REVERSE CONTESTS

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Abstract

We study a *reverse contest* with n agents, each of whom has both a linear reward function that increases in the agent's effort and an effort constraint. However, since the effort (output) of the players has a negative effect on society the designer imposes a punishment such that the agent with the highest effort who caused the greatest damage is punished. We analyze the equilibrium of this model with either symmetric or asymmetric agents. At all the equilibrium points, all the agents are active and all have positive expected payoffs. We characterize the properties of the agents' equilibrium strategies and compare them to the well-known equilibrium strategies of the all-pay auction in which the agent with the highest effort wins a prize.

1 Introduction

In winner-take-all contests the agent who exerts the highest effort (output) wins the contest (see, Tullock 1980, Rosen 1986, Hillman and Samet 1987 and Hillman and Riley 1989). In such environments, the agents have an incentive to win or to be first by exerting the highest effort or by producing the best output in order to win the prize. However, there are environments in which being the winner is not necessarily the goal of the agents and they might not want to be first nor the winner. An example of an environment in which it is not necessarily desirable to be the winner is common value auctions with incomplete information. In such

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auctions, the phenomenon of winner's curse is likely to occur according to which the winner will have the tendency to overpay (see, among others, Kagel and Levin 1986, Thaler 1988 and Bajari and Hortascu 2003). In this paper, we study a model in which the disadvantage of winning is even stronger than in common value auctions since we assume that only the first (best) agent is punished. To illustrate, consider n firms that produce a homogenous product where the production process yields some damage such as pollution. Then, the regulator imposes a punishment on the firm with the highest production since this obviously causes the greatest damage. In that case, the goal of each firm is to maximize its profit by maximizing its production but at the same time not producing more than all the other firms. Another illustration would be a group of cars speeding on a highway who wish to arrive at some final destination. The fastest car is the one most likely to be caught and fined, the goal of each driver would be to minimize his travel time but at the same time not to be fastest.

Formally, we study a model with n agents, each of whom has a linear reward function which is a combination of his production and cost functions which increases in the agent's effort. Each agent has an effort constraint which restricts his ability to exert an effort above his constraint. However, since the effort of the agents has a negative effect on society the designer imposes a punishment such that the agent with the highest effort gets a negative payoff according to his value for this punishment. The agents are not necessarily symmetric and they may have asymmetric reward functions, asymmetric values of the punishment as well as asymmetric effort constraints.

We first analyze the equilibrium of this model when the agents are symmetric, namely, they have the same reward function, the same value for the punishment, and the same effort constraint. In this case, independent of the number of agents, all of them use the same mixed strategy equilibrium where the minimal effort (the left point of the support of the agents' mixed strategy) of all the agents is positive while the maximal effort (the right point of the support of the agents' mixed strategy) is equal to their effort constraint. In equilibrium, the expected payoff of all the agents is the same and is positive.

We then study this model when the agents have asymmetric reward functions, where an agent will be called stronger than his opponent if his marginal reward function is larger than that of his opponent. In the two-agent case we show that both agents use a mixed strategy equilibrium and that both of them have a positive expected payoff where the stronger agent has a higher expected payoff than his opponent. When there are $n > 2$ asymmetric agents, the strongest agent as well as one of the other agents, not necessarily the second strongest agent, use a mixed strategies, while all the other agents use a pure strategy which is the effort that is equal to the minimal possible effort of the agents with the mixed strategy, namely, the smallest point of the support of their strategies. All the agents have positive expected payoffs and these payoffs are fixed and do not depend on the identity of the agents who use the mixed strategy. An interesting point to consider is that the agents' expected total effort when they have asymmetric reward functions is higher than when these agents are symmetric and have the same reward function as one of the asymmetric ones.

We then analyze the equilibrium in our model when the agents have asymmetric values for the punishment imposed on the agent with the highest effort. Similar to the case with asymmetric reward functions, in the two-agent model both agents use a mixed strategy equilibrium and in the n-agent model, two of the agents (one of them is the agent with the lowest value for the punishment and the other agent could be anyone else) use mixed strategies. All the other agents use a pure strategy which is an effort that is equal to the minimal possible effort of the two agents with the mixed strategies, namely the smallest point of the support of their strategies. What is interesting here is that regardless of the asymmetry of the agents, they have the same expected payoff. Similar to the case with asymmetric reward functions, when agents have asymmetric values for the punishments, their expected total effort is higher than when they are symmetric and they will have the same value for the punishment as one of the asymmetric agents.

Last, we study this model when agents have asymmetric effort constraints. In equilibrium, as in the previous cases, in the two-agent model both agents use mixed strategies and in the n-agent model, two agents, those with the highest effort constraints also use mixed strategies in which each of them chooses the effort that is equal to his effort constraint with a positive probability. Each of the other agents chooses a pure strategy that is equal to the minimum of his effort constraint and the smallest point in the support of the agents' mixed strategies. Similar to the case with asymmetric values for the punishments, regardless of the asymmetry of the agents, all the agents that choose efforts below their effort constraints either by a mixed or a pure strategy have the same expected payoff.

The equilibrium analysis of our model has features that are in common with the standard model of the all-pay auction under complete information in which each player submits a bid (effort) for the prize and the player who submits the highest bid receives the prize, but, independently of success, all players bear the cost of their bids. In the economic literature, all-pay auctions are usually studied either under complete information where each player's value for the prize is common knowledge (see, for example, Baye, Kovenock and de Vries 1993, 1996, Che and Gale 1998, Konrad 2006, Konrad and Kovenock 2009, Siegel 2009, Sela 2012, and Hart 2016), or under incomplete information where each player's value for the prize is private information to that player and only the distribution of the players' values is common knowledge (see, for example, Amman and Leininger 1996, Krishna and Morgan 1997, Moldovanu and Sela 2001, 2006, Gavious et al. 2003, Cohen et al. 2008, and Moldovanu et al. 2012). To elucidate this similarity between our model and the all-pay auction, in Appendix A we provide an analysis of the two-player all-pay auction under complete information and in our concluding remarks we compare the equilibrium in our model with the equilibrium in the all-pay auction under complete information.

2 The model

Consider n agents, each of whom has both a production function $\beta_i(x_i) = \beta_i x_i, \beta_i > 1$ (where x_i is contestant i's effort) and an effort cost function $c(x_i) = x_i, i = 1, 2, ..., n$. The designer imposes a punishment on the agent with the highest effort. Let $P_i, i = 1, 2, ..., n$, be agent i's value for this punishment. If we define agent i's reward function as $\alpha_i(x_i) = \beta_i(x_i) - c(x_i) = (\beta_i - 1)x_i = \alpha_i x_i, i = 1, 2, ..., n$, agent i's expected payoff is

$$u_i(x_i) = \begin{cases} \alpha_i x_i - P_i & \text{if } x_i > \max_j x_j \\ \alpha_i x_i - \frac{P_i}{m} & \text{if } x_i = \max_j x_j \\ \alpha_i x_i & \text{if } x_i < \max_j x_j \end{cases}$$

where $m \geq 2$ is the number of agents who exert the highest effort. In addition, agent i has an effort constraint of d_i such that $x_i \leq d_i, i = 1, 2, ..., n$. Each agent chooses his effort in order to maximize his expected payoff given the efforts of the other agents. Since no one wants to be first in this model we refer to it as the *reverse contest.*

3 The symmetric contest

We assume here that agents are symmetric, namely, they have the same reward function, $\alpha_i = \alpha, i = 1, ..., n$, the same value for the punishment in the case of winning, $P_i = P$, $i = 1, ..., n$, and the same effort constraint $d_i = d, i = 1, ..., n$.

