# THE OPTIMAL ALLOCATION OF PUNISHMENTS IN TULLOCK CONTESTS

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# The Optimal Allocation of Punishments in Tullock Contests

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**Abstract**

We study the role of punishments in Tullock contests with symmetric players. We first characterize the players' equilibrium strategies in a contest with either multiple identical prizes or multiple identical punishments (negative prizes). Given that a prize and a punishment have the same absolute value, we show that if the number of prizes is equal to the number of punishments and is lower (higher) than or equal to half the number of players, a designer who wishes to maximize the players' efforts will prefer to allocate punishments (prizes) over prizes (punishments). We also demonstrate that if the sum of the punishments is constrained, then in a contest without an exit option for the players, it is optimal for the designer who maximizes the players' efforts to allocate a single punishment that is equal to the punishment sum. However, in a contest with an exit option the optimal number of punishments depends on the value of the punishment sum and, in particular, the optimal number of punishments does not monotonically increase in the value of the punishment sum.

*JEL Classifications*: D44, J31, D72, D82

Keywords: Tullock contests; prizes; punishments.

# 1 Introduction

Prizes have a key role in contests as they provide the incentive for players to participate and exert efforts. Thus, the contest literature has focused on what the optimal prize structure is.[1] While most of these studies have concentrated on the incentive role of prizes, punishments (negative prizes) which are also part of many existing incentive contracts have been ignored. The reason for this is that a contest that has some punishments can be replicated by one that uses only prizes and that yields the same incentives. But this equivalence breaks down if the contest designer faces a budget constraint and if the punishments are costly.[2] Under these circumstances, the designer will prefer to distribute the entire punishment sum to the punishment with the highest marginal effect on the participants' total effort relative to its cost. In other words, subject to his budget constraint, a designer will determine the number and the size of punishments in order to maximize the participants' expected total effort. This problem of determining the optimal allocation of punishments is the goal of this paper.

Moldovanu Sela and Shi (2012) have dealt with the issue of punishment allocation and showed that in the all-pay contest under incomplete information the designer should only punish the player with the lowest performance if the distribution of the players' abilities has an increasing hazard (or failure) rate. If this last condition is not satisfied then more punishments may be optimal. Recently, Kamijo (2016) showed how prizes as well as punishments can be implemented in all-pay contests when the designer wishes to maximize either the players' expected highest effort or their expected minimal effort. We study here the role of punishments in the Tullock contest which is the most common rent seeking contest (see Tullock 1980 and Konrad 2009). In this contest each player's probability of winning is the ratio of his effort and the total effort exerted by all the players.[3]

[1] Most of the literature on optimal prize allocation in contests focuses on theree classes of models: (1) the all-pay contest (see Barut and Kovenock 1998, and Moldovanu and Sela 2001, 2006); (2) the Tullock contest (see Clark and Riis 1996, 1998, Schweinzer and Segev 2009, Szymanski 2003, Szymanski and Valleti 2005 and Fu and Lu 2012); and (3) the rank-order tournament (see Lazear and Rosen 1981, Green and Stokey 1983, Nalebuff and Stiglitz 1983 and Akerlof and Holden 2012).

[2] For example, a costly punishment can occur if a firm fires workers who perform poorly and then spends some resources to search for alternative replacements.

[3] A number of studies provided axiomatic justification for the Tullock contest (see, for example, Skaperdas 1966). Baye and Hoppe (2003) have identified conditions under which a variety of rent-seeking contests, innovation tournaments, and patent-race

We first analyze the equilibrium strategies of $n$ symmetric players with either multiple identical prizes or multiple identical punishments. We assume that the players exert their efforts once in the first stage and the prizes are determined sequentially. The first winner is determined by a probability success function that is based on the efforts of all the players. The second winner is determined by a probability success function that is based on the efforts of all the players excluding the effort of the first winner. This sequential process continues until all the prizes are allocated. Likewise, the allocation of punishments are sequentially determined. We show that the equilibrium effort in a Tullock contest with $n$ symmetric players and $k$ identical prizes is identical to the equilibrium effort in a Tullock contest with $n$ symmetric players and $n-k$ identical punishments with the same absolute value. Then, we show that if the absolute value of a prize is equal to that of a punishment and that, if the number of prizes that is equal to the number of punishments is smaller than half the number of players, then the punishments' effect on the players' equilibrium effort is higher than the prizes' effect. The opposite is true when the number of prizes that is equal to the number of punishments is larger than half the number of players.

