GENERALIZED COLEMAN-SHAPLEY INDICES AND TOTAL-POWER MONOTONICITY

Ori Haimanko

Discussion Paper No. 18-13

November 2018

Monaster Center for
Economic Research
Ben-Gurion University of the Negev
P.O. Box 653
Beer Sheva, Israel

Fax: 972-8-6472941
Tel: 972-8-6472286

Generalized Coleman-Shapley Indices and Total-Power Monotonicity

Ori Haimanko*

November 2018

Abstract

We introduce a new axiom for power indices, which requires the total (additively aggregated) power of the voters to be nondecreasing in response to an expansion of the set of winning coalitions; the total power is thereby reflecting an increase in the collective power that such an expansion creates. It is shown that total-power monotonic indices that satisfy the standard semivalue axioms are probabilistic mixtures of generalized Coleman-Shapley indices, where the latter concept extends, and is inspired by, the notion introduced in Casajus and Huettner (2018). Generalized Coleman-Shapley indices are based on a version of the random-order pivotality that is behind the Shapley-Shubik index, combined with an assumption of random participation by players.

JEL Classification Numbers: C71, D72.

Keywords: Simple Games, Shapley-Shubik Index, Banzhaf Index, Coleman-Shapley Index, Semivalues, Power of Collectivity to Act, Total-Power Monotonicity Axiom, Probabilistic Mixtures.

*Department of Economics, Ben-Gurion University of the Negev, Beer Sheva 84105, Israel. E-mail: orih@exchange.bgu.ac.il

1 Introduction

The Shapley-Shubik power index[1] and the Banzhaf power index[2] enjoy a near-universal recognition as valid measures of a priori voting power. The two indices quantify the power held by individual voters under a given decision rule by assigning each individual the *probability of being pivotal* in a certain mode of random voting. The Shapley-Shubik power index views voters as "aligned in order of their enthusiasm for the proposal" over which the vote is held, with all orders being possible and equally likely a priori; an individual is pivotal if "by joining his more enthusiastic colleagues, [he] brings [that] coalition up to winning strength."[3] In the Banzhaf power index, the pivotal status of an individual is defined as his ability to affect the outcome of the vote in a random set of voters, assuming that all sets are equally likely. Thus, the assumption behind the Shapley-Shubik index is that all voters will ultimately vote "yes," and the pivotality of a voter only arms him with some bargaining advantage in demanding adjustments in the content of the proposal; the Banzhaf power index, on the other hand, views pivotality as being in a position to single-handedly push the proposal through.

The two indices, in their narrow interpretation, measure the voting power of each *individual* voter, but the individual power is often additively aggregated across individuals in order to compute the implied power of *sets* of voters. There is somewhat less clarity as to what an aggregation of power over a set represents, compared to the rather straightforward concept of individual power that is behind the two indices, but such an aggregation is taken quite seriously. A need for comparison of power of different sets arises both in practice (see, e.g., Brahms (2013, Chapter 5)), and in axiomatic treatment of power indices. Indeed, the original axiomatizations of the two indices contain references to the *total*, or *combined*, *power* of voters. Dubey (1975), who axiomatically characterized the Shapley-Shubik power index, imposed the efficiency axiom – whereby the total power of the voter set is 1, independently of

[1] Defined in Shapley and Shubik (1954).

[2] As if often done in the literature, we use the term "Banzhaf index" for brevity, although the origin of this power index lies in multiple works (Penrose (1946), Banzhaf (1965, 1966, 1968), Coleman (1971)). The specific variant of the Banzhaf index used in this work is referred to as the "Banzhaf measure" in Felsenthal and Machover (1998).

[3] All quotations in this sentence are taken from Dubey and Shapley (1979, p. 103).

the particular decision rule. Dubey and Shapley (1979), who axiomatized the Banzhaf power index, assumed the total power of all voters to be equal to the expected number of swing voters, or "swingers,"[4] in the voter set; this number is also known as the "sensitivity of the decision rule" (see Felsenthal and Machover (1998), Section 3.3). Following the discovery of the 2-efficiency of the Banzhaf index by Lehrer (1988), whereby the combined power of any two voters remains unchanged if the two voters "merge" and act as a single bloc, multiple axiomatizations of the Banzhaf index were offered based on relaxed versions of that property.[5]

Recently, Casajus and Huettner (2018) suggested a new power index, which they named the Coleman-Shapley index, with an underlying probability model that very naturally combines the assumptions behind the Shapley-Shubik and the Banzhaf indices. Specifically, the power of an individual voter is, again, his probability of being pivotal, but in the following hybrid situation. Similarly to the Banzhaf scenario, each individual votes "yes" with probability $\frac{1}{2}$, independently of the other voters; an alternative, equivalent, assumption would be that an individual is only interested in/capable of voting with probability $\frac{1}{2}$. As in the Shapley-Shubik scenario, the pivotality of a voter is defined with respect to a random order (reflecting the "enthusiasm") of all active[6] voters; the probability of being pivotal is now conditional on the voter being active. Thus, the notion of pivotality here still implies the ability to affect the content of a proposal, but the passage of the proposal is now viewed as uncertain. The characteristic feature of the Coleman-Shapley index is that the total power of all voters coincides with a well-recognized concept – the power attributed to individual voters sums up to (twice) the Coleman's (1971) "power of a collectivity to act," defined as the proportion of winning sets among all (sub)sets of voters.

The total power of voters according to the Banzhaf power index, being the definition of the sensitivity of a decision rule, quantifies "the ease with which [the decision rule] responds to voters' wishes."[7] It is not surprising that the total Banzhaf power

[4] See Dubey et al. (1979, p. 103). A swinger is defined w.r.t. a random set of voters (with the uniform distribution over all subsets of the voting body) by the requirement that his vote affects the voting outcome of that set.

[5] See, e.g., Lehrer (1988), Nowak (1997), Casajus (2012), Haimanko (2018).

[6] Depending on the previous assumption, *active voters* are either yes-voters, or those that are interested in/capable of voting.

[7] See Felsenthal and Machover (1998, p. 52).

favors games where the outcome of the vote appears, a priori, to be very uncertain, as this is when individual voters have a good chance to be pivotal. Indeed, as shown in Dubey and Shapley (1979), the total Banzhaf power of a given voter set is maximal for the simple majority rule, as that rule creates the greatest instability in the outcome of the vote, under the assumption that votes are cast completely at random and independently across individuals.

The total power behaves in a notably different fashion, however, under the other two indices. The total Shapley-Shubik power is fixed at 1, and hence the simple majority rule has the same standing as the rest. Under the Coleman-Shapley index, the total power, which is identifiable with the aforementioned Coleman power of collectivity to act, is at the intermediate level for the simple majority rule, *falling* with an increase in the majority quota. Indeed, Coleman's measure of the power of collectivity is concerned with the ease of a *collective* achievement. Thus, higher quotas mean a lower number of winning sets, and, accordingly, lower collective power.

The above monotonicity feature of the total Coleman-Shapley power extends from the simple majority to general, not necessarily symmetric and quota-based, decision rules: the smaller is the set of winning sets (as in the particular case of a rising majority quota), the lower is the total power of the voters. This property appears to be quite reasonable if one wishes the *total* power, obtained by additive aggregation of the individual power, to measure, or at least be highly correlated with, some form of *collective* power held by the voters (naturally, collective power should respond positively to an expansion of the set of winning coalitions).

