THE OPTIMAL REWARDS IN CONTESTS

Chen Cohen, Todd R. Kaplan and Aner Sela

Discussion Paper No. 05-01

January 2005

Monaster Center for Economic Research
Ben-Gurion University of the Negev
P.O. Box 653
Beer Sheva, Israel

Fax: 972-8-6472941
Tel: 972-8-6472286

The Optimal Rewards in Contests

Chen Cohen, Todd R. Kaplan and Aner Sela*

January, 2005

Abstract

We study all-pay contests under incomplete information where the reward is a function of the contestant's type and also of his effort. We analyze the optimal reward for the designer when the reward is either multiplicatively separable or additively separable in effort and type. In the multiplicatively separable environment the optimal reward is always positive while in the additively separable environment it may also be negative. In both environments, depending on the designer's utility, the optimal reward may either increase or decrease in the contestants' effort. Finally, in both environments, the designer's payoff depends only upon the expected value of the effort-dependent rewards and not the number of rewards.

Keywords: Contests, All-Pay Auctions, Optimal Design.

JEL classification: D44, O31.

*Kaplan: Department of Economics, University of Exeter, UK. Cohen, Sela: Department of Economics, Ben-Gurion University of the Negev, Israel. Corresponding Author: Sela, anersela@bgumail.bgu.ac.il.

1 Introduction

The X prize is a ten-million-dollar competition created to jumpstart the space tourism industry by attracting the attention of the most talented entrepreneurs and rocket experts in the world. The cash prize will be awarded to the first team that privately finances, builds and launches a spaceship that is able to (a) carry three people to 100 kilometers (62.5 miles), (b) returns safely to earth, (c) repeats the launch with the same ship within two weeks. The X prize was inspired by the early aviation prizes of the 20th century, primarily the spectacular trans-Atlantic flight of Charles Lindbergh in the Spirit of St. Louis which captured the $25,000 Orteig prize in 1927.

The X-prize competition is an example of an R&D contest in which all contestants including those that do not win any prize incur costs as a result of their efforts but only the winner gets the prize. These winner-take-all contests appear in many different forms: only the first to invent gets a patent, the hedge fund that finds the arbitrage opportunity the quickest gets the rewards, the first to cross the finish line wins a marathon, only one worker may get the promotion. In many cases, the sponsor has at least a limited control over the design used: the government can determine scope and length of patents, the SEC can regulate hedge funds, the organizers of the marathon can set the size of the prize, and the company can set rules with a promotion contest.

This potential control has led to research in contest design. Initial research in contest design analyzed limiting the number of contestants. Taylor (1995) and Fullerton and McAfee (1999) study the optimal number of participants in contests and methods of restricting entry. Baye et al. (1993) look at the optimal set of contestants in all-pay contests. They find that under complete information, it is sometimes advantageous to exclude the contestant with the highest valuation for winning the contest.

A different approach to contest design is restricting the contestants' strategies. Che and Gale (1998) show that in all-pay contests under complete information, if agents have linear cost functions bid cap may be profitable for the designer who wishes to maximize the total effort; Gavious et al. (2003) show that in all-pay contests under incomplete information, if agents have convex cost functions then effectively capping the bids is profitable for a designer with a large number of contestants.

Further investigation into contest design allowed the designer control over the reward structure. In fixed-prize contests, the designer can determine the number of prizes having positive value and the distribution of the fixed total prize sum among the different prizes. In symmetric all-pay contests under complete information, Barut and Kovenock (1998) show that the revenue maximizing prize structure allows any combination of $K-1$ prizes, where K is the number of contestants. In particular, allocating the entire amount to a first prize is among the optimal designs. In an all-pay contests with incomplete information, Moldovanu and Sela (2001) show that when cost functions are linear or concave in effort, it is optimal to allocate the entire prize sum to a single "first" prize, and when cost functions are convex, several positive prizes may be optimal. Che and Gale (2003) study a contest where the contestants choose a prize from a menu of fixed prizes and the winner is determined according to the best combination of effort and prize.

In this paper we go one step forward and allow the designer to determine a structure of a variable reward such that there is a relationship between the efforts incurred in the contest and the size of the reward collected by the winning contestant. That is, a larger effort changes not only the probability of winning the contest but also the size of the reward gained by winning. Such a relationship is already present in many examples of contests: In the X prize as with patent races, the winning firm choosing a larger effort leads to an earlier innovation time. This in present value terms leads to a larger reward. A hedge fund not only faces competition from other hedge funds, but from market forces eliminating opportunities. Earlier detection can lead to larger profits. In the marathon, harder training can lead to a quicker winning time. This can result in a larger prize (such as if a course or world record is broken). Also, in work promotions, greater effort can result in a larger raise to the winner.

We study the optimal reward structure in all-pay contests under incomplete information where the designer wishes to maximize either the total expected effort or the expected highest effort. The optimal effort-dependant reward is not clear-cut. The reward affects the designer's payoff in two ways: indirectly through influencing the effort exerted by the contestants and directly by the payments made to the winner. Moreover, Kaplan et al. (2002) show that in all-pay contests under incomplete information an increase in the reward function may decrease the contestants' efforts.[1] Indeed, as we show in this paper the optimal reward structure is surprising in several environments.

