TWO STAGE CONTESTS WITH EFFORT-DEPENDENT REWARDS

Aner Sela

Discussion Paper No. 16-12

November 2016

Monaster Center for
Economic Research
Ben-Gurion University of the Negev
P.O. Box 653
Beer Sheva, Israel

Fax: 972-8-6472941
Tel: 972-8-6472286

Two-stage contests with effort-dependent rewards

Aner Sela *

September 2, 2016

Abstract

We study two-stage all-pay contests where there is synergy between the stages. The reward for each contestant is fixed in the first stage while it is effort-dependent in the second one. We assume that a player's effort in the first stage either increases (positive synergy) or decreases (negative synergy) his reward in the second stage. The subgame perfect equilibrium of this contest is analyzed with either positive or negative synergy. We show, in particular, that whether the contestants are symmetric or asymmetric their expected payoffs may be higher under negative synergy than under positive synergy. Consequently, they prefer smaller rewards (negative synergy) over higher ones (positive synergy).

Keywords: Two-stage all-pay contests, effort-dependent rewards.

JEL classification: C70, D44, L12, O32

1 Introduction

Multi-stage contests are situations in which agents spend resources in order to win one or more prizes. The prizes are allocated either in each of the stages or only in some of them, usually in the last one. There are many architectures of multi-stage contests. Examples include best-of-k contests (see Klumpp and Polborn 2006, Harris and Vickers 1987 and Konrad and Kovenock 2009), elimination contests (see Gradstein and Konrad 1999, Groh et al. 2009 and Fu and Lu 2012) and contests which are repeated a finite number of times. The analysis of multi-stage contests is quite complicated and challenging particularly when there is

*Department of Economics, Ben-Gurion University of the Negev, Beer–Sheva 84105, Israel. Email: anersela@bgu.ac.il

synergy between the various stages of the contest. Such a synergy could occur when contestants have a fixed budget of resources and have to decide how to allocate them over the stages of the contest. Amegashie et al. (2007), for example, showed that in a two-stage elimination all-pay contest if contestants have fixed equal resources, they spend more resources in the initial rounds than in the subsequent ones. Likewise, Sela and Erez (2013) studied a dynamic contest between two contestants who compete against each other in n different stages and have heterogeneous resource budgets that decrease from a given stage to the next proportionally to the resources allocated in that stage. They showed that when the winning value is equal between the stages, the contestants' resource allocations are weakly decreasing over the stages. Another well known dynamic contest with resource allocation is the Colonel Blotto game in which two contestants compete against each other in n different contests. Each contestant distributes a fixed amount of resource over the n contests without knowing his opponent's resource distribution. In each contest, the contestant who allocates the higher level of resource wins where each contestant's payoff is a function of the sum of wins across the individual contests (see, for example, Snyder 1989, Roberson 2006, Kvasov 2007 and Hart 2008).

The literature suggests various reasons for the occurrence of synergy in multi-stage contests other than a fixed resource budget. Ryvkin (2011), for example, studied a best-of k contest in which the contestants' probabilities of winning in each stage depend on the contestants' efforts in that stage as well as their efforts in the previous stages and found that agents are more likely to exert higher efforts in the later stages of the contest. Kovenock and Roberson (2009) studied a two-stage campaign resource allocation game in which the players' difference campaign expenditures in the first stage serve as a head start advantage to the contestants in the second stage.[1]

In this paper, we study a two-stage all-pay contest in which the contestants' efforts in the first stage do not affect the contestants' success functions in the second stage (as in the above two studies) but instead affect their rewards in the later stage. There are numerous examples of rewards in contests which are not necessarily fixed and where there is a relationship between the efforts made and the size of the rewards. To illustrate, the greater the effort a student exerts for an exam at a university, the greater is his chance to

[1] The one-stage all-pay contest with discrimination in the form of a head-start advantage has been analyzed by Konrad (2002).

achieve a higher grade. Similarly, the greater the effort a firm exerts to produce a new product, the greater is the probability that the quality of the final product will improve. A last example is patent races, where the more effort a firm invests, the earlier is the innovation time and therefore the larger is the reward.

In one-stage contests, the effort–dependent rewards have been shown to have a complicated effect on contestants' behavior. For example, in one-stage all-pay contests with effort-dependent rewards under incomplete and complete information, Kaplan et al. (2002, 2003) showed that substantial qualitative changes can occur in the behavior of the contestants compared to their behavior in the same contests with constant rewards. Cohen et al. (2008) studied all-pay contests with effort-dependent rewards under incomplete information in which the value of winning the contest for each contestant depends not only on his type but also on the effort-dependent reward chosen by the designer. They showed that when the designer maximizes the contestants' expected total effort and there is a sufficiently large number of contestants, the optimal reward decreases in the contestants' effort. However, when the designer maximizes the contestants' expected highest effort, the optimal reward may increase in the contestants' effort for any number of contestants. A last example is Kaplan and Wettstein (2015) who studied the optimal effort-dependent reward in all-pay contests under complete information. Their results indicate that for asymmetric environment with two firms, it is optimal to set different rewards for each firm.