We also study the optimal allocation of punishments without an exit option, namely, the players have to participate in the contest even if their ex-ante expected payoff is negative. We assume that punishments are costly and the designer has a fixed punishment sum and show that, similar to allocating prizes, it is optimal to allocate the entire punishment sum to a single punishment. Afterwards, we study the optimal allocation of punishments when players have an exit option, and thus may choose not to participate in the contest. Then, in contrast to the case without an exit option, the value of the punishment sum has a meaningful effect on the optimal distribution of punishments. In particular, for a relatively low punishment sum, in order to maximize the players' equilibrium effort, it is optimal to allocate only one punishment as all the players still have an incentive to participate in the contest. On the other hand, when the punishment sum increases allocating two punishments may be better than allocating one prize since in a contest with one punishment the number of the players will be smaller. However, if the punishment sum keeps increasing we show that the number of players in both contest forms with either one or two punishments is equalized and therefore allocating one punishment will then be optimal. In sum, we demonstrate that in the Tullock

games are strategically equivalent to the Tullock contest.

contest with an exit option, the optimal number of punishments neither increases nor decreases in the value of the punishment sum.

The rest of the paper is organized as follows: In Section 2 we analyze the symmetric equilibrium effort with multiple identical prizes and punishments. In Section 3 we study the optimal allocation of punishments with and without an exit option. Section 4 concludes. The proofs appear in an Appendix.

## 2 Contests with multiple identical prizes or punishments

We consider a Tullock contest with $n$ risk-neutral players where each player $j$ makes an effort $x_j$. For simplicity, we postulate a deterministic relation between effort and output, and assume these to be equal. Efforts are submitted simultaneously. If only one prize with a value of $v$ is awarded, then if $x$ is the effort of player 1 and $y$ is the symmetric effort of all the other players, player 1's probability of winning the prize is

$$\frac{x}{x+(n-1)y}$$

If, on the other hand, two identical prizes with a value of $v$ are awarded, player 1's probability of winning one of the prizes (each player can win only one prize) is

$$\frac{x}{x+(n-1)y}+\frac{(n-1)y}{x+(n-1)y}\cdot\frac{x}{x+(n-2)y}$$

Suppose now that the designer allocates $k < n$ identical prizes with a value of $v$. Thus, the maximization problem of player 1 is given by

$$\max_x v\sum_{i=1}^{k}\frac{y^{i-1}x\left(\frac{n-1)!}{(n-i)!}\right)}{\Pi_{j=1}^{i}\left((n-j)y+x\right)}-x \tag{1}$$

where $x$ is the effort of player 1 and $y$ is the symmetric effort of all the other players. Then, we obtain that

**Proposition 1** *In the Tullock contest with $n$ symmetric players and $k$ identical prizes with a value of $v$, the symmetric equilibrium effort is*

$$x = v\sum_{i=1}^{k}\frac{1-(H_n-H_{n-i})}{n} \tag{2}$$

*where $H_n=\sum_{i=1}^{n}\frac{1}{i}$ is the Harmonic series.*

**Proof.** See Appendix. ■

Let $x_k$ be the the players' symmetric equilibrium effort with $k$ identical prizes given by (2). Then we have

$$x_k - x_{k-1} = v\frac{1-(H_n - H_{n-k})}{n}$$

Thus, the optimal number of prizes $k_{\max}$ that maximizes the players' symmetric equilibrium effort satisfies

$$H_n - H_{n-k_{\max}} \leq 1$$

and

$$H_n - H_{n-k_{\max}-1} > 1$$

Now suppose that there are no prizes but that there is one punishment with a value of $-s$ where $s > 0$. Then if $x$ is the effort of player 1 and $y$ is the symmetric effort of all the other players, player 1's probability to be punished is

$$\frac{(n-1)y}{(n-1)y+x} \cdot \frac{(n-2)y}{(n-2)y+x} \cdot ... \cdot \frac{2y}{2y+x} \cdot \frac{y}{y+x}$$

If, on the other hand, there are two punishments with a value of $-s$, then player 1's probability to be punished is

$$\frac{(n-1)y}{(n-1)y+x} \cdot \frac{(n-2)y}{(n-2)y+x} \cdot ... \cdot \frac{2y}{2y+x} \cdot (\frac{y}{y+x} + \frac{x}{y+x})$$

Thus, if the designer allocates $l$ identical punishments with a value of $-s$, the maximization problem of player 1 is

$$\max_x -s\left(\frac{y^{n-1}(n-1)!}{\Pi_{i=1}^{n-1}(iy+x)}\right) - s\sum_{i=1}^{l}\frac{y^{n-i}x(\frac{(n-1)!}{(i-1)!})}{\Pi_{j=i-1}^{n-1}(jy+x)} - x \tag{3}$$

where $x$ is the effort of player 1 and $y$ is the symmetric effort of all the other players. Then, we obtain that