We find the equilibrium effort and the optimal reward in two different environments: where the reward is multiplicatively separable in effort and type and where the reward is additively separable. The environments are similar since in both of them, if the contest

[1] Also under complete information, Kaplan et al. [2003] show that effort-dependent rewards introduce substantial qualitative changes to the behavior of the contestants compared with constant reward all-pay auctions.

designer wishes to maximize the expected total effort, the equilibrium effort is independent of the number of contestants. In this case, for sufficiently large number of contestants, the optimal reward function in both environments decreases in effort, that is, a larger effort decreases the size of the reward gained by winning. On the other hand, in both environments, if the contest designer wishes to maximize the expected highest effort, the equilibrium effort depends on the number of contestants. Then, for any number of contestants, the optimal reward function may increase in effort.

Our environments are distinguished since in the multiplicatively separable environment the optimal reward is always positive while in the additively separable environment it may also be negative. Furthermore, in the multiplicatively separable environment, for every optimal reward function, all contestants choose to participate in the auction. On the other hand, the optimal reward function in the additively separable environment may limit the number of contestants that choose to participate in the contest. That is the optimal reward function serves the role of entry fees or alternatively reserve prices in the standard contests (auctions). It is interesting to note that the optimal reward does not necessarily eliminate participation of the contestants with the lowest valuations (it never eliminate the contestants with the highest valuations). In all cases, more types are included when the designer cares about the total effort.

Finally, we allow the designer the additional control of the number of effort-dependent rewards (as mentioned above, this control is analyzed in Barut & Kovenock (1998) and Moldovanu & Sela (2001)). We find that it does not matter upon how many prizes the reward is distributed; only the expected value of the reward matters. We present an example where an optimal design is to give an effort-dependent prize to the loser of a two-contestant contest. This further shows the consequence of this work – that an effort-dependent reward can be an efficient tool for the contest designer but its structure as well as its effects on the contest are sometimes unusual.

The paper proceeds as follows. In Section 2 we present the model. We analyze the optimal reward in the multiplicatively separable environment in Section 3 and in the additively separable environment in Section 4. In Section 5, we revisit the question of multi-prize contests and in Section 6 we conclude. The Appendix contains the proofs.

2 The model

Consider an all-pay contest with effort-dependent rewards. The set of contestants is $N = \{1, 2, ..., n\}$. Each contestant's type $\theta_i, i = 1..., n$, is independently drawn from the interval $[\underline{\theta}, \overline{\theta}]$ according to the distribution function F.[2] While F is common knowledge, each contestant is privately informed about his own type. Each contestant i exerts an effort x_i and, by doing so, incurs a disutility (or cost) denoted by $c(\theta_i, x_i)$, where $c : [\underline{\theta}, \overline{\theta}] \times \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly increasing in x, strictly decreasing in θ and twice continuously differentiable function with $c(\theta, 0) = 0$, and $c_x(\theta, 0) = 0$. We have additional sufficient conditions to guarantee a monotonic solution to our problem: $c_{\theta x} < 0$, $c_{xx} > 0$, $c_{x^2\theta} \leq 0$ and $c_{x\theta^2} \geq 0$.

The contestant i that chooses the highest effort x_i wins a reward $R(x_i) : \mathbb{R}_+ \to \mathbb{R}_+$ and values this reward according to the value function $V(\theta_i, R(x_i)) : [\underline{\theta}, \overline{\theta}] \times \mathbb{R}_+ \to \mathbb{R}_+$ which is

[2] It is assumed that the hazard rate $F'/(1 - F)$ is increasing.

a twice continuously differentiable function. Each contestant chooses his effort in order to maximize his expected utility given the other contestants' actions and the form of the reward function. We consider here two forms of utility for the designer: (1) the designer maximizes the expected value of total effort $E(\sum_{i=1}^{n} x_i)$ minus the expected reward he must pay out, and (2) the designer maximizes the expected value of the highest effort $E(\max\{x_i\})$ minus the expected reward he must pay out.

2.1 Equilibrium

Consider first the equilibrium in the contest that results after the designer sets the reward function. In a Bayesian equilibrium, the effort function $x(\theta)$ chosen by each contestant maximizes his expected utility given the effort functions chosen by the other contestants. Hence, for each θ, a symmetric equilibrium effort function $x(\theta)$ (assumed to be monotonic increasing and differentiable) solves the following maximization problem

$$\pi(\theta) = \max_{x} F(\theta(x))^{n-1} \cdot V(\theta, R(x)) - c(\theta, x) \tag{1}$$

Proposition 1 *Any equilibrium strategy $x(\theta)$ is given by the implicit function*

$$F(\theta)^{n-1} V(\theta, R(x(\theta))) - c(\theta, x(\theta)) = \int_{\underline{\theta}}^{\theta} [F(\tilde{\theta})^{n-1} V_{\theta}(\tilde{\theta}, R(x(\tilde{\theta}))) - c_{\theta}(\tilde{\theta}, x(\tilde{\theta}))] d\tilde{\theta} \tag{2}$$

while the RHS of (2) is the expected payoff of a contestant given this strategy.

Proof. See Appendix.

The designer decides the exact form of the reward function in order to maximize his expected payoff subject to the choice of contestants' effort in equilibrium. When the designer wishes to maximize the expected value of total effort, his expected payoff is given by

$$n \int_{\underline{\theta}}^{\overline{\theta}} x(\theta) dF - \int_{\underline{\theta}}^{\overline{\theta}} R(x(\theta)) dF^n \tag{3}$$

The left term of (3) is the expected total effort exerted by the contestants and the right term of (3) is the designer' expected payment to the contestant with the highest effort.

