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ASYMMETRIC ALL-PAY AUCTIONS**

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Abstract

We study asymmetric all-pay auctions with multiple objects where players' values for the objects are common knowledge. The players have different values for the objects but they have the same ranking. The contest designer may award one prize including all the objects to the player with the highest bid, or, alternatively, he may allocate several prizes, each prize including one object such that the first prize is awarded to the player with the highest bid, the second prize to the player with the second-highest bid, and so on until all the objects are allocated. We analyze the distribution of effort in one-prize and multiple-prize contests and show that allocation of several prizes may be optimal for a contest designer who maximizes the total effort.

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1 Introduction

In all-pay auctions each player submits a bid for the object being sold, the player who submits the highest bid receives the object, but, independently of success, all players bear the cost of their bids. Common applications of all-pay auctions include rent-seeking, lobbying in organizations, R&D races, political contests, promotions in labor markets, arm races and sport competitions. In the economic literature, all-pay auctions are usually studied under complete information where the players' valuations for the object are common knowledge (see, for example, Hillman and Riley (1989), Baye et al. (1993) and Che and Gale (1998)), or under incomplete information where each player's valuation for the object is private information to that player and only the distribution of the players' valuations is common knowledge (see, for example, Amman and Leininger (1996), Gavious et al. (2003) and Moldovanu and Sela (2004)). Most of this literature has focused on all-pay auctions with a unique prize that is awarded to the player with the highest effort. In the real world, however, we can find numerous contests with several prizes. Sometimes in fact, there are similar contests which are distinguished only by the number of prizes. For example, players in sport competitions may compete for a unique prize or they may compete for several prizes, i.e., gold, silver or bronze medals awarded in the Olympic games. In political races the winner may hold a position with several titles, or several winners may hold these titles separately. One last example would be the Nobel prize which may be awarded to either a single person or to several persons who would share the prize for the same year.

In the literature on all-pay auctions only a few studies deal with the question of what is the optimal number of prizes in contests and particularly in all-pay auctions.¹ Moldovanu and Sela (2001) showed that in all-pay auctions under incomplete information when cost functions are linear or concave in effort, it is optimal to allocate the entire prize sum to a single first prize, but when cost functions are convex, several positive prizes may be optimal. This explanation, however, cannot be generalized to the case of all-pay auctions under complete information. In symmetric all-pay auctions under complete information, Barut and Kovenock (1998) showed that the revenue maximizing prize structure allows any combination of $k - 1$ prizes, where k is the number of players. That is, the contest designer is indifferent to whether he should allocate one prize or several prizes. In this paper we show that in asymmetric all-pay auctions under complete information allocation of several prizes might be profitable for the contest designer who maximizes the total effort.

Baye et al. (1996) provided a complete characterization of equilibrium behavior in the complete information all-pay auction with one object. Clark and Riis (1998) analyzed multiple object all-pay auctions where the objects are identical but players may have different valuations for the objects. On the other hand, Barut and Kovenock (1998) studied multiple object all-pay auctions where the objects are not identical but for each object the players have the same valuation. The study of all-pay auctions with asymmetry in both objects and

¹Szymanski and Valletti (2004) studied the optimal number of prizes in Tullock's model where the chance of a player to win a prize depends on the ratio of his effort with respect to the other players' efforts.

players' valuations is more complicated and there is no complete characterization of the equilibrium behavior in this environment. In our model of multiple object all-pay auctions both the players and the objects are asymmetric. The players have different values for the objects but they have the same ranking, and the players' valuations for the objects are common knowledge. The contest designer may award one prize, including all the objects, to the player with the highest bid or, alternatively, he may allocate several prizes, each prize including one object such that the first prize is awarded to the player with the highest bid, the second prize to the player with the second-highest bid, and so on until all the objects are allocated. Although we are not able to provide a complete characterization of a designer's preferences among one-prize and multiple-prize contests, we do show that if players have different valuations for the objects, allocation of several prizes may be optimal for a contest designer who wishes to maximize the expected total effort.

The contest designer may have other goals in addition to maximizing the expected total effort. For example, he may wish to determine the identity of the winners of the contest.² We demonstrate that by allocating several prizes, the contest designer can drastically change the effort distribution of the players and accordingly the players' probabilities of winning such that the player with the lowest probability (zero) of winning the one-prize contest may have the highest probability of winning the highest prize and also the highest probability

²Cohen and Sela (2004) studied Tullock's classical model and showed that by a simple non-discriminating rule the contest designer is able to manipulate the outcome of the contest such that the probabilities to win are not ordered according to the contestants' abilities (valuations).

of winning one of the prizes in the multiple-prize contest. Hence, if the contest designer wishes to influence the identity of the winners of the contest, it might be worthwhile to allocate several prizes.