**Proposition 2** *In the Tullock contest with $n$ symmetric players and $l$ identical punishments with a value of $s$, the symmetric equilibrium effort is*

$$x = s\left(\frac{(H_n - 1) - \Sigma_{i=2}^{l}(H_n - H_{i-1}) - 1}{n}\right) \tag{4}$$

*where $H_n = \sum_{i=1}^{n}\frac{1}{i}$ is the Harmonic series.*

**Proof.** See Appendix. ■

Let $x_l$ be the players' symmetric equilibrium effort with $l$ identical punishments given by (4). Then we have

$$x_l - x_{l-1} = s\frac{(H_n - H_{l-1}) - 1}{n}$$

Thus, the optimal number of punishments $l_{\max}$ that maximizes the players' equilibrium effort satisfies

$$H_n - H_{l_{\max}-1} \geq 1$$

and

$$H_n - H_{l_{\max}} < 1$$

The similarity of the conditions of the optimal number of prizes and punishments indicates that there is a relationship between contests with identical prizes and punishments with the same absolute values. Indeed, we have

**Proposition 3** *The symmetric equilibrium effort in the Tullock contest with $n$ symmetric players and $k$ identical prizes with a value of $v$ is identical to the symmetric equilibrium effort in the Tullock contest with $n$ symmetric players and $n-k$ identical punishments with the same (absolute) value of $v$.*

**Proof.** See Appendix.

Given the assumption that punishments as well as prizes are costly, it is of interest to ask whether a prize or a punishment has a higher effect on the players' equilibrium strategies. By (2), the symmetric equilibrium effort with one prize of a value of $v$ is

$$x = v\frac{1-(H_n - H_{n-1})}{n}$$

and by (4), the symmetric equilibrium effort with one punishment of a value of $-v$ is

$$x = v\frac{H_n - 1}{n}$$

Since for all $n \geq 2$, we have $H_n - 1 \geq 1-(H_n - H_{n-1}) = 1 - \frac{1}{n}$ we obtain that a punishment has a higher effect on the symmetric equilibrium effort than a prize with the same absolute value. In general, punishments do not necessarily have a larger effect than prizes on the players' equilibrium efforts. Indeed, the next result shows that if the number of prizes and punishments is relatively low the allocation of punishments is then more effective and if their number is relatively high then the allocation of prizes is more effective.

**Proposition 4** *The symmetric equilibrium effort in the Tullock contest with $k < (>) \frac{n}{2}$ punishments of a value $-v$ is higher (lower) than in a Tullock contest with $k$ prizes of a value of $v$.*

**Proof.** See Appendix. ■

If we combine Propositions 3 and 4 we obtain that in order to maximize the players' symmetric equilibrium effort, allocating $n-1$ prizes are better than allocating one prize with the same value. Similarly, allocating $n-2$ prizes is better than allocating two prizes with the same value and so on. In other words, although the players' equilibrium effort is not monotonic in the number of prizes, $n-s$ prizes are better than $s$ prizes for every $0 < s < \frac{n}{2}$.

# 3 The optimal allocation of punishments with and without an exit option

We assume now that the punishment sum is constrained and is equal to $v$. Denote by $v_i \geq 0$ the cost of the $i$-th punishment which is equal to its (absolute) value, namely $-v_1$ is the first punishment, $-v_2$ is the second punishment and so on, such that $v_1 \geq v_2 \geq v_3 ..... \geq v_n$ and $\sum_{i=1}^{n} v_i = v$. We also assume that the players cannot exit, namely, even if a player has an ex-ante negative expected payoff he cannot stay out of the contest. Let $x$ be the effort of player 1 and by symmetry $y$ be the effort of each of the other players. Then, the maximization problem of player 1 is

$$\max_{x} -v_1 \frac{y^{n-1}(n-1)!}{\Pi_{i=1}^{n-1}(iy+x)} - \sum_{i=2}^{n-1} v_i \frac{y^{n-i} x \frac{(n-1)!}{(i-1)!}}{\Pi_{j=i-1}^{n-1}(jy+x)} - x \tag{5}$$

In that case, we obtain that the optimal allocation of punishments is $v_1 = v$ and $v_i = 0$ for all $i \neq 1$.