Similarly, when the designer wishes to maximize the expected value of the highest effort, his expected payoff is given by

$$\int_{\underline{\theta}}^{\overline{\theta}} [x(\theta) - R(x(\theta))] dF^n \tag{4}$$

3 Multiplicatively separable case

In this section we assume that the value function is multiplicatively separable with the form

$$V(\theta, R(x)) = \theta \cdot R(x)$$

The equilibrium effort $x(\theta)$ is the solution of the following maximization problem

$$\arg\max_x F(\theta(x))^{n-1} \cdot \theta \cdot R(x) - c(\theta, x)$$

Equivalently, the equilibrium effort $x(\theta)$ is also the solution of the problem

$$\arg\max_x F(\theta(x))^{n-1} \cdot R(x) - \hat{c}(\theta, x)$$

where $\hat{c}(\theta, x) = c(\theta, x)/\theta$. Hence, without loss of generality, we can consider the independent-type case in which the value function is given by

$$V(\theta, R(x)) = R(x) \tag{5}$$

We now analyze the two cases of designer's objective: maximization of total effort and maximization of highest effort.

3.1 Maximization of the total effort

Proposition 2 *Consider a multiplicatively separable environment and a designer that wishes to maximize the expected value of total effort. Then, the optimal reward is given by*

$$R(x) = \left(\hat{c}(\theta(x), x) - \int_0^x \hat{c}_\theta(\theta(\tilde{x}), \tilde{x}) d\theta(\tilde{x}) \right) / F(\theta(x))^{n-1} \tag{6}$$

where $\hat{c}(\theta, x) = c(\theta, x)/\theta$ *and* $\theta(x)$ *is the inverse of the equilibrium effort* $x(\theta)$ *which is given by*

$$1 + \hat{c}_{\theta x}(\theta, x(\theta)) \frac{1 - F(\theta)}{f(\theta)} = \hat{c}_x(\theta, x(\theta)) \tag{7}$$

Proof. See Appendix.

By equation (7), we have

Corollary 1 *In the multiplicatively separable environment when the designer wishes to maximizes the expected value of total effort, the equilibrium effort* $x(\theta)$ *is independent of the number of contestants* n*.*

The independence of the equilibrium effort in the number of contestants becomes clear if we notice that the optimal reward $R(x)$ in the multiplicatively separable environment when the designer wishes to maximize the expected value of total effort is comparable to the optimal wage contract in a Principal-Agent (PA) model where the principal offers the agent a wage $w(x)$ that depends upon output x (which can be sold at a price of one). In the PA model the agent's maximization problem is

$$\max_x w(x) - c(\theta, x)$$

The principal's expected payoff given the solution $x(\theta)$ of the agent's problem is the expected output that he receives (the price of which is one) minus the expected wage that he must pay:

$$n \int_{\underline{\theta}}^{\overline{\theta}} [x(\theta) - w(x(\theta))] dF$$

The substitution of $w(x) \equiv R(x) \cdot F(\theta(x))^{n-1}$ yields the same problems of (1) and (3).

However, while in the PA model the optimal wage is always increasing, in our model, we have

Proposition 3 *In the multiplicatively separable environment, when the designer wishes to maximize the expected value of total effort, for large enough* n, *the optimal reward is decreasing.*

Proof. See Appendix.

Example 1 *Consider the multiplicatively separable environment where the designer maximizes the total effort. The cost function is* $c(x, \theta) = x^2$ *and the distribution of the contestants' types* F *is uniform on* $[0, 1]$.

From this specification, $\hat{c}(x, \theta) = x^2/\theta$ and we can rewrite (7) as

$$1 - (2x(\theta)/\theta^2)(1 - \theta) = 2x(\theta)/\theta$$

Thus, the equilibrium effort is

$$x(\theta) = \frac{\theta^2}{2}$$

Notice that the optimal effort does not depend upon n. The inverse of the equilibrium effort is $\theta(x) = \sqrt{2x}$. By (6) the optimal reward is

$$R(x) = (2x)^{2-n/2}/3$$

For two bidders the optimal reward is $R(x) = \frac{2x}{3}$. For four bidders, we have an independent-effort optimal reward $R(x) = \frac{1}{3}$. For six bidders, we have already a decreasing reward function $R(x) = \frac{1}{6x}$. ■

3.2 Maximization of the highest effort

Proposition 4 *In the multiplicatively separable environment, when the designer wishes to maximize the expected value of the highest effort, the optimal reward is given by*

$$R(x) = \left(\hat{c}(\theta(x), x) - \int_0^x \hat{c}_\theta(\theta(\tilde{x}), \tilde{x}) d\theta(\tilde{x})\right) / F(\theta(x))^{n-1} \tag{8}$$

where $\hat{c}(\theta, x) = c(\theta, x)/\theta$ *and* $\theta(x)$ *is the inverse of the equilibrium effort* $x(\theta)$ *which is given by*

$$F(\theta)^{n-1} + \hat{c}_{\theta x}(\theta, x(\theta)) \frac{1 - F(\theta)}{f(\theta)} = \hat{c}_x(\theta, x(\theta)) \tag{9}$$

Proof. See Appendix.

As a function of the equilibrium effort, the reward formula where the designer maximizes the highest effort (8) is the same as in the case where the designer maximizes the total effort

(6). However, since the equilibrium efforts are not identical in both cases (equations (7) and (9) are not equal) we obtain that the optimal rewards are different.