Our concern in this work will be with power indices whose implied total power reflects collective power in the sense indicated above. We will call the property whereby the total power in nondecreasing when winning sets are added *total-power monotonicity*, or TP-monotonicity. Following the approach pioneered in Shapley and Shubik (1954) and adopted in much of the literature on power indices, we will model voting situations/decision rules as cooperative games known as *simple* (or *voting*) *games*, and view a power index as a map defined on the domain of simple games. The focus will be on power indices that are *semivalues*, a term that was borrowed by Einy (1987) from the realm of value maps considered by Dubey et al. (1981), and applied to power indices that satisfy four axioms that are quite standard and figure prominently in the literature on axiomatizations. These axioms are: *transfer* (or *valuation*), which has been a routine substitute for the additivity axiom for value maps in the context of simple games since its introduction in Dubey (1975); *positivity*, or non-negativity of the power index; *symmetry*, which requires covariance under permutations of the player (voter) set; and *dummy*, whereby the power of a dummy player (which can only be a null player, or a dictator, in a simple game) equals to the payoff of his stand-alone coalition.

Our first contribution will be the identification of a one-parametric class of TP-monotonic semivalues. These semivalues constitute a natural generalization of the Coleman-Banzhaf index of Casajus and Huettner (2018). For any $q \in (0, 1]$, we will define the q-Coleman-Banzhaf index (or q-CS index, for short) in the same way as the Coleman-Banzhaf index above, but with the following change: each voter's probability to be active (i.e., to vote "yes," or to be interested in/capable of voting, depending on the interpretation) is now taken to be q, and not $\frac{1}{2}$. As before, different individuals are independent in their activity status, and the q-CS index assigns each voter his probability (conditional on being an active voter) of being a pivot in a uniformly distributed random order of all active voters. The total q-CS power is a ($\frac{1}{q}$-scaled) version of the Coleman power of collectivity to act, defined as the probability that the coalition of all active voters is winning. Obviously, such a probability responds monotonically to any expansion of the set of winning coalitions in the game, and hence the q-CS index is TP-monotonic; it is moreover a semivalue by standard arguments. The class of q-CS indices contains the Shapley-Shubik index (for $q = 1$) and the Coleman-Banzhaf index (for $q = \frac{1}{2}$) as special cases, and in particular captures all scenarios lying on the spectrum defined by the two indices, where the voter's probability of being active (which is either his natural propensity to support a proposal, or the likelihood of being interested in/capable of voting, depending on the interpretation) is given by the parameter $q \in (0, 1]$. For completeness, we will also admit the limit case of $q = 0$, when no one supports the proposal and the power of a voter is the same as the winning status of his stand-alone coalition.

Although the generalized Coleman-Banzhaf indices are not the only TP-monotonic semivalues, we will show, via a somewhat indirect approach, that they generate all such semivalues. Our main tool will be Einy's (1987) characterization of semivalues of simple games as probabilistic mixtures of x-values. For $x \in [0,1]$, the x-value is a power index (in fact, a semivalue itself) that assigns each voter i in a simple game v the probability that he is pivotal[8] for a random coalition of other players, joined by each player with probability x independently of the rest. Einy's result states that any semivalue is obtained by integrating over x-values w.r.t. a uniquely determined probability distribution ξ. Our first result, Theorem 1, studies the effect of imposing the TP-monotonicity assumption on a semivalue in terms of the implied conditions on the representing distribution ξ. It turns out that a semivalue is TP-monotonic if and only if the c.d.f. of the distribution ξ is a concave function.[9]

The structural implication of TP-monotonicity in Theorem 1 appears rather technical from first glance. However, it contains a much more explicit message, initially hidden from view. Our Theorem 2 uses the concavity of the c.d.f. of the representing distribution of a TP-monotonic semivalue to show that the latter is a probabilistic mixture of generalized Coleman-Banzhaf indices. This characterization of the TP-monotonic semivalues is a complete one: a semivalue is TP-monotonic if and only if it is obtained by integrating q-CS indices over $q \in [0,1]$ w.r.t. a uniquely defined probability measure on $[0,1]$. In particular, any TP-monotonic semivalue that is not a convex combination of generalized Coleman-Banzhaf indices can be approximated by such combinations, since integrals in our characterization are approximable by weighted averages.

The paper is organized as follows. Section 2 recalls the basic definitions pertaining to games and power indices, lists the semivalue axioms, and calls attention to the known characterization of semivalues as mixtures of x-values. Generalized Coleman-Shapley indices are defined in Section 3, and are shown to be attainable by the "random arrival" and random-order approaches. Section 3 also introduces the axiom of TP-monotonicity, and checks that it is satisfied by all q-CS indices. Section 4 contains our main results: Theorem 1, which characterizes TP-monotonic semivalues in terms of their underlying probablity distribution, and Theorem 2, which represents TP-monotonic semivalues as mixtures of generalized Coleman-Shapley indices.

[8] In the context of a simple game, a pivot for a coalition is a player whose presence switches that coalition from losing to winning.

[9] Theorem 1 contains the difficult "only if" direction of this statement. The simpler "if" direction is proved in Remark 1.

2 Preliminaries

2.1 Finite games and simple (voting) games

Let U be an infinite universe of *players* (or *voters*), and assume, w.l.o.g., that U includes the set $\mathbb{N}$ of positive integers. Denote the collection of all *coalitions* (subsets of U) by 2^U, and the empty coalition by $\emptyset$. A *game* on U is given by a map $v : 2^U \rightarrow \mathbb{R}$ with $v(\emptyset) = 0$. A coalition $N \subset U$ is called a *carrier* of v if $v(S) = v(S \cap N)$ for any $S \in 2^U$. We say that v is a *finite game* if it has a finite carrier; the minimal carrier of such v is, in effect, its true player set. The space of all finite games on U is denoted by $\mathcal{G}$. The domain $\mathcal{SG}$ of *simple* (or *voting*) *games* on U consists of all $v \in \mathcal{G}$ such that: (i) $v(S) \in \{0, 1\}$ for all $S \in 2^U$; (ii) $v(U) = 1$; and (iii) v is *monotonic*, i.e., if $S \subset T$ then $v(S) \leq v(T)$. If $v \in \mathcal{SG}$, a coalition S is *winning* if $v(S) = 1$, and *losing* otherwise. Thus, as in Shapley and Shubik (1954), any $v \in \mathcal{SG}$ describes a voting system or a decision rule, with a full account of all possible coalitions of yes-voters that can win the vote.

The space $\mathcal{AG}$ of *additive games* consists of all $v \in \mathcal{G}$ satisfying $v(S \cup T) = v(S) + v(T)$ whenever $S \cap T = \emptyset$. Any $w \in \mathcal{AG}$ with a finite carrier N is identifiable with the vector[10] $\{w(i) \mid i \in N\}$, and thus may be thought of as a payoff vector to the players in N.

2.2 Power indices and Semivalues

A *power index* φ is a map $\varphi : \mathcal{SG} \rightarrow \mathcal{AG}$, where $\varphi(v)(i)$ is interpreted as the voting power of player i in a simple game v. The following four axioms – plausible requirements that a general power index φ may be expected to obey – are quite routinely assumed in analyzing and designing power indices, either in their entirety or in part. As in Einy (1987), who was the first to look at the conjunction of these four axioms, we will use the term *semivalue*[11] in reference to any power index φ that satisfies all the axioms.[12]

[10] We shall henceforth omit braces when indicating one-player sets.

