existence of the symmetry in an optimal allocation in Proposition 6 below.

Proposition 6 *The optimal complex in a solution of the model that satisfies Condition A is located in a symmetric structure (as described above).*

Proof: To prove the Proposition we need to show that the necessary conditions are consistent with a symmetric structure. We do so by assuming that the first-order conditions are fulfilled in a symmetric structure and show that it does not lead to any contradiction. When checking the consistency of the necessary conditions we need to concentrate mainly on their spatial aspects. Due to Condition A it is sufficient to analyze a representative complex only.

We designate by L the right boundary and length of the representative complex and by o its left boundary. Accordingly, the center of the representative complex is $L/2$, it turns out that $L/2 = x_{i,m_i+1}$ for all i and if m_i is odd, x_{i,m_i+1} is the location of the median facility $\frac{m_i+1}{2}$ and if m_i is even, x_{i,m_i+1} is the boundary between the two middle facilities, $\frac{m_i}{2}$ and $\frac{m_i}{2}+1$ in the complex. For each x, $o \le x \le L/2$ there is a point x', symmetric to x with respect to $L/2$, such that $x' = L - x$, then $L/2 \le x' \le L$.

We start the consistency check with equation (18) by showing that the (bid) housing price function(s) calculated from the equation is symmetric. We first show that the function $Tr\left(x\right)$ in (17) is symmetric with respect to $L/2$. If x, $o \le x \le L/2$, is in the market area of facility $j = \frac{m_i+1}{2}$ of club i, where m_i is odd, then so is x' and $t_i\left(|x - L/2|\right) = t_i\left(|x' - L/2|\right)$. If x, $o \le x \le L/2$ is in a facility of club i with an even m_i or in facility j, $j \le m_i$ *and* $j \ne \frac{m_i+1}{2}$ of club i that has an odd m_i, then$(x_{i,2j} - x) = \left(x' - x_{i,2j'}\right)$ and $t_i\left(|x_{i,2j} - x|\right) = t_i\left(\left|x' - x_{i,2j'}\right|\right)$, where $j + j' - 1 = m_i$, $x_{i,2j'} = L - x_{i,2j}$ and $x = L - x'$. So far we have shown that commuting costs of a household at x to facilities of all I clubs are the same as they are to a household at x', where $x' = L - x$.

Symmetry also implies equality of patronage in symmetric facilities, i.e., $N_{i,j} = N_{i,j'}$, as well as equality between the CGs. Hence $G_{i,j} = G_{i,j'}$, where $j + j' - 1 = m_i$. The equalities of the patronage and CGs between symmetric facilities imply that so are the congestion tolls, i.e., $c_2^i\left(G_{i,j}, N_{i,j}\right) = c_2^i\left(G_{i,j'}, N_{i,j'}\right)$. If x is a point in the market area of facility $j = \frac{m_i+1}{2}$ of club i that has an odd m_i, then so is its symmetric point $x' = L - x$, and households in the two locations pay the same congestion toll in facility $j = \frac{m_i+1}{2}$ of club i.

The implications from the arguments above are that two households, one residing at x and the other at x', travel to symmetric facilities, consume the same amounts of CGs and pay the same commuting costs and congestion tolls. This implies that $Tr\left(\cdot\right)$ in (17) fulfills

$$Tr\left(x\right) = Tr\left(x'\right),\ \forall x, x',\ s.t.\ x + x' = L \tag{27}$$

We are now able to show that (18) is consistent with a symmetric structure. From (27) it follows that $Y + v$ -$Tr(x) = Y + v$ -$Tr(x')$. Therefore, $p_h(x) h(p_h(x)) + z(p_h(x))$, the term equal to $Y + v$ -$Tr(x)$ in (18) after moving $Tr(x)$ to the other side of the equation, must also be equal to $p_h(x') h(p_h(x')) + z(p_h(x'))$. The term $p_h(\cdot) h(\cdot) + z(\cdot)$ is a monotonic increasing function of $p_h(\cdot)$, its only independent argument once all the CGs are given.[23] It follows that the bid housing price function solved from $p_h(\cdot) h(\cdot) + z(\cdot)$ are symmetric, i.e., $p_h^b(x, j^1, .., j^I) = p_h^b(x', j^{1\prime}, .., j^{I\prime})$, where $j^i + j^{i\prime} - 1 = m_i$; and $x + x' = L$, $j^{i\prime}, j^i \in (1, ..., m_i)$, $0 \leq x \leq L/2$,$L/2 \leq x' \leq L$. This in turn implies that the housing price function itself is symmetric, i.e., $p_h(x) = p_h(x')$, where $x + x' = L$. The symmetry of $p_h(\cdot)$ implies that $n(x) (= n(p_h(x)))$ is also symmetric around $L/2$, since it too is a monotonic increasing function of $p_h(\cdot)$ when the facility locations are given (see Appendix 7.2). For given facility locations we can obtain from (24) the boundaries of symmetric facilities by using bid housing price functions. The symmetry of these functions together with the symmetry of facility locations imply the symmetry of the boundaries. When the boundaries are given, (20) implies the symmetry of facility locations. The consistency of a symmetric complex with the rest of the necessary conditions and constraints follows immediately. ■

Propositions 5 and 6 state that a first-best allocation is symmetric and that its population density is never uniform. The question is whether agglomerations of facilities of various clubs actually take place in an optimal allocation. To answer this question we characterize two optimal allocations, each with a simple yet different complex configuration, where we show that the concentration of households and agglomeration of clubs' facilities do occur in an edgeless economy.

5.2. Perfect Agglomeration

The term *perfect agglomeration* refers to an agglomeration of facilities of different clubs located at the same place.[24] An allocation with a central location pattern (CLP) in which all the facilities in a complex are located in the center of the complex is an example of perfect agglomeration. In what follows we show that our model with the configuration $\overbrace{(1, ..., 1)}^{I}$ has an optimal solution with a CLP that satisfies the necessary conditions specified in Section 3. From Proposition 6 we know that all the facilities in a CLP allocation are located in the center of a symmetric complex. By the superscript c we designate the optimal values of variables of the model with the configuration $\overbrace{(1, ..., 1)}^{I}$. In the Proposition below we investigate properties of the model's solution.

Proposition 7 *The optimal allocation of a spatial clubs economy with the configuration* $\left(\overbrace{1, ..., 1}^{I}\right)$ *that satisfies Condition A consists of k complexes, each with a CLP, in which facilities of all the clubs are located in the center of the complex and the population is distributed symmetrically around the complex center. The population density function and the price of housing function are both symmetrical around the complex's center, continuous and differentiable everywhere except at the nodes where both functions are continuous but not differential. Both functions are declining with distance from the center and the housing price function has a positive second*

[23] To see this, note that since h and z are compensated demand functions, $p_h \frac{\partial h}{\partial p_h} + \frac{\partial z}{\partial p_h} = 0$. When we substitute the above expression into the differential of the term $p_h h + z$ with respect to p_h we obtain $\frac{\partial (p_h h + z)}{\partial p_h} = h > 0$.

[24] In a model where facilities occupy space, perfect agglomeration means that the areas occupied by the facilities are adjacent to each other with no households in between.

derivative.

Proof: In a CLP $x_{i,2j} = x^c_{2j}, \forall i$, since all facilities are located in the center of the j-th complex, for all $j = 1, ..., k$. Substituting x^c_{2j} into equation $(B1)$ in Appendix 7.2 yields,

$$\dot{p}_h(x) = -\frac{1}{h(x)} \sum_{i=1}^{I} t'_i(|x - x^c_{2j}|) sign(x - x_{2j});$$
$$\forall x,\ x^c_{2j-1} < x < x^c_{2j+1}, and\ \forall j,\ j \in (1, ..., k)\,, \tag{28}$$

where a dot above a function designates differentiation with respect to distance and x^c_{2j-1} and x^c_{2j+1} are the boundaries of complex j and of all the facilities in the complex. Let $y_j \overset{def}{=} |x - x^c_{2j}|$; s.t., $x^c_{2j-1} \le x \le x^c_{2j+1}, \forall j,\ j \in (1, ..., k)$, be the distance of a point x in complex j from the center of the complex, x^c_{2j}. Then $0 \le y_j \le L/2, \forall j$. If we designate points in complex j by y_j then $y_j = 0$ is the center of complex j, and $y_j = L/2$ are the boundaries of the complex. Note that each y_j, $0 < y_j \le L/2$, stands for two symmetric points in the complex. In what follows we avoid using the index j or superscript c, unless there is a possibility of confusion. From (28), after substituting into it y for $|x - x^c_{2j}|$, we obtain

$$\frac{\partial p_h(y)}{\partial y} = -\frac{1}{h(y)} \sum_{i=1}^{I} t'_i(y). \tag{29}$$

Equation (29) implies that the first derivative of $p_h(y)$ is negative. This means that $p_h(y)$ is monotonically decreasing, its highest value is at the center where $y = 0$ and its lowest value at the boundaries where $y = L/2$. The second derivative of $p_h(y)$ is always positive (see $(B2)$ in Appendix 7.2), which means that the rate of decline decreases with distance from the center. From $(B4)$ in Appendix 7.2, we learn that $n\,(p_h)$ for a given set of CGs, (G_i), is a monotonic increasing function of p_h. This implies that n^c, like p^c_h, is monotonically decreasing with y, and attains its highest value in the center of the complex where $y = 0$, and its lowest value at the boundaries where $y = L/2$. ■

Corollary 8 *In an optimal allocation with a CLP, the agglomeration of facilities of different clubs in the center of each complex is accompanied by a concentration of households around the center.*

The proof follows directly from Proposition 7.