A simple comparison of the equilibrium efforts (equations (7) and (9)) yields

Corollary 2 *In the multiplicatively separable environment, the equilibrium effort function when the designer maximizes the expected highest effort is point-wise smaller than the equilibrium effort when the designer maximizes the expected total effort. Thus, the expected value of the highest effort is smaller when the designer maximizes the highest effort.*

It is important to notice that in the case when the designer maximizes the expected highest effort the optimal reward $R(x)$ in the multiplicatively separable environment is not comparable to any variable in the classical Principal-Agent (PA) model. In contrast to the case where the designer maximizes the expected value of total effort we have

Corollary 3 *In the multiplicatively separable environment, when the designer maximizes the expected value of the highest effort, the equilibrium effort $x(\theta)$ depends on the number of players n.*

It is shown that in the case when the designer maximizes the expected value of total effort, the optimal reward function decreases in effort if the number of players is sufficiently large. In this case where the designer maximizes the expected value of the highest effort the optimal reward may be increasing for any number of players.

Example 2 *Consider the multiplicatively separable environment where the designer maximizes the highest effort. The cost function is $c(x, \theta) = x^2$ and the distribution of the contestants' types F is uniform on $[0, 1]$.*

Thus $\hat{c}(x, \theta) = x^2/\theta$. From this specification, we can rewrite (9) as

$$\theta^{n-1} - (2x(\theta)/\theta^2)(1-\theta) = 2x(\theta)/\theta$$

This implies the equilibrium effort

$$x(\theta) = \frac{\theta^{n+1}}{2}$$

The inverse of the equilibrium effort is $\theta(x) = (2x)^{1/(n+1)}$. By (8) the optimal reward is then

$$R(x) = \frac{2n+2}{4(2n+1)}(2x)^{\frac{n+2}{n+1}}$$

It can be verified that for large n this reward approaches $R(x) = x/2$.

Notice that the expected highest effort in this case is $\frac{n}{4n+2}$ while the expected highest effort in the case where the designer maximizes the total effort is always larger and equal to $\frac{n}{2n+4}$. Thus, the expected payment when the designer maximizes the highest effort must be smaller than the expected payment when the designer maximizes the total effort, otherwise, there is a contradiction to the optimality of the reward function in this example.

4 Additively separable case

In this section we assume that the value function is additively separable with the form

$$V(\theta, R(x)) = \theta + R(x)$$

Here we again analyze the two cases of the designer's objective.

4.1 Maximization of the total effort

Proposition 5 *In the additively separable environment, when the designer wishes to maximize the expected value of total effort, the optimal reward is given by*

$$R(x) = \left(c(\theta(x), x) + \int_{\theta^*}^{\theta(x)} [F(\theta)^{n-1} - c_\theta(\theta, x(\theta))] d\theta \right) / F(\theta(x))^{n-1} - \theta(x) \tag{10}$$

where $\theta(x)$ is the inverse of the equilibrium effort $x(\theta)$ which is given by

$$1 + c_{\theta x}(\theta, x(\theta)) \frac{1 - F(\theta)}{f(\theta)} = c_x(\theta, x(\theta)) \tag{11}$$

and the cutoff θ^* is the θ that maximizes the designer's profit from the set $\{\theta \in [\underline{\theta}, \overline{\theta}] : x(\theta) - c(\theta, x(\theta)) + \left(\theta - \frac{1-F(\theta)}{f(\theta)}\right) F(\theta)^{n-1} + c_\theta(\theta, x(\theta)) \frac{1-F(\theta)}{f(\theta)} = 0\}$ (if the set is empty $\theta^* = \underline{\theta}$).

Proof. See Appendix.

The designer's payoff when he maximizes the total effort is

$$n \int_{\underline{\theta}}^{\overline{\theta}} \left[x(\theta) - c(\theta, x(\theta)) + \left(\theta - \frac{1 - F(\theta)}{f(\theta)}\right) F(\theta)^{n-1} + c_\theta(\theta, x(\theta)) \frac{1 - F(\theta)}{f(\theta)} \right] dF$$

The reward function in this case limits the participation of players for which the expression within the integral is negative. Notice that if the cost function does not depend on the contestant type, i.e., $c(\theta, x) = x$, then (11) holds for all $x(\theta)$. This is because our environment is converted into the standard auction environment and revenue equivalence holds– where profit of the designer depends only upon which types are included determined by when $\left(\theta - \frac{1-F(\theta)}{f(\theta)}\right)$ is positive.

The equilibrium effort is the same in the additively separable case as in the multiplicatively separable case when the value function is independent-type. However, the optimal reward functions are different in these two cases. The difference between these cases is an added benefit for winning that depends upon θ in the additively-separable case. This added benefit yields additional rents for the contestant, but the induced equilibrium effort does not influence these rents.

Proposition 6 *In the additively-separable case when the designer wishes to maximize the expected value of total effort, for large enough* n *the optimal reward is decreasing.*

Proof. See Appendix.

Example 3 *Consider the additively separable environment where the designer maximizes the total effort. The cost function is* $c(\theta, x) = x^2/\theta$ *and the distribution of the contestants' types* F *is uniform on* $[0, 1]$.

By (11) the optimal effort is

$$x(\theta) = \theta^2/2$$

Notice, that also here the optimal effort function does not depend upon n. The cutoff equation is

$$\theta^2/4 + 2\theta^n - \theta^{n-1} = 0$$

For $n = 2$, this has a solution of $\theta^* = 4/9$. By (10) this yields an optimal reward function of

$$R(x) = 2x/3 - (2x)^{1/2}/2 - \frac{8 \cdot 29}{3^7(2x)^{1/2}}$$

Notice that the expected payment is $-\frac{1145}{4374}$ which is negative.