**Proposition 5** *In the Tullock contest with $n$ symmetric players, the players' symmetric equilibrium effort is maximized when the entire punishment sum is allocated to a single punishment.*

**Proof.** See Appendix. ■

By Propositions 5 and 3 we obtain that in a Tullock contest with $n$ symmetric players an allocation of $n-1$ identical prizes of a value $v$ yields a higher symmetric equilibrium effort than an allocation of $n-j$ identical prizes of a value $\frac{v}{j}$ , $j = 2, 3, .....$

Now, suppose that a player can exit, namely, if he has a negative ex-ante expected payoff he does not participate in the contest. However, it is important to emphasize that although the players are symmetric there is no equilibrium in which all the players decide not to participate, since when some of the players quit, the ex-ante expected payoffs of the other players increase. If the players have no incentive to participate, players randomly and sequentially quit until the rest of the players again have an incentive to participate in the contest. Since in a contest with only punishments players obviously have no incentive to participate, we assume that there is one prize of a value $v$ and a punishment sum that is equal to $\beta v$ where $\beta > 0$. Then, for any allocation of the punishment sum into two punishments, the maximization problem of player 1 is

$$\max_x v\frac{x}{(n-1)y+x} - \alpha\beta v\left(\frac{y^{n-1}(n-1)!}{\Pi_{i=1}^{n-1}(iy+x)}\right) - (1-\alpha)\beta v\frac{y^{n-2}x(n-1)!)}{\Pi_{i=1}^{n-1}(iy+x)} - x \tag{6}$$

where $x$ is the effort of player 1, $y$ is the symmetric effort of all the other players, and $0 \leq \alpha \leq 0.5$. Then we obtain

**Proposition 6** *In the Tullock contest with $n$ symmetric players, one prize of a value $v$ and two punishments of a values $\beta v$ and $(1-\alpha)\beta v, \beta > 0$, $0 \leq \alpha \leq 0.5$, the symmetric equilibrium effort of each player is*

$$x = \frac{v(n-1)}{n^2} - \frac{(1-\alpha)\beta v - (H_n - 1)\beta v}{n} \tag{7}$$

**Proof.** See Appendix. ■

In order to analyze the players' symmetric equilibrium effort in a Tullock contest with an exit option we need to define the critical punishment sums for which players decide to exit. By (6) and (7) a player's expected payoff in the contest with $n$ symmetric players, one prize of a value $v$ and one punishment of a value $\beta v$ is

$$u_{1,n} = \frac{v(1-\beta)}{n} - \frac{v(n-1)}{n^2} - \frac{\beta v(H_n - 1)}{n} \tag{8}$$

Define the critical value $\beta^*$ for which the expected payoff of each of the $n$ symmetric players in the contest with one punishment of $\beta v$ is equal to zero. Then by (8) we have

$$\beta^* = \frac{1}{nH_n}$$

Similarly, by (6) and (7) a player's expected payoff in a contest with one prize of a value $v$ and two punishments of a values $\alpha\beta v$ and $(1-\alpha)\beta v$, $0 < \alpha \leq 0.5$, is

$$u_{\alpha,n} = \frac{v(1-\beta)}{n} - \frac{v(n-1)}{n^2} - \frac{\beta v((H_n - 1) - (1-\alpha))}{n} \tag{9}$$

Define the critical value $\beta^{**}$ for which the expected payoff of each of the $n$ symmetric players in the contest with two punishments of values $\alpha\beta v$ and $(1-\alpha)\beta v$ is equal to zero. Then by (4) we have

$$\beta^{**} = \frac{1}{n(H_n - 1 + \alpha)} \tag{10}$$

Note that for all $n > 2$, $\beta^{**} > \beta^*$, that is, in the contest with one punishment, the players have an incentive to quit for a smaller value of the punishment sum than in the contest with two punishments.

By Proposition 5, for a relatively low punishment sum it is optimal to allocate only one punishment since all the players have an incentive to participate in the contest. However, using the above critical values, the next lemma shows that when the punishment sum increases, an allocation of two punishments will be better than an allocation of only one punishment since when there is only one punishment that is equal to the punishment sum, one of the players will exit, but when there are two smaller punishments whose sum is equal to the punishment sum, all the players will participate in the contest. Then, a contest with $n$ players and two relatively small punishments yields a higher total effort than a contest with $n-1$ players and one large punishment.

**Lemma 1** *For every $\beta^{**} > \beta \geq \beta^*$, if $1 - \frac{1}{n} - \frac{1}{n(n-1)\beta} < \alpha < 1$, then the players' symmetric equilibrium effort in the Tullock contest with one punishment of $\beta v$ is smaller than in the Tullock contest with two punishments of $\alpha\beta v$ and $(1-\alpha)\beta v$.*

**Proof.** See Appendix. ■

By Lemma 1, we obtain that for all $\beta^{**} > \beta \geq \beta^{*}$, if

$$\alpha^{**} = 1 - \frac{1}{n} - \frac{1}{n(n-1)} < \alpha < 1$$

the players' symmetric equilibrium effort in a Tullock contest with one punishment of $\beta v$ is smaller than in a Tullock contest with two punishments of $\alpha\beta v$ and $(1-\alpha)\beta v$.