In our two-stage contest the effect of the effort-dependent reward on the contestants' behavior is even more complicated than in one-stage contests since this reward implies a synergy between the two stages. This synergy might be either positive or negative, namely, the contestants' efforts in the first stage may either increase or decrease their rewards in the second stage. The reason for a positive effect is that the effort in the first stage may increase the value of winning in the second stage. This can occur, for example, in a two-stage R&D contest when a contestant acquires some knowledge and experience in the first stage which increases the benefit from winning in the second stage. The reason for a negative effect is that the effort in the first stage may decrease the value of winning in the second stage. This can happen when the contestant's reward in the second stage is a function of his budget of effort which decreases within the stages. Thus, the contestant's effort in the first stage decreases his budget constraint in the second stage and as such decreases his reward in that stage.

In our two-stage all-pay contest, in each stage each contestant exerts an effort and the contestant who exerts the highest effort receives the reward, but, independently of success, all the contestants bear the cost of their efforts.[2] The contestants compete in each stage for a different reward. The reward in the first stage is fixed but in the second stage it is variable with a constant marginal increasing (decreasing) rate; namely, a contestant's reward in the second stage is a linear function of his effort in the first stage.

With symmetric contestants who have the same rewards, we show that if there is positive synergy between the stages (each contestant's effort in the first stage increases his reward in the second stage) both contestants have an expected payoff of zero, but, if there is negative synergy between the stages (each contestant's effort in the first stage decreases his reward in the second stage), both contestants have positive expected payoffs. This result is in contrast to the standard one-stage all-pay contest in which symmetric contestants always have an expected payoff of zero. However, with asymmetric contestants who have different fixed prizes in the first stage, we show that if there is positive synergy between the stages, the stronger contestant (the contestant with the higher reward in the first stage) has a positive expected payoff while the weaker contestant has an expected payoff of zero. However, if the synergy between the stages is negative, both contestants have positive expected payoffs.

Furthermore, regardless of whether the contestants are symmetric or asymmetric and whether the effort-dependent reward in the second stage is increasing or decreasing in the contestants' efforts, we find that the contestants' expected payoffs are non-decreasing in the (absolute) value of the marginal increasing (decreasing) rate of the effort-dependent reward. That is, when the synergy is either negative or positive, the higher the effect of the contestant's effort in the first stage is on his reward in the second stage, the higher is his expected payoff. In particular, when the synergy is negative, lower rewards in the second stage lead to higher expected payoffs. The reason is that when all contestants have lower rewards, the contestants' expected payoffs which are based on the difference of their rewards do not necessarily decrease.

We also compare the contestants' expected payoffs under positive and negative synergies when the marginal increasing rate and the marginal decreasing rate of the effort-dependent reward have the same absolute value. We find that if the contestants are symmetric, they have either the same expected payoff or a higher

[2] The all-pay contest has been analyzed, among others, by Kovenock and de Vries (1993), Che and Gale (1998), Amman and Leininger (1996), Krishna and Morgan (1997), Moldovanu and Sela (2001, 2006), Gavious et al. (2003) and Siegel (2009).

expected payoff under negative synergy than under positive one. However, if the contestants are asymmetric, the weaker contestant (the contestant with the lower type in the first stage) has either the same expected payoff or a higher expected payoff under negative synergy than under positive one. On the other hand, the stronger contestant (the contestant with the higher value in the first stage), depending on the value of the marginal increasing/decreasing rate, has either a lower or a higher expected payoff under negative synergy than under positive synergy. Thus, even when contestants are asymmetric, they both may prefer that their rewards in the second stage will decrease and not increase in their efforts of the previous stage. In other words, each contestant prefers lower rewards for all the contestants over higher ones. The reason is that under negative synergy the contestants have lower rewards than under positive synergy, but, on the other hand, the contestants exert smaller efforts. Last, when the value of the marginal increasing/decreasing rate is sufficiently high (close enough to 1) then the preferences of the asymmetric contestant are no longer identical, since the stronger contestant has a higher expected payoff under positive synergy and the weaker contestant has a higher expected payoff under negative synergy.

In a related paper (Sela 2012), we studied sequential two-prize all-pay contests under complete information where the prizes are identical; each contestant may win more than one prize; and each contestant's marginal values for the first and the second prize are either decreasing, constant or increasing. In such a case the contestants' strategies in the first stage affect what their prizes will be in the second stage. In contrast, the synergy between the stages in our present model does not depend on the identity of the winner in the first stage but instead on each contestant's effort in that stage.

The rest of the paper is organized as follows: In sections 2 and 3, we analyze the two-stage all-pay contest with positive and negative synergies. In section 4, we compare between the results of the analyses in the two previous sections and in section 5 we conclude.