4.2 Maximization of the highest effort

In a similar way to the case where the designer maximizes the total effort we obtain,

Proposition 7 *In the additively separable environment, when the designer wishes to maximize the expected value of the highest effort, the optimal reward is given by*

$$R(x) = \left(c(\theta(x), x) + \int_{\theta^*}^{\theta(x)} [F(\theta)^{n-1} - c_\theta(\theta(\tilde{x}), \tilde{x}(\theta)] d\theta \right) / F(\theta(x))^{n-1} - \theta(x) \quad (12)$$

where $\theta(x)$ is the inverse function of the equilibrium effort $x(\theta)$ which is given by

$$F(\theta)^{n-1} + c_{\theta x}(\theta, x(\theta)) \frac{1 - F(\theta)}{f(\theta)} = c_x(\theta, x(\theta)) \quad (13)$$

and the cutoff θ^* is the θ that maximizes the designer's profits from the set $\{\theta \in [\underline{\theta}, \overline{\theta}] : x(\theta)F(\theta)^{n-1} - c(\theta, x(\theta)) + \left(\theta - \frac{1-F(\theta)}{f(\theta)}\right) F(\theta)^{n-1} + c_\theta(\theta, x(\theta))\frac{1-F(\theta)}{f(\theta)} = 0\}$ (if the set is empty $\theta^* = \underline{\theta}$).

Example 4 *Consider the additively separable case where the designer maximizes the highest effort. The cost function is $c(x, \theta) = x^2/\theta$ and the distribution of the contestants' types F is uniform on $[0, 1]$.*

From this specification, we can rewrite (13) as

$$\theta^{n-1} - \frac{2x(\theta)}{\theta^2}(1 - \theta) = \frac{2x(\theta)}{\theta}$$

This implies the equilibrium effort function

$$x(\theta) = \frac{\theta^{n+1}}{2}$$

The inverse function is $\theta(x) = (2x)^{1/(n+1)}$. The cutoff equation is given by

$$\theta^{2n}/4 + 2\theta^n - \theta^{n-1} = 0$$

(For $n = 2$, this has a solution at $\theta^* = .486$.)

The optimal reward is then

$$R(x) = \frac{2n+2}{4(2n+1)} (2x)^{\frac{n+2}{n+1}} - (2x)^{\frac{1}{n+1}} \left(\frac{n-1}{n}\right) - (2x)^{-\frac{n-1}{n+1}} \left[\frac{\theta^{*n}}{n} + \frac{\theta^{*2n+1}}{4(2n+1)}\right]$$

For large n this reward approaches to the increasing reward function $R(x) = \frac{x}{2} - 1$.

Again we can compare equilibrium efforts. Since the equilibrium effort equations, (7) and (9), do not change, the result of Corollary (2) holds here (for those that choose to participate).

Corollary 4 *In the additively separable case, the equilibrium effort function when the designer maximizes the expected highest effort is point-wise smaller than the equilibrium effort when the designer maximizes the expected total effort.*

When the designer cares about total effort his payoff for including each individual type is higher than when the designer cares about maximum effort.[3] Therefore, we have the following.

Corollary 5 *In the additively separable case, the set of types of participants when the designer maximizes the expected total effort is larger (and includes) than the set of types of participants when the designer maximizes the expected highest effort.*

[3]It may be possible that a type with a negative payoff to the designer is included in order to include lower types with higher payoffs. This does not matter since when the designer cares about total effort the benefit for including lower types will be higher as well as the cost for including those negative types will be lower.

5 Multiple Rewards

So far, we analyzed the optimal reward when an effort-dependent reward $R(x)$ is awarded only to the winner. We now extend the analysis to the case of multiple effort-dependent rewards where the contestant with the highest effort wins the reward $R_1(x)$, the contestant with the second highest effort wins the reward $R_2(x)$, and so on until all the rewards are allocated. That is, $R_k(x)$ is the reward for the contestant with the k-highest effort who exerts an effort of x. In this extended environment we can ask what are the optimal structures and the optimal number of effort-dependent rewards. Moldovanu and Sela [2001], using our environmental assumption of convexity of the cost function, show that it may be optimal to allocate several prizes. Since their rewards were not only fixed but independent of effort, it is interesting to examine if their result holds in our environments where the designer has more flexibility.

Proposition 8 *A multi-reward contest* $\{R_i(x)\}_{i\geq 1}$ *has ex-ante equivalent payoffs to a single reward contest* $R(x)$ *if* $E[R_i(x)|\theta(x)] = F(\theta(x))^{n-1}R(x)$ *for the following environments:*[4]

(i) the multiplicative-separable case, when a contestant of type θ *with the* $k-$*highest effort receives payoff of* $R_k(x)\cdot\theta$.

(ii) the independent case (the reward is independent of θ*), when a contestant of type* θ *with the* $k-$*highest effort receives payoff of* $R_k(x)$.

[4]The expression $E\left[R_i(x)|\theta(x)\right]$ is the expected reward of each contestant given the equilibrium bid function. For example, if there is only one reward, this expected value is $F(\theta(x))^{n-1}R_1(x)$, and if there are two rewards, this expected value is $F(\theta(x))^{n-1}R_1(x) + (n-1)F(\theta(x))^{n-2}(1-F(\theta(x)))R_2(x)$.

(iii) the additively separable case, when the winner receives payoff of $R_1(x) + \theta$, and the contestant of type θ with the k-highest effort receives payoff of $R_k(x)$ when $k > 1$.

Proof. See Appendix.