Definition: An optimal allocation of the model with a given set of functions and the basic configuration M, is a global optimum if any optimal allocation of the model with the same functions but with a basic configuration other than M has a lower value of the goal function.

In the example below we present additional specifications to the model's general functions introduced in Section 2. The allocation with the CLP of Proposition 7 together with the complex configuration $\left(\overbrace{1, ..., 1}^{I}\right)$ is the global optimum solution of the model whose functions fulfill the specifications in the example below.

EXAMPLE: Functions Specifications For A Global Perfect Agglomeration

Consider a model of an economy with spatial clubs which in addition to the conditions on the functions set in Section 2, satisfies the following more specific conditions;

1. The utility function is of the form $u = U(H, Z, \psi(G_1, \ldots, G_I))$, *where* $\psi(G_1, \ldots, G_I)$ *is invariant for permutations of the set* $(G_1, \ldots, G_I)$, *e.g.,* $\psi(G_1, \ldots, G_I) = \prod_{i=1}^{I} G_i$.

2. All clubs share the same transportation cost function, i.e., $t_i(y) = t(y), \forall i$ *an the same provision cost function, i.e.,* $c^i(G_{ij}, N_{ij}) = c(G_{ij}, N_{ij}), \forall i, j$.

The housing price function of a CLP in an optimal complex is depicted in Figure 3.

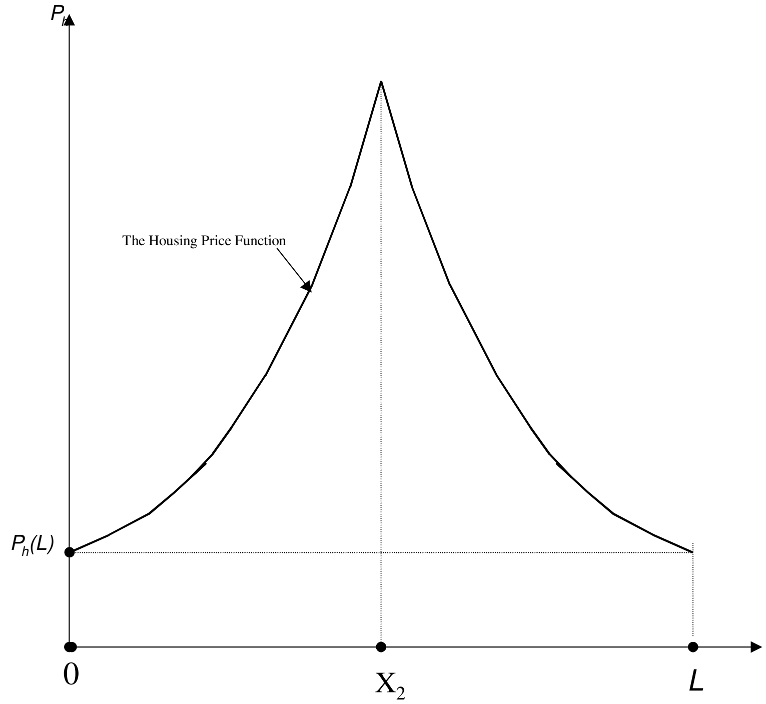

Figure 3: The Housing Price Function of an Optimal Complex with the Configuration (1,...,1).

Lemma 4 *An optimal CLP allocation as described in Proposition 7 is a global optimum solution to the model with functions from the above Example.*

Proof: To show that the global optimum with functions from the Example has a CLP, we show that such a global optimum has the configuration $\left(\overbrace{1,...,1}^{I}\right)$. If in a global optimum solution to the model with functions in the Example above, the sizes of a particular club's market areas, the quantity of CGs in each of the club's facilities and the club's facility locations and boundaries satisfy the necessary conditions specified in section 3 and are therefore optimal for this one club. These same values also satisfy the necessary conditions of all other clubs and the same values are optimal for all clubs. The reason for this is that all clubs have the same cost functions and utility function. This implies that every market area is common to all clubs, which in turn, implies that each market area is a separate complex. The configuration of such a complex is $\left(\overbrace{1,...,1}^{I}\right)$ and an allocation with a CLP is, according to 7, its optimal solution.■

It should be noted that a marginal change in the number of facilities in a complex is impossible and the smallest change is of one more (or less) facility. Therefore, sufficiently small variations in the specifications of the functions would leave the basic configuration of the global optimum unchanged. For example, if instead of using the utility function $u = H \cdot Z \cdot \prod_{i=1}^{I} G_i$

in the Example, the utility function used is $u = H \cdot Z \cdot \prod_{i=1}^{I} G_i^{1+\alpha_i}$, where $|\alpha_i|$ are sufficiently small yet different from each other, the global optimum allocation would still be a CLP. The same is true for small variations in the transportation cost functions of the different clubs or small differences in their provision cost functions. However, while the basic configuration of the global optimum may not change due to small variations, all other variables change continuously.

5.3. Imperfect Agglomeration

In this section we characterize the optimal allocation of the model with the basic configuration given below,

$$(m_i = 1,\ \forall i = 1, ..., I_1\ \ \&\ \ m_i = 2\,, \forall i = I_1 + 1, ..., I)\,;\ 1 \leq I_1 < I < \infty. \tag{30}$$

While in perfect agglomeration we investigate the agglomeration of facilities of all clubs in a single location at the center of the complex, in this section we investigate an example of *an imperfect agglomeration.* Such an agglomeration takes place when facilities of different clubs agglomerate in clusters around the center of the complex but steer clear of it. By saying that the agglomeration is imperfect we mean that the clusters may contain dwellings between the facilities.

We first introduce the symmetric structure of the allocation with the configuration given in (30) as specified in Proposition 6. The symmetric structure possesses the following properties: (1) Each of the clubs $i \in 1, ..., I_1$, (henceforth SF clubs) have one facility located in the middle of the complex and its market area is the whole complex. (2) The two facilities of each of the clubs $i \in (I_1 + 1, ..., I)$ (henceforth DF clubs), are symmetrically located on each side of the center of the complex and each of their market areas is extended between a complex boundary and the center. Altogether, the complex has I_1 facilities of SF clubs and $2\,(I - I_1)$ facilities of DF clubs, $(I - I_1)$ facilities of DF clubs on each side of the complex center. The properties of the allocation with the configuration (30) are depicted in the series of lemmas presented in the rest of this section.

Lemma 5 *In the optimal allocation of the model with the configuration (30) discussed above, all the facility locations of the DF clubs are in the second and third quarters of the complex area.*[25] *The average density of the population residing between the two facilities of the DF club located farthest from the center (one facility to the left and one to the right of the center), is higher than it is between these two DF facilities and the boundaries.*

Proof: Without loss of generality, we consider only the first half of the representative complex, namely the segment $(0, L/2)$ (an equivalent analysis would apply to the other half $(L/2, L)$). Let $\underline{x}_2$ in $(0, L/2)$ be the facility location of the DF club closest to the origin, 0, and farthest from the center $L/2$ (if there is more than one such club, one of them is chosen arbitrarily). This means that all facility locations of the DF clubs in $(0, L/2)$ are located between $\underline{x}_2$ and the center, $L/2$. We show below that $\underline{x}_2$ is in the second quarter of the complex area and therefore so are the rest of the DF locations.