2 The two-stage all-pay contest with positive synergy

We begin by considering a two-stage all-pay contest with two contestants, 1 and 2 that compete against each other in the first stage where contestant i's reward in that stage is $v_1^i, i = 1, 2$. Contestant j's expected utility is $u_1^i = v_1^i - x_1^i$ if $x_1^i > x_1^{-i}$ and otherwise $u_1^i = -x_1^i$, where x_1^i, x_1^{-i} are the contestants' efforts in that stage. We assume that there is positive synergy between the two stages, namely, the effort of each contestant positively affects his reward in the second stage such that contestant i's effort-dependent reward is $v_2^i + \alpha x_1^i$, $i = 1, 2$ where $0 < \alpha < 1$ is the marginal increasing rate and x_1^i is contestant i's effort in the first stage. The contestants observe the efforts in the first stage and then choose their efforts x_2^1, x_2^2 in the second stage such that contestant i's expected utility in the second stage is $u_2^i = v_2^i + \alpha x_1^i - x_2^i$ if $x_2^i > x_2^{-i}$ and otherwise $u_2^i = -x_2^i$. The goal of each contestant is to maximize his expected utility. Henceforth, we refer to this model as a two-stage all-pay contest with positive synergy.

In the following we assume that the contestants may have different rewards in the first stage, where $v_1^1 \geq v_1^2$, but have the same effort-dependent reward function $v_2 + \alpha x_2^i$ in the second stage. In order to analyze a subgame perfect equilibrium of the symmetric two-stage all-pay contest we begin with the second stage and go backwards to the first one.

2.1 The second stage

Assume without loss of generality that the contestants' strategies in the first stage are x_1^1, x_1^2 where $x_1^1 \geq x_1^2$. Then, according to Hillman and Riley (1989) and Baye, Kovenock and de Vries (1996, 2012), there is always a unique mixed-strategy equilibrium in which contestants 1 and 2 randomize on the interval $[0, v_2 + \alpha x_1^2]$ according to their effort cumulative distribution functions $G_{1-p}(y)$, $G_{2-p}(y)$ which are given by

$$
\begin{aligned}
(v_2 + \alpha x_1^1)G_{2-p}(y) - y &= \alpha(x_1^1 - x_1^2) \\
(v_2 + \alpha x_1^2)G_{1-p}(y) - y &= 0
\end{aligned}
$$

where $v_2 + \alpha x_1^i$ is contestant i's effort-dependent reward in the second stage. Thus, contestant 1's equilibrium effort in the second stage is distributed according to the cumulative distribution function

$$G_{1-p}(y) = \frac{y}{v_2 + \alpha x_1^2}$$

while contestant 2's equilibrium effort is distributed according to the cumulative distribution function

$$G_{2-p}(y) = \frac{\alpha(x_1^1 - x_1^2) + y}{v_2 + \alpha x_1^1}$$

The respective expected payoffs are

$$\begin{aligned} E^1_{2-p} &= \alpha(x^1_1 - x^2_1) \\ E^2_{2-p} &= 0 \end{aligned} \tag{1}$$

Thus, contestant 1's expected payoff is positive while contestant 2's expected payoff is zero. The contestants' probabilities to win are

$$\begin{aligned} p^1_{2-p} &= 1 - \frac{v_2 + \alpha x^2_1}{2v_2 + \alpha(x^1_1 + x^2_1)} \\ p^2_{2-p} &= \frac{v_2 + \alpha x^2_1}{2v_2 + \alpha(x^1_1 + x^2_1)} \end{aligned}$$

The contestant who wins the reward in the first stage (contestant 1) has a higher probability ($p^1_2 \geq 0.5$) to win the reward in the second stage. The contestants' strategies in the second stage are well known from the analysis of the one-stage all-pay contest, but their strategies in the first stage have yet to be determined which we proceed to do in the following section.

2.2 The first stage

If contestant 1 wins in the first stage, his reward is v^1_1 and then the contestants' effort-dependent rewards in the second stage satisfy $v_2 + \alpha x^1_1 > v_2 + \alpha x^2_1$. Thus, if contestant 1 wins in the first stage, by (1) his expected payoff in the second stage is positive and equals $\alpha(x^1_1 - x^2_1)$. On the other hand, by (1), if contestant 1 loses in the first stage his expected payoff in the second stage is zero. A similar argument holds for contestant 2. Thus, there is always a mixed-strategy equilibrium in which contestants 1 and 2 randomize on the interval interval $[0, x_{\max - p_a}]$ according to their effort cumulative distribution functions $F_{1-p}(x), F_{2-p}(x)$ which are given by

$$\begin{aligned} v^1_1 F_{2-p}(x) + \int_0^x \alpha(x-t) f_{2-p}(t)dt - x &= c_1 \\ v^2_1 F_{1-p}(x) + \int_0^x \alpha(x-t) f_{1-p}(t)dt - x &= c_2 \end{aligned} \tag{2}$$

where $f_{i-p}(x) = F'_{i-p}(x), i = 1, 2$; $x_{\max - p_a}$ is the highest possible effort of each of the contestants, and $c_i, i = 1, 2$, is contestant i's expected payoff. The derivative of (2) yields

$$\begin{aligned} v_1^1 f_{2-p}(x) + \alpha F_{2-p}(x) - 1 &= 0 \\ v_1^2 f_{1-p}(x) + \alpha F_{1-p}(x) - 1 &= 0 \end{aligned} \tag{3}$$