The three environments are chosen to ensure that surplus is not created nor destroyed simply by the mere fact of giving a non-first place prize – there is no intrinsic value to being runner-up.[5] Proposition 8 implies that all prize structures that have $E[R_i(x)|\theta(x)] = F(\theta(x))^{n-1}R(x)$ where $R(x)$ is the optimal reward in the case of unique dependent-effort reward are optimal. The proof simply comes from the fact that these are substitutable in both the contestant's expected surplus and the designer's expected profits. And since we use both equations to eliminate the rewards (to solve for the optimal effort) they do not appear in the form of the designer's expected payoff that we maximize.

Proposition 8 shows that general all-pay auctions with effort-dependent rewards over a number of prizes have equivalence if the equilibrium bid function is monotonic. This allows us to easily analyze a range of problems including those with a disadvantage to the winner: everyone may want to try to beat the fastest gunfighter in town, the tallest building may be a clearer target for terrorism, etc. In the following example we generate a peculiar example of an optimal contest with two contestants where there is only a prize for the loser.

Example 5 *Consider the independent case with two contestants where the designer maximizes the total effort. The cost function is $c(x, \theta) = x^2/\theta$ and F is uniform on $[0, 1]$.*

[5] Some other possibilities do not share this property: For example, in the additively-separable case if a contestant of type θ with the i-highest effort receives $R_i(x) + \theta$ for all i. In this case, the equivalence would disappear since giving additional prizes (say of value ϵ) would create surplus.

In this example we have shown that the optimal reward function for the winner is $R(x) = 2x/3$ and the equilibrium effort is given by $\theta(x) = \sqrt{2x}$. We then have $F(\theta(x))^{n-1}R(x) = (2x)^{3/2}/3$. One can maintain the same revenue by giving a prize of zero to the "winner" and an effort-dependent reward to the loser. This would be set such that $(1-\theta(x))R_2(x) = (2x)^{3/2}/3$. Thus, the optimal rewards are

$$R_1(x) = 0, \quad R_2(x) = \frac{(2x)^{3/2}/3}{1-\sqrt{2x}}$$

Notice that this reaches infinity as $x \to 1/2$ (this is the effort chosen by the highest type, $\theta(1/2) = 1$), since there is an almost certain chance of winning and getting paid nothing.

6 Concluding remarks

In this paper, we study the design of contests when the designer has full flexibility over what reward function to use. We solve our problem of finding the optimal reward by indirect means. First, we solve for the optimal effort function. This is done by looking at the virtual cost of increasing an effort for a specific type. Second, we solve for the reward that induces the effort function. Here, an increase in the effort is reflected by an increase in the overall reward paid to contestants. This at first glance appears to contradict Kaplan et al. (2002), who found an increase in the rewards may lead to a decrease in the expected effort. However, by the analysis here one can see how such an example can exist: increase the effort when it is costly (marginal cost is high) and decrease the efforts when such an increase would be cheap (marginal cost is low). Hence, one can increase the rewards while leading to a decrease in the expected effort.

We find new surprising results such as the reward to winning may be not only increasing, but decreasing in the efforts. It is easy to envision contests where the reward to winning is increasing in the results. These bonuses for good performances may be external rewards to winning, extra payment from the designer, or simply getting the reward sooner. On the other hand, it is not so obvious to dream up a scenario where the reward is actually decreasing in effort. This comes in the case of contests where the reward is increasing over time. This can happen if money is raised for the winner of a contest similar to the X prize where not only is the prize money kept aside earning interest, but where the organizers continue to raise funds. The reason that this is in fact, a decreasing reward is that inventing early requires more effort.

While the environment we study here is restricted to contests, it is possible to use the same tools to study optimal design with effort-dependent rewards in the classical auction mechanisms.

References

[1] Barut, Y., and Kovenock, D. (1998), "The Symmetric Multiple Prize All-Pay Auction with Complete Information," *European Journal of Political Economy*, 14, 627-644.

[2] Baye, M., Kovenock, D., and de Vries, C. (1993), "Rigging the Lobbying Process," *American Economic Review*, 83, 289-294.

[3] Che, Y-K, and Gale, I. (1998), "Caps on Political Lobbying." *American Economic Review,* Vol. 88, pp. *643-651.*

[4] Che, Y-K, and Gale, I. (2003), "Optimal Design of Research Contests," *American Economic Review,* 93, 646-671.

[5] Fullerton, R. and McAfee, P. (1999), "Auctioning Entry into Tournaments," *Journal of Political Economy* 107, 573-605.

[6] Gavious, A., Moldovanu, B., and Sela, A. (2003) "Bid Costs and Endogenous Bid Caps," *Rand Journal of Economics,* 33(4), 709-722.

[7] Kaplan, T.R., Luski, I, Sela, A. and Wettstein, D. (2002) "All-Pay Auctions with Variable Rewards," *Journal of Industrial Economics*, L (4): 417-430.

[8] Kaplan, T.R., Luski, I. and Wettstein, D. (2003) "Innovative Activity and Sunk Costs," *International Journal of Industrial Organization*, 21:1111-1133.

[9] Moldovanu, B., and Sela, A. (2001), "The Optimal Allocation of Prizes in Contests," *American Economic Review*, 91, 542-558.

[10] Taylor, C. (1995), "Digging for Golden Carrots: An Analysis of Research Tournaments," *American Economic Review*, 85, 873-890.