[11] The term "semivalue" was originally coined in Dubey et al (1981) in the context of value maps on $\mathcal{G}$ (see Remark 2).

[12] Variants of semivalue axioms have been present in the original axiomatizations of the Shapley-Shubik and the Banzhaf power indices (see Dubey (1975) and Dubey and Shapley (1979)).

Axiom I: Transfer. For any $v, w \in \mathcal{SG}$, $\varphi(\max\{v,w\}) + \varphi(\min\{v,w\}) = \varphi(v) + \varphi(w)$.

As was shown in Dubey et al. (2005, p. 24), **Tran** can be restated in an equivalent but conceptually clearer form, amounting to a requirement that the change in power depends only on the change in the voting game.[13]

Axiom II: Symmetry. For any $v \in \mathcal{SG}$, $i \in U$, and a permutation π of U, $\varphi(\pi v)(i) = \varphi(v)(\pi(i))$, where $\pi v \in \mathcal{G}$ is given by $(\pi v)(S) = v(\pi(S))$ for all $S \in 2^U$.

According to **Symmetry**, if players are relabeled in a game, their power indices will be relabeled accordingly. Thus, irrelevant characteristics of the players, outside of their role in the game v, have no influence on the power index.

Axiom III: Positivity. For any $v \in \mathcal{SG}$ and $i \in U$, $\varphi(v)(i) \geq 0$.

Positivity is natural, as every $v \in \mathcal{SG}$ is monotonic by assumption, and hence no player that joins a coalition can affect its winning status negatively.

Axiom IV: Dummy. If $v \in \mathcal{SG}$ and i is a dummy player in v, i.e. $v(S \cup i) = v(S) + v(i)$ for every $S \subset U \setminus i$, then $\varphi(v)(i) = v(i)$.

A dummy player in a simple game can be either a dictator (if $v(i) = 1$), in which case $\{i\}$ is the minimal carrier of v, or a null player (if $v(i) = 0$), that does not belong to the minimal carrier of v. **Dummy** can be viewed as a normalization requirement, assigning power 1 to a dictator and power 0 to a null player.

2.3 Characterization of Semivalues

Dubey et al. (1981) defined a family of semivalues[14] $\left(\phi_\xi\right)_\xi$, parameterized by $\xi \in M([0,1]) \equiv$the set of probability measures on $[0,1]$, as follows: given $\xi \in M([0,1])$,

[13] The possibility of such a restatement has been mentioned in Dubey and Shapley (1979, p. 106). A special version of the restatement also appeared in Laruelle and Valenciano (2001).

[14] Dubey at al. (1981) considered semivalues on $\mathcal{G}$ and not on $\mathcal{SG}$ (for further discussion, see Remark 2). The family of power indices with the forthcoming description is obtained by restricting those semivalues to games in $\mathcal{SG}$.

for every $v \in \mathcal{SG}$ with some finite carrier N,

$$\phi_\xi(v)(i) = \sum_{S \subset N \setminus i} p_{|S|}^{|N|}(\xi)\left[v(S \cup i) - v(S)\right] \tag{1}$$

if $i \in N$, where

$$p_s^n(\xi) = \int_0^1 x^s (1-x)^{n-s-1} d\xi(x); \tag{2}$$

and $\phi_\xi(v)(i) = 0$ if $i \in U \setminus N$. The definition is independent of the choice of a carrier N.

Einy (1987) showed that the set of semivalues on $\mathcal{SG}$ *coincides* with the family $\left(\phi_\xi\right)_{\xi \in M([0,1])}$. Formally, a power index φ is a semivalue if and only if $\varphi = \phi_\xi$ for some $\xi \in M([0,1])$, and ξ is uniquely determined by φ. Relying on this equivalence, the term *semivalue* will henceforth be used in reference to some member of the family $\left(\phi_\xi\right)_{\xi \in M([0,1])}$.

Each semivalue ϕ_ξ has a simple probabilistic interpretation. Assume that player i believes that players other than himself have the same probability x of voting "yes" (thereby joining the coalition of yes-voters), and that they do so independently of each other; however, i may be uncertain about the parameter x, with his prior belief being the distribution ξ over x. Then $\phi_\xi(v)(i)$ represents i's a priori likelihood to switch a random coalition of yes-voters from losing to winning by joining it.

If the parameter x is known, one may refer to the corresponding semivalue, for which ξ is the Dirac measure concentrated on x, as x*-value,* which will be denoted ϕ_x for simplicity. A general ϕ_ξ is then a probabilistic mixture of x-values: the definition of ϕ_ξ implies that, for every $v \in \mathcal{SG}$ and $i \in U$,

$$\phi_\xi(v)(i) = \int_0^1 \phi_x(v)(i) d\xi(x). \tag{3}$$

The family $\left(\phi_\xi\right)_{\xi \in M([0,1])}$ includes the two best-known and widely used semivalues: the Banzhaf power index $\phi_{\frac{1}{2}}$, corresponding to ξ that is the Dirac measure concentrated on $\frac{1}{2}$, and the Shapley-Shubik power index, corresponding to the uniform distribution on $[0,1]$. The Coleman-Shapley index, introduced in Casajus and Huettner (2018), is precisely ϕ_ξ for ξ that corresponds of the uniform distribution on $\left[0, \frac{1}{2}\right]$. Its probabilistic interpretation will be discussed in the next section, in a unifying set-up that will single out a subfamily of semivalues in $\left(\phi_\xi\right)_{\xi \in M([0,1])}$.

3 Generalized Coleman-Shapley Indices and Total-Power Monotonicity

The definition of the Coleman-Shapley index in Casajus and Huettner (2018) allows to conjure up a more general framework, in which the Shapley-Shubik and the Coleman-Shapley indices are included as particular cases. We will define generalized Coleman-Shapley indices as a one-pramateric family of semivalues, and will then show how these indices arise in two models of random voting.

3.1 Generalized Coleman-Shapley Indices as Semivalues

For any $0 \leq q \leq 1$, consider the probability measure $\xi_q \in M([0,1])$ that is concentrated on the interval $[0, q]$ and, when $q > 0$, corresponds to the uniform distribution on $[0, q]$, i.e.,

$$d\xi_q(x) = \frac{1}{q} I_{x \leq q} dx, \tag{4}$$

where I_A denotes the indicator function of the set A. Denote $\varphi_q = \phi_{\xi_q}$, and call it *q-Coleman-Shapley index*, or *q-CS index* for short.

3.2 Random-arrival interpretation of q-CS Indices

When $q > 0$, the definition of the q-CS index by means of (1), (2) and (4) lends itself to the following probabilistic interpretation, which is a version of the "random arrival times" view that has usually been reserved for the Shapley value and the weighted Shapley value (starting with Owen (1968)). Let $v \in \mathcal{SG}$ be a game with some finite carrier N, and let $\{X_i\}_{i \in N}$ be i.i.d. random variables with the uniform distribution on $[0, 1]$. Think of X_i as measuring the *dissatisfaction* of player i with a certain proposal that stands for vote; the given parameter q represents the cut-off value of dissatisfaction above which a player will never vote in favor of a proposal. Players whose dissatisfaction falls below or is equal to q will, on the other hand, ultimately vote "yes", but their turn to join the support of the proposal depends on their measure of dissatisfaction: the higher is X_i, the later will i join the other yes-voters. It stands to reason that, in such a scenario, the influence of player i over the vote should be quantified as the probability (conditional on i being a yes-voter, having $X_i \leq q$) that the coalition of the proposal supporters switches from losing to winning precisely when i's turn arrives and he declares his support for the proposal.