To see that $\underline{x}_2$ is in the second quarter, consider a location x that is to the left of $\underline{x}_2$, i.e., $0 \leq x < \underline{x}_2$. The point x has a point x', $\underline{x}_2 \leq x' \leq L/2$, symmetric to x with respect to $\underline{x}_2$, so that $\underline{x}_2 - x = x' - \underline{x}_2$. Note that all the individuals residing in the segment $(0, L/2)$ travel to

[25] By the term 'quarter' we refer to a segment which results from a division of the complex's length into four equal consecutive segments. The first quarter is the segment farthest to the left and the other three quarters are numbered consecutively in the clockwise direction.

the same facilities. An individual residing in x' travels to the same facilities as the individual at x, however, the distances from x' to all the facilities that are not located in $\underline{x}_2$ are shorter than from x, because all the facilities are located to the right of $\underline{x}_2$. Even if all the DF clubs are located in $\underline{x}_2$, the distance to travel from $x' = 2\underline{x}_2 - x$ to the SF-clubs in the center is still shorter. Accordingly, commuting costs from x' to all facilities are lower than from x, while all other arguments of $p_h^{ia}(x)$ are the same. In $(B1)$ in Appendix 7.2 it is shown that an increase in transportation costs, while all arguments of $p_h(x)$ are kept constant, causes $p_h(x)$ to decline. Therefore, $p_h^{ia}(x)$, the housing price at x, $0 \leq x < \underline{x}_2$, must be lower than $p_h^{ia}(x')$, the housing price at x'. Consequently, because $n(p_h)$ is monotonic increasing in p_h while the rest of the variables remain constant it follows that $n^{ia}(x) < n^{ia}(x')$. Consider now (20) with respect to facility ($\underline{i}$,1) located in $\underline{x}_2$. On the left-hand side of the equation, for every x which has a symmetric x', $t'_{\underline{i}}(\underline{x}_2 - x)$ is equal to $t'_{\underline{i}}(x' - \underline{x}_2)$ on the right-hand side. However, $t'_{\underline{i}}(\underline{x}_2 - x)$ on the left-hand side of the equation is weighted by $n^{ia}(x)$, which is lower than $n^{ia}(x')$, the weight of $t'_{\underline{i}}\left(x' - \underline{x}_2\right)$ on the right-hand side. This implies that for the equality in (20) to hold, the interval $(0, \underline{x}_2)$, in which the weights are lower, must be longer than the interval $(\underline{x}_2, L/2)$ in which the weights are higher. This, in turn, implies that $\underline{x}_2 - 0 > L/2 - \underline{x}_2$, which means that $\underline{x}_2 > L/4$ (see Figure 4). This is the first item we have to prove in the lemma.

We just proved that $n^{ia}(x') > n^{ia}(x)$ for all x located to the left of $\underline{x}_2$ and having a symmetric point x' with respect to $\underline{x}_2$. Additionally, there are points between the origin and the point $(2\underline{x}_2 - L/2)$, the point symmetric to $L/2$ with respect to $\underline{x}_2$, which is located to the left of $\underline{x}_2$. These points left of $(2\underline{x}_2 - L/2)$ have no matching symmetric points and since the transportation costs of residents in these locations are higher than in any point x that has a matching symmetric point, the value of $p_h^{ia}(x)$ for $x < (2\underline{x}_2 - L/2)$ must be lower than it is for any $x>(2\underline{x}_2 - L/2)$. Furthermore, as $x < (2\underline{x}_2 - L/2)$ approaches the origin, $p_h^{ia}(x)$ continues to decline. To see this, consider equation $(B1)$ in Appendix 7.2. In this equation, $\dot{Tr}(x)$ is positive (negative) when x increases (declines) since $sign(x - x_{i,2ji})$ is negative for all i. Therefore, $\dot{p}_h^{ia} = -\frac{\dot{Tr}}{h(x)}$ is positive, i.e., $p_h^{ia}(x)$ declines when x approaches zero. In turn, it follows from (B4) in Appendix 7.2 that the density of population at the unmatched points to the left of $(2\underline{x}_{i2} - L/2)$ is lower than at any other point to the right of $(2\underline{x}_{i2} - L/2)$. Thus, the average density of population between $\underline{x}_2$ and the origin must be lower than between $\underline{x}_2$ and $L/2$.■

In what follows we show that all the DF clubs agglomerate in two clusters, one in the second quarter of the complex and the other in the third. Let $\overline{x}_2$ designate the facility location of the DF club in $(0, L/2)$, which is located closest to $L/2$. At the DF club closest to the center of the complex (20) is satisfied with the point $\overline{x}_2$ as the location of the facility. This means that there is a positive distance between $\overline{x}_2$ and the boundary at $L/2$, equal to the short side of the market area of the facility closest to the center. It follows that all the facilities of the DF clubs in the first half of the complex are located between $\overline{x}_2$ $(< L/2)$ and $\underline{x}_2$ $(> L/4)$ and are clustered together close to each other in the second quarter of the complex (and consequently, the DF clubs in the second half of the complex are clustered in the third quarter of the complex). We term such close grouping of facility locations, a *cluster of DF facilities.* In the lemma above we showed that such clusters of DF clubs are located closer to the center of the complex than to the boundaries. In such cases we say that the DF clusters *gravitate towards the center of the complex.*

To clarify the role of transportation costs in an imperfect agglomeration of DF clubs, consider the following Lemma;

Lemma 6 *In an allocation with the basic configuration specified in (30), different DF clubs with* *proportional transportation cost functions share the same facility locations.*[26]

Proof: Suppose that i and i' are two clubs with proportional transportation costs, i.e., $t_i(x)/t_{i'}(x) = \alpha_{ii'}, \forall x$, where $\alpha_{ii'}$, the factor of proportionality, is constant. Then the proportionality is retained by the derivatives as well as the functions and $t_i'(x) = \alpha_{ii'} t_{i'}'(x)$ .Thus, if (20) holds for club i it holds for its proportional club i' at the same facility location as well. To see this consider (20) for club i, in which we substitute $\alpha_{ii'} t_{i'}'(x)$ for $t_i'(x)$ and then we eliminate the proportionality factor $\alpha_{ii'}$ from the equation to obtain (20) for club i' at the same facility location as club i. ■

Note that all linear transportation cost functions are proportional and therefore DF clubs with linear transportation cost functions agglomerate perfectly at a single location. Examples can be constructed of non-proportional transportation cost functions of DF clubs that yield different facility locations for each club.

Figure 4 depicts the housing price function in an optimal complex with the configuration (1,2).

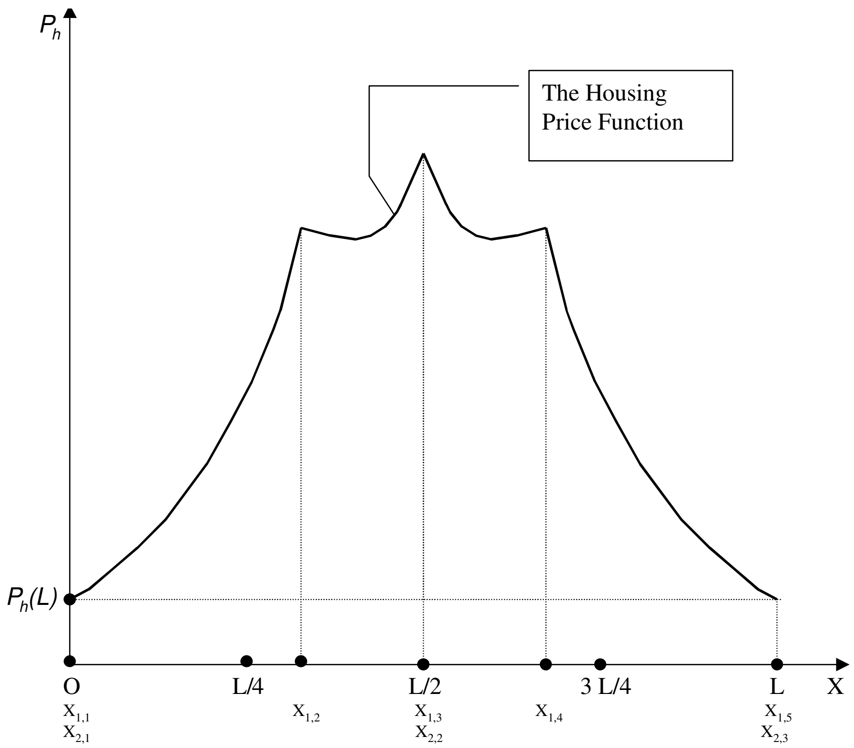

Figure 4: The Housing Price Function in a Complex with the Configuration (1,2)

We can now summarize the analysis of imperfect agglomeration performed in this section by the following Proposition;

Proposition 9 *In an optimal allocation with the basic configuration specified in* (30)

(*i*) *The facilities of SF clubs in the complex, i.e., of clubs* $i \in (1, ..., I_1)$, *are all perfectly agglomerated in the center of the complex;*

(*ii*) *The facilities of DF clubs, i.e., of clubs* $i \in (I_1 + 1, ..., I)$, *agglomerate imperfectly in clusters that gravitate towards the center of the complex, i.e., the clusters agglomerate in the second and third quarters of the complex.*

[26] Recall that in our model, if two facilities of different club types share the same location, it means that they are adjacent to each other with no residential area between them.