3.1 The symmetric two-agent contest

Consider first that there are only two agents. Then the agents' mixed strategy equilibrium is given by

Proposition 1 *In the reverse contest with two symmetric agents, if $\alpha d > P$, there is a mixed strategy equilibrium in which agents 1 and 2 randomize on the interval $[d - \frac{P}{\alpha}, d]$ according to their effort cumulative distribution function $F(x)$ which is given by*

$$-PF(x) + \alpha x = -P + \alpha d$$

Thus, each agent's equilibrium effort is distributed according to the cumulative distribution function

$$F(x) = \frac{-P + \alpha(d - x)}{-P} \ , \ d - \frac{P}{\alpha} \leq x \leq d \tag{1}$$

Proof. See Appendix B. ■

The expected payoff of each agent is

$$\pi_i = -P + \alpha d \ , \ i = 1, 2$$

The agents' expected total effort is

$$TE = 2\int_{d-\frac{P}{\alpha}}^{d} xF'(x)dx = 2\int_{d-\frac{P}{\alpha}}^{d} \frac{\alpha}{P} x dx = 2d - \frac{P}{\alpha} \tag{2}$$

3.2 The n-agent symmetric contest

We consider now that there are $n > 2$ symmetric agents. Then, the agents' mixed strategy equilibrium is given by

Proposition 2 *In the reverse contest with* n *symmetric agents, if* $\alpha d > P$*, there is a mixed strategy equilibrium in which agents* $1,...,n$ *randomize on the interval* $[d - \frac{P}{\alpha}, d]$ *according to their effort cumulative distribution function* $F(x)$ *which is given by*

$$-PF(x)^{n-1} + \alpha x = -P + \alpha d$$

Thus, each agent's equilibrium effort is distributed according to the cumulative distribution function

$$F(x) = \sqrt[n-1]{\frac{-P+\alpha(d-x)}{-P}} \ , \ \ d - \frac{P}{\alpha} \leq x \leq d \tag{3}$$

Proof. See Appendix B. ■

The expected payoff of each agent is

$$\pi_i = -P + \alpha d \ , \ i = 1, ..., n$$

The agents' expected total effort is

$$\begin{aligned} TE &= n\int_{d-\frac{P}{\alpha}}^{d} xF'(x)dx = n\int_{d-\frac{P}{\alpha}}^{d} x\frac{1}{n-1}(\frac{-P+\alpha(d-x)}{-P})^{\frac{2-n}{n-1}}\frac{\alpha}{P}dx \\ &= nd - \frac{P}{\alpha}(n-1) \end{aligned} \tag{4}$$

4 Contests with asymmetric reward functions

We assume here that agents are asymmetric such that they have different reward functions. Without loss of generality, $\alpha_i \geq \alpha_{i+1}, i = 1, ..., n-1$. We also assume that agents have the same value for the punishment in the case of winning, $P_i = P$, and the same effort constraint $d_i = d$, $i = 1, ..., n$.

4.1 The two-agent contest

Consider first that there are only two agents. Then, the agents' mixed strategy equilibrium is given by

Proposition 3 *In the reverse contest with two agents with asymmetric reward functions, if* $\alpha_1 > \alpha_2$ *and* $\alpha_1 d > P$*, there is a mixed strategy equilibrium in which agents* 1 *and* 2 *randomize on the interval* $[d - \frac{P}{\alpha_1}, d]$

according to their effort cumulative distribution functions $F_1(x)$, $F_2(x)$ which are given by

$$\begin{aligned} -PF_2(x) + \alpha_1 x &= -P + \alpha_1 d \\ -PF_1(x) + \alpha_2 x &= -\frac{\alpha_2}{\alpha_1} P + \alpha_2 d \end{aligned}$$

Thus, agent 1's equilibrium effort is distributed according to the cumulative distribution function

$$F_1(x) = \begin{cases} \frac{-\frac{\alpha_2}{\alpha_1}P + \alpha_2(d-x)}{-P} & , d - \frac{P}{\alpha_1} \leq x < d \\ 1 & , \quad x \geq d \end{cases} \tag{5}$$

and agent 2's equilibrium effort is distributed according to the cumulative distribution function

$$F_2(x) = \frac{-P + \alpha_1(d-x)}{-P}, \; d - \frac{P}{\alpha_1} \leq x \leq d \tag{6}$$

Proof. See Appendix B. ■

The agents' expected payoffs are

$$\begin{aligned} \pi_1 &= -P + \alpha_1 d \\ \pi_2 &= -\frac{\alpha_2}{\alpha_1} P + \alpha_2 d \end{aligned}$$

Note that in contrast to the standard all-pay auction (see Appendix A), the expected payoffs of both contestants are positive and we can see that

$$\pi_1 - \pi_2 = (\alpha_1 - \alpha_2)(d - \frac{P}{\alpha_1}) > 0$$

Thus, when $\alpha_1 > \alpha_2$, agent 1's expected payoff is higher than that of agent 2. The agents' probabilities to be punished are

$$\begin{aligned} p_2 &= \int_{d-\frac{P}{\alpha_1}}^{d} \int_{d-\frac{P}{\alpha_1}}^{x} F_2'(x) F_1'(y) dy dx \\ &= \int_{d-\frac{P}{\alpha_1}}^{d} \int_{d-\frac{P}{\alpha_1}}^{x} \frac{\alpha_2 \alpha_1}{P^2} dy dx = \frac{\alpha_2}{2\alpha_1} \end{aligned}$$

and

$$p_1 = 1 - p_2 = \frac{2\alpha_1 - \alpha_2}{2\alpha_1}$$

Thus, if $\alpha_1 > \alpha_2$, agent 1's probability to be punished is higher than that of agent 2. Then, the agents' expected total effort is

$$TE = \int_{d-\frac{P}{\alpha_1}}^{d} xF_2'(x)dx + \int_{d-\frac{P}{\alpha_1}}^{d} xF_1'(x)dx + d\frac{\alpha_1-\alpha_2}{\alpha_1} \tag{7}$$

$$= \int_{d-\frac{P}{\alpha_1}}^{d} \frac{\alpha_1}{P}xdx + \int_{d-\frac{P}{\alpha_1}}^{d} \frac{\alpha_2}{P}xdx + d\frac{\alpha_1-\alpha_2}{\alpha_1} = 2d - P\frac{\alpha_1+\alpha_2}{2\alpha_1^2}$$

By comparing the expected total effort in the symmetric (2) and asymmetric (7) contests with two agents, the expected total effort in the asymmetric contest with reward functions of $\alpha_1 x$ and $\alpha_2 x$ is higher than the expected total effort in the symmetric contest with a reward function of either $\alpha_1 x$ or $\alpha_2 x$.