Define now another critical value $\beta^{***}$ for which the expected payoff of each of the $n-1$ players in the contest with one punishment of $\beta v$ is equal to zero. Then, by (8), we have

$$\beta^{***} = \frac{1}{(n-1)H_{n-1}} \tag{11}$$

By (10) and (11) we obtain that

$$\beta^{***} = \frac{1}{(n-1)H_{n-1}} > \frac{1}{n(H_n - 1 + \alpha)} = \beta^{**} \tag{12}$$

iff

$$\alpha > \frac{n-1}{n} H_{n-1} - H_n + 1 = \alpha^{***}$$

The inequality of the critical values (12) implies that a player will exit the contest with $n$ players and two punishments of $\alpha\beta v$ and $(1-\alpha)\beta v$ earlier (namely, when the punishment sum will be lower) than he will with $n-1$ players and one punishment of $\beta v$ . Thus, for all $\beta^{***} > \beta > \beta^{**}$, the number of players will be $n-1$ in both contests either with one punishments of $\beta v$ or with two punishments of $\alpha\beta v$ and $(1-\alpha)\beta v$. In that case, by Proposition 5 it is optimal to allocate one prize and therefore we have

**Lemma 2** *For every $\beta^{**} > \beta \geq \beta^{***}$ and $1 > \alpha > \alpha^{***}$, the symmetric equilibrium effort of a player in a Tullock contest with one punishment of $\beta v$ is higher than in a Tullock contest with two punishments of $\alpha\beta v$ and $(1-\alpha)\beta v$.*

By Lemmas 1 and 2 since for all $n > 2$, $\alpha^{**} > \alpha^{***}$, we obtain that for every two punishments of $\alpha\beta v$ and $(1-\alpha)\beta b$ that yield a higher symmetric equilibrium effort than one punishment of a value $\beta v$ there is a higher $\beta$ for which these two punishments yield a lower symmetric equilibrium effort than one punishment of a value $\beta v$. Thus, we have

**Proposition 7** *In the Tullock contest with an exit option, the optimal number of punishments neither increase nor decrease in the value of the punishment sum.*

We showed that in contrast to the Tullock contest without an exit option in which it is optimal to allocate the entire punishment sum to a single punishment, in a Tullock contest with an exit option, there are two intervals of values of the entire punishment sum for which it is optimal to allocate it to a single punishment but these intervals do not interact. Moreover, for the values of the punishment sum between these two intervals it is optimal to allocate the entire punishment sum to more than one punishment.

# 4 Concluding remarks

We analyzed the effect of prizes and punishments on the players' performances in the Tullock contest, and derived the optimal punishment structures in several environments with both costly and costless punishments. We showed that when the number of prizes and punishments is small (large) rewarding the top performers is less (more) effective than punishing the worst performers . Finally, we showed that if punishments are costly and players do not have the option not to participate then it is optimal to allocate the entire punishment sum to a single punishment. However, when players have the option not to participate, allocating several punishments may be optimal. Moreover, the optimal number of punishments is not a monotonic function of the level of the entire punishment sum.

# 5 Appendix

## 5.1 Proof of Proposition 1

The F.O.C. of (1) yields

$$v\sum_{i=1}^{k}\frac{y^{i-1}\left(\frac{n-1)!}{(n-i)!}\right)\Pi_{j=1}^{i}\left((n-j)y+x\right)}{\left[\Pi_{j=1}^{i}\left((n-j)y+x\right)\right]^{2}}$$
$$-\frac{\Pi_{j=1}^{i}\left((n-j)y+x\right)\Sigma_{j=1}^{i}\frac{1}{((n-j)y+x)}y^{i-1}x\left(\frac{n-1)!}{(n-i)!}\right)}{\left[\Pi_{j=1}^{i}\left((n-j)y+x\right)\right]^{2}}$$
$$= \quad v\sum_{i=1}^{k}\frac{y^{i-1}\left(\frac{n-1)!}{(n-i)!}\right)-\Sigma_{j=1}^{i}\frac{1}{((n-j)y+x)}y^{i-1}x\left(\frac{n-1)!}{(n-i)!}\right)}{\Pi_{j=1}^{i}\left((n-j)y+x\right)}=1$$