(*iii*) *The average density of population between the clusters of the DF clubs and the center of the complex is higher than the average density between these clusters and the boundaries of the complex.*

(*iv*) *If, in a cluster, two DF clubs have proportional transportation cost functions, they share the same facility location.*

6. Summary and Concluding Remarks

The purpose of this paper was to characterize optimal allocations of an economy with spatial clubs and to investigate agglomeration trends of households and club facilities in it. Our results showed that each local optimum could be decentralized, sometimes in more than one way, although most were difficult, if not impossible, to implement. In an optimal allocation of clubs, the primary agglomeration was of club goods into facilities due to scale economies in their provision to the population. The primary agglomeration led to a secondary agglomeration of population which, in turn, led to the tertiary agglomeration of facilities of different clubs in centers. The three types of agglomerations occurred simultaneously and their ordering is due to causality not timing: Without the primary agglomeration there would not be a secondary one, and without the secondary one there would not be a tertiary agglomeration. Furthermore, an optimal allocation would never have a uniform population distribution and neither would an allocation with a uniform distribution of population have an efficient agglomeration of facilities. We then showed that the price of housing as well as the supply and demand for housing functions were continuous functions of the distance, x, as was the density of population function. We also showed that the optimum complex was symmetric with respect to its center.

We characterized two allocations, each with a specific complex configuration: in the first allocation, each complex contained one facility of each club and in the second allocation each complex contained both: clubs that had one facility per complex and clubs that had two facilities per complex. We identified two distinct types of agglomerations of club facilities: the perfect agglomeration and the imperfect one. In the perfect agglomeration, facilities of different clubs agglomerated perfectly in the center of the complex, where they are adjacent to each other without residential activity between them. In the second allocation, besides the perfect agglomeration of facilities of some of the clubs in the center of the complex, facilities of the rest of the clubs agglomerated imperfectly in clusters, but households still may were residing between them. While the clusters as a whole were away from the center of the complex, but drawn towards it.

In respect to the issue of global optimum solutions of the model, the solutions specified were mostly to the model with a predetermined complex configuration. We termed such an allocation a local optimum since the global optimum included a solution to the configuration variables as well. The only global optimum was the solution to the model with functions specified in the Example in section 5.2 together with the configuration $(1, ..., 1)$.

The purposes of this paper were completely satisfied so far. One avenue for future research may focus on the relation between certain costs and utility functions and their global optimal configuration. Such a research may shade light on questions like what functions would result in a hierarchy of clubs in a global optimum or what causes certain types of clubs to be imbedded in other club facilities.

REFERENCES

1. Berglas, E. (1976) "On the Theory of Clubs," *American Economic Review,* May (Paper and Proceedings), 66(2), 116-121.

2. Berliant, M. and T. ten Raa (1994) "Regional Science: the State of the Art," *Regional Science and Urban Economics*, 24, 631-647.
3. Fujita, M. (1989) *Urban Economic Theory*, Cambridge: Cambridge University Press.
4. Fujita, M. and H. Ogawa (1982) "Multiple Equilibria and Structural Transition of Non-monocentric Urban Configurations," *Regional Science and Urban Economics*, 12, 161-196.
5. Fujita, M. and J. F. Thisse (1986) "Spatial Competition with Land Market: Hotelling and Von Thunen United," Review of Economic Studies53,819-41.
6. Fujita, M. and J. F. Thisse (1996) "Economics of Agglomeration," *Journal of the Japanese and International Economies*, 10, 339-378.
7. Fujita, M. and J. F. Thisse (2002) *Economics of Agglomeration,* Cambridge, Cambridge University Press.
8. Henderson, J. V. (1974) "The Sizes and Types of Cities," *American Economic Review*, 64, 640-56.
9. Hochman, O., D. Pines and J.-F. Thisse (1995) "On the Optimal Structure of Local Governments," *American Economic Review*, 85, 1224-1240.
10. Knight, F. H. (1924) "Some Fallacies in the Interpretation of Social Cost," The Quarterly Journal of Economics, 38, 4, 582-606.
11. Mills, E. S. (1967) "An Aggregative Model of Resource Allocation in a Metropolitan Area," *American Economic Review,* 57, 197-210.
12. Mills, E. S. and B. W. Hamilton (1989) *Urban Economics*, Scott, Foresman, 4th ed.
13. Ogawa, H. and M. Fujita, (1980) "Equilibrium Land Use Patterns in a Nonmonocentric City," *Journal of Regional Science*, 20:4, 455-475.
14. Muth R. F. (1969) *Cities and Housing,* Chicago, University of Chicago Press.
15. Lucas, R. E. Jr. and E. Rosi-Hansberg (2002) "On The Internal Structure of Cities," *Econometrica,* 70,4,1445-1476.
16. Papageorgiou, Y. and D. Pines (1999) *An Essay on Urban Economic Theory*, Kluwer Academic Press, Ch. 10.
17. Scotchmer, S. (2002) "Local Public Goods and Clubs" in *Handbook of Public Economics,* Vol. IV, (A. Anerbach and M. Feldstein, Eds), North-Holland.
18. Scotchmer, S. and M. H. Wooders (1987) "Competitive Equilibrium and the Core in Club Economies with Anonymous Crowding," *Journal of Public Economics,* 34(2), 159-174.
19. Solow, R. W. and W. S. Vickrey (1971) "Land Use in a Long Narrow City," *Journal of Economic Theory*, 3, 430-447.
20. Thisse, J.-F. and D. E. Wildasin (1992) "Public Facility Location and Urban Spatial Structure: Equilibrium and Welfare Analysis," *Journal of Public Economics*, 48, 83-118.

7. Appendices

7.1. Deriving the Necessary Conditions for a Pareto Optimum

The Lagrangian, $\mathbb{L}$, of the problem set up in Section 2 is given below in $(A1)$. The Lagrange multipliers are: $[\lambda(x), \alpha(x), \delta_{ij}, \gamma_i, \rho, \omega, \eta, \ell_i, \theta_i]$. The constraints multiplied by ω, ℓ_i, θ_i in the Lagrangian are effective equalities and their multipliers can obtain any sign. The rest of the constraints are inequalities and their multipliers must be non-negative. When these multipliers are positive the constraint they multiply is effective and when a constraint is not effective, its multiplier vanishes. Thus,

$$\begin{aligned}\mathbb{L} &= U - \int_0^L \lambda(x)\left(U - u(Z(x), H(x), G_{1,j_1(x)}, \dots, G_{I,j_I(x)})\right)dx - \sum_i \gamma_i \left(N - \sum_{j=1}^{m_i} N_{i,j}\right) \\ &- \int_0^{\mathsf{L}} \alpha(x)\left(n(x)H(x) - H^s(x)\right)dx - \sum_i \sum_j \delta_{ij}\left(N_{ij} - \int_{x_{i,2j-1}}^{x_{i,2j+1}} n(x)dx\right) \\ &-\eta\left[\int_0^L [n(x)Z(x) + c_h(H^s)]dx + \sum_{i=1}^{I}\sum_{j=1}^{m_i}\int_{x_{i,2j-1}}^{x_{i,2j+1}} n(x)t_i(|x - x_{i,2j}|)dx + \sum_{i=1}^{I}\sum_{j=1}^{m_i} c^i(i,j) - NY\right] \\ &-\sum_i \ell_i\left(x_{i,2m_i+1} - L\right) - \sum_i \theta_i x_{i,1} - \rho\left(L - \mathcal{L}/k\right) - \omega\left(\mathcal{N}/k - N\right). \qquad (A1)\end{aligned}$$

The following equations are first-order conditions for the maximization of the model. To obtain a first-order condition we differentiate the Lagrangian with respect to a variable of the model and equate the result to zero. In each of the first-order conditions, the particular derivative is written at the beginning of each equation and to the left of the double slashes.