The paper is organized as follows: In Section 2 we present our model of multiple-prize all-pay auctions. In Section 3 we review the results of all-pay auctions with a unique prize, and then analyze asymmetric all-pay auctions with two prizes. In Section 5 we compare the effort distribution in one and two-prize contests.

2 The Model

We consider an all-pay auction with n players and p objects. The value of the j -th object for player i is v_i^j , where for all i , $v_i^1 \geq v_i^2 \geq \dots \geq v_i^p \geq 0$, that is, the players have different values for the objects but they have the same ranking. It is assumed that the value of several objects for player i is the sum of his values for these objects. The values of the objects are common knowledge.

Each player i makes an effort x_i . These efforts are submitted simultaneously, and all contestants incur the cost of their effort. The contest designer who wishes to maximize the expected value of the total effort ($E(\sum_{i=1}^n x_i)$) exerted by the players determines the number of prizes, where each prize may include a combination of objects. We restrict the designer's decision such that he can decide between two designs: a one-prize contest in which the player with the highest effort wins all the p objects, and p -prize contests in which the player

with the highest effort wins the first object (object 1), the player with the second highest effort wins the second object (object 2), and so on, until all the objects are allocated.

3 Equilibrium

In order to decide what the optimal number of prizes is in our model, we first analyze the players' strategies in equilibrium. We consider an all pay auction with a single object in which the player with the highest effort wins the object. Let v_i be the value of player i for the object. The effort distribution functions of the players $F_i(x)$, $i = 1, 2, \dots, n$, are given by the following system of equations:

$$v_i \prod_{j \neq i} F_j(x) - x = c_i \quad i = 1, 2, \dots, n$$

Assume that the players' valuations satisfy, $v_1 \geq v_2 \geq v_l$ for all $l \neq 1, 2$. According to Hillman and Riley (1989) and Baye et al. (1993), there is always a mixed-strategy equilibrium in which all the players except players 1 and 2 stay out of the contest. Players 1 and 2 randomize on the interval $[0, v_2]$ according to their effort distribution functions which are given by:

$$v_1 F_2(x) - x = v_1 - v_2$$

$$v_2 F_1(x) - x = 0$$

Thus, player 1's effort is uniformly distributed, while player 2's effort is distributed according to the cumulative distribution function $F_2(x) = (v_1 - v_2 + x)/v_1$.

Given these mixed strategies, player 1's winning probability against 2 is

$$q_{12} = 1 - \frac{v_2}{2v_1}. \quad (1)$$

Player 1's expected effort is $\frac{v_2}{2}$, and player 2's expected effort is $\frac{(v_2)^2}{2(v_1)}$. Therefore the total expected effort is

$$\frac{v_2}{2} \left(1 + \frac{v_2}{v_1}\right). \quad (2)$$

and the respective expected payoffs are $u_1 = v_1 - v_2$ and $u_2 = 0$.

Next we consider an all-pay auction with two objects in which the player with the highest effort wins the first object and the player with the second highest effort wins the second one. Let v_i^j be the value of player i for object $j, j = 1, 2$. The effort distribution functions of the players $F_i(x), i = 1, 2, \dots, n$, are given by the following system of equations:

$$v_i^1 \Pi_{j \neq i} F_j(x) + v_i^2 \sum_{j \neq i} (1 - F_j(x)) (\Pi_{k \neq j, i} F_k(x)) - x = c_i \quad i = 1, 2, \dots, n \quad (3)$$

Since it is very complex to solve this system of equations for the general case of asymmetric players, we consider the case of $n - 1$ symmetric players with the same valuations for the objects and only one player with different valuations than the others. In formal terms, player 1 has values of v_1^1 and v_1^2 for objects 1 and 2, respectively, and all the other players are symmetric with values of v_m^1 and v_m^2 for objects 1 and 2. We assume that player 1 is the dominant player such that $v_1^1 \geq v_m^1$. In this case, it can be easily verified that there is always an equilibrium with only three players: player 1 and two symmetric players, the latter denoted by 2 and 3. We denote the effort distribution function of

players 2 and 3 by F_{23} and their values for the objects by v_{23}^1 and v_{23}^2 . The effort distribution function of players 2 and 3 is given by

$$v_1^1 F_{23}(x) + v_1^2 2F_{23}(x)(1 - F_{23}(x)) - x = v_1^1 - v_{23}^1$$

Thus the efforts of players 2 and 3 are distributed according to the cumulative distribution function

$$F_{23}(x) = \frac{-2v_1^2 + \sqrt{4(v_1^2)^2 + 4(v_1^1 - 2v_1^2)(x + v_1^1 - v_{23}^1)}}{2(v_1^1 - 2v_1^2)} \quad (4)$$

It can be verified that $F_{23}(x)$ increases for all $0 \leq x \leq v_{23}^1$ and satisfies $F_{23}(0) = 0$ and $F(v_{23}^1) = 1$.