The solution of (3) implies that

Proposition 1 *In the subgame perfect equilibrium of the two-stage all-pay contest, if the contestants are asymmetric, $v_1^1 > v_1^2$, and there is positive synergy, player 1's equilibrium effort in the first stage is distributed according to*

$$F_{1-p}(x) = \frac{1}{\alpha} - \frac{1}{\alpha} e^{-\frac{\alpha x}{v_1^2}}$$

player 2's equilibrium strategy in the first stage is distributed according to

$$F_{2-p}(x) = \frac{1}{\alpha} - \frac{(1-\alpha)^{\frac{v_1^1 - v_1^2}{v_1^1}}}{\alpha} e^{-\frac{\alpha x}{v_1^1}}$$

where the highest possible effort exerted by each of the contestants is

$$x_{\max - p_a} = \frac{-v_1^2}{\alpha} \ln(1-\alpha)$$

Proof. We can see that the function $F_{i-p}(x)$, $i = 1, 2$, is well defined, strictly increasing on $[0, x_{\max - p_a}]$, continuous, and that $F_{1-p}(0) = 0$, $F_{2-p}(0) = \frac{1}{\alpha} - \frac{(1-\alpha)^{\frac{v_1^1 - v_1^2}{v_1^1}}}{\alpha} > 0$ and $F_{1-p}(x_{\max - p_a}) = F_{2-p}(x_{\max - p_a}) = 1$. Thus, $F_{i-p}(x)$, $i = 1, 2$ are cumulative distribution functions of continuous probability distributions supported on $[0, x_{\max - p_a}]$. In order to see that the above strategies are an equilibrium, note that when contestant 2 uses the mixed strategy $F_{2-p}(x)$, contestant 1's expected payoff is

$$EP_{1-p} = v_1^1 \left(\frac{1}{\alpha} - \frac{(1-\alpha)^{\frac{v_1^1 - v_1^2}{v_1^1}}}{\alpha}\right) \tag{4}$$

for any effort $x \in [0, x_{\max - p_a}]$. Since it can be easily shown that efforts above $x_{\max - p_a}$ would lead to a lower expected payoff than $v_1^1 (\frac{1}{\alpha} - \frac{(1-\alpha)^{\frac{v_1^1 - v_1^2}{v_1^1}}}{\alpha})$ for contestant 1, any effort in $[0, x_{\max - p_a}]$ is a best response of contestant 1 to $F_{2-p}(x)$. Similarly, when contestant 1 uses the mixed strategy $F_{1-p}(x)$, contestant 2's expected payoff is

$$EP_{2-p} = 0$$

for any effort $x \in [0, x_{\max - p_a}]$. Again, since it can be easily shown that efforts above $x_{\max - p_a}$ would result in a non-positive expected payoff for contestant 2, any effort in $[0, x_{\max - p_a}]$ is a best response of contestant 2 to $F_{1-p}(x)$. Hence, the pair $(F_{1-p}(x), F_{2-p}(x))$ is a mixed strategy equilibrium. ■

In the symmetric case when $v_1^1 = v_1^2 = v_1$, we obtain that

$$F_{1-p}(x) = F_{2-p}(x) = \frac{1}{\alpha} - \frac{1}{\alpha} e^{-\frac{\alpha x}{v_1}}$$

and both contestants' expected payoff is

$$EP_p = v_1 + \int_0^{\frac{-v_1}{a}\ln(1-a)} a(x-t)\frac{1}{v_1}e^{-\frac{at}{v_1}}\,dt + \frac{v_1}{a}\ln(1-\alpha) = 0$$

By (4), we have

$$\frac{dEP_{1-p}}{d\alpha} = \frac{1}{\alpha^2 (1-\alpha)^{\frac{v_1^2}{v_1^1}}} \left(v_1^1 - v_1^1 (1-\alpha)^{\frac{v_1^2}{v_1^1}} - \alpha v_1^2 \right) \tag{5}$$

In order to find whether $\frac{dEP_{1-p}}{d\alpha}$ is positive or negative, define $g(\alpha) = \left(v_1^1 - v_1^1 (1-\alpha)^{\frac{v_1^2}{v_1^1}} - \alpha v_1^2 \right)$. Since $g(0) = 0$ and $g'(\alpha) = v_1^2 \left(\frac{1}{(1-\alpha)^{\frac{v_1^1 - v_1^2}{v_1^1}}} - 1 \right) \geq 0$, we obtain that $g(\alpha) \geq 0$ for all $0 < \alpha < 1$ and therefore $\frac{dEP_{1-p}}{d\alpha} \geq 0$. In other words, the stronger contestant's (contestant 1) expected payoff increases in the marginal increasing rate α.

Let $v_1^1 = 2$ and $v_1^2 = 1$. Then, in the following figure we can present the stronger contestant's expected payoff as a function of the marginal increasing rate α.