$$\frac{\partial \mathbb{L}}{\partial x_{i,2m_i+1}}\left\| n(L)\frac{\delta_{i,m_i}}{\eta} - n(L)\left[t_i(|L - x_{i,2m_i}|)\right] = \frac{\ell_i}{\eta}. \right. \qquad (A2)$$

$$\frac{\partial \mathbb{L}}{\partial L}\left\| \sum_i \frac{\ell_i}{\eta} - \left[n(L)Z(L) + c_h(H^s(L))\right] = \frac{\rho}{\eta}. \right. \qquad (A3)$$

$$\frac{\partial \mathbb{L}}{\partial N_{ij}}\left\| \frac{\gamma_i}{\eta} - c_2^i(G_{ij}, N_{ij}) = \frac{\delta_{ij}}{\eta} \right. \qquad (A4)$$

Substituting $(A4)$ for $j = m_i$ into $(A2)$ and the result into $(A3)$, yields

$$n(L)\sum_i \frac{\gamma_i}{\eta} - \frac{\rho}{\eta} = n(L)\sum_{i=1}^{I}\left[t_i(|L - x_{i,2m_i}|) + c_2^i(i, m_i)\right] + n(L)z(L) + c_h(H^s(L)). \qquad (A5)$$

$$\frac{\partial \mathbb{L}}{\partial n(x)}\left\| \sum_i \frac{\delta_{ij_i(x)}}{\eta} - \frac{\alpha(x)}{\eta}H(x) - z(x) - \sum_{i=1}^{I} t_i(|x - x_{i,2j_i(x)}|) = 0. \right. \qquad (A6)$$

Substituting $(A4)$ into $(A6)$ yields

$$\sum_i \frac{\gamma_i}{\eta} - \left[z(x) + \frac{\alpha(x)}{\eta}H(x) + \sum_i c_2^i(i, j_i(x)) + \sum_{i=1}^{I} t_i(|x - x_{i,2j_i(x)}|)\right] = 0, \forall x,\ 0 \le x \le L. \qquad (A7)$$

$$\frac{\partial \mathbb{L}}{\partial Z(x)}\left\| \lambda(x) = \frac{\eta n(x)}{u_1(x)}. \right. \qquad (A8)$$

$$\left.\frac{\partial \mathbb{L}}{\partial H(x)}\right\| \begin{array}{l} \lambda(x)\, u_2(x) - \alpha(x)\, n(x) = 0 \\ \Longrightarrow n(x) \left[\frac{\alpha(x)}{\eta} - \frac{u_2(x)}{u_1(x)}\right] = 0. \end{array} \tag{A9}$$

$$\left.\frac{\partial \mathbb{L}}{\partial H^s(x)}\right\| \frac{\alpha(x)}{\eta} - c_h'(H^s) = 0. \tag{A10}$$

We multiply $(A7)$ for $x = L$ by $n(L)$ and then substitute the result together with $(A10)$ and $n(L)\, H(L) = H^s(L)$ into $(A5)$, to obtain;

$$H^s(L)\, c_h'(H^s(L)) - c_h(H^s(L)) = \frac{\rho}{\eta}. \tag{A11}$$

Equation $(A7)$ implies that $\frac{\rho}{\eta}$, the shadow price of the occupied land constraint, is also the land rent at L, the boundary of the complex. By applying the Kunn-Tucker rule to (A11) we obtain: $\frac{\rho}{\eta} > 0 \Longrightarrow L = \mathcal{L}/k$ and $L < \mathcal{L}/k \Longrightarrow \frac{\rho}{\eta} = 0$. This, in turn, implies that $\frac{\rho}{\eta}$ is the alternative land rent (R_A in the text proper). We continue now with the derivation of the rest of the first-order conditions

$$\left.\frac{\partial \mathbb{L}}{\partial G_{i,j}}\right\| \int_{x_{i,2j-1}}^{x_{i,2j+1}} n(x) \left[u_{i+2}(x)/\, u_1(x)\right] dx - c_1^i(i,j)\ = 0. \tag{A12}$$

$$\left.\frac{\partial \mathbb{L}}{\partial x_{i,2j}}\right\| \int_{x_{i,2j-1}}^{x_{i,2j}} n(x) t_i'(x_{i,2j} - x) dx = \int_{x_{i,2j}}^{x_{i,2j+1}} n(x) t_i'(x - x_{i,2j}) dx. \tag{A13}$$

$$\left.\frac{\partial \mathbb{L}}{\partial N}\right\| Y + \frac{\omega}{\eta} - \sum_i \frac{\gamma_i}{\eta} = 0. \tag{A14}$$

$$\left.\frac{\partial \mathbb{L}}{\partial k}\right\| \rho \mathcal{L} = \omega \mathcal{N}. \tag{A15}$$

Since ρ is non-negative according to $(A15)$, so must ω be non-negative. We now substitute $(A14)$ into $(A7)$ to obtain:

$$Y + \frac{\omega}{\eta} - \left[z(x) + \frac{\alpha(x)}{\eta} H(x) + \sum_i c_2^i(i, j_i(x)) + \sum_{i=1}^{I} t_i(|x - x_{i,2j_i(x)}|) \right] = 0. \tag{A16}$$

Note that $(A9)$ and $(A10)$ imply that $\frac{\alpha(x)}{\eta}$ is equal to $p_h(x)$, the housing price function in the text proper, and equation $(A16)$ is the so-called 'household's optimal budget constraint', where $\sum_i c_2^i(i, j_i(x))$ is the sum of all 'congestion tolls' to be paid by the household at x to each of the I facilities it patronizes. The term $\frac{\omega}{\eta}$ is the household's share in the overall alternative land rents, as can be verified from $(A11)$ and $(A15)$.

Now we substitute $(A11)$ into $(A15)$ to replace ρ with ω, and substitute the result into $(A16)$ to eliminate ω. We multiply the result by $n(x)$, integrate between $[0, L]$ and into the result we add the resource constraint (12) with (13) substituted into it. We then substitute $(A10)$ and $(A11)$ into the result to obtain

$$\int_0^L \left[c_h'(H^s(x)) H^s(x) - c_h(H^s(x)) - \frac{\rho}{\eta} \right] dx + \sum_i \sum_j \left[N_{i,j} c_2^i(G_{i,j}, N_{i,j}) - c^i(G_{i,j}, N_{i,j}) \right] = 0. \tag{A17}$$

This is the Henry George rule for the complex.

7.1.1. Continuity of Functions at Boundaries

The derivations below concern variables at the boundaries of clubs. In the text we use bid rent functions to derive these relations. In this Appendix we use direct differentiation and show that the technique of bid functions used in the text satisfies the necessary conditions derived in this Appendix.

First we note that households residing to the left of a boundary travel to the left in order to consume the CG, while residents to the right of a boundary travel to the right. This implies residents in a neighborhood of a boundary consume each CG whose market areas are separated by the boundary, at one of two different facilities. These two facilities provide the same CG but they may be located at different distances from the boundary and contain different quantities of the CG. Therefore the quantity of housing and composite good consumed by households at the vicinity of a boundary may be discontinuous and when approaching a boundary from the left the consumption basket may differ from the one when approaching the boundary from the right. Consider such a boundary point x_o between facilities $j_k \& \; j_k + 1 \in (1, ..., m_{i_k})$, of clubs $i_k \in (1, ..., I)$, $k = 1, ..., K$; $1 \leq K < I$, i.e., $x_o = x_{i_1,2j_1+1} = ... = x_{i_K,2j_K+1}$ and x_o is an interior point to all clubs i, s.t., $i \neq i_k \& i \in (1, ..., I)$. In the Lagrangian $(A1)$, every integral which contains x_o in its domain can be split at x_o into two integrals: in one integral x_o is the upper limit and in the other integral x_o is the lower limit, i.e., $\int_0^L f(x)\,dx = \int_0^{x_o} f(x)\,dx + \int_{x_o}^L f(x)\,dx$. When the integrand is continuous at x_o, splitting the integral does not affect the outcome. However, there are control variables that may be discontinuous at such a boundary point, x_o. Households residing at x_0 may commute either to the left of x_0 or to its right to consume the CGs for which x_0 is a boundary. These two types of households, in addition to possibly using different quantities of CGs, may differ in commuting costs and congestion tolls as well as in the amounts of housing and composite good they consume. Thus, the variables that may be discontinuous at x_o besides G_{i_k}, are $H(x_o)$, $H^s(x_o)$, $n(x_o)$ and $Z(x_o)$. We designate by the superscript $+$ the limit at $x_o$of these variables when approaching x_o from the left and by the superscript $-$ the limit of the variables when approaching x_o from the right. Accordingly, at x_o the variables $H^+(x_o)$, $Z^+(x_o)$, $n^+(x_o)$ and $H^{s+}(x_o)$ are each left-continuous and $H^-(x_0)$, $Z^-(x_0)$, $n^-(x_o)$ and $H^{s-}(x_0)$ are each right-continuous. Note that Lagrange multipliers that are functions of x, such as $\lambda(x)$ and $\alpha(x)$, may also split at the boundaries.