The effort distribution function of player 1 is given by

$$v_{23}^1 F_1(x) F_{23}(x) + v_{23}^2 (F_{23}(x)(1 - F_1(x)) + F_1(x)(1 - F_{23}(x))) - x = 0$$

Thus the effort of player 1 is distributed according to the cumulative distribution function

$$F_1(x) = \frac{x - v_{23}^2 F_{23}(x)}{(v_{23}^1 - 2v_{23}^2) F_{23}(x) + v_{23}^2} \quad (5)$$

where $F_{23}(x)$ is given by (4). It can be verified that $F_1(x)$ increases for all $0 \leq x \leq v_{23}^1$ and satisfies $F_1(0) = 0$ and $F_1(v_{23}^1) = 1$. The respective expected payoffs of the players are $u_1 = v_1^1 - v_{23}^1$ and $u_2 = u_3 = 0$.

4 One-Prize Contests Versus Two-Prize Contests

We now compare two contest designs with two objects in order to show that allocation of a single prize may be not profitable for a contest designer who

maximizes the total effort exerted by the players. In the two-prize contest, the player with the highest effort wins object 1 and the player with the second highest effort wins object 2. In the one-prize contest, the player with the highest effort wins objects 1 and 2. Without loss of generality, we assume that three players participate (exert positive efforts) in the contests. According to Barut and Kovenock (1998), if the players have the same valuations for the objects, i.e., $v_i^1 = v^1$ and $v_i^2 = v^2$ for $i = 1, 2, 3$, then the expected total effort in the one-prize and two-prize contests are the same. In the following we show that if the players have different valuations for the objects, then these contest designs are not necessarily equivalent, however, the one-prize contest is not always the optimal design for the contest designer who wishes to maximize the expected total effort.

Proposition 1 *The expected total effort in the two-prize contest may be higher than the expected total effort in the one-prize contest.*

Proof. Suppose that in the status quo all the players have the same valuations for the objects such that $v_i^1 = v$ and $v_i^2 = 0$ for $i = 1, 2, 3$. Obviously in this case both designs coincide and the total effort is equal to v . Now we change the valuation of player 1 for object 2 such that $v_1^2 = \alpha v$, $0 < \alpha < 1$.

Lemma 2 *In the one-prize contest, an increase of player 1's valuation for object 2 yields a decrease of the expected total effort.*

The players' valuations in the one-prize contest are now $v_1 = (1 + \alpha)v$ and $v_2 = v_3 = v$. Thus, by (2), the change of total effort with respect to the status

quo in the one-prize contest is

$$\Delta E_1 = \frac{d}{d\alpha} \left(\frac{v}{2} + \frac{v^2}{2(1+\alpha)v} \right) = \frac{-v}{2(1+\alpha)^2}$$

Note that for all $\alpha \geq 0$, $\Delta E_1 < 0$ and for $\alpha = 0$, we obtain that $\Delta E_1 = \frac{-v}{2}$.

Lemma 3 *In the two-prize contest, a small increase of player 1's valuations for object 2 yields an increase of the expected total effort.*

The players' valuations in the two-prize contest are now $v_1^1 = v_2^1 = v_3^1 = v$, $v_1^2 = \alpha v$, $0 < \alpha < 1$ and $v_2^2 = v_3^2 = 0$. The above change of player 1's valuation for object 2 affects the effort distributions of all the players with respect to the status quo. The change in the efforts of players 2 and 3 is

$$\Delta E_{23} = \frac{d}{d\alpha} \int_0^v x \frac{dF_{23}}{dx}(x, \alpha) dx \quad (6)$$

where

$$F_{23}(x, \alpha) = \frac{-2\alpha v + \sqrt{4(\alpha v)^2 + 4(v - 2\alpha v)x}}{2(v - 2\alpha v)}$$

A simple calculation gives

$$\frac{d^2 F_{23}}{d\alpha dx}(x, 0) = \frac{1}{2} \frac{1}{\sqrt{v \cdot x}} \quad (7)$$