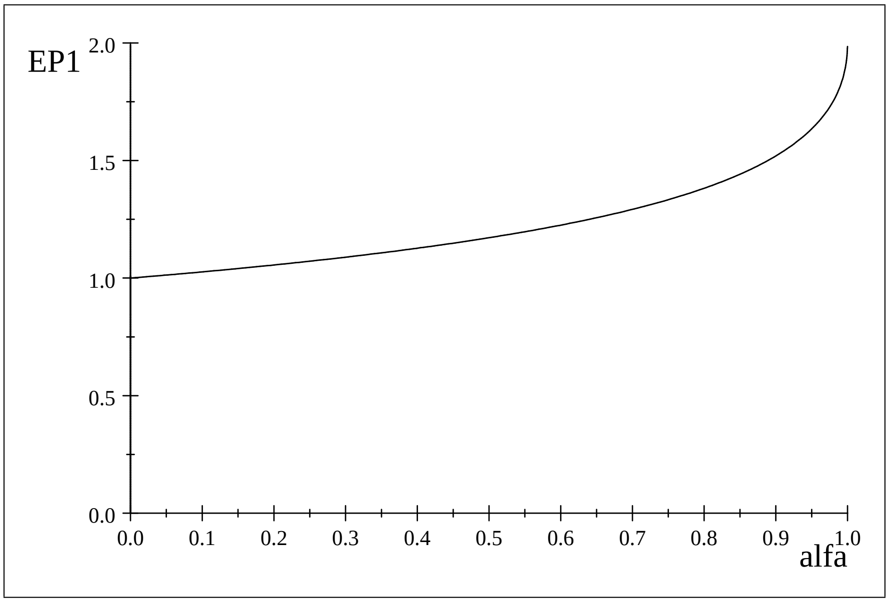

Figure 1: Contestant 1's expected payoff as a function of the marginal increasing rate α.

Figure 1 shows that when the marginal increasing rate α approaches 1, contestant 1's expected payoff converges to the value of his reward in the first stage.

The contestants' probabilities to win in the first stage are given by

$$\begin{aligned} p_{1-p} &= \int_{0}^{\frac{-v_1^2}{a}\ln(1-\alpha)} \frac{1}{v_1^2} e^{-\frac{\alpha x}{v_1^2}} [\frac{1}{\alpha} - \frac{(1-\alpha)^{\frac{v_1^1-v_1^2}{v_1^1}}}{\alpha} e^{-\frac{\alpha x}{v_1^1}}] dx \\ &= \frac{1}{\alpha}\left(\frac{1}{\alpha} v_1^1 \frac{(1-\alpha)^{\frac{v_1^1-v_1^2}{v_1^1}}}{v_1^1+v_1^2}\left((1-\alpha)^{\frac{v_1^1+v_1^2}{v_1^1}} - 1\right) + 1\right) \end{aligned}$$

and by $p_{2-p} = 1 - p_{1-p}$. It can be easily verified that when $v_1^1 = v_1^2$, we obtain that $p_{1-p} = p_{2-p} = \frac{1}{2}$. We also have that

$$\frac{dp_{1-p}}{d\alpha} = -\frac{1}{\alpha^3 (1-\alpha)^{\frac{v_1^2}{v_1^1}} (v_1^1 + v_1^2)} \left((2v_1^1 - \alpha v_1^1 + \alpha v_1^2)(1-\alpha)^{\frac{v_1^2}{v_1^1}} - (2-\alpha)v_1^1 + \alpha v_1^2 \right)$$

In order find whether $\frac{dp_{1-p}}{d\alpha}$ is positive or negative define

$$g(\alpha) = \left((2v_1^1 - \alpha v_1^1 + \alpha v_1^2)(1-\alpha)^{\frac{v_1^2}{v_1^1}} - (2-\alpha)v_1^1 + \alpha v_1^2 \right)$$

Since $v_1^1 \geq v_1^2$, we have

$$
\begin{aligned}
g(\alpha) &\leq \left((2v_1^1 - \alpha v_1^1 + \alpha v_1^1)\,(1-\alpha)^{\frac{v_1^2}{v_1^1}} - (2-\alpha)v_1^1 + \alpha v_1^1 \right) \\
&= 2v_1^1((1-\alpha)^{\frac{v_1^2}{v_1^1}} - 1) \leq 0
\end{aligned}
$$

Thus, we obtain that $\frac{dp_{1-n}}{d\alpha} > 0$ for all $0 < \alpha < 1$. In other words, the probability of the stronger contestant (contestant 1) to win in the first stage increases in the marginal increasing rate α.

Let $v_1^1 = 2$ and $v_1^2 = 1$. Then, in the following figure we can present the stronger contestant's probability to win in the first stage as a function of the marginal increasing rate α.

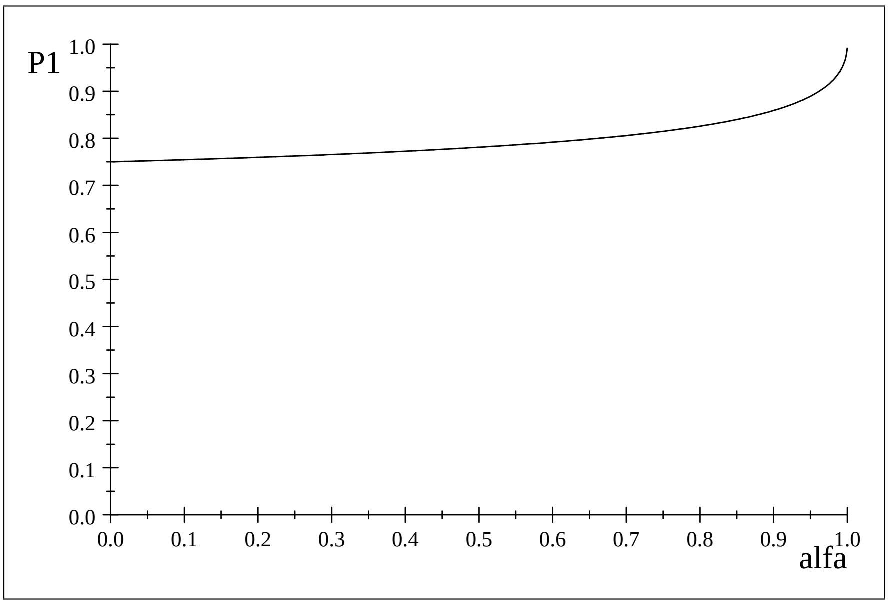

Figure 2: Contestant 1's probability to win in the first stage as a function of the marginal increasing rate α.