The measure of voting power given by $\varphi_q(v)(i) = \phi_{\xi_q}(v)(i)$ does exactly that. Formally, (1), (2) and (4) mean that

$$\varphi_q(v)(i) = \sum_{S \subset N \setminus i} \left(\int_0^q x^{|S|} (1-x)^{|N|-|S|-1} \frac{1}{q} dx \right) [v(S \cup i) - v(S)],$$

for every $i \in N$ (and $\varphi_q(v)(i) = 0$ for every $i \in U \setminus N$), which can be readily seen to be a restatement in terms of integrals of the equality

$$\varphi_q(v)(i) = E\left[v(\{j \in N \mid X_j \leq X_i\}) - v(\{j \in N \mid X_j < X_i\}) \;\mid\; X_i \leq q\right], \tag{5}$$

where E stands for the expectation operator. The last equality is itself equivalent to

$$\varphi_q(v)(i) = \Pr\left[v(\{j \in N \mid X_j < X_i\}) = 0 \text{ and } v(\{j \in N \mid X_j \leq X_i\}) = 1 \mid X_i \leq q\right]. \tag{6}$$

3.3 Random-order interpretation of q-CS Indices

The following alternative description of a q-CS index can be derived from (5). Given $v \in \mathcal{SG}$ with a finite carrier N, consider a random coalition $\overline{S}_N \subset N$ defined by the property that, for each $i \in N$, $\Pr\left(i \in \overline{S}_N\right) = q$, and the events $\left\{i \in \overline{S}_N\right\}_{i \in N}$ are independent. We can think of $\overline{S}_N$ as the coalition of players who are interested in, or capable of, voting for a specific proposal.[15] Call the players in $\overline{S}_N$ *active*. Additionally, let $\mathcal{R}_N$ be a random linear order of players in N, chosen w.r.t. to the uniform distribution over all such orders, and assume that the choice of order is made independently of the realization of $\overline{S}_N$. $\mathcal{R}_N$ can be thought of as the ranking of players w.r.t. their eagerness to vote in favor of the proposal; note that $\mathcal{R}_N$ ranks all players, including those who might not be active. For any such $\mathcal{R}_N$ and $i \in N$, denote by $S_i(\mathcal{R}_N)$ the (random) coalition of players in N that precede i in $\mathcal{R}_N$ (according to our interpretation, $S_i(\mathcal{R}_N)$ consists of players who like the proposal more than i). Then (5) is equivalent to

$$\varphi_q(v)(i) = E\left[v((S_i(\mathcal{R}_N) \cup i) \cap \overline{S}_N) - v(S_i(\mathcal{R}_N) \cap \overline{S}_N) \;\mid\; i \in \overline{S}_N\right], \tag{7}$$

[15] As mentioned in the Introduction, an alternative, equivalent interpretation views $\overline{S}_N$ as the coalition of all yes-voters.

or

$$\varphi_q(v)(i) = \Pr\left[v(S_i(\mathcal{R}_N) \cap \overline{S}_N) = 0 \text{ and } v((S_i(\mathcal{R}_N) \cup i) \cap \overline{S}_N) = 1 \mid \ i \in \overline{S}_N\right], \quad (8)$$

for every $i \in N$.

Just as in the random-arrival approach, here $\varphi_q(v)(i)$ is expressed as the probability that i is switches from losing to winning the coalition of active voters who are ranked below i (i.e., are stronger than i) in their support, conditional on that i is active. In order to see how (7) is obtained from (5) of the random-arrival set-up, take $\mathcal{R}_N$ be the order induced by the relative positions of the players in $\{X_i\}_{i\in N}$, and let $\overline{S}_N = \{i \in N \mid X_i \leq q\}$; notice that even though such $\mathcal{R}_N$ is *not* independent of $\overline{S}_N$, the random coalition $S_i(\mathcal{R}_N) \cap \overline{S}_N$ is distributed *as if* $\mathcal{R}_N$ is independent of $\overline{S}_N$ when there is a conditioning on $i \in \overline{S}_N$.[16]

Note that 1-CS index, φ_1, is the Shapley-Shubik power index, as (6) or (8) boil down to its usual definition as the (unconditional) probability of being pivotal in a random order. Also, when $q = \frac{1}{2}$, (8) is, in effect, the definition of the Coleman-Shapley index in Casajus and Huettner (2018), and hence $\varphi_{\frac{1}{2}}$ is precisely that index.

3.4 The Total Power in q-CS Indices

The total power of players under a given q-CS index can be computed directly, but we will find it as an upshot of a more general exercise. It turns out, as has been already observed by Casajus and Huettner (2018) in the case of $\varphi_{\frac{1}{2}}$, that for *any* $0 < q \leq 1$ the q-CS index of $v \in \mathcal{SG}$ can be expressed as the Shapley (1953) value of an appropriately modified game. Indeed, let $v_q \in \mathcal{G}$ be the game in which the payoff to any coalition S is the $\frac{1}{q}$-scaled probability that the coalition of active players in S is winning in the game v, i.e., $v_q(S) = \frac{1}{q}E\left[v(S \cap \overline{S}_N)\right]$. Also recall that, for any game $w \in \mathcal{G}$ with a finite carrier N, its Shapley value $Sh(w)$ is defined as $Sh(w)(i) = E\left[w(S_i(\mathcal{R}_N) \cup i) - w(S_i(\mathcal{R}_N))\right]$ for every $i \in N$ (and $Sh(w)(i) = 0$ for every $i \in U \setminus N$).

[16] In (7), $\mathcal{R}_N$ can be replaced by $\mathcal{R}_{\overline{S}_N}$ (a random, uniformly distributed order of players in $\overline{S}_N$), i.e., it suffices to rank only the active players. Such an equation would have been the reduced form of both (5) and (7), consistent with our description of the q-CS index in the Introduction. The current (7) is preferable, however, as it is used in the proof of our forthcoming Proposition 1.

Proposition 1. For any q, v as above and $i \in U$, $\varphi_q(v)(i) = Sh(v_q)(i)$.

Proof. For any $i \in N$, by using the independence of $S_i(\mathcal{R}_N)$ and $\overline{S}_N$, (7) can be transformed into

$$\begin{aligned}
\varphi_q(v)(i) &= E_{\mathcal{R}_N}\left(E_{\overline{S}_N}\left[v((S_i(\mathcal{R}_N) \cup i) \cap \overline{S}_N) - v(S_i(\mathcal{R}_N) \cap \overline{S}_N) \;\mid\; i \in \overline{S}_N\right]\right) \\
&= E_{\mathcal{R}_N}\left(\frac{1}{q} E_{\overline{S}_N}\left[\left(v((S_i(\mathcal{R}_N) \cup i) \cap \overline{S}_N) - v(S_i(\mathcal{R}_N) \cap \overline{S}_N)\ \right) \cdot I_{i \in \overline{S}_N}\right]\right) \\
&= E_{\mathcal{R}_N}\left(\frac{1}{q} E_{\overline{S}_N}\left[v((S_i(\mathcal{R}_N) \cup i) \cap \overline{S}_N) - v(S_i(\mathcal{R}_N) \cap \overline{S}_N)\ \right]\right) \\
&= E_{\mathcal{R}_N}(v_q(S_i(\mathcal{R}_N) \cup i) - v_q(S_i(\mathcal{R}_N)) = Sh(v_q)(i).
\end{aligned}$$

Finally, when $i \in U \setminus N$, $\varphi_q(v)(i) = 0$ and $Sh(v_q)(i) = 0$ by definition. ■

Proposition 1 and the efficiency of the Shapley value imply that, for $0 < q \leq 1$ and $v \in \mathcal{SG}$ with a finite carrier N,

$$\varphi_q(v)(U) = v_q(N) = \frac{1}{q} E\left[v(\overline{S}_N)\right], \tag{9}$$

which is equivalent to

$$\varphi_q(v)(U) = \frac{1}{q} \Pr\left[v(\overline{S}_N) = 1\right]. \tag{10}$$

That is, the total power in the game v, as measured by φ_q, is a constant multiple $(\frac{1}{q})$ of the probability that the coalition of all active players is winning.