4.2 The n-agent contest

We now consider n agents with asymmetric reward functions. Assume first that $\alpha_1 \geq \alpha_2 = ... = \alpha_n$. Then, the agents' mixed strategy equilibrium is given by

Proposition 4 *In the reverse contest with n agents where $n-1$ of them are symmetric such that $\alpha_1 \geq \alpha_2 = ... = \alpha_n = \alpha$, if $\alpha_1 d > P$, there is a mixed strategy equilibrium in which all agents randomize on the interval $[d - \frac{P}{\alpha_1}, d]$ according to their effort cumulative distribution functions $F_1(x)$, $F(x) = F_i(x), i = 2, ..., n$ which are given by*

$$-PF^{n-1}(x) + \alpha_1 x = -P + \alpha_1 d$$

$$-PF_1(x)F^{n-2}(x) + \alpha x = -\frac{\alpha}{\alpha_1}P + \alpha d$$

Thus, the equilibrium effort of agent $i, i = 2, ..., n$, is distributed according to the cumulative distribution function

$$F(x) = \sqrt[n-1]{\frac{-P + \alpha_1(d-x)}{-P}} \text{ , } d - \frac{P}{\alpha_1} \leq x \leq d \tag{8}$$

and agent 1's equilibrium effort is distributed according to the cumulative distribution function

$$F_1(x) = \begin{cases} \dfrac{-\frac{\alpha}{\alpha_1}P + \alpha(d-x)}{-P(\frac{-P+\alpha_1(d-x)}{-P})^{\frac{n-2}{n-1}}} & , \; d - \frac{P}{\alpha_1} \leq x < d \\ 1 & , \; x \geq d \end{cases} \tag{9}$$

Proof. See Appendix B. ■

The agents' expected payoffs in this case are

$$\begin{aligned}\pi_1 &= -P + \alpha_1 d \\ \pi_2 &= ... = \pi_n = -\frac{\alpha}{\alpha_1} P + \alpha d\end{aligned}$$

Now, if we assume that $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_n$, then the agents' hybrid equilibrium is given by

Proposition 5 *In the reverse contest with n asymmetric agents such that $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_n$, if $\alpha_1 d > P$, there is a hybrid equilibrium in which agents 1, 2, randomize on the intervals $[d - \frac{P}{\alpha_1}, d]$ according to their effort cumulative distribution functions $F_1(x), F_2(x)$ which are given by*

$$\begin{aligned}-PF_2(x) + \alpha_1 x &= -P + \alpha_1 d \\ -PF_1(x) + \alpha_2 x &= -\frac{\alpha_2}{\alpha_1} P + \alpha_2 d\end{aligned}$$

Thus, agent 1's equilibrium effort is distributed according to the cumulative distribution function

$$F_1(x) = \begin{cases} \frac{-\frac{\alpha_2}{\alpha_1} P + \alpha_2 (d-x)}{-P} & , d - \frac{P}{\alpha_1} \leq \ x < d \\ 1 & , \quad x \geq d \end{cases} \tag{10}$$

and agent 2's equilibrium effort is distributed according to the cumulative distribution function

$$F_2(x) = \frac{-P + \alpha_1 (d - x)}{-P} \ , \ \ d - \frac{P}{\alpha_1} \leq x \leq d \tag{11}$$

The other agents choose a pure strategy of $x_i = d - \frac{P}{\alpha_1}, i = 3, ..., n$.[1]

Proof. See Appendix B. ■

The agents' expected payoffs are given by

$$\begin{aligned}\pi_1 &= -P + \alpha_1 d \\ \pi_i &= -\frac{\alpha_i}{\alpha_1} P + \alpha_i d \ , \ i = 2, 3, ..., n\end{aligned}$$

Note that $\alpha_i > \alpha_j$ implies that $\pi_i > \pi_j$.

The probabilities of agents 1 and 2 to be punished are the same as in the asymmetric two-agent contest when only these agents compete against each other, and the probability of the other agents to be punished

[1] In this contest, we have $n-1$ profiles of a hybrid equilibrium in which a different player $i = 2, ..., n$ chooses a mixed strategy together with player 1 and all the others choose the pure strategy $x = d - \frac{P}{\alpha_1}$.

is zero. The expected total effort is

$$TE = 2d - P\frac{\alpha_1 + \alpha_2}{2\alpha_1^2} + (n-2)(d - \frac{P}{\alpha_1}) = nd - P\frac{(2n-3)\alpha_1 + \alpha_2}{2\alpha_1^2} \tag{12}$$

As with two agents, by comparing the expected total effort in the symmetric (4) and asymmetric (12) contests with n agents, the expected total effort in the asymmetric contest with reward functions of $\alpha_i x$, $i = 1, ..., n$ is higher than the expected total effort in the symmetric contest with a reward function $\alpha_i x$ for all $i = 1, 2, ..., n$.

5 Contests with asymmetric punishments

We assume here that the agents are asymmetric such that they have different values for the punishment imposed by the designer where $P_i \leq P_{i+1}, i = 1, 2, ..., n-1$. We also assume that agents have the same reward function, $\alpha_i = \alpha$, and the same effort constraint $d_i = d$ for all $i = 1, ..., n$.

5.1 The two-agent contest

Consider first that there are only two agents. Then, the agents' mixed strategy equilibrium is given by

Proposition 6 *In the reverse contest with two agents with asymmetric values for the punishment $P_1 < P_2$, if $\alpha d > P_1$, there is a mixed strategy equilibrium in which agents 1 and 2 randomize on the interval $[d - \frac{P_1}{\alpha}, d]$ according to their effort cumulative distribution functions $F_1(x)$, $F_2(x)$ which are given by*

$$\begin{aligned} -P_1 F_2(x) + \alpha x &= -P_1 + \alpha d \\ -P_2 F_1(x) + \alpha x &= -P_1 + \alpha d \end{aligned}$$

Thus, agent 1's equilibrium effort is distributed according to the cumulative distribution function

$$F_1(x) = \begin{cases} \frac{-P_1 + \alpha(d-x)}{-P_2} & , \ d - \frac{P_1}{\alpha} \leq x < d \\ 1 & , \quad x \geq d \end{cases} \tag{13}$$

and agent 2's equilibrium effort is distributed according to the cumulative distribution function

$$F_2(x) = \frac{-P_1 + \alpha(d-x)}{-P_1} \ , \ d - \frac{P_1}{\alpha} \leq x \leq d \tag{14}$$