We now introduce these split variables into the Lagrangian at boundary points and then derive the necessary conditions associated with them. First, note that the differentiation with respect to the split variables themselves yields the same equations as the derivation with respect to the same continuous variables but with the split variables replacing the continuous ones. New necessary conditions are obtained only when differentiating with respect to the location of the boundary, x_o. In the Lagrangian, besides the integrals with limits in boundaries such as $x_{i,2j\pm1}$, there are three additional places where these split variables may appear: one is in the integrals of the resource constraint that are multiplied by η; the second is in the utility constraint that is multiplied by $\lambda(x)$; and the third is in the equality of demand and supply of housing at each location that is multiplied by $\alpha(x)$.

When differentiating with respect to x_o either the utility constraint or the equality between the demand and supply of housing, we obtain the constraint multiplied by the Lagrange multiplier at the boundary. This expression vanishes and therefore can be ignored. It should be noted that households at a boundary $x_o = x_{i_k,j_k+1}$, may either commute to their left or to their right. If a household commutes to his left to facility i_k, j_k at $x_{i_k,2j_k}$ he bears commuting costs of $t_{i_k}(x_o - x_{i_k,2j_k})$, congestion tolls $c_2^{i_k}(i_k, j_k)$ and the variables associated with it at x_o are right-continuous (with superscript $+$). If, however, a household commutes to his right to facility $i_k, j_k + 1$ at $x_{i_k,2(j_k+1)}$, his commuting costs are $t_{i_k}(x_{i_k,2j_k+1} - x_o)$, congestion tolls $c_2^{i_k}(i_k, j_k + 1)$

and the variables associated with the household at x_o are left-continuous (with superscript -). Accordingly, the differentiation of the Lagrangian with respect to x_o $(= x_{i_k,2j_k+1})$ is given below,

$$-n^+(x_o)\frac{\sum_k \delta_{i_k,j_k}}{\eta} + n^+(x_o)Z^+(x_o) + c_h(H^{s+}(x_o)) + n^+(x_o)\sum_k t_{i_k}(x_o - x_{i_k,2j_k}) =$$
$$-n^-(x_o)\frac{\sum_k \delta_{i_k,j_k+1}}{\eta} + n^-(x_o)Z^-(x_o) + c_h(H^{s-}(x_o) + n^-(x_0)\sum_k t_{i_k}(|x_{i_k,2(j_k+1)} - x_o).$$

After substituting $(A4)$ into the above equation and rearranging terms, we obtain the following necessary condition for efficiency,

$$n^+(x_o)\left[\sum_k \left(c_2^{i_k}(i_k,j_k) + t_{i_k}(x_o - x_{i_k,2j_k}) - \frac{\gamma_{i_k}}{\eta}\right) + Z^+(x_o)\right] + c_h(H^{s+}(x_o)) =$$
$$n^-(x_o)\left[\sum_k \left(c_2^{i_k}(i_k,j_k+1) + t_{i_k}(x_{i_k,2(j_k+1)} - x_o) - \frac{\gamma_{i_k}}{\eta}\right) + Z^-(x_o)\right] + c_h(H^{s-}(x_o)). \quad (A18)$$

$P_h(x)$ in the text is a continuous function of x (see Corollary 1) and in this Appendix it is equal to $\frac{\alpha(x)}{\eta}$ (see $(A9)$), hence $\frac{\alpha(x)}{\eta}$ is continuous as well. Condition $(A10)$ here, implies that $H^s(x_o)$ is also continuous, i.e., $H^s(x_o) = (c_h')^{-1}\left(\frac{\alpha(x_o)}{\eta}\right)$. This, in turn, implies that $i)$ $c_h(H^{s\pm}(x_o)) = c_h(H^s(x_o))$. and $ii)$ $n^\pm(x_o)\,H^\pm(x_o) = H^s(x_o)$.

We now subtract from the square brackets in the left-hand side of $(A18)$ the null-valued left-hand side of $(A7)$ for the variables at x_o with the superscript $+$. With these right-continuous variables the commuting is to the left to clubs $i_k, j_k, k = 1, ..., K$. Next, we subtract from the square brackets on the right-hand side of $(A18)$, the null-valued left-hand side of $(A7)$ for the variables at x_o with the superscript $-$, where commuting to clubs $i_k, j_k + 1$ is to the right of x_o to $x_{i_k,2(j_k+1)}$. Since x_o is an interior point of all clubs other than i_k, commuting costs to and congestion tolls at any club $i, (i \in 1, ..., I \cap i \neq i_k, k = 1, ..., K)$, are the same for all households at x_o. In addition, we substitute $i)$ and $ii)$ above into the result to obtain, after rearranging terms

$$n^-(x_o)\left[\sum_{i\neq i_o}\frac{\gamma_i}{\eta} - \sum_{i\neq i_o} c_2^i(i,j^i(x_o)) - \sum_{i\neq i_o}^{I} t_i(|x_o - x_{i,2j_i(x_o)}|)\right] + c_h(H^s(x_o)) - \frac{\alpha(x_o)}{\eta}H^s(x_o) =$$
$$n^+(x_o)\left[\sum_{i\neq i_k}\frac{\gamma_i}{\eta} - \sum_{i\neq i_k} c_2^i(i,j^i(x_o)) - \sum_{i\neq i_k}^{I} t_i(|x_o - x_{i,2j_i(x_o)}|)\right] + c_h(H^s(x_o)) - \frac{\alpha(x_o)}{\eta}H^s(x_o).$$

By reducing equal terms from both sides of the above equation we obtain $n^+(x_o) = n^-(x_o)$, i.e., the density of the population is a continuous function of x in the boundary x_o. This implies that $H(x_o) = \frac{H^s(x_o)}{n(x_o)}$ is continuous as well at x_o.

To summarize, in this section we proved that the continuity of the housing price function at a boundary implies the continuity of the population density function, the housing supply function and the housing demand function at the boundary. ■

7.1.2. Necessary Conditions for Boundaries and Bid Housing Price Functions

We now substitute the equality between the variables $n(x_0)$, $H^s(x_0)$ and $H(x_0)$ with superscript $+$ to their counterparts with superscript $-$ into $(A18)$ and after reducing equal terms we obtain,

$$\sum_k \left[c_2^{i_k}(i_k, j_k) + t_{i_k}(x_o - x_{i_k,2j_k})\right] + Z^+(x_o) = \sum_k \left[c_2^{i_k}(i_k, j_k+1) + t_{i_k}(x_{i_k,2(j_k+1)} - x_o)\right] + Z^-(x_o). \quad (A19)$$

Condition $(A19)$ is the necessary condition for x_o to be a boundary point between facilities j_k and j_k+1 of club i_k, for all $k = 1, ..., K$, i.e., $x_o = x_{i_1,2j_1+1} = ... = x_{i_K,2j_K+1}$ and an interior point in all other clubs. In the Lemma below we prove that this condition is equivalent to the determination of boundary points by the intersection of bid rent functions (see section 3.3.3).

Lemma 7 *Equation $(A19)$ holds for x_o if and only if $(A20)$ below holds.*

$$P_h^b\left(x_o; \left[j_k^{i_k}, \forall k \in (1, ..., K)\right] \cup \left[\ \left(j^i(x_o), \forall i \in (1, ..., I)\right) \cap (i \neq i_k, \forall k)\right]\right) =$$
$$P_h^b\left(x_o; \left[j_k^{i_k}+1, \forall k \in (1, ..., K)\right] \cup \left[\ \left(j^i(x_o), \forall i \in (1, ..., I)\right) \cap (i \neq i_k, \forall k)\right]\right). \quad ((A20))$$

It should be noted that the indexes $j_k^{i_k}$ above are identical to the pairs i_k, j_k.