When $q = 1$, (9) is the usual efficiency property of the Shapley-Shubik index. When $q = \frac{1}{2}$, (10) is precisely the 2CPCA-efficiency of Casajus and Huettner (2018), whereby the total power in v equals to twice the Coleman (1971) *power of a collectivity to act* (≡the proportion of the winning coalitions among all coalitions in a carrier N of v). Casajus and Huettner (2018) used the 2CPCA-efficiency to characterize the Coleman-Shapley index $\varphi_{\frac{1}{2}}$ using 2CPCA-efficiency as a replacement of efficiency in the set of axioms of Dubey (1975) (originally devised for the Shapley-Shubik index). Similar axiomatizations can be obtained for any φ_q, with (10) as a substitute for 2CPCA-efficiency, where the right-hand side is viewed as an alternative measure of the power of a collectivity to act.

For $q = 0$, $\varphi_0(v)(i) = v(i)$ for every $i \in U$; hence,

$$\varphi_0(v)(U) = \varphi_0(v)(N) = \sum_{i \in N} v(i), \tag{11}$$

which can be seen as a limit version of (10), obtained by letting $q > 0$ tend to 0.

3.5 The Axiom of Total-Power Monotonicity

When the set of winning coalitions in the game expands, there is no guarantee that the individual power of every (or even most) players will not be affected negatively. Indeed, think of the change in the Shapley-Shubik power index φ_1 when a unanimity game $v = u_T$ (where $u_T(S) = 1$ if and only if $T \subset S$) sees its carrier T shrink to a strict subset, $T' \subsetneq T$, and the game becomes $u_{T'}$. The power of every $i \in T \backslash T'$ then falls from $\varphi_1(u_T) = \frac{1}{|T|}$ to $\varphi_1(u_{T'}) = 0$, which is to be expected as the players in $T \backslash T'$ become null in $u_{T'}$, despite there being more winning coalitions. However, the *total power* of players, $\varphi_1(v)(U) = 1$, remains constant regardless of changes in v.

When a general q-CS index φ_q is concerned, the total power $\varphi_q(v)(U)$ may depend on v, but if the set of winning coalitions in v expands, i.e., if $v \in \mathcal{SG}$ is replaced by $w \in \mathcal{SG}$ that satisfies $v \leq w$, the total power cannot go down:

$$\varphi_q(v)(U) \leq \varphi_q(w)(U). \tag{12}$$

This fact is immediate from (9) when $q > 0$, and from (11) when $q = 0$.

The property embodied in (12) seems compelling, and, at the same time, sufficiently selective – it is not possessed by all semivalues. The Banzhaf power index, for instance, attains the maximal total power on a given carrier of odd size at the simple majority game (see Theorem 2 in Dubey and Shapley (1979)). We shall state the monotonicity requirement in (12) as an axiom on the behavior of a general power index φ, and study its implication in the next section.

Axiom V: Total-Power Monotonicity (TP-Mon). If $v, w \in \mathcal{SG}$ and $v \leq w$, then $\varphi(v)(U) \leq \varphi(w)(U)$.

4 Results

4.1 Total-Power Monotonicity of a Semivalue

In this section we will characterize the effect of imposing the axiom of **TP-Mon** on the family of semivalues. On a technical level, **TP-Mon** reduces to concavity of the c.d.f. of the representing distribution.

Theorem 1. If a semivalue $\varphi = \phi_\xi$ satisfies **TP-Mon**, then the c.d.f. F_ξ of the distribution corresponding to ξ is concave on $[0,1]$.

Proof of Theorem 1. Let $\xi \in M([0,1])$ be such that ϕ_ξ satisfies **TP-Mon**. We start with the following claim.

Claim. Let $0 < a < b < 1$ and $0 < c < d \leq 1$ be such that $c - a = d - b > 0$. Then

$$\xi((a,b]) \geq \xi((c,d]). \tag{13}$$

Proof of the claim. We shall first establish (13) under the assumption that

$$\xi(\{a,b,c,d\}) = 0. \tag{14}$$

Fix $\delta > 0$, and let $0 < \varepsilon < \frac{b-a}{2}$ be such that $\xi([t^-, t^+]) < \delta$ for $t \in \{a,b,c,d\}$, where $t^+ = \min(t+\varepsilon, 1)$, $t^- = \max(t-\varepsilon, 0)$. Also, for any $n \in \mathbb{N}$ such that $\frac{1}{n} < \frac{\varepsilon}{2}$ and any $x \in [0,1]$, let Y_x^n be a random variable with the binomial distribution $B(n,x)$. Then

$$\begin{aligned}
\xi((a,b]) &\geq \xi\left((a^-,b^+)\right) - 2\delta = \int_{(a^-,b^+)} d\xi(x) - 2\delta \geq \int_{(a^-,b^+)} \left(\sum_{k=[an]}^{[bn]} \Pr(Y_x^n = k)\right) d\xi(x) - 2\delta \\
&\quad \text{(where } [t] \text{ stands for the integer part of } t\text{)} \\
&= \sum_{k=[an]}^{[bn]} \left(\int_0^1 \Pr(Y_x^n = k) d\xi(x)\right) - \int_{(a^-,b^+)^c} \left(\sum_{k=[an]}^{[bn]} \Pr(Y_x^n = k)\right) d\xi(x) - 2\delta \\
&\geq \sum_{k=[an]}^{[bn]} \left(\int_0^1 \Pr(Y_x^n = k) d\xi(x)\right) - \int_{(a^-,b^+)^c} \Pr(\left|\frac{Y_x^n}{n} - x\right| > \frac{\varepsilon}{2}) d\xi(x) - 2\delta. \qquad (15)
\end{aligned}$$

By the Chebishev's inequality,

$$\Pr(\left|\frac{Y_x^n}{n} - x\right| > \frac{\varepsilon}{2}) \leq \frac{1}{n\varepsilon^2}, \tag{16}$$

and hence the expression in (15) is bound from below by

$$\sum_{k=[an]}^{[bn]} \left(\int_0^1 \Pr(Y_x^n = k) d\xi(x)\right) - \frac{1}{n\varepsilon^2} - 2\delta = \sum_{k=[an]}^{[bn]} \left(\int_0^1 \binom{n}{k} x^k (1-x)^{n-k} d\xi(x)\right) - \frac{1}{n\varepsilon^2} - 2\delta. \tag{17}$$