Figure 2 shows that when the marginal increasing rate α approaches 1, contestant 1's probability to win in the first stage converges to 1.

3 The two-stage contest with negative synergy

We consider again a two-stage all-pay contest with two contestants, 1 and 2 that compete against each other in the first stage where contestant i's reward is $v_1^i, i = 1, 2$. Contestant i's expected utility is $u_1^i = v_1^i - x_1^i$ if $x_1^i > x_1^{-i}$ and otherwise $u_1^i = -x_1^i$ where x_1^i, x_1^{-i} are the contestants' efforts in the first stage. We assume now that there is negative synergy between the two stages, namely, the effort of each contestant negatively affects his reward in the second stage such that contestant i's reward is $v_2^i - \alpha x_1^i, i = 1,2$ where $0 < \alpha < 1$ is the marginal decreasing rate and x_1^i is contestant i's effort in the first stage. The contestants observe the efforts in the first stage and then choose their efforts x_2^1, x_2^2 in the second stage such that contestant i's expected utility is $u_2^i = v_2^i - \alpha x_1^i - x_2^i$ if $x_2^i > x_2^{-i}$ and otherwise $u_2^i = -x_2^i$. Henceforth, we refer to this model as a two-stage all-pay contest with negative synergy.

In the following we assume that the contestants may have different rewards in the first stage, $v_1^1 \geq v_1^2$, but have the same effort-dependent reward function $v_2 - \alpha x_2^i$ in the second stage. In order to analyze a subgame perfect equilibrium of the symmetric two-stage all-pay contest we begin with the second stage and go backwards to the first stage.

3.1 The second stage

Similarly to the previous section, the contestants' strategies in the first stage are x_1^1, x_1^2 where $x_1^1 \geq x_1^2$. Then, according to Hillman and Riley (1989) and Baye, Kovenock and de Vries (1996, 2012), there is always a unique mixed-strategy equilibrium in which contestants 1 and 2 randomize on the interval $[0, v_2 - \alpha x_1^1]$ according to their effort cumulative distribution functions $G_{1-n}(y)$, $G_{2-n}(y)$ which are given by

$$\begin{aligned}(v_2 - \alpha x_1^1)G_{2-n}(y) - y &= 0 \\ (v_2 - \alpha x_1^2)G_{1-n}(y) - y &= \alpha(x_1^1 - x_1^2)\end{aligned}$$

where $v_2 - \alpha x_1^i$ is contestant i's effort-dependent reward in the second stage. Thus, contestant 2's equilibrium effort is distributed according to the cumulative distribution function

$$G_{2-n}(y) = \frac{y}{v_2 - \alpha x_1^1}$$

while contestant 1's equilibrium effort is distributed according to the cumulative distribution function

$$G_{1-n}(y) = \frac{\alpha x_1^1 - \alpha x_1^2 + y}{v_2 - \alpha x_1^2}$$

The respective expected payoffs are

$$\begin{aligned}E_{2-n}^1 &= 0 \\ E_{2-n}^2 &= \alpha(x_1^1 - x_1^2)\end{aligned} \tag{6}$$

In other words, the expected payoff of contestant 1 (the winner in stage 1) in the second stage is zero while the expected payoff of contestant 2 (the loser in stage 1) is positive. The contestants' probabilities to win in the second stage are

$$\begin{aligned} p^1_{2-n} &= \frac{v_2 - \alpha x^1_1}{2v_2 - \alpha(x^1_1 + x^2_1)} \\ p^2_{2-n} &= 1 - \frac{v_2 - \alpha x^1_1}{2v_2 - \alpha(x^1_1 + x^2_1)} \end{aligned}$$

Thus, the contestant who wins the reward in the first stage (contestant 1) has a lower probability ($p^1_{2-n} < 0.5$) to win the reward in the second stage. Now, given the equilibrium strategies in the second stage we can analyze the equilibrium strategies in the first stage, as done in the following section.