Proof: First we show that $(A20)$ implies $(A19)$. From (22) we obtain at x_o for facilities $j_k^{i_k}$:

$$H(x_o)\, P_h^b\left(x_o; \left[j_k^{i_k}, \forall k \in (1, ..., K)\right] \cup \left[\ \left(j^i(x_o), \forall i \in (1, ..., I)\right) \cap (i \neq i_k, \forall k)\right]\right) = Y + v -$$
$$\left\{Z^+(x_o) + \sum_k \left[c_2^{i_k}(j_k^{i_k}) + t_{i_k}(x_o - x_{i_k,2j_k})\right] + \sum_{i \neq i_k} \left[c_2^i\left(j^i(x_o)\right) + t_i\left(\left|x_o - x_{i,2j(x_o)}\right|\right)\right]\right\}, \quad (i)$$

whereas for facilities $j_k^{i_k}+1$ we obtain,

$$H(x_o)\, P_h^b\left(x_o; \left[j_k^{i_k}+1, \forall k \in (1, ..., K)\right] \cup \left[\ \left(j^i(x_o), \forall i \in (1, ..., I)\right) \cap (i \neq i_k, \forall k)\right]\right) = Y + v -$$
$$\left\{Z^-(x_o) + \sum_k \left[c_2^{i_k}(j_k+1) + t_{i_k}(x_{i_k,2(j_k+1)} - x_o)\right] + \sum_{i \neq i_k} \left[c_2^i\left(j^i(x_o)\right) + t_i\left(\left|x_o - x_{i,2j(x_o)}\right|\right)\right]\right\} (ii)$$

We now multiply $(A20)$ by $H(x_o)$ and then substitute into its left-hand side the right-hand side of (i) above, and into the right-hand side of the extended $(A20)$, we substitute the right-hand side of (ii) above. Then by reducing identical terms from both sides of the equation we obtain $(A19)$. Thus we showed that $(A20)$ implies $(A19p)$.

To show that $(A19)$ implies $(A20)$ we claim the following : from the right-hand side of (22) for facilities $j_k^{i_k}$ at the boundary x_o, we subtract the left-hand side of $(A19)$ for i_k and $j_k^{i_k}$. We then equate the result to the right-hand side of (22) for facilities $j_k^{i_k}+1$ at the boundary x_o, from which we subtracted the right-hand side of $(A19)$ for i_k and $j_k^{i_k}$. By reducing identical terms from both sides of the resulting equality, we obtain $(A20)$.■

7.1.3. Prices and Shadow Prices

The prices we used in all sections of the paper and their shadow counterparts in this Appendix are presented in the following table.

Variable in text	Variable in Appendix	Description of variable
R_A	$\frac{\varrho}{\eta}$	The alternative land rent
$p_h(x)$	$\frac{\alpha(x)}{\eta}$	The housing price function
υ	$\frac{\omega}{\eta}$	The share of a household in alternative land rents

7.2. Characterizing the Bid Housing Price and Other Related Functions

The following differentiation of the (bid) housing price function proves Lemma 2. Differentiating (22) with respect to distance, bearing in mind that no facility is located in x, yields the Muthian spatial equilibrium condition,[27]

$$h(x,p_h^b)\dot{p}_h^b(x,j^1,...,j^I) + \dot{T}r\,(x,j^1,...,j^I) \equiv 0 \; where \; \dot{T}r\,(x,j^1,...,j^I) =$$
$$= \sum_{i=1}^{I} t_i'(|x - x_{i,2ji}|)sign(x - x_{i,2ji}). \qquad ((B1))$$

A dot above a function designates differentiation with respect to x. The reader should bear in mind that according to our assumptions $t_i'(y) = \frac{dt_i(y)}{dy} > 0$ and $t_i''(y) = \frac{d^2t_i(y)}{dy^2} \leq 0$.

Equation $(B1)$ implies that a marginal displacement at a given location causes a marginal change in the bid-housing-price function proportional to the sum of all marginal changes in the home-facility commuting costs to the facilities of clubs $j_1, ..., j_I$. The factor of proportionality is $-1/h(x|\,p_h^b(x))$, i.e., minus the reciprocal of the amount of housing consumed by a household at x, provided $p_h^b(x)$ is the price of housing. Note that since $t_i(|y|)$ is not differentiable at $y = 0$, at the facility locations, $x_{i,2j_i}$, $p_h^b\left(x/\left(j^i\right)\right)$ is continuous but not differentiable. For an x that is not a facility location, the second derivative of the bid housing price is obtained by differentiating $(B1)$ with respect to distance, thus

$$\ddot{p}_h^b = -\frac{\frac{\partial h}{\partial p_h^b}\left(\dot{p}_h^b\right)^2 + \sum_{i=1}^{I} t_i''(|x - x_{i,2j_i}|)}{h(\cdot)} \geq 0. \qquad (B2)$$

Thus, $(B2)$ implies that $p_h^b(x)$ is a concave function of x.

Since the housing price function, $p_h(x)$, at a location x that is not a node coincides with one of the bid rent functions, it has all the properties of a bid housing price function, except at boundaries and facility locations where it is continuous but not differentiable. We now turn to other continuous functions that depend on $p_{h(x)}$ (see Appendix 7.1.1). By differentiating (14) we obtain

$$\frac{dH^s}{dp_h} = \frac{1}{c_h''(H^s)} > 0 \Rightarrow \dot{H}^s = \frac{dH^s}{dp_h}\dot{p}_h = sign\,(\dot{p}_h)\,\frac{dH^s}{dp_h}\,|\dot{p}_h| \qquad (B3)$$

Equation $(B3)$ implies that the supply of housing at a given location is an increasing function of its product's price there, and that $\dot{H}^s$ has the same sign as $\dot{p}_h$.

[27] The function $sign(x)$ is given by $sign(x) = \begin{cases} 1 \; for \; x > 0 \\ 0 \; for \; x = 0 \\ -1 \; for \; x < 0 \end{cases}$.

The function $sign(x)$ is differentiable everywhere except at $x = 0$. Furthermore, $|x| = x \cdot sign(x)$ and $\partial|x|/\partial x = sign(x)$, except atr $x = 0$, where it is not defined.

The density function, $n(p_h) = H^s(p_h(x))/h(p_h(x))$ (defined as the number of households per unit of land) increases with the price of housing. To see this, we make the following differentiation:

$$\frac{\partial n(x)}{dp_h} = \frac{d(H^s/h)}{dp_h} = \frac{h\,\partial H^s/\partial p_h - H^s \partial h/\partial p_h}{h^2} > 0. \qquad (B4)$$

The sign of $(B4)$ follows from $(B3)$ and from the substitution effect which implies that $\partial h(\cdot)/\partial p_h(\cdot) < 0$ in (10). It follows from $(B4)$ that the density $n(x) = H^s(x)/h(x)$ increases with distance the same way that $p_h(x)$ does.

By differentiating the land rent function in (15) and using $(B1)$ as well as (14), we obtain

$$\dot{R}(x) = H^s(x)\dot{p}_h(x), \qquad (B5)$$

which implies that $R(x)$ varies with distance in the same way that $p_h(x)$ does. By differentiating $\dot{R}(x)$, we obtain

$$\ddot{R}(x) = \dot{H}^s\dot{p}_h + H^s\ddot{p}_h \geq 0 \qquad (B6)$$

Together, equations $(B6)$ and $(B2)$ imply that, in the general case, R, like p_h, is a concave function of x.

The functions $p_h^b(x)$ and $p_h(x)$, are also functions of the parameters U, Y and G_{ij}. By differentiation of (18) as well as (5), with respect to Y, taking into account that only variables controlled by the consumer may be indirectly affected, namely $H(x)$ and $Z(x)$, we obtain

$$\frac{\partial p_h(x)}{\partial Y} = \frac{1}{h(x)} \geq 0 \qquad (B7)$$

In the same way we obtain for G_{ij}

$$\frac{\partial p_h(x)}{\partial G_{ij}} = \frac{1}{h(x)}\frac{U_{i+2}}{U_1} > 0, x_{i,2j-1} \leq x \leq x_{i,2j+1} \qquad (B8)$$

7.3. Proof for Section 3

7.3.1. Proof of Lemma 3

The proof is by contradiction. We assume that the market area is not connected and show that this assumption leads to a contradiction. Without loss of generality, let the disconnected market area be of club 1(not necessarily the industrial club).

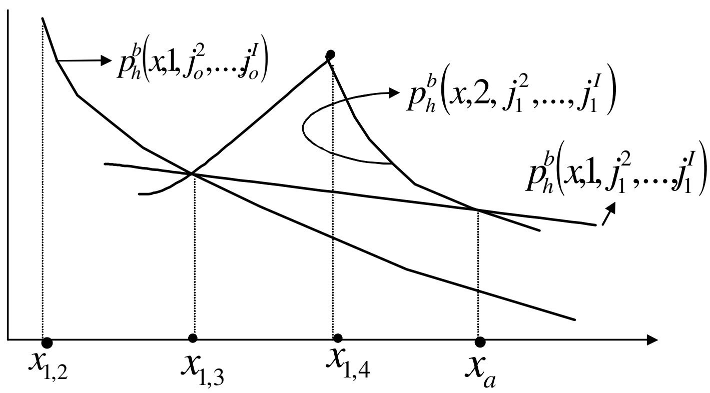

Figure 5: Bid housing price functions in a disconnected market area.