By symmetry $x=y$ and then we obtain

$$v\sum_{i=1}^{k}\frac{x^{i-1}\left(\frac{n-1)!}{(n-i)!}\right)-\Sigma_{j=1}^{i}\frac{1}{(n-j+1)x}x^{i}\left(\frac{n-1)!}{(n-i)!}\right)}{\Pi_{j=1}^{i}\left((n-j+1)x\right)}=1$$

By some simple calculations this implies

$$v\sum_{i=1}^{k}\frac{1-\Sigma_{j=1}^{i}\frac{1}{(n-j+1)}}{nx}=1$$

Then we have

$$v\sum_{i=1}^{k}\frac{1-(H_n-H_{n-1})}{nx}=1$$

where $H_n=\sum_{i=1}^{n}\frac{1}{i}$ is the harmonic series. Hence, the players' symmetric equilibrium effort in a contest with $k$ identical prizes of a value $v$ is

$$x=v\sum_{i=1}^{k}\frac{1-(H_n-H_{n-1})}{n}$$

*Q.E.D.*

## 5.2 Proof of Proposition 2

The F.O.C. of (3) yields

$$
\begin{aligned}
& -s\left(\frac{-\Pi_{i=1}^{n-1}(iy+x)\Sigma_{i=1}^{n-1}\frac{1}{(iy+x)}y^{n-1}(n-1)!}{(\Pi_{i=1}^{n-1}(iy+x))^2}\right) \\
& -s\left(\sum_{i=2}^{l}\frac{y^{n-i}(\frac{(n-1)!}{(i-1)!})\Pi_{j=i-1}^{n-1}(jy+x)-\Pi_{j=i-1}^{n-1}(jy+x)\Sigma_{j=i-1}^{n-1}\frac{1}{jy+x}y^{n-i}x(\frac{(n-1)!}{(i-1)!})}{(\Pi_{j=i-1}^{n-1}(iy+x))^2}\right) \\
= & -s\left(\frac{\Sigma_{i=1}^{n-1}\frac{1}{(iy+x)}y^{n-1}(n-1)!}{\Pi_{i=1}^{n-1}(iy+x)}\right)-s\left(\sum_{i=2}^{l}\frac{y^{n-i}(\frac{(n-1)!}{(i-1)!})-\Sigma_{j=i-1}^{n-1}\frac{1}{jy+x}y^{n-i}x(\frac{(n-1)!}{(i-1)!})}{\Pi_{j=i-1}^{n-1}(iy+x)}\right) \\
= & \ 1
\end{aligned}
$$

By symmetry $x = y$ and then we have

$$
s\left(\frac{\Sigma_{i=1}^{n-1}\frac{1}{(i+1)x}x^{n-1}(n-1)!}{\Pi_{i=1}^{n-1}(i+1)x}\right)-s\left(\sum_{i=2}^{l}\frac{x^{n-i}(\frac{(n-1)!}{(i-1)!})-\Sigma_{j=i-1}^{n-1}\frac{1}{(j+1)x}x^{n}(\frac{(n-1)!}{(i-1)!})}{\Pi_{j=i-1}^{n-1}(i+1)x}\right)=1
$$

By some simple calculations this implies

$$
s\left(\frac{H_n-1}{nx}\right)-s\left(\sum_{i=2}^{l}\frac{1-(H_n-H_{i-1})}{nx}\right)=1
$$

where $H_n = \sum_{i=1}^{n}\frac{1}{i}$ is the harmonic series. Hence, the players' symmetric equilibrium effort in a contest with $l$ identical punishments of a value $s$ is

$$
x = s\left(\frac{(H_n-1)-\Sigma_{i=2}^{l}(H_n-H_{i-1})-1}{n}\right)
$$

*Q.E.D.*

# 6 Proof of Proposition 3

By (2) the symmetric equilibrium effort with $k$ identical prizes of a value of $v$ is

$$
x = v\sum_{i=1}^{k}\frac{1-(H_n-H_{n-i})}{n}
$$

and by (4) the symmetric equilibrium effort with $l$ identical punishments of a value of $v$ is

$$
x = v\left(\frac{(H_n-1)-\Sigma_{i=2}^{l}(H_n-H_{i-1})-1}{n}\right)
$$

Thus, in order to show the equivalence between a contest with $k$ identical prizes and a contest with $n-k$ identical punishment we need to show that

$$\sum_{i=1}^{k} 1-(H_n-H_{i-1}) = (H_n-1)-\sum_{i=2}^{n-k}((H_n-H_{i-1})-1)$$

or alternatively that

$$H_n+\sum_{i=2}^{k}(H_n-H_{i-1}) = n-\sum_{i=1}^{n-k}(H_n-H_{n-i})$$

This equation is equivalent to

$$\begin{aligned} n-H_n &= \sum_{i=2}^{k}(H_n-H_{i-1})+\sum_{i=1}^{n-k}(H_n-H_{n-i}) \\ &= (n-1)H_n-(\sum_{i=2}^{k}H_{i-1}+\sum_{i=1}^{n-k}H_{n-i}) \\ &= (n-1)H_n-(\sum_{i=1}^{k-1}H_i+\sum_{i=1}^{n-k}H_{n-i}) \end{aligned}$$