3.2 The first stage

If contestant 1 wins in the first stage his reward is v^1_1 and then the contestants' effort-dependent rewards in the second stage satisfy $v_2 - \alpha x^1_1 < v_2 - \alpha x^2_1$. Thus, by (6) his expected payoff in the second stage is zero. On the other hand, if contestant 1 loses in the first stage, by (6) his expected payoff in the second stage is $\alpha(x^1_1 + x^2_1)$. A similar argument holds for contestant 2. Thus, there is always a mixed-strategy equilibrium in which contestants 1 and 2 randomize on the interval $[0, x_{\max -n_a}]$ according to their effort cumulative distribution functions $F_{1-n}(x), F_{2-n}(x)$ which are given by

$$\begin{aligned} v^1_1 F_{2-n}(x) + \int_x^{x_{\max -n}} \alpha(t-x) f_{2-n}(t)dt - x &= c_1 \\ v^2_1 F_{1-n}(x) + \int_x^{x_{\max -n}} \alpha(t-x) f_{1-n}(t)dt - x &= c_2 \end{aligned} \tag{7}$$

where $f_{i-n}(x) = F'_{i-n}(x), i = 1, 2$; $x_{\max -n_a}$ is the highest possible effort of both contestants and $c_i, i = 1, 2$ is contestant i's expected payoff in the two-stage contest. The derivative of (7) yields

$$\begin{aligned} v^1_1 f_{2-n}(x) - \alpha + \alpha F_{2-n}(x) - 1 &= 0 \\ v^2_1 f_{1-n}(x) - \alpha + \alpha F_{1-n}(x) - 1 &= 0 \end{aligned} \tag{8}$$

The solution of (8) implies that

Proposition 2 *In the subgame perfect equilibrium of the two-stage all-pay contest, if the contestants are asymmetric, $v^1_1 > v^2_1$, and there is negative synergy, player 1's equilibrium effort in the first stage is distributed* *according to*

$$F_{1-n}(x)=\frac{1+\alpha}{\alpha}-\frac{1+\alpha}{\alpha}e^{-\frac{\alpha x}{v_1^2}}$$

player 2's equilibrium effort in the first stage is distributed according to

$$F_{2-n}(x)=\frac{1+\alpha}{\alpha}-\frac{(1+\alpha)^{\frac{v_1^2}{v_1^1}}}{\alpha}e^{-\frac{\alpha x}{v_1^1}}$$

where the highest possible effort exerted by each of the contestants is

$$x_{\max-n_a}=\frac{v_1^2}{\alpha}\ln(1+\alpha)$$

Proof. It is clear that the functions $F_{i-n}(x)$, $i=1,2$, are well defined, strictly increasing on $[0,x_{\max-n_a}]$, continuous, and that $F_{1-n}(0)=0$, $F_{2-n}(0)=\frac{1+\alpha}{\alpha}-\frac{(1+\alpha)^{\frac{v_1^2}{v_1^1}}}{\alpha}$ and $F_{1-n}(x_{\max-n})=F_2(x_{\max-n})=1$. Thus, $F_{i-n}(x)$, $i=1,2$ are cumulative distribution functions of continuous probability distributions supported on $[0,x_{\max-n}]$.

In order to see that the above strategies are an equilibrium, note that when contestant 2 uses the mixed strategy $F_{2-n}(x)$, contestant 1's expected payoff is

$$EP_{1-n}=v_1^1-\frac{v_1^2}{\alpha}\ln(1+\alpha) \tag{9}$$

for any effort $x\in[0,x_{\max-n_a}]$. Since it can be easily shown that efforts above $x_{\max-n_a}$ would lead to a lower expected payoff than $v_1^1-\frac{v_1^2}{\alpha}\ln(1+\alpha)$ for contestant 1, any effort in $[0,x_{\max-n_a}]$ is a best response of contestant 1 to $F_{2-n}(x)$. Similarly, when contestant 1 uses the mixed strategy $F_{1-n}(x)$, contestant 2's expected payoff is

$$EP_{2-n}=v_1^2-\frac{v_1^2}{\alpha}\ln(1+\alpha)$$

for any effort $x\in[0,x_{\max-n_a}]$. Again, since it can be easily shown that efforts above $x_{\max-n}$ would result in a lower expected payoff than $v_1^2-\frac{v_1^2}{\alpha}\ln(1+\alpha)$ for contestant 2, any effort in $[0,x_{\max-n_a}]$ is a best response of contestant 2 to $F_{1-n}(x)$. Hence, the pair $(F_{1-n}(x),F_{2-n}(x))$ is a mixed strategy equilibrium. ■

In the symmetric case when $v_1^1=v_1^2=v_1$, we obtain that

$$F_{1-n}(x)=F_{2-n}(x)=\frac{1+\alpha}{\alpha}-\frac{1+\alpha}{\alpha}e^{-\frac{x\alpha}{v_1}}$$

and both contestants' expected payoff is

$$EP_n = v_1 - \frac{v_1}{\alpha}\ln(1+\alpha) > 0$$

Moreover, even when the contestants are asymmetric, by (7), in contrast to the case with positive synergy, both contestants' expected payoffs are positive. For $i = 1, 2$ we have

$$\begin{aligned} \frac{dEP_{i-n}}{d\alpha} &= \frac{1}{\alpha^2}\frac{v_1^2}{\alpha+1}\left(\ln(\alpha+1) - \alpha + \alpha\ln(\alpha+1)\right) \\ &= \frac{1}{\alpha^2}\frac{v_1^2}{\alpha+1}\left(\ln(\alpha+1)(1+\alpha) - \alpha\right) \end{aligned} \tag{10}$$

In order to find whether $\frac{dEP_{i-n}}{d\alpha}$ is positive or negative, define $g(\alpha) = (\ln(\alpha+1)(1+\alpha) - \alpha)$. Also note that $g(0) = 0$ and $g'(\alpha) = \ln(\alpha+1) \geq 0$. Thus, we obtain that $g(\alpha) \geq 0$ and therefore for all $0 < \alpha < 1$, $\frac{dEP_{i-n}}{d\alpha} \geq 0$, $i = 1, 2$. In other words, both contestants' expected payoffs increase in the absolute value of the marginal decreasing rate α.