For $k = 0, \ldots, n$, let $w_{n+1,k+1} \in \mathcal{SG}$ be the $k+1$-majority game with carrier $N = \{1, \ldots, n+1\}$, i.e., $w_{n+1,k+1}(S) = 1$ if and only if $|S \cap N| \geq k+1$. It follows from the definition of ϕ_ξ in (1) and (2) that

$$\phi_\xi\left(w_{n+1,k+1}\right)(U)=(n+1)\int_0^1\binom{n}{k}x^k(1-x)^{n-k}d\xi\left(x\right), \tag{18}$$

and so the right-hand side of (17) is equal to

$$\frac{1}{n+1}\sum_{k=[an]}^{[bn]}\phi_\xi\left(w_{n+1,k+1}\right)(U)-\frac{1}{n\varepsilon^2}-2\delta.$$

We have thereby established that

$$\xi\left((a,b]\right)\geq\frac{1}{n+1}\sum_{k=[an]}^{[bn]}\phi_\xi\left(w_{n+1,k+1}\right)(U)-\frac{1}{n\varepsilon^2}-2\delta. \tag{19}$$

Since: (i) ϕ_ξ satisfies **TP-Mon;** (ii) $w_{n+1,k+1}\geq w_{n+1,k'+1}$ whenever $k\leq k'$; and (iii) $c-a=d-b>0$, we obtain

$$\sum_{k=[an]}^{[bn]}\phi_\xi\left(w_{n+1,k+1}\right)(U)\geq\sum_{k=[cn]+1}^{[dn]}\phi_\xi\left(w_{n+1,k+1}\right)(U).$$

From this, (18) and (19),

$$\begin{aligned}\xi\left((a,b]\right) &\geq \frac{1}{n+1}\sum_{k=[cn]+1}^{[dn]}\phi_\xi\left(w_{n+1,k+1}\right)(U)-\frac{1}{n\varepsilon^2}-2\delta\\ &= \sum_{k=[cn]+1}^{[dn]}\int_0^1\binom{n}{k}x^k(1-x)^{n-k}d\xi\left(x\right)-\frac{1}{n\varepsilon^2}-2\delta\\ &\geq \sum_{k=[cn]+1}^{[dn]}\int_{[c^+,d^-]}\binom{n}{k}x^k(1-x)^{n-k}d\xi\left(x\right)-\frac{1}{n\varepsilon^2}-2\delta\\ &= \sum_{k=[cn]+1}^{[dn]}\int_{[c^+,d^-]}\Pr(Y_x^n=k)d\xi\left(x\right)-\frac{1}{n\varepsilon^2}-2\delta\\ &= \sum_{k=0}^{n}\int_{[c^+,d^-]}\Pr(Y_x^n=k)d\xi\left(x\right)-\sum_{k<[cn]+1\text{ or }k>[dn]}\int_{[c^+,d^-]}\Pr(Y_x^n=k)d\xi\left(x\right)-\frac{1}{n\varepsilon^2}-2\delta.\end{aligned}$$

As

$$\sum_{k=0}^{n}\int_{[c^+,d^-]}\Pr(Y_x^n=k)d\xi\left(x\right)=\int_{[c^+,d^-]}\left(\sum_{k=0}^{n}\Pr(Y_x^n=k)\right)d\xi\left(x\right)=\xi\left(\left[c^+,d^-\right]\right)\geq\xi((c,d])-2\delta,$$

we obtain

$$
\begin{aligned}
\xi\left((a,b]\right) &\geq \xi\left((c,d]\right) - \sum_{k<[cn]+1 \text{ or } k>[dn]} \int_{[c^+,d^-]} \Pr(Y_x^n = k)d\xi\left(x\right) - \frac{1}{n\varepsilon^2} - 4\delta \\
&= \xi\left((c,d]\right) - \int_{[c^+,d^-]} \left(\sum_{k<[cn]+1 \text{ or } k>[dn]} \Pr(Y_x^n = k) \right) d\xi\left(x\right) - \frac{1}{n\varepsilon^2} - 4\delta \\
&\geq \xi\left((c,d]\right) - \int_{[c^+,d^-]} \Pr(\left| \frac{Y_x^n}{n} - x \right| > \frac{\varepsilon}{2}) d\xi\left(x\right) - \frac{1}{n\varepsilon^2} - 4\delta.
\end{aligned}
$$

By using the Chebishev's inequality (16) again, the last expression is bound from below by $\xi\left((c,d]\right) - \frac{2}{n\varepsilon^2} - 4\delta$. We have thus shown that

$$
\xi\left((a,b]\right) \geq \xi\left((c,d]\right) - \frac{2}{n\varepsilon^2} - 4\delta.
$$

By letting $n \to \infty$, this turns into $\xi\left((a,b]\right) \geq \xi\left((c,d]\right) - 4\delta$, and since the fixed $\delta > 0$ was arbitrary, the desired inequality (13) is established under the assumption (14).

We will now show that assumption (14) can be dispensed with. First, notice that when $d = 1$, all the arguments above work without the need to pass from d to d^-. Hence, it is not necessary to assume that $\xi\left(\{d\}\right) = 0$ when $d = 1$ (and, in addition, $\xi\left(\{a,b,c\}\right) = 0$), in order to obtain (13).

Next, for any $0 < x < y \leq 1$, there exists a sequence $\{(a_n, b_n, c_n, d_n)\}_{n=1}^{\infty}$ such that $0 < a_n < x < b_n < 1$, $0 < c_n < y \leq d_n \leq 1$, $c_n - a_n = d_n - b_n > 0$, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = x$, $\lim_{n\to\infty} c_n = \lim_{n\to\infty} d_n = y$, $\xi\left(\{a_n, b_n, c_n\}\right) = 0$, and $\xi\left(\{d_n\}\right) = 0$ (unless $d_n = 1$). As (13) holds for such a_n, b_n, c_n, d_n by what has been shown, we have $\xi\left((a_n, b_n]\right) \geq \xi\left((c_n, d_n]\right)$, which translates into $\xi\left(\{x\}\right) \geq \xi\left(\{y\}\right)$ by letting $n \to \infty$. Since the latter inequality holds for all $0 < x < y \leq 1$, ξ cannot have atoms in $(0,1]$. It follows that (14) always holds, and hence (13) holds for *any* a, b, c, d as in the premise of the claim. $\square$

Proof of Theorem 1 (continued). As has been argued in the last part of the preceding proof, ξ has no atoms in $(0,1]$. It follows that the c.d.f. F_ξ that corresponds to ξ, given by $F_\xi\left(x\right) = \xi\left([0,x]\right)$ for any $x \in [0,1]$, is continuous on $(0,1]$. Because a c.d.f. is right-continuous, F_ξ is continuous on the entire closed interval $[0,1]$. By (13),

$$
F_\xi\left(b\right) - F_\xi\left(a\right) \geq F_\xi\left(d\right) - F_\xi\left(c\right) \tag{20}
$$

for any $0 < a < b < 1$ and $0 < c < d \leq 1$ such that $c - a = d - b > 0$. The continuity of F_ξ on $[0,1]$ implies that, furthermore, (20) holds even if $a = 0$.