Using the identity

$$\sum_{i=1}^{k-1}H_i+\sum_{i=1}^{n-k}H_{n-i}=\sum_{i=1}^{n-1}H_i$$

we obtain that we need to show that

$$n = nH_n-\sum_{i=1}^{n-1}H_i$$

Since

$$\sum_{i=1}^{n-1}H_i=(n-1)1+(n-2)\frac{1}{2}+...+(n-(n-1))\frac{1}{n-1}$$

We indeed obtain that

$$\begin{aligned} nH_n-\sum_{i=1}^{n-1}H_i &= (n+\frac{n}{2}+\frac{n}{3}+...+\frac{n}{n-1}+\frac{n}{n})-(\frac{n-1}{1}+\frac{n-2}{2}+...+\frac{1}{n-1}) \\ &= (n-1)1+\frac{n}{n}=n \end{aligned}$$

$Q.E.D.$

## 6.1 Proof of Proposition 4

We first assume that $k < \frac{n}{2}$ (a similar proof holds when $k > \frac{n}{2}$ ) and show that the symmetric equilibrium effort in the Tullock contest with $k < \frac{n}{2}$ punishments of a value $-v$ is higher than in a Tullock contest with $k$ prizes of a value of $v$. By (2) and (4) we need to show that

$$v\left(\frac{(H_n-1)-\Sigma_{i=2}^{k}1-(H_n-H_{i-1})}{n}\right) \geq v\sum_{i=1}^{k}\frac{1-(H_n-H_{n-i})}{n}$$

By some calculations this inequality is equivalent to

$$2k(H_n-1) > \sum_{i=1}^{k}H_{n-i}+\sum_{i=2}^{k}H_{i-1}=\sum_{i=1}^{k-1}H_i+\sum_{i=n-k}^{n-1}H_{i-1}$$

or alternatively to

$$\begin{aligned} & 2k(\frac{1}{2}+\frac{1}{3}+...+\frac{1}{k-1}+\frac{1}{k}+...+\frac{1}{(n-k)}+...+\frac{1}{n-1}+\frac{1}{n}) \\ > \; & (2k-1)+(2k-2)\frac{1}{2}+...+(k+1)\frac{1}{k-1}+ \\ & +k\frac{1}{k}+...+k\frac{1}{n-k}+(k-1)\frac{1}{n-k+1}+...+2\frac{1}{n-2}+\frac{1}{n-1} \end{aligned}$$

By moving elements from the RHS to the LHS we have

$$\frac{k}{k+1}+\frac{k}{k+2}+...+\frac{k}{n-k}+\frac{k+1}{n-k+1}+...+\frac{2k-1}{n-1}+\frac{2k}{n} > k \tag{13}$$

Note that for all $1 \leq j < n-k$,

$$\frac{k}{n-k} < \frac{k}{k+j}$$

and for all $1 \leq j \leq k$

$$\frac{k}{n-k} < \frac{k+j}{n-k+j}$$

Then, we have

$$\begin{aligned}
&\frac{k}{k+1}+\frac{k}{k+2}+...+\frac{k}{n-k}+\frac{k+1}{n-k+1}+...+\frac{2k-1}{n-1}+\frac{2k}{n} \\
> \quad &(n-k)\frac{k}{n-k}=k
\end{aligned}$$

That is, the inequality (13) holds. *Q.E.D.*

## 6.2 Proof of Proposition 5

The F.O.C. of (5) is

$$\begin{aligned}
&-v_1\left(\frac{\Pi_{i=1}^{n-1}(iy+x)\Sigma_{i=1}^{n-1}\frac{1}{(iy+x)}y^{n-1}(n-1)!}{(\Pi_{i=1}^{n-1}(iy+x))^2}\right) \\
&-\sum_{i=2}^{n-1}v_i\left(\frac{y^{n-i}\frac{(n-1)!}{(i-1)!}\Pi_{j=i-1}^{n-1}(jy+x)-\Pi_{j=i-1}^{n-1}(jy+x)\Sigma_{j=i-1}^{n-1}\frac{1}{(jy+x)}y^{n-i}x\frac{(n-1)!}{(i-1)!}}{(\Pi_{j=i-1}^{n-1}(jy+x))^2}\right)
\end{aligned}$$