Let $v_1^2 = 1$ and $v_1^1 = 2$. Then, in the following figure we can present the contestants' expected payoffs as functions of the marginal decreasing rate α.

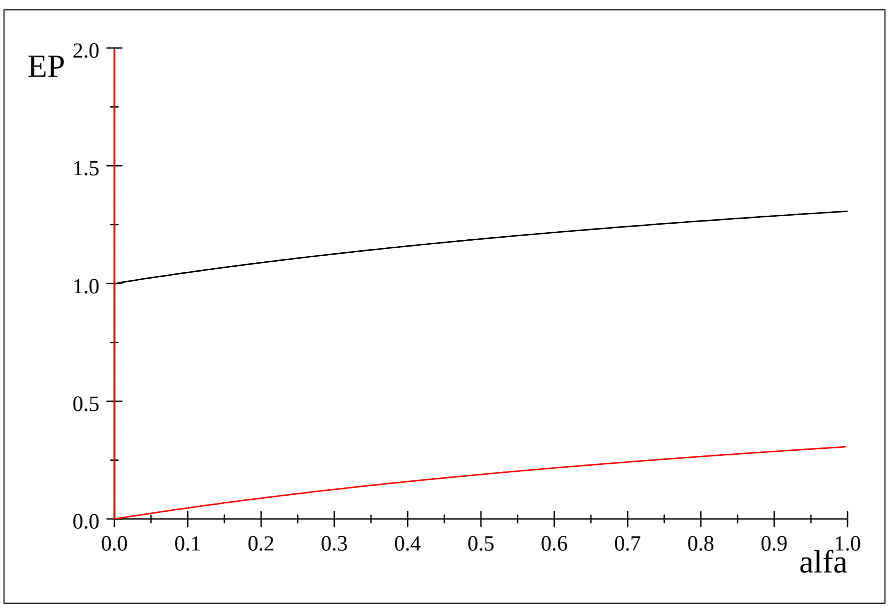

Figure 3: Contestant 1's expected payoff (the black curve) and contestant 2's expected payoff (the red curve) as functions of α.

The contestants' probabilities to win in the first stage are given by

$$
\begin{aligned}
p_{1-n} &= \int\limits_{0}^{\frac{v_1^2}{a}\ln(1+\alpha)} \frac{1+\alpha}{v_1^2} e^{-\frac{\alpha x}{v_1^2}} \left[\frac{1+\alpha}{\alpha} - \frac{(1+\alpha)^{\frac{v_1^2}{v_1^1}}}{\alpha} e^{-\frac{\alpha x}{v_1^1}} \right] dx \\
&= \frac{1+\alpha}{\alpha} - \frac{v_1^1}{\alpha^2(v_1^1+v_1^2)}((1+\alpha)^{\frac{v_1^1+v_1^2}{v_1^1}} - 1)
\end{aligned}
$$

and by $p_{2-n} = 1 - p_{1-n}$. It can be easily verified that when $v_1^1 = v_1^2$ then $p_{1-n} = \frac{1+\alpha}{\alpha} - \frac{1}{2\alpha^2}((1+\alpha)^2 - 1) = \frac{1}{2} = p_{2-n}$. We also have that

$$
\frac{dp_{1-n}}{d\alpha} = -\frac{1}{\alpha^3 (v_1^1 + v_1^2)}((\alpha+1)^{\frac{v_1^2}{v_1^1}} (-2v_1^1 - \alpha v_1^1 + \alpha v_1^2) + 2v_1^1 + \alpha v_1^1 + \alpha v_1^2)
$$

In order to find whether $\frac{dp_{1-n}}{d\alpha}$ is positive or negative, define

$$
g(\alpha) = (\alpha+1)^{\frac{v_1^2}{v_1^1}} (-2v_1^1 - \alpha v_1^1 + \alpha v_1^2) + 2v_1^1 + \alpha v_1^1 + \alpha v_1^2
$$

Then we have that for all $0 < \alpha < 1$

$$
\begin{aligned}
g(\alpha) &\leq (\alpha+1)(-2v_1^1 - \alpha v_1^1 + \alpha v_1^2) + 2v_1^1 + \alpha v_1^1 + \alpha v_1^2 \\
&= (2\alpha + \alpha^2)(v_1^2 - v_1^1) \leq 0
\end{aligned}
$$

Therefore we obtain that for all $0 < \alpha < 1$, $\frac{dp_{1-n}}{d\alpha} > 0$. In other words, the probability of contestant 1 to win in the first stage increases in the absolute value of the marginal decreasing rate α.

Let $v_1^2 = 1$ and $v_1^1 = 2$. Then, in the following figure we can present the contestant 1's probability to win as a function of the marginal decreasing rate α.