Now, given $0 \leq x < y \leq 1$, consider any rational number $0 < r < 1$, which has the form $r = \frac{m}{n}$ for some $n > m \in \mathbb{N}$. Successive applications of (20) yield

$$\begin{aligned}
F_\xi(\frac{m}{n}x + \frac{n-m}{n}y) - F_\xi(x) &= \sum_{k=m+1}^{n} \left(F_\xi(\frac{k-1}{n}x + \frac{n-k+1}{n}y) - F_\xi(\frac{k}{n}x + \frac{n-k}{n}y) \right) \\
&\geq (n-m) \left(F_\xi(\frac{m}{n}x + \frac{n-m}{n}y) - F_\xi(\frac{m+1}{n}x + \frac{n-m-1}{n}y) \right) \\
&\geq (n-m) \left(F_\xi(\frac{m-1}{n}x + \frac{n-m+1}{n}y) - F_\xi(\frac{m}{n}x + \frac{n-m}{n}y) \right) \\
&\geq \frac{n-m}{m} \sum_{k=1}^{m} \left(F_\xi(\frac{k-1}{n}x + \frac{n-k+1}{n}y) - F_\xi(\frac{k}{n}x + \frac{n-k}{n}y) \right) \\
&= \frac{n-m}{m} \left(F_\xi(y) - F_\xi(\frac{m}{n}x + \frac{n-m}{n}y) \right),
\end{aligned}$$

and hence

$$F_\xi(rx + (1-r)y) \geq rF_\xi\left(x\right) + (1-r)F_\xi\left(y\right) \tag{21}$$

holds for $r = \frac{m}{n}$. Since F_ξ is continuous on $[0,1]$, the inequality (21) holds for *any* $0 < r < 1$, which shows that F_ξ is indeed concave on $[0,1]$. ■

4.2 Mixing Generalized CS Indices: The Only Way to Achieve Total-Power Monotonicity

Theorem 1 contains an implication that is somewhat hidden from sight. The next theorem uncovers it, and points to a tight link between the **TP-Mon** property and the family of generalized CS indices.

Theorem 2. A semivalue φ satisfies **TP-Mon** if and only if it is a *mixture* of generalized CS indices, i.e., there exist a probability measure $\mu \in M\left([0,1]\right)$, uniquely determined by φ, such that for every $v \in \mathcal{SG}$ and $i \in U$,

$$\varphi\left(v\right)\left(i\right) = \int_0^1 \varphi_q\left(v\right)\left(i\right) d\mu\left(q\right). \tag{22}$$

Proof. The fact that any q-CS index φ_q satisfies **TP-Mon** has already been noted (see (12)), and it is obvious that any mixture φ of generalized CS indices, given by (22), inherits this property. This establishes the "if" direction. To prove the "only if" direction, fix a semivalue φ that satisfies **TP-Mon**. By Theorem 1,

$\varphi = \phi_\xi$ for some $\xi \in M([0,1])$ whose c.d.f. F_ξ is continuous[17] and concave on $[0,1]$. The last two properties of F_ξ and its monotonicity as a c.d.f. imply that there exists a nonincreasing function $f_\xi \geq 0$ on $(0,1]$ such that[18] $F_\xi(t) = F_\xi(0) + \int_0^t f_\xi(x)dx$ for every $t \in (0,1]$, where $F_\xi(0) = \xi(\{0\})$.

Next, let $g \geq 0$ be any continuous function on $[0,1]$, and assume that $F_\xi(0) < 1$. Notice that

$$\begin{aligned}\int_0^1 g(x)d\xi(x) &= g(0)F_\xi(0) + \int_0^1 g(x)f_\xi(x)dx = g(0)F_\xi(0) + \int_0^1 g(x)\left(\int_0^{f_\xi(x)} ds\right)dx \\ &= g(0)F_\xi(0) + \int_0^\infty \left(\int_0^1 g(x)I_{s\leq f_\xi(x)}dx\right)ds. \qquad (23)\end{aligned}$$

Denote $a_\xi = \lim_{x\to 0+} f_\xi(x) > 0$,[19] and let h_ξ be a nonincreasing function on $[0,a_\xi)$ defined by $h_\xi(s) = \sup\{x \in [0,1] \mid s \leq f_\xi(x)\}$ for every $s \in [0,a_\xi)$; notice that $h_\xi > 0$. The expression in (23) is then equal to

$$\begin{aligned}& g(0)F_\xi(0) + \int_{[0,a_\xi)} \left(\int_0^1 g(x)I_{x\leq h_\xi(s)}dx\right)ds \\ = \;& g(0)F_\xi(0) + \int_{[0,a_\xi)} h_\xi(s)\left(\int_0^1 g(x)\frac{1}{h_\xi(s)}I_{x\leq h_\xi(s)}dx\right)ds.\end{aligned}$$

We have thereby shown that

$$\int_0^1 g(x)d\xi(x) = g(0)F_\xi(0) + \int_{[0,a_\xi)} h_\xi(s)\left(\int_0^1 g(x)\frac{1}{h_\xi(s)}I_{x\leq h_\xi(s)}dx\right)ds, \qquad (24)$$

where $\int_{[0,a_\xi)} h_\xi(s)\,ds = 1 - F_\xi(0)$. Now recall the definition of the probability measure $\xi_q \in M([0,1])$ as the one that is concentrated on $[0,q]$, with $d\xi_q(x) = \frac{1}{q}I_{x\leq q}dx$ when $q > 0$. The equality (24) then becomes

$$\int_0^1 g(x)d\xi(x) = \int_0^1 \left(\int_0^1 g(x)d\xi_q(x)\right)d\nu_\xi(q), \qquad (25)$$

[17]Continuity of F_ξ was established in the proof of Theorem 1, but we did not need to claim both continuity and concavity in the statement of that theorem because concavity of F_ξ on $[0,1]$ implies its continuity on that interval. Indeed, the only discontinuity of a concave function on $[0,1]$ might occur at the end-points, but that is impossible because F_ξ is right-continuous and nondecreasing as a c.d.f.

[18]One may take f_ξ to be the left-hand derivative of F_ξ on $(0,1]$. If $\lim_{x\to 0+} f_\xi(x) = \infty$, then all integrals in the proof that have the form $\int_0^t ...dx$ (for $0 < t \leq 1$), and in which the integrand involves $f_\xi(x)$, should be regarded as improper integrals.

[19]The limit a_ξ exists because f_ξ is nondecreasing, and its positivity follows from the assumption that $F_\xi(0) < 1$. It may, furthermore, be equal to ∞.

where $\nu_\xi \in M([0,1])$ is the probability measure determined by the following properties: $\nu_\xi(\{0\}) = F_\xi(0)$, and $\nu_\xi((0,x]) = \int_{[0,a_\xi)} I_{h_\xi(s)\leq x} h_\xi(s)\, ds$ for any $x \in (0,1]$.

The measure $\mu = \nu_\xi$ turns out to be the one that is required in (22). Indeed, given $v \in \mathcal{SG}$ and $i \in U$, by using (3) we obtain

$$\begin{aligned}
\varphi(v)(i) &= \phi_\xi(v)(i) = \int_0^1 \phi_x(v)(i)\, d\xi(x) \\
\text{(by (25))} &= \int_0^1 \left(\int_0^1 \phi_x(v)(i)\, d\xi_q(x) \right) d\nu_\xi(q) \\
&= \int_0^1 \phi_{\xi_q}(v)(i) d\nu_\xi(q) = \int_0^1 \varphi_q(v)(i) d\nu_\xi(q).
\end{aligned}$$

Lastly, if $F_\xi(0) = 1$ then ξ is supported on $\{0\}$, i.e., $\xi = \xi_0$, implying that $\varphi = \phi_{\xi_0} = \varphi_0$, and hence (22) holds trivially. Thus, the existence of μ that satisfies (22) has been established for any given $\varphi = \phi_\xi$.