A Appendix

A.1 Proof of Proposition 1

Using the envelope theorem on the contestant's maximization problem (1) yields

$$\pi'(\theta) = F(\theta)^{n-1}V_\theta(\theta, R(x(\theta))) - c_\theta(\theta, x(\theta))$$

Assume that all contestants with value $\theta \geq \underline{\theta}$ take part in the auction and that $\pi(\underline{\theta}) = 0$. Then by integration we obtain

$$\pi(\theta) = \int_{\underline{\theta}}^{\theta} [F(\tilde{\theta})^{n-1} V_{\theta}(\tilde{\theta}, R(x(\tilde{\theta}))) - c_{\theta}(\tilde{\theta}, x(\tilde{\theta}))] d\tilde{\theta}$$

From the maximization problem, we also have

$$\pi(\theta) = F(\theta)^{n-1} \cdot V(\theta, R(x(\theta))) - c(\theta, x(\theta))$$

The comparison of the contestant's expected payoffs gives us the desired result. ■

A.2 Proof of Proposition 2

Straightforward substitution of (5) into (2) implies that an equilibrium strategy $x(\theta)$ must be given by the implicit function

$$F(\theta)^{n-1} R(x(\theta)) - \hat{c}(\theta, x(\theta)) = \int_{\underline{\theta}}^{\theta} -\hat{c}_{\theta}(\tilde{\theta}, x(\tilde{\theta}))] d\tilde{\theta} \tag{14}$$

while RHS of (14) is the expected payoff of a contestant given this strategy.

Substituting (14) in the designer's expected payoff (3) yields the following designer's problem

$$\max_{x} n \int_{\underline{\theta}}^{\overline{\theta}} \left[x(\theta) - \hat{c}(\theta, x(\theta)) + \int_{\underline{\theta}}^{\theta} \hat{c}_{\theta}(\tilde{\theta}, x(\tilde{\theta}))] d\tilde{\theta} \right] dF \tag{15}$$

By using of integration by parts, we can rewrite the last term as follows

$$\int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\theta} \hat{c}_{\theta}(\widetilde{\theta}, x(\widetilde{\theta})) d\widetilde{\theta} dF = \int_{\underline{\theta}}^{\overline{\theta}} \hat{c}_{\theta}(\theta, x(\theta)) d\theta - \int_{\underline{\theta}}^{\overline{\theta}} F(\theta) \hat{c}_{\theta}(\theta, x(\theta)) d\theta = \int_{\underline{\theta}}^{\overline{\theta}} \hat{c}_{\theta}(\theta, x(\theta)) \frac{1 - F(\theta)}{f(\theta)} dF$$

Thus, the designer's problem is

$$\max_x n \int_{\underline{\theta}}^{\overline{\theta}} \left[x(\theta) - \hat{c}(\theta, x(\theta)) + \hat{c}_\theta(\theta, x(\theta)) \frac{1 - F(\theta)}{f(\theta)} \right] dF$$

Since the designer is indirectly choosing $x(\theta)$ through the reward function. We can look at the first-order condition to find the induced optimal effort

$$1 + \hat{c}_{\theta x}(\theta, x(\theta)) \frac{1 - F(\theta)}{f(\theta)} = \hat{c}_x(\theta, x(\theta)) \tag{16}$$

Notice that our assumptions on c imply the same assumptions on $\hat{c} = \frac{c}{\theta}$. These assumptions imply that as θ increases the LHS of (16) increases and the RHS increases. When x increases, the LHS of (16) decreases and the RHS increases. Thus, there is a monotonic solution to this equation.

Given the optimal effort $x(\theta)$, the optimal reward is obtained by changing variables from θ to x in equation (14). Therefore, the optimal reward is simply

$$R(x) = \left(\hat{c}(\theta(x), x) - \int_0^x \hat{c}_\theta(\theta(\tilde{x}), \tilde{x}) d\theta(\tilde{x}) \right) / F(\theta(x))^{n-1}$$

where $\theta(x)$ is the inverse of $x(\theta)$. ■

A.3 Proof of Proposition 3

The equilibrium effort does not depend on the number of contestants n. The reward must be strictly positive for participation. Thus, the optimal reward given by (6) can be written as a fraction of two functions, $z_1(x)/z_2(x)^{n-1}$ where

$$\begin{aligned} z_1(x) &= \left(\hat{c}(\theta(x), x) - \int_0^x \hat{c}_\theta(\theta(\tilde{x}), \tilde{x}) d\theta(\tilde{x}) \right) \\ z_2(x) &= F(\theta(x)) \end{aligned}$$

Then, the derivative of the reward function is given by

$$\frac{z_2\left(x\right)^{n-2}\left[z_2\left(x\right)z_1'(x)-(n-1)z_2'\left(x\right)z_1(x)\right]}{z_2\left(x\right)^{2n-2}}$$

Since by our assumptions all the parameters here are finite, for large enough n this derivative must be negative. ■

A.4 Proof of Proposition 4

As in the case of maximization of total effort, we can use equation (14) to substitute for $F(\theta)^{n-1}R(x(\theta))$ in the designer's expected payoff (4) and use integration by parts to simplify. Now the designer's expected payoff becomes

$$n\int_{\underline{\theta}}^{\overline{\theta}}\left[x(\theta)F(\theta)^{n-1}-\hat{c}(\theta,x(\theta))+\hat{c}_\theta(\theta,x(\theta))\frac{1-F(\theta)}{f(\theta)}\right]dF \tag{17}$$

The first-order condition of this yields the optimal (profit-maximizing) $x(\theta)$

$$F(\theta)^{n-1}+\hat{c}_{\theta x}(\theta,x(\theta))\frac{1-F(\theta)}{f(\theta)}=\hat{c}_x(\theta,x(\theta)) \tag{18}$$