Substituting (7) in (6) implies that for $\alpha = 0$, the overall change in the efforts of players 2 and 3 is

$$2\Delta E_{23} = \frac{2v}{3}$$

Similarly, the change in the effort of player 1 is

$$\Delta E_1 = \frac{d}{d\alpha} \int_0^v x \frac{dF_1}{dx}(x, \alpha) dx \quad (8)$$

where

$$F_1(x, \alpha) = \frac{x - \alpha v F_{23}(x, \alpha)}{[F_{23}(x, \alpha)(v - 2\alpha v) + \alpha v]}$$

A simple calculation gives

$$\frac{d^2 F_1}{d\alpha dx}(x, 0) = -\frac{1}{2\sqrt{vx}} \quad (9)$$

Substituting (9) in (8) implies that for $\alpha = 0$, the change in the effort of player 1 is

$$\Delta E_1 = -\frac{v}{3}$$

Consequently, around $\alpha = 0$, the overall change in the total effort is positive and equals

$$\Delta E_1 + 2\Delta E_{23} = \frac{v}{3}$$

Since the total expected effort in the status quo where the players are symmetric is the same for both one-prize and two-prize contests, by lemmas 2 and 3 we obtain that the total expected effort in the two-prize contest is larger than in the one-prize contest. ■

By Proposition 1 we proved that a small asymmetry of the players' valuation implies that the total effort in the two-prize contest may be larger than in the one-prize contest. From Figure 1, which shows the total effort in both contest designs for any change of player 1's valuation for object 2, we can see that the result of Proposition 1 is not restricted only to weak asymmetry of the players' valuations and holds, independent of the size of asymmetry. Moreover, since all the players have the same value for object 1 in Proposition 1, the result of

Proposition 1 holds independently of player 1's value for object 2 (see Figure 1), that is, independently of whether player 1 is stronger or weaker than the other players.

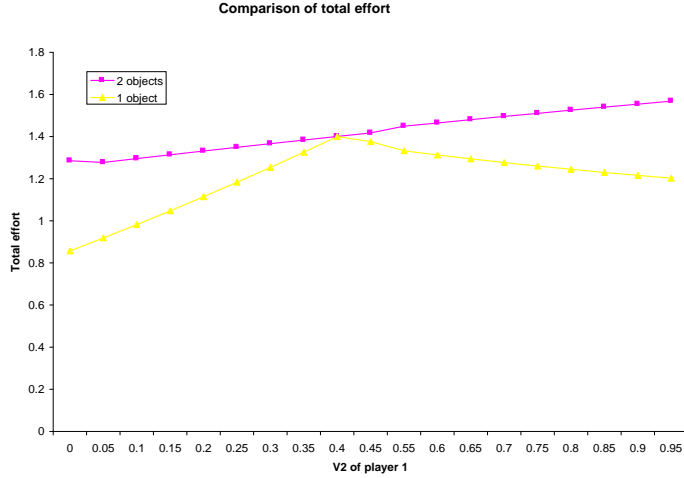


Figure 1

In contrast to Proposition 1, it can be shown that the expected total effort in the two-prize contest may also be lower than the expected total effort in the one-prize contest. However, since complete characterization of the equilibrium strategy in the multiple-prize contest is not possible, we cannot define exactly under which conditions the total effort is larger in each of the contest designs.

So far we have assumed that the goal of a contest designer is to maximize the expected total effort. But he may also have other goals. Suppose that he wishes to determine the identity of the winners in the contest. The following result shows that the number of prizes allocated by the designer has a significant

effect on determining the identity of the winner(s) of the contest.

Proposition 4 *The player with the lowest probability to win the one-prize contest may have the highest probability to win the two-prize contest.³*

Proof. Assume that in the status quo all the players have the same valuations for the objects, i.e., $v_1^1 = v_2^1 = v_3^1 = v^1$ and $v_1^2 = v_2^2 = v_3^2 = v^2$.

Lemma 5 *An identical increase of players 2 and 3' valuations for object 2 implies that these players will have the same probability to win the one-prize contest, while player 1 will have a probability of zero to win this contest.*

If we slightly change the valuation of players 2 and 3 for object 2 ($\Delta v_{23}^2 = \epsilon$) we obtain that the players' valuations in the one-prize contest are $v_1 = v^1 + v^2$, $v_2 = v_3 = v^1 + v^2 + \epsilon$. According to (1) Players 2 and 3's probability to win the contest will be $\frac{1}{2}$, whereas player 1's probability to win the contest will be zero.