In Figure 5, facility $1,1$ (facility 1 of club 1) providing $G_{1,1}$ of the CG of club 1 is located in $x_{1,2}$, and facility $1,2$ providing $G_{1,2}$ is located in $x_{1,4}$. The locations $x_{1,3}$ and x_a are boundaries between the market areas of facility $1,1$ and facility $1,2$. There are two parts of the market area of facility $1,1$: the first lies to the left of $x_{1,3}$ and includes $x_{1,2}$ and the second is spread to the right of x_a. The market area of facility $1,2$ is between $x_{1,3}$ and x_a. Thus, the market area of facility $1,1$ is disconnected and we show here that such a layout leads to a contradiction when transportation costs are linear. To avoid the question of where people residing in a boundary use the CGs, n what follows we assume that market areas consist of half-closed segments, e.g., $(x_{i,2j-1}, x_{i,2j+1}]$.

The proof is divided into two parts. In the first part, the housing price function in the connected segment of the market area $\left(x_{1,3}, x_{\alpha}\right)$ coincides with a single bid housing price function. In the second part, we extend the proof to the more general case.

The function $p_h^b\left(x, 1, j_o^2, ..., j_o^I\right)$ (see Figure 5) is the bid housing price of residents who travel to $x_{1,2}$ to consume $G_{1,1}$ and to $x_{i,2j_o^i}$, $i = 2, ..., I$ (the points are not depicted in Figure 5) to consume the rest of the CGs. $p_h^b\left(x, 1, j_o^2, ..., j_o^I\right)$ coincide with the housing price function in the segment $(x_{1,2}, x_{1,3}]$. The function $p_h^b\left(x, 2, j_1^2, ..., j_1^I\right)$ is the bid of residents at x who travel to $x_{1,4}$, as well as to $x_{i,2j_1^i}$, $i = 2, ..., I$ (these points are also not depicted in Figure 5) and this bid function coincides with the housing price function in the segment $(x_{1,3}, x_a]$. In addition to the above two bid housing price functions, we consider the bid housing price function, $p_h^b\left(x, 1, j_1^2, ..., j_1^I\right)$. In the segment $(x_{1,3}, x_a]$ this function is the highest bid for housing that residents are willing to pay for housing while patronizing facility $(1,1)$. The reason is that the optimal vector of facilities of all clubs other than 1 at $(x_{1,3}, x_a]$ is $j_1^2, ..., j_1^I$, the same vector as is in the housing price function in the segment $(x_{1,3}, x_a]$. In other words, among all bid housing price functions with households that patronize facility $(1,1)$, the bid function $p_h^b\left(x, 1, j_1^2, ..., j_1^I\right)$ is the highest in $(x_{1,3}, x_a]$. In this case the market area of facility $(1,1)$ would exist to the right of x_a only if the functions $p_h^b\left(x, 1, j_1^2, ..., j_1^I\right)$ and $p_h^b\left(x, 2, j_1^2, ..., j_1^I\right)$ intersect at x_a.

From (21) we learn that the only difference between $Tr\left(x, 1, j_1^2, ..., j_1^I\right)$ and $Tr\left(x, 2, j_1^2, ..., j_1^I\right)$, is the cost terms associated with facilities of club 1. At the point $x_{1,4}$, the function $Tr\left(x_{1,4}, 1, j_1^2, ..., j_1^I\right)$ must be higher than $Tr\left(x_{1,4}, 2, j_1^2, ..., j_1^I\right)$ otherwise no one would travel to facility $(1,2)$ and the market area of facility $(1,1)$ would be connected. From equation $(B1)$ in Appendix 7.2, it follows that in locations to the right of $x_{1,4}$, the equality $\dot{Tr}\left(x, 2, j_1^2, ..., j_1^I\right) = \dot{Tr}\left(x, 1, j_1^2, ..., j_1^I\right)$ must hold. The reason for the equality of the two $\dot{Tr}$-s is that since from all locations to the right of $x_{1,4}$ households commute in both cases to the same facilities of clubs i, $t_i'(x)$ for $i > 1$ is the same in both the above Tr functions. In addition, $sign\left(x - x_{1,2}\right) = sign\left(x - x_{1,4}\right), \forall x > x_{1,4}$, and by assumption, $t_1'(y)$ =Constant. Hence, the two bid functions, $p_h^b\left(x, 2, j_1^2, ..., j_1^I\right)$ and $p_h^b\left(x, 1, j_1^2, ..., j_1^I\right)$ at x_a have the same slopes as is shown below in $(D1)$ and calculated from equation $(B1)$ in Appendix 7.2.

$$\dot{p}_h^b\left(x_a, 1, j_1^2, ..., j_1^I\right) = -\frac{\dot{Tr}\left(x_a, 1, j_1^2, ..., j_1^I\right)}{H(x_a)} = -\frac{\dot{Tr}\left(x_a, 2, j_1^2, ..., j_1^I\right)}{H(x_a)} = \dot{p}_h^b\left(x_a, 2, j_1^2, ..., j_1^I\right).. \quad ((D1))$$

In the middle of $(D1)$, the two $\dot{Tr}$-functions in the numerators above are equal, due to the linearity of the transportation cost functions and so is $H(x_a)$ (the two p_h^b functions in the compensated demand for housing in the denominator are the same at their intersection point). Thus, at x_a both p_h^b functions, as well as their derivatives, are the same. This means that the two bid functions do not intersect at x_a but are tangent to each other. This is a contradiction, which means that facility 1,1 is connected. This completes the first part of the proof.

In the second stage we prove the lemma for the case in which the housing price function $p_h(x)$, $x_{1,3} < x < x_a$ consists of segments of different bid functions, each with a different vector $\left(J^2, ..., j^I\right)$. All of these bid functions of which p_h consists, have $j^1 = 2$, i.e., in all of them residents travel to facility 2 of club 1. In this case, let $\bar{p}_h^b\left(x, 2, j_1^2, ..., j_1^I\right)$ be the bid function that in $(x_{1,4}, x_a)$ coincides with the last segment of the price function, namely, the segment that ends in x_a. By $\bar{p}_h^b\left(x, 1, j_1^2, ..., j_1^I\right)$ we designate the bid function of a household that patronizes $\left(1, j_1^2, ..., j_1^I\right)$. This function is the highest bid of patrons of $(1,1)$ in a sufficiently small segment to the left of x_a. The two bid functions, $\bar{p}_h^b\left(x, 2, j_1^2, ..., j_1^I\right)$ and $\bar{p}_h^b\left(x, 1, j_1^2, ..., j_1^I\right)$ intersect at x_a. From here on the proof proceeds exactly as the proof in part one. This completes the proof of the Lemma.■

7.4. Proofs for Section 4

7.4.1. Proof of Proposition 4

The profit function of a contractor at x facing the given price of housing $p_h^*(x)$ is $\pi(x) = p_h^*(x) H^s(x) - c_h(H^s(x))$. Hence, contractors maximizing their profits at x by choosing $H^s(x)$, lead to the fulfillment of $p_h^*(x) = c_h'(H^s(x))$. Thus causes $H^s(x)$ to equal $H^{s*}(x)$. The subsidy $S(x)$ ensures that a household at x can purchase the optimal consumption basket at the given prices. The price of housing being optimal implies that so is $H(x)$. Together $H^*(x)$ and $H^{s*}(x)$ imply in turn that $n(x) = n^*(x) = \frac{H^{s*}(x)}{H^*(x)}$ and upon integration that indeed, $N_{ij} = N_{ij}^*$.

Let $\pi_{ij} = p_{G_{ij}} G_{ij} - c^i\left(G_{ij}, N_{ij}^*\right)$ be the profit function that an operator of facility ij maximizes by choosing G_{ij} for given $p_{G_{ij}}$ and N_{ij}^*. The necessary condition for this maximization is $= c_1^i\left(G_{ij}, N_{ij}^*\right)$ and since $p_{G_{ij}} \overset{def}{=} c_1^i(G_{ij}^*, N_{ij}^*)$, this condition yields $G_{ij} = G_{ij}^*$. Each individual pays $p_{G_{ij}}^d = \frac{p_{G_{ij}}}{N_{ij}^*}$. This ensures that the overall payments paid for $G_{i,j}$ by residents of the market area of facility (i,j), i.e., $N_{ij}^* p_{G_{ij}}^d = p_{G_{ij}}\left(= c_1^i(G_{ij}^*, N_{ij}^*)\right)$, are sufficient to induce the facility operator to provide the optimal CG.

The Henry George rule ensures that aggregate land rents, in addition to aggregate clubs' profits, that are all within the complex's jurisdiction, exactly match the funds needed to finance the required transfers to residents. ■