Since $F^{n-1}(\theta)$ is increasing in θ, the same arguments as before guarantees a monotonic solution. From equation (14), we find the optimal reward:

$$R(x)=\left(\hat{c}(\theta(x),x)-\int_0^x\hat{c}_\theta(\theta(\tilde{x}),\tilde{x})d\theta(\tilde{x})\right)/F(\theta(x))^{n-1} \tag{19}$$

where $\theta(x)$ is the inverse of $x(\theta)$ that satisfies (18). ■

A.5 Proof of Proposition 5

The equilibrium strategy $x(\theta)$ is given by the implicit function

$$F(\theta)^{n-1}[\theta + R(x(\theta))] - c(\theta, x(\theta)) = \int_{\underline{\theta}}^{\theta} [F(\tilde{\theta})^{n-1} - c_\theta(\tilde{\theta}, x(\tilde{\theta}))]d\tilde{\theta} \tag{20}$$

while the expected payoff of a contestant given this strategy is

$$\pi(\theta) = \int_{\underline{\theta}}^{\theta} [F(\tilde{\theta})^{n-1} - c_\theta(\tilde{\theta}, x(\tilde{\theta}))]d\tilde{\theta}$$

As before, we can use (20) to find the reward as a function of the equilibrium effort

$$R(x) = \left(c(\theta(x), x) + \int_0^x [F(\theta(\tilde{x}))^{n-1} - c_\theta(\theta(\tilde{x}), \tilde{x})]d\theta(\tilde{x}) \right) / F(\theta(x))^{n-1} - \theta(x)$$

By substituting this reward into the designer's payoff and using integration by parts, we obtain

$$\begin{aligned} & n \int_{\underline{\theta}}^{\overline{\theta}} \left[x(\theta) - c(\theta, x(\theta)) + \theta F(\theta)^{n-1}\right] dF - n \int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\theta} \left[F(\tilde{\theta})^{n-1} - c_\theta(\theta, x(\theta))\right] d\tilde{\theta} dF \quad (21) \\ = \; & n \int_{\underline{\theta}}^{\overline{\theta}} \left[x(\theta) - c(\theta, x(\theta)) + \left(\theta - \frac{1 - F(\theta)}{f(\theta)}\right) F(\theta)^{n-1} + c_\theta(\theta, x(\theta)) \frac{1 - F(\theta)}{f(\theta)}\right] dF \end{aligned}$$

The first-order condition of this yields the optimal effort function

$$1 + c_{\theta x}(\theta, x(\theta)) \frac{1 - F(\theta)}{f(\theta)} = c_x(\theta, x(\theta))$$

As before, our assumptions on c satisfy the second-order conditions. The designer also has the option of having a cutoff type in order to not include lower types for when the expression within the integral is negative. It is important to notice that this expression within the integral does not necessarily increases in θ.■

A.6 Proof of Proposition 6

Since the optimal equilibrium effort is the same as in the multiplicatively separable case when the value function is independent-type, the difference between the two rewards is that now the reward is larger by

$$\int_0^x [F(\theta(\tilde{x}))^{n-1} d\theta(\tilde{x}) / F(\theta(x))^{n-1} - \theta(x).$$

The derivative of this with respect to x is

$$\begin{aligned} & \frac{F(\theta(x))^{2n-2}\theta'(x) - \int_0^x [F(\theta(\tilde{x}))^{n-1} d\theta(\tilde{x}) \cdot (n-1) F(\theta(x))^{n-2} F'(\theta(x))\theta'(x)}{F(\theta(x))^{2n-2}} - \theta'(x) \\ = \quad & \frac{-\int_0^x [F(\theta(\tilde{x}))^{n-1} d\theta(\tilde{x}) \cdot (n-1) F'(\theta(x))\theta'(x)}{F(\theta(x))^{n}} < 0 \end{aligned}$$

Thus, the reward is also decreasing for large n. ■

A.7 Proof of Proposition 8

We can rewrite the contestant's expected surplus, (14) and (20), as the following two equations (the first holds for environments (1) and (2), while the second holds for environment (3))

$$E[R_i(x(\theta))] - \hat{c}(\theta, x(\theta)) = \int_{\underline{\theta}}^{\theta} -\hat{c}_\theta(\tilde{\theta}, x(\tilde{\theta}))] d\tilde{\theta}$$

$$F(\theta)^{n-1}\theta + E[R_i(x(\theta))] - c(\theta, x(\theta)) = \int_{\underline{\theta}}^{\theta} [F(\tilde{\theta})^{n-1} - c_\theta(\tilde{\theta}, x(\tilde{\theta}))] d\tilde{\theta}$$

The designer's payoff changes from (3) and (4) to the following two formulas, respectively,

$$n \int_{\underline{\theta}}^{\overline{\theta}} x(\theta) dF - n \int_{\underline{\theta}}^{\overline{\theta}} E[R_i(x(\theta))] dF$$

$$\int_{\underline{\theta}}^{\overline{\theta}} x(\theta)dF^n - n\int_{\underline{\theta}}^{\overline{\theta}} E[R_i(x(\theta))]dF$$

When we use the contestant's surplus equations to substitute for the expected rewards in the above two formulas (depending upon the environment and whether the designer's goal is total or maximum effort), we arrive at exactly the same formulas for the designer's payoff as before: (15), (17) and (21). Thus, both the induced effort and the respective payoffs will remain the same. ■