Proof. See Appendix B. ■

The agents' expected payoffs are

$$\pi_1 = \pi_2 = -P_1 + \alpha d$$

The probabilities of the agents to be punished are

$$\begin{aligned} p_2 &= \int_{d-\frac{P_1}{\alpha}}^{d} \int_{d-\frac{P_1}{\alpha}}^{x} F_2'(y) F_1'(y) dy dx \\ &= \int_{d-\frac{P_1}{\alpha}}^{d} \int_{d-\frac{P_1}{\alpha}}^{x} \frac{\alpha^2}{P_1 P_2} dy dx = \frac{P_1}{2P_2} \end{aligned}$$

and

$$p_1 = 1 - p_2 = \frac{2P_2 - P_1}{2P_2}$$

Thus, if $P_1 < P_2$, the probability of agent 1 to be punished is higher than that of agent 2. Then, the agents' expected total effort is

$$\begin{aligned} TE &= \int_{d-\frac{P_1}{\alpha}}^{d} x F_2'(x) dx + \int_{d-\frac{P_1}{\alpha}}^{d} x F_1'(x) dx + d \frac{P_2 - P_1}{P_2} \\ &= \int_{d-\frac{P_1}{\alpha}}^{d} \frac{\alpha}{P_1} x dx + \int_{d-\frac{P_1}{\alpha}}^{d} \frac{\alpha}{P_2} x dx + d \frac{P_2 - P_1}{P_2} = 2d - \frac{P_1(P_1 + P_2)}{2\alpha P_2} \end{aligned} \tag{15}$$

By comparing the expected total effort in the symmetric (2) and asymmetric (15) contests with two agents, the expected total effort in the asymmetric contest with values of punishments $P_1$and P_2 is higher than the expected total effort in the symmetric contest with a value of punishment P_1 or P_2.

5.2 The n-agent contest

Consider now that there are n asymmetric agents. Without loss of generality, $P_i \leq P_{i+1}, i = 1, 2, ..., n-1$. Then, the agents' hybrid equilibrium is given by

Proposition 7 *In the reverse contest with n asymmetric agents where $P_1 \leq P_2 \leq ... \leq P_n$, if $\alpha d > P_1$, there is a hybrid equilibrium in which agents $1, 2,$ randomize on the intervals $[d - \frac{P_1}{\alpha}, d]$ according to their effort cumulative distribution functions $F_1(x), F_2(x)$ which are given by*

$$\begin{aligned} -P_1 F_2(x) + \alpha_1 x &= -P_1 + \alpha d \\ -P_2 F_1(x) + \alpha_2 x &= -P_1 + \alpha d \end{aligned}$$

Thus, agent 1's equilibrium effort is distributed according to the cumulative distribution function

$$F_1(x) = \begin{cases} \frac{-P_1+\alpha(d-x)}{-P_2} & , \; d - \frac{P_1}{\alpha} \leq \; x < d \\ 1 & , \qquad\qquad x \geq d \end{cases} \tag{16}$$

and agent 2's equilibrium effort is distributed according to the cumulative distribution function

$$F_2(x) = \frac{-P_1 + \alpha_1(d-x)}{-P_1} \;, \; d - \frac{P_1}{\alpha} \leq x \leq d \tag{17}$$

and all the other agents choose the pure strategy of $x_i = d - \frac{P_1}{\alpha}, i = 3, ..., n$.

Proof. See Appendix B. ■

Then the agents' expected payoffs are

$$\pi_i = -P_1 + \alpha_1 d \;, \; i = 1, 2, ..., n$$

and the probabilities of agents 1 and 2 to be punished are the same as in the two-agent contest when they compete against each other. All the other agents have a probability of zero to be punished. Then, the agents' expected total effort is

$$TE = 2d - \frac{P_1(P_1+P_2)}{2\alpha P_2} + (n-2)(d - \frac{P_1}{\alpha}) = nd - \frac{(2n-3)P_1P_2 + P_1^2}{2\alpha P_2} \tag{18}$$

As in the case with two agents, by comparing the expected total effort in the symmetric (4) and asymmetric (18) contests with n agents, the expected total effort in the asymmetric contest with values of the punishments P_i, $i = 1, ..., n$ is higher than the expected total effort in the symmetric contest with a value of the punishment P_i, for all $i = 1, 2, ..., n$.

6 Contests with asymmetric effort constraints

We assume here that agents have asymmetric effort constraints where $d_i \geq d_{i+1}, i = 1, 2, ..., n-1$. We also assume that agents have the same value of the punishment $P_i = P$, and the same reward function $\alpha_i = \alpha$ for all $i = 1, ..., n$.

6.1 The two-agent contest

Consider first that there are only two agents. Then, the agents' mixed strategy equilibrium is given by

Proposition 8 *In the reverse contest with two asymmetric agents where* $d_1 > d_2$ *, if* $\alpha d_1 > P$ *and* $P > \alpha(d_1 - d_2)$*, there is a mixed strategy equilibrium in which agent* $i, i = 1, 2$ *randomizes on the interval* $[d_1 - \frac{P}{\alpha}, d_i]$ *according to his effort cumulative distribution function* $F_i(x), i = 1, 2,$ *which is given by*

$$\begin{aligned} -PF_2(x) + \alpha x &= -P + \alpha d_1 \\ -PF_1(x) + \alpha x &= -P + \alpha d_1 \end{aligned}$$

Thus, agent 1*'s equilibrium effort is distributed according to the cumulative distribution function*

$$F_1(x) = \begin{cases} \frac{-P+\alpha(d_1-x)}{-P} & , x \leq d_2 \\ \frac{-P+\alpha(d_1-d_2)}{-P} & , d_2 < x < d_1 \\ 1 & , \quad x \geq d_1 \end{cases} \tag{19}$$

and agent 2*'s equilibrium effort is distributed according to the cumulative distribution function*

$$F_2(x) = \begin{cases} \frac{-P+\alpha(d_1-x)}{-P} & x < d_2 \\ 1 & x \geq d_2 \end{cases} \tag{20}$$

Proof. See Appendix B. ∎

Then the agents' expected payoffs are

$$\pi_1 = \pi_2 = -P + \alpha d_1$$

The probabilities of the agents to be punished are

$$\begin{aligned} p_1 &= \int_{d_1-\frac{P}{\alpha}}^{d_2} \int_{d_1-\frac{P}{\alpha}}^{x} F_2'(y)F_1'(y)dydx + \frac{\alpha(d_1-d_2)}{P} \\ &= \int_{d_1-\frac{P}{\alpha}}^{d_2} \int_{d_1-\frac{P}{\alpha}}^{x} \frac{\alpha^2}{P^2}dydx + \frac{\alpha(d_1-d_2)}{P} = \frac{1}{2} + \frac{\alpha^2(d_1-d_2)^2}{2P^2} \end{aligned}$$

and $p_2 = 1 - p_1$. Note that since we assume that $P > \alpha(d_1 - d_2)$ then p_1 is necessarily smaller than 1. The agents' expected total effort is

$$\begin{aligned} TE &= \int_{d_1-\frac{P}{\alpha}}^{d_2} xF_2'(x)dx + \int_{d_1-\frac{P}{\alpha}}^{d_2} xF_1'(x)dx + \frac{\alpha(d_1-d_2)}{P}(d_1+d_2) \\ &= 2\int_{d_1-\frac{P}{\alpha}}^{d_2} \frac{\alpha}{P}xdx + \frac{\alpha(d_1-d_2)}{P}(d_1+d_2) = 2d_2 - \frac{P}{\alpha} \end{aligned} \tag{21}$$

By comparing the expected total effort in the symmetric (2) and asymmetric (21) contests with two agents, the expected total effort in the asymmetric contest with effort constraints d_1 and d_2 is lower than the expected total effort in the symmetric contest with an effort constraint of d_1 and is the same as the expected total effort in the symmetric contest with an effort constraint of d_2.

6.2 The n-agent contest

Consider now that there are n asymmetric agents with effort constraints that satisfy $d_i \geq d_{i+1}, i = 1, 2, ..., n-1$. Then, the agents' hybrid equilibrium is given by

Proposition 9 *In the reverse contest with n asymmetric agents where $d_1 \geq d_2 \geq ... \geq d_n$, if $\alpha d_1 > P$ and $P > \alpha(d_1 - d_n)$, there is a hybrid equilibrium in which agent $i, i = 1, 2$ randomizes on the interval $[d_1 - \frac{P}{\alpha}, d_i]$ according to his effort cumulative distribution function $F_i(x), i = 1, 2$, which is given by*

$$\begin{aligned} -PF_2(x) + \alpha x &= -P + \alpha d_1 \\ -PF_1(x) + \alpha x &= -P + \alpha d_1 \end{aligned}$$

Thus, agent 1's equilibrium effort is distributed according to the cumulative distribution function

$$F_1(x) = \begin{cases} \frac{-P+\alpha(d_1-x)}{-P} \quad , & x \leq d_2 \\ \frac{-P+\alpha(d_1-d_2)}{-P} \quad , & d_2 < x < d_1 \\ 1 \quad , & x \geq d_1 \end{cases} \tag{22}$$

agent 2's equilibrium effort is distributed according to the cumulative distribution function

$$F_2(x) = \begin{cases} \frac{-P+\alpha(d_1-x)}{-P} \quad , & x < d_2 \\ 1 \quad , & x \geq d_2 \end{cases} \tag{23}$$

and all the other agents choose the pure strategy of $x_i = d_1 - \frac{P}{\alpha}, i = 3, ..., n$.

Proof. *See Appendix B.* ■

Then the agents' expected payoffs are given by

$$\pi_i = -P + \alpha d_1 \ , \ i = 1, 2, ..., n$$

and the probabilities of agents 1 and 2 to be punished are the same as in the two-agent contest. All the other agents have a probability of zero to be punished. The agents' expected total effort is

$$TE = 2d_2 - \frac{P}{\alpha} + (n-2)(d_1 - \frac{P}{\alpha}) = 2d_2 + (n-2)d_1 - (n-1)\frac{P}{\alpha} \tag{24}$$

As in the case with two agents, by comparing the expected total effort in the symmetric (4) and asymmetric (24) contests with n agents, the expected total effort in the asymmetric contest with effort constraints d_i, $i = 1, ..., n$ is lower than the expected total effort in the symmetric contest with an effort constraint of d_1 and is the same as the expected total effort in the symmetric contest with an effort constraint of d_2.

Remark 1 *If there exists $k < n$ such that $P > \alpha(d_1 - d_i), i = 3, ..., k$ and $P < \alpha(d_1 - d_i), i = k+1, ..., n$ then there is an equilibrium in which agents 1 and 2 use the mixed strategies given by (22) and (23); agent $i, i = 3, ..., k$ chooses $x_i = d_1 - \frac{P}{\alpha}$; and agent $j, j = k+1, ..., n$ chooses $x_j = d_j$.*

Then the agents' expected payoffs are given by

$$\begin{aligned} \pi_i &= -P + \alpha d_1 \quad , i = 1, ..., k \\ \pi_j &= \alpha d_j \quad , j = k+1, ..., n \end{aligned}$$

The agents' expected total effort is then

$$TE = 2d_2 + kd_1 - (k+1)\frac{P}{\alpha} \tag{25}$$

7 Concluding remarks

The equilibrium in our model has a structure that is similar to the equilibrium of the all-pay auction under complete information (see Appendix A). However, there are also prominent differences as follows:

- The smallest point of the support of the players' mixed strategies in the all-pay auction is zero while in our model it is larger than zero.
- In the all-pay auction with two asymmetric players one of the players chooses an effort with a positive probability that is equal to the smallest point of the support of the players' mixed strategies, while in our model one of the players chooses an effort with a positive probability that is equal to the largest point of the support of the players' mixed strategies.
- In the symmetric two-player all-pay auction the expected payoff of all the players is zero, while in our model all the players have positive expected payoffs.

- In the asymmetric two-player all-pay auction only one player has a positive expected payoff and in our model both players have positive expected payoffs.
- In the symmetric all-pay auction with n players all the players have an expected payoff of zero, while in our model all the players have a positive expected payoff.
- In the asymmetric all-pay auction with n players if the players have different values only one player has a positive expected payoff, while in our model all the players who use mixed strategies as well as those who use a pure strategy have positive expected payoffs
- In the asymmetric all-pay auction the players' expected payoffs are not the same, while in our model with asymmetric values for the punishment as well as with asymmetric effort constraints, the players' expected payoffs are the same.
- In the all-pay auction the expected total effort in the symmetric model is higher than in the asymmetric one, while in our model the players' expected effort in the asymmetric model might be higher than in the symmetric one.

The mixed-strategy equilibrium as well as the hybrid equilibrium in our model seem more plusible than the mixed strategy equilibrium in the all-pay auction. This is because in the asymmetric all-pay auction the incentive of all the players except the one who has an expected payoff larger than zero to participate in the contest is not clear, while in our model all the players, whether symmetric or asymmetric, have positive expected payoffs and therefore they all have an incentive to participate.

8 Appendix A

Consider the standard all-pay auction with two players, 1 and 2. Player i's reward is v_i, where $v_1 \geq v_2$, and his expected utility is $u_i = v_i - x_i$ if $x_i > x_j$. Otherwise $u_i = -x_i$, where x_i is the effort of player i. The goal of each contestant is to maximize his expected payoff. According to Hillman and Riley (1989) and Baye, Kovenock and de Vries (1996), there is always a unique mixed strategy equilibrium in which players 1 and 2 randomize on the interval $[0, v_2]$ according to their effort cumulative distribution functions, which are given by

$$\begin{aligned} v_1F_2(x) - x &= v_1 - v_2 \\ v_2F_1(x) - x &= 0 \end{aligned}$$

Thus, player 1's equilibrium effort is uniformly distributed; that is

$$F_1(x) = \frac{x}{v_2}$$

while player 2's equilibrium effort is distributed according to the cumulative distribution function

$$F_2(x) = \frac{v_1 - v_2 + x}{v_1}$$

The players' expected payoffs are

$$\begin{aligned} \pi_1 &= v_1 - v_2 \\ \pi_2 &= 0 \end{aligned}$$

and their probabilities of winning are

$$\begin{aligned} p_1 &= 1 - \frac{v_2}{2v_1} \\ p_2 &= \frac{v_2}{2v_1} \end{aligned}$$

The players' expected total effort is

$$TE = \frac{v_2}{2}(1 + \frac{v_2}{v_1})$$

9 Appendix B

9.1 Proof of Proposition 1

The function $F(x)$ given by (1) is well-defined, strictly increasing on $[d-\frac{P}{\alpha}, d]$, continuous, satisfies $F(d-\frac{P}{\alpha}) = 0$ and $F(d) = 1$. Thus, $F(x)$ is a cumulative distribution function of a continuous probability distribution supported on $\left[d - \frac{P}{\alpha}, d\right]$. In order to see that the above strategies are an equilibrium, note that when agent 2 uses the mixed strategy $F(x)$, agent 1's expected payoff is $\pi_1 = -P + \alpha d$ for any effort $x \in [d - \frac{P}{\alpha}, d]$.

Since it can be easily shown that for agent 1, efforts below $d - \frac{P}{\alpha}$ would lead to a lower expected payoff than $-P + \alpha d$, and since efforts above d are infeasible, any effort in $[d - \frac{P}{\alpha}, d]$ is a best response of agent 1 when agent 2 uses $F(x)$. By symmetry, any effort in $[d - \frac{P}{\alpha}, d]$ is a best response of agent 2 when agent 1 uses $F(x)$. Hence, $F(x)$ is a symmetric mixed strategy equilibrium.

9.2 Proof of Proposition 2

The function $F(x)$ given by (3) is well-defined, strictly increasing on $[d-\frac{P}{\alpha}, d]$, continuous, satisfies $F(d-\frac{P}{\alpha}) = 0$ and $F(d) = 1$. Thus, $F(x)$ is a cumulative distribution function of a continuous probability distribution supported on $\left[d - \frac{P}{\alpha}, d\right]$. In order to see that the above strategies are an equilibrium, note that when agents $2, 3, ..., n$ use the mixed strategy $F(x)$, agent 1's expected payoff is $\pi_1 = -P + \alpha d$ for any effort $x \in [d - \frac{P}{\alpha}, d]$. Since it can be easily shown that for agent 1, efforts below $d - \frac{P}{\alpha}$ would lead to a lower expected payoff than $-P + \alpha d$, and since efforts above d are infeasible, any effort in $[d - \frac{P}{\alpha}, d]$ is a best response of agent 1 when all the other agents use $F(x)$. By symmetry, any effort in $[d - \frac{P}{\alpha}, d]$ is a best response of agent $i, i = 2, ..., n$ when all the other agents use $F(x)$. Hence, $F(x)$ is a symmetric mixed strategy equilibrium.

9.3 Proof of Proposition 3

The functions $F_i(x)$, $i = 1, 2$ given by (5) and (6) are well-defined, strictly increasing on $[d-\frac{P}{\alpha_1}, d]$, continuous, satisfy $F_1(d - \frac{P}{\alpha}) = F_2(d - \frac{P}{\alpha}) = 0$, and $F_2(d) = 1$, $F_1(d) = 1$, where agent 1 chooses the effort that is equal to d with a probability of $\frac{\alpha_1 - \alpha_2}{\alpha_1} > 0$. Thus, $F_i(x)$, $i = 1, 2$ are cumulative distribution functions of continuous probability distributions supported on $\left[d - \frac{P}{\alpha_1}, d\right]$. In order to see that the above strategies are an equilibrium, note that when agent 2 uses the mixed strategy $F_2(x)$, agent 1's expected payoff is $\pi_1 = -P + \alpha_1 d$ for any effort $x \in [d - \frac{P}{\alpha_1}, d]$. Since it can be easily shown that for agent 1, efforts below $d - \frac{P}{\alpha_1}$ would lead to a lower expected payoff than $-P + \alpha_1 d$, and efforts above d are infeasible, any effort in $[d - \frac{P}{\alpha_1}, d]$ is a best response of agent 1 when agent 2 uses $F_2(x)$. Similarly, when agent 1 uses the mixed strategy $F_1(x)$, agent 2's expected payoff is $\pi_2 = -\frac{\alpha_2}{\alpha_1} P + \alpha_2 d$ for any effort $x \in [d - \frac{P}{\alpha_1}, d)$. Since it can be easily shown that for agent 2, efforts below $d - \frac{P}{\alpha_1}$ as well as an effort that is equal to d would result in a lower expected payoff, and efforts above d are infeasible, any effort in $[d - \frac{P}{\alpha_1}, d)$ is a best response of agent

2 when agent 1 uses $F_1(x)$. Hence, the pair $(F_1(x), F_2(x))$ is a mixed strategy equilibrium.

9.4 Proof of Proposition 4

The functions $F_1(x)$, $F(x)$, given by (9) and (8) are well-defined, strictly increasing on $[d-\frac{P}{\alpha_1}, d]$, continuous, satisfy $F(d-\frac{P}{\alpha_1}) = F_1(d-\frac{P}{\alpha_1}) = 0$, and $F(d) = 1, F_1(d) = 1$, where agent 1 chooses the effort that is equal to d with a probability of $\frac{\alpha_1-\alpha}{\alpha_1} > 0$. Thus, $F(x), F_1(x)$ are cumulative distribution functions of continuous probability distributions supported on $\left[d-\frac{P}{\alpha_1}, d\right]$. In order to see that the above strategies are an equilibrium, note that when agents $2, ..., n$ use the mixed strategy $F(x)$, agent 1's expected payoff is $\pi_1 = -P + \alpha_1 d$ for any effort $x \in [d-\frac{P}{\alpha_1}, d]$. Since it can be easily shown that for agent 1, efforts below $d-\frac{P}{\alpha_1}$ would lead to a lower expected payoff than $-P + \alpha_1 d$, and efforts above d are infeasible, any effort in $[d-\frac{P}{\alpha_1}, d]$ is a best response of agent 1 when all the other agents use $F(x)$. Similarly, when agent 1 uses the mixed strategy $F_1(x)$, and all the other $n-2$ agents, $i = 2, ..., n$ and $i \neq j$, use the mixed strategy $F(x)$, agent j's expected payoff is $\pi_j = -P + \alpha d$ for any effort $x \in [d-\frac{P}{\alpha_1}, d)$. Since it can be easily shown that for agent j,$j = 2, 3, ..., n$ efforts below $d-\frac{P}{\alpha_1}$ as well as an effort that is equal to d would result in a lower expected payoff, and efforts above d are infeasible, any effort in $[d-\frac{P}{\alpha_1}, d)$ is a best response of agent j when agent 1 uses $F_1(x)$ and all the other agents $i = 2, ..., n$ and $i \neq j$ use $F(x)$. Hence, $(F_1(x), F(x), ..., F(x))$ is a mixed strategy equilibrium.

9.5 Proof of Proposition 5

The functions $F_i(x)$, $i = 1, 2$, given by (10) and (11) are well-defined, strictly increasing on $[d-\frac{P}{\alpha_1}, d]$, continuous, satisfy $F_1(d-\frac{P}{\alpha}) = F_2(d-\frac{P}{\alpha}) = 0$, and $F_2(d) = 1, F_1(d) = 1$, where agent 1 chooses the effort that is equal to d with a probability of $\frac{\alpha_1-\alpha_2}{\alpha_1} > 0$. Thus, $F_i(x)$, $i = 1, 2$ are cumulative distribution functions of continuous probability distributions supported on $\left[d-\frac{P}{\alpha_1}, d\right]$. In order to see that the above strategies with $x_i = d-\frac{P}{\alpha_1}$, $i = 3, ..., n$, are an equilibrium, note that when agent 2 uses the mixed strategy $F_2(x)$ and all the other agents choose the pure strategy $x_i = d-\frac{P}{\alpha_1}$, $i = 3, ..., n$, agent 1's expected payoff is $\pi_1 = -P + \alpha_1 d$ for any effort $x \in [d-\frac{P}{\alpha_1}, d]$. Since it can be easily shown that for agent 1, efforts below $d-\frac{P}{\alpha_1}$ would lead to a lower expected payoff than $-P + \alpha_1 d$ and efforts above d are infeasible, any effort in $[d-\frac{P}{\alpha_1},d]$ is a best response of agent 1 when agent 2 uses $F_2(x)$ and all the other agents choose the pure strategy $x_i=d-\frac{P}{\alpha_1}, i=3,...,n$. Similarly, when agent 1 uses the mixed strategy $F_1(x)$, and all the other agents choose the pure strategy $x_i=d-\frac{P}{\alpha_1}, i=3,...,n$, agent 2's expected payoff is $\pi_2=-\frac{\alpha_2}{\alpha_1}P+\alpha_2 d$ for any effort $x\in[d-\frac{P}{\alpha_1},d)$. Since it can be easily shown that for agent 2, efforts below $d-\frac{P}{\alpha_1}$ and an effort that is equal to d would result in a lower expected payoff, and efforts above d are infeasible, any effort in $[d-\frac{P}{\alpha_1},d)$ is a best response of agent 2 when agent 1 uses $F_1(x)$ and all the other agents choose the pure strategy $x_i=d-\frac{P}{\alpha_1}, i=3,...,n$. In addition, for every agent $i, i=3,...,n$, the expected payoff is $\pi_i=-\frac{\alpha_i}{\alpha_1}P+\alpha_2 d$, $i=3,...,n$. It can be shown that efforts below and above $d-\frac{P}{\alpha_1}$ would result in a lower expected payoff, and therefore $x_i=d-\frac{P}{\alpha_1}$ is a best response for agent i when agents 1 and 2 use the mixed strategies $F_1(x)$, $F_2(x)$ and all the other agents choose the same strategy as agent i. Hence, the mixed strategies $(F_1(x),F_2(x))$ with the pure strategies $x_i=d-\frac{P}{\alpha_1}, i=3,...,n$ are an hybrid equilibrium.

9.6 Proof of Proposition 6

The functions $F_i(x)$, $i=1,2$, given by (13) and (14), are well-defined, strictly increasing on $[d-\frac{P_1}{\alpha},d]$, continuous, satisfy $F_1(d-\frac{P_1}{\alpha})=F_2(d-\frac{P_1}{\alpha})=0$, and $F_2(d)=1, F_1(d)=1$, where agent 1 chooses the effort that is equal to d with a probability of $\frac{P_2-P_1}{P_2}>0$. Thus, $F_i(x)$, $i=1,2$ are cumulative distribution functions of continuous probability distributions supported on $\left[d-\frac{P_1}{\alpha},d\right]$. In order to see that the above strategies are an equilibrium, note that when agent 2 uses the mixed strategy $F_2(x)$, agent 1's expected payoff is $\pi_1=-P_1+\alpha d$ for any effort $x\in[d-\frac{P_1}{\alpha},d]$. Since it can be easily shown that for agent 1, efforts below $d-\frac{P_1}{\alpha}$ would lead to a lower expected payoff than $-P_1+\alpha d$ and efforts above d are infeasible, any effort in $[d-\frac{P_1}{\alpha},d]$ is a best response of agent 1 when agent 2 uses $F_2(x)$. Similarly, when agent 1 uses the mixed strategy $F_1(x)$, agent 2's expected payoff is $\pi_2=-P_1+\alpha d$ for any effort $x\in[d-\frac{P_1}{\alpha},d)$. Since it can be easily shown that for agent 2, efforts below $d-\frac{P_1}{\alpha}$ as well as an effort that is equal to d would result in a lower expected payoff, and efforts above d are infeasible, any effort in $[d-\frac{P_1}{\alpha},d)$ is a best response of agent 2 when agent 1 uses $F_1(x)$. Hence, the pair $(F_1(x),F_2(x))$ is a mixed strategy equilibrium.

9.7 Proof of Proposition 7

The functions $F_i(x)$, $i = 1, 2$, given by (16) and (17), are well defined, strictly increasing on $[d - \frac{P_1}{\alpha}, d]$, continuous, satisfy $F_1(d - \frac{P_1}{\alpha}) = F_2(d - \frac{P_1}{\alpha}) = 0$, and $F_2(d) = 1, F_1(d) = 1$, where agent 1 chooses the effort that is equal to d with a probability of $\frac{P_2 - P_1}{P_2} > 0$. Thus, $F_i(x)$, $i = 1, 2$ are cumulative distribution functions of continuous probability distributions supported on $\left[d - \frac{P_1}{\alpha}, d\right]$. In order to see that the above strategies with $x_i = d - \frac{P_1}{\alpha}$, $i = 3, ..., n$, are an equilibrium, note that when agent 2 uses the mixed strategy $F_2(x)$ and all the other agent choose the pure strategy $x_i = d - \frac{P_1}{\alpha}$, $i = 3, ..., n$, agent 1's expected payoff is $\pi_1 = -P_1 + \alpha d$ for any effort $x \in [d - \frac{P_1}{\alpha}, d]$. Since it can be easily shown that for agent 1, efforts below $d - \frac{P_1}{\alpha}$ would lead to a lower expected payoff than $-P_1 + \alpha d$ and efforts above d are infeasible, any effort in $[d - \frac{P_1}{\alpha}, d]$ is a best response of agent 1 when agent 2 uses $F_2(x)$ and all the other agents choose the pure strategy $x_i = d - \frac{P_1}{\alpha}, i = 3, ..., n$. Similarly, when agent 1 uses the mixed strategy $F_1(x)$, and all the other agents choose the pure strategy $x_i = d - \frac{P_1}{\alpha}, i = 3, ..., n$, agent 2's expected payoff is $\pi_2 = -P_1 + \alpha d$ for any effort $x \in [d - \frac{P_1}{\alpha}, d)$. Since it can be easily shown that for agent 2, efforts below $d - \frac{P_1}{\alpha}$ as well as an effort that is equal to d would result in a lower expected payoff, and efforts above d are infeasible, any effort in $[d - \frac{P_1}{\alpha}, d)$ is a best response of agent 2 when agent 1 uses $F_1(x)$ and all the other agents choose the pure strategy $x_i = d - \frac{P_1}{\alpha}, i = 3, ..., n$. In addition, for every agent $i, i = 3, ..., n$, the expected payoff is $\pi_i = -P_1 + \alpha d$. It can be shown that efforts below and above $d - \frac{P_1}{\alpha}$ would result in a lower expected payoff and therefore $x_i = d - \frac{P_1}{\alpha}$ is a best response for agent i when agents 1 and 2 use the mixed strategies $F_1(x)$, $F_2(x)$ and all the other agents choose the same strategy as agent i. Hence, the mixed strategies $(F_1(x), F_2(x))$ with the pure strategies $x_i = d - \frac{P_1}{\alpha}, i = 3, ..., n$ are a hybrid equilibrium.