By some simple calculations we obtain that the symmetric equilibrium effort in a contest with punishments of the absolute values of $v_1 \geq v_2 \geq .... \geq v_n$ is

$$x = v_1\frac{H_{n-1}}{n}+\sum_{i=2}^{n-1}v_i\frac{H_n-H_{i-1}-1}{n}$$

Thus, the designer who wishes to maximize the symmetric equilibrium effort has the following maximization problem

$$\begin{aligned}
&\max_{v_1,\ldots,v_n} v_1\frac{H_{n-1}}{n}+\sum_{i=2}^{n-1}v_i\frac{H_n-H_{i-1}-1}{n} \\
&s.t. \\
\sum_{i=1}^{n-1}v_i &= 1
\end{aligned}$$

Since

$$H_{n-1} > H_n - H_1 > H_n - H_2 > ... > H_n - H_{n-2}$$

we obtain that the symmetric equilibrium effort $x$ is maximized for $v_1 = 1$ and $v_j = 0$ , $2 \leq j \leq n-1$. *Q.E.D.*

## 6.3 Proof of Proposition 6

The F.O.C. of (6) is

$$
\begin{aligned}
& \frac{v(n-1)y}{((n-1)y+x)^2} - \left[\frac{(1-\alpha)\beta v y^{n-2}(n-1)!\Pi_{i=1}^{n-1}(iy+x)}{(\Pi_{i=1}^{n-1}(iy+x))^2}\right] \\
& + \left[\frac{\Pi_{i=1}^{n-1}(iy+x)\Sigma_{i=1}^{n-1}\frac{1}{(iy+x)}(\alpha\beta v y^{n-1}(n-1)! + (1-\alpha)\beta v y^{n-2}x(n-1)!)}{(\Pi_{i=1}^{n-1}(iy+x))^2}\right] \\
= \quad & 1
\end{aligned}
$$

By symmetry, $x = y$ and then we have

$$
\begin{aligned}
& \frac{v(n-1)x}{(nx)^2} - \left[\frac{(1-\alpha)\beta v x^{n-2}(n-1)!}{(\Pi_{i=1}^{n-1}((i+1)x))}\right] \\
& + \left[\frac{\Sigma_{i=1}^{n-1}\frac{1}{((i+1)x)}(\alpha\beta v x^{n-1}(n-1)! + (1-\alpha)\beta v y^{n-1}(n-1)!)}{(\Pi_{i=1}^{n-1}((i+1)x))}\right] \\
= \quad & 1
\end{aligned}
$$

By some simple calculations we obtain

$$
\frac{v(n-1)}{n^2 x} - \left[\frac{(1-\alpha)\beta v x^{-1} - \Sigma_{i=2}^{n}\frac{1}{ix}\beta v}{n}\right]
$$

and therefore the symmetric equilibrium effort is

$$
x = \frac{v(n-1)}{n^2} - \frac{(1-\alpha)\beta v - (H_n - 1)\beta v}{n}
$$

*Q.E.D.*

## 6.4 Proof of Lemma 1

If $\beta^* < \beta$, by (7) the total effort in a contest with one prize of a value $v$ and one punishment of a value $\beta v$, $\beta^* < \beta$, is

$$
E(1, n-1) = (n-1)x_{n-1} = \frac{v(n-2)}{n-1} + \beta v(H_{n-1} - 1)
$$

and the total effort in a contest with one prize of $v$ and two punishments of values $\alpha\beta v$ and $(1-\alpha)\beta v, 0 < \alpha \leq 0.5$, is

$$E(2,n) = nx_n = \frac{v(n-1)}{n} + \beta v((H_n - 2 + \alpha))$$

Then, we have

$$E(2,n) - E(1,n-1) = \frac{v}{n(n-1)} + \beta v(H_n - H_{n-1} - 1 + \alpha)$$

Thus, the total effort in a contest with one punishment of $\beta v$ is lower than in a contest with two punishments of $\alpha\beta v$ and $(1-\alpha)\beta b$ iff

$$\alpha \geq 1 - H_n + H_{n-1} - \frac{1}{n(n-1)\beta}$$

Note also that for all $\beta \leq 1$,

$$1 - H_n + H_{n-1} - \frac{1}{n(n-1)\beta} \leq \frac{n-1}{n} - \frac{1}{n(n-1)} < 1$$

Thus, there is always $0 < \alpha < 1$ such that the players' total effort in a contest with $n-1$ symmetric players and one punishment of a value $\beta v$ is smaller than in a contest with $n$ symmetric players and two punishments of values $\alpha\beta v$ and $(1-\alpha)\beta v$. *Q.E.D.*

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