Lemma 6 *An identical increase of the valuations of players 2 and 3 for object 2 implies that the probability of each of them to win the two-prize contest is smaller than the probability of player 1 to win this contest.*

In the status quo, the three players use the same strategies in the two-prize contest and therefore the probability of winning is the same for all the players. When we slightly increase the valuations of players 2 and 3, players 2 and 3' strategies are not affected (see equation (4)), but player 1's strategy is. Then,

³Having the highest probability to win the two-prize contest means that the player has the highest probability to win the first prize as well as to win one of the prizes.

by (5) the derivative of the effort distribution of player 1 with respect to v_{23}^2 is

$$\frac{dF_1}{dv_{23}^2}(x) = \frac{-F_{23}(x) [F_{23}(x) (v^1 - 2v_{23}^2) + V_{23}^2] - [x - v_{23}^2 F_{23}(x)] [-2F_{23}(x) + 1]}{[F_{23}(x) (v^1 - 2v_{23}^2) + V_{23}^2]^2} \quad (10)$$

Since $F_1(x)$ is positive for all x we obtain that around $x = 0$ the nominator of (10) is negative and therefore the effort distribution of player 1 decreases in v_{23}^2 . Thus, the function $H(x) = F_{23}(x) - F_1(x)$ is negative around $x = 0$. Since $H(x)$ is a quadratic function whose roots are $x = 0$ and $x = v^1$, we obtain that $F_1(x) \leq F_{23}(x)$ for all $0 \leq x \leq v^1$, namely, $F_1(x)$ stochastically dominates $F_{23}(x)$. Hence, player 1's probability of winning must be higher than players 2 and 3's probability of winning the two-prize contest. Figure 2 shows how player 1's effort distribution is changed as a result of an identical change of players 2 and 3's valuations. Note that an increase of players 2 and 3's valuations increases player 1's probability of winning the contest.

By lemmas 5 and 6, in the one-prize contest player 1 has a probability of zero to win while in the two-prize contest he has the highest probability of winning the first prize and also the highest probability of winning one of the two prizes.

■

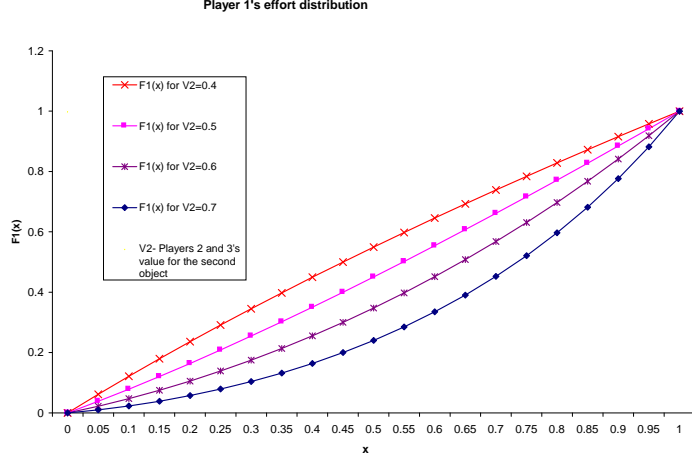


Figure 2

Under the assumptions in Proposition 4, if the contest designer does not want player 1 to win he should allocate only one prize. On the other hand, if he wants player 1 to win he should allocate two prizes. Thus, the contest designer can influence the identity of the winners by his decision on the number of prizes that will be allocated at the contest.

5 Concluding Remarks

The study of asymmetric all-pay auctions with multiple-prizes under complete information is very complex. Even for the simplest case assumed in this paper where three players compete to acquire two different prizes, the players' equilibrium strategies cannot be explicitly calculated independently of the players' values for the prizes. Accordingly, the expected total effort exerted by the

players in these contests cannot be explicitly calculated. Thus, to compare the distributions of effort in one-prize and two-prize asymmetric all-pay auctions we analyzed the change of the players' distributions of effort with respect to the symmetric case where the total effort is the same for both kinds of contests. One result showed that the total effort exerted by the players may be larger in the two-prize contest than in the one-prize contest. However, the exact relation between any two different designs would depend on the specific parameters of these models, that is, the players' exact valuations for the prizes. The second result showed that in contrast to the one-prize all-pay auction in which the player with the highest value has the highest probability to win, in multiple-prize contests the identity of the winners as well as the order of the winners according to their probabilities of winning are ambiguous. Therefore, by making a decision on the number of prizes, the contest designer can significantly affect the results of any multiple-prize contest.

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