9.8 Proof of Proposition 8

The functions $F_i(x)$, $i = 1, 2$, given by (19) and (20), are well-defined, strictly increasing on $[d_1 - \frac{P}{\alpha}, d_2]$, continuous, satisfy $F_1(d_1 - \frac{P}{\alpha}) = F_2(d_1 - \frac{P}{\alpha}) = 0$. Agent 2's mixed strategy satisfies $F_2(d_2) = 1$, where agent 2 chooses the effort that is equal to d_2 with a probability of $\frac{\alpha(d_1 - d_2)}{P} > 0$. Agent 1's mixed strategy satisfies $F_1(d_2) < F_1(d_1) = 1$, where agent 1 chooses the effort that is equal to d_1 with a probability of $\frac{\alpha(d_1 - d_2)}{P} > 0$. Thus, $F_1(x)$ is a cumulative distribution function of a continuous probability distribution supported on $\left[d_1 - \frac{P}{\alpha}, d_1\right]$, and $F_2(x)$ is a cumulative distribution function of a continuous probability distribution supported on $\left[d_1 - \frac{P}{\alpha}, d_2\right]$. In order to see that the above strategies are an equilibrium, note that when agent 2 uses the mixed strategy $F_2(x)$, agent 1's expected payoff is $\pi_1 = -P + \alpha d_1$ for any effort $x \in [d_1 - \frac{P}{\alpha}, d_2] \cup \{d_1\}$. Since it can be easily shown that for agent 1, efforts below $d_1 - \frac{P}{\alpha}$ and between d_2 and $d_1$would lead to a lower expected payoff than $-P + \alpha d_1$, and efforts above d_1 are infeasible, any effort in $[d_1 - \frac{P}{\alpha}, d_2] \cup \{d_1\}$ is a best response of agent 1 when agent 2 uses $F_2(x)$. Similarly, when agent 1 uses the mixed strategy $F_1(x)$, agent 2's expected payoff is $\pi_2 = -P + \alpha d_1$ for any effort $x \in [d_1 - \frac{P}{\alpha}, d_2]$. Since it can be easily shown that for agent 2, efforts below $d_1 - \frac{P}{\alpha}$ would result in a lower expected payoff, and efforts above d_2 are infeasible, any effort in $[d_1 - \frac{P}{\alpha}, d_2]$ is a best response of agent 2 when agent 1 uses $F_1(x)$. Hence, the pair $(F_1(x), F_2(x))$ is a mixed strategy equilibrium.

9.9 Proof of Proposition 9

The functions $F_i(x)$, $i = 1, 2$, given by (22) and (23) are well-defined, strictly increasing on $[d_1 - \frac{P}{\alpha}, d_2]$, continuous, and satisfy $F_1(d_1 - \frac{P}{\alpha}) = F_2(d_1 - \frac{P}{\alpha}) = 0$. Agent 2's mixed strategy satisfies $F_2(d_2) = 1$, where agent 2 chooses the effort that is equal to d_2 with a probability of $\frac{\alpha(d_1-d_2)}{P} > 0$. Agent 1's mixed strategy satisfies $F_1(d_2) < F_1(d_1) = 1$, where agent 1 chooses the effort that is equal to d_1 with a probability of $\frac{\alpha(d_1-d_2)}{P} > 0$.

Thus, $F_i(x), i = 1, 2$, is a cumulative distribution function of continuous probability distributions supported on $\left[d_1 - \frac{P}{\alpha}, d_i\right]$. In order to see that the above strategies with $x_i = d_1 - \frac{P}{\alpha}, i = 3, ..., n$, are an equilibrium, note that when agent 2 uses the mixed strategy $F_2(x)$ and all the other agents choose the pure strategy $x_i = d_1 - \frac{P}{\alpha}, i = 3, ..., n$, agent 1's expected payoff is $\pi_1 = -P + \alpha d_1$ for any effort $x \in [d_1 - \frac{P}{\alpha}, d_2] \cup \{d_1\}$. Since it can be easily shown that for agent 1, efforts below $d_1 - \frac{P}{\alpha}$ and between d_2 and $d_1$would lead to a lower expected payoff than $-P + \alpha d_1$ and efforts above d_1 are infeasible, any effort in $[d_1 - \frac{P}{\alpha}, d_2] \cup \{d_1\}$ is a best response of agent 1 when agent 2 uses $F_2(x)$ and all the other agent choose the pure strategy $x_i = d_1 - \frac{P}{\alpha}, i = 3, ..., n$. Similarly, when agent 1 uses the mixed strategy $F_1(x)$ and all the other agents choose the pure strategy $x_i = d_1 - \frac{P}{\alpha}, i = 3, ..., n$, agent 2's expected payoff is $\pi_2 = -P + \alpha d_1$ for any effort $x \in [d_1 - \frac{P}{\alpha}, d_2]$. Since it can be easily shown that for agent 2, efforts below $d_1 - \frac{P}{\alpha}$ would result in a lower expected payoff and efforts above d_2 are infeasible, any effort in $[d_1 - \frac{P}{\alpha}, d_2]$ is a best response of agent 2 when agent 1 uses $F_1(x)$ and all the other agent choose the pure strategy $x_i = d_1 - \frac{P}{\alpha}, i, i = 3, ..., n$. Last, for every agent $i, i = 3, ..., n$, the expected payoff is $\pi_i = -P + \alpha d_1$. It can be shown that efforts below and above $d_1 - \frac{P}{\alpha}$ would result in a lower expected payoff and therefore $x_i = d_1 - \frac{P}{\alpha}$ is a best response for agent i when agents 1 and 2 use the mixed strategies $F_1(x)$, $F_2(x)$ and all the other agents choose the same strategy as agent i. Hence, the mixed strategies $(F_1(x), F_2(x))$ with the pure strategies $x_i = d_1 - \frac{P}{\alpha}, i = 3, ..., n$ are a hybrid equilibrium.

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