In order to show that μ in (22) is determined uniquely by the given φ, note that for any $v \in \mathcal{SG}$ and $i \in U$,

$$\begin{aligned}
\varphi(v)(i) &= \int_0^1 \varphi_q(v)(i) d\mu(q) = \int_0^1 \left(\int_0^1 \phi_x(v)(i)\, d\xi_q(x) \right) d\mu(q) \\
&= \phi_0(v)(i)\mu(\{0\}) + \int_{(0,1]} \left(\int_{(0,1]} \phi_x(v)(i) \frac{1}{q} I_{x\leq q} dx \right) d\mu(q) \\
&= \phi_0(v)(i)\mu(\{0\}) + \int_{(0,1]} \phi_x(v)(i) \left(\int_{(0,1]} \frac{1}{q} I_{x\leq q} d\mu(q) \right) dx \\
&= \phi_0(v)(i)\mu(\{0\}) + \int_{(0,1]} \phi_x(v)(i) \left(\int_x^1 \frac{1}{q} d\mu(q) \right) dx.
\end{aligned}$$

Thus, $\varphi = \varphi_\xi$ for ξ that is given by $\xi(\{0\}) = \mu(\{0\})$ and the equation $d\xi(x) = \left(\int_x^1 \frac{1}{q} d\mu(q) \right) dx$ on $(0,1]$. Because ξ is uniquely determined by φ (see Section 2.3), the Radon-Nikodym derivative of ξ w.r.t. the Lebesgue measure, namely $\int_x^1 \frac{1}{q} d\mu(q)$, is uniquely determined for almost every $x \in (0,1]$, and $\mu(\{0\})$ is also uniquely determined.

Now define a σ-finite measure η on $(0,1]$ by $d\eta(q) = \frac{1}{q} d\mu(q)$. Thus $\eta([x,1]) = \int_x^1 \frac{1}{q} d\mu(q)$ for every $x \in (0,1]$, and so, as claimed above, $\eta([x,1])$ is uniquely determined for almost every $x \in (0,1]$. The latter fact implies that the entire η is uniquely determined. But, since μ (restricted to $(0,1]$) is determined by η via its Radon-Nikodym derivative $\frac{d\mu}{d\eta}(q) = \left(\frac{d\eta}{d\mu}(q) \right)^{-1} = q$, i.e., $d\mu(q) = q d\eta(q)$, μ is also determined uniquely by the given φ. ■

We conclude with two remarks.

Remark 1. It is easy to see, using Theorem 2 and its proof, that the assertion in Theorem 1 holds in both directions: a semivalue $\varphi = \phi_\xi$ satisfies **TP-Mon** *if and only if* the c.d.f. F_ξ of the distribution corresponding to ξ is concave on $[0, 1]$. Indeed, Theorem 1 provides the "only if" direction. As for the "if" direction, assume that the c.d.f. F_ξ of the distribution corresponding to ξ is concave on $[0, 1]$. The *proof* of the "only if" direction of Theorem 2 shows that, in such a case, $\varphi = \phi_\xi$ has the representation (22), and thus φ satisfies **TP-Mon** by the "if" assertion of Theorem 2.

Remark 2. Dubey et al. (1981) defined a semivalue on the space of all finite games, $\mathcal{G}$, as a linear projection[20] $\varphi : \mathcal{G} \to \mathcal{AG}$ that satisfies the **Symmetry** and **Positivity** axioms of Section 2.2 (in the context of general games in $\mathcal{G}$, **Symmetry** needs to be stated for any $v \in \mathcal{G}$, and **Positivity** for any monotonic $v \in \mathcal{G}$). Their characterization of semivalues on $\mathcal{G}$ as the family $(\phi_\xi)_{\xi \in M([0,1])}$ (defined by (1) and (2) for all $v \in \mathcal{G}$) is identical to the one surveyed in Section 2.3 in the context of simple games. Using this characterization, generalized Coleman-Shapley *values* for games in $\mathcal{G}$ can be defined in the same way as the corresponding indices on $\mathcal{SG}$ in Section 3.1, and, with **TP-Mon** stated for games in $\mathcal{G}$, all our results (Proposition 1, Theorems 1 and 2, and Remark 1) hold for semivalues on $\mathcal{G}$ instead of $\mathcal{SG}$, by identical arguments.

[20] I.e., φ acts as the identity map when restricted to $\mathcal{AG}$.

References

1. Banzhaf JF (1965) Weighted Voting Doesn't Work: A Mathematical Analysis. Rutgers Law Review 19:317-343

2. Banzhaf JF (1966) Multi-member Electoral Districts – Do They Violate the "One Man, One Vote" Principle. Yale Law Journal 75:1309-1338

3. Banzhaf JF (1968) One Man, 3.312 Votes: A Mathematical Analysis of the Electoral College. Vilanova Law Review 13:304-332

4. Brams SJ (2013) Game Theory and Politics. Dover Books on Mathematics, Dover Publications, Mineola, NY

5. Casajus A (2012) Amalgamating Players, Symmetry, and the Banzhaf Value. International Journal of Game Theory 41:497–515

6. Casajus A, Huettner F (2018) The Coleman-Shapley-Index: Being Decisive Within the Coalition of the Interested. ESMT working paper 18-03.

7. Coleman JS (1971) Control of Collectives and the Power of a Collectivity to Act. In: Lieberman, Bernhardt (eds) Social Choice. Gordon and Breach, New York. pp 192–225

8. Dubey P (1975) On the Uniqueness of the Shapley Value. International Journal of Game Theory 4:131-139

9. Dubey P, Neyman A, Weber RJ (1981) Value Theory Without Efficiency. Mathematics of Operations Research 6:122-128

10. Dubey P, Shapley LS (1979) Mathematical Properties of the Banzhaf Power Index. Mathematics of Operations Research 4:99-131

11. Einy E (1987) Semivalues of Simple Games. Mathematics of Operations Research 12:185-192

12. Felsenthal DS, Machover M (1998) The Measurement of Voting Power: Theory and Practice, Problems and Paradoxes. Edward Elgar Publishers, London, UK

13. Laruelle A, Valenciano F (2001) Shapley-Shubik and Banzhaf Indices Revisited. Mathematics of Operations Research 26:89-104

14. Lehrer E (1988) Axiomatization of the Banzhaf Value. International Journal of Game Theory 17:89-99

15. Nowak AS (1997) On an Axiomatization of the Banzhaf Value Without the Additivity Axiom. International Journal of Game Theory 26:137–141.

16. Owen G (1968) A Note on the Shapley Value. Management Science 14:731-732

17. Penrose LS (1946) The Elementary Statistics of Majority Voting. Journal of the Royal Statistical Society 109:53-57

18. Shapley LS (1953) A Value for n-person Games. In: Kuhn H. W. and Tucker A. W. (eds.), Contributions to the Theory of Games II (Annals of Mathematical Studies 28). Princeton University Press, Princeton

19. Shapley LS, Shubik M (1954) A Method for Evaluating the Distribution of Power in a Committee System. The American Political Science Review 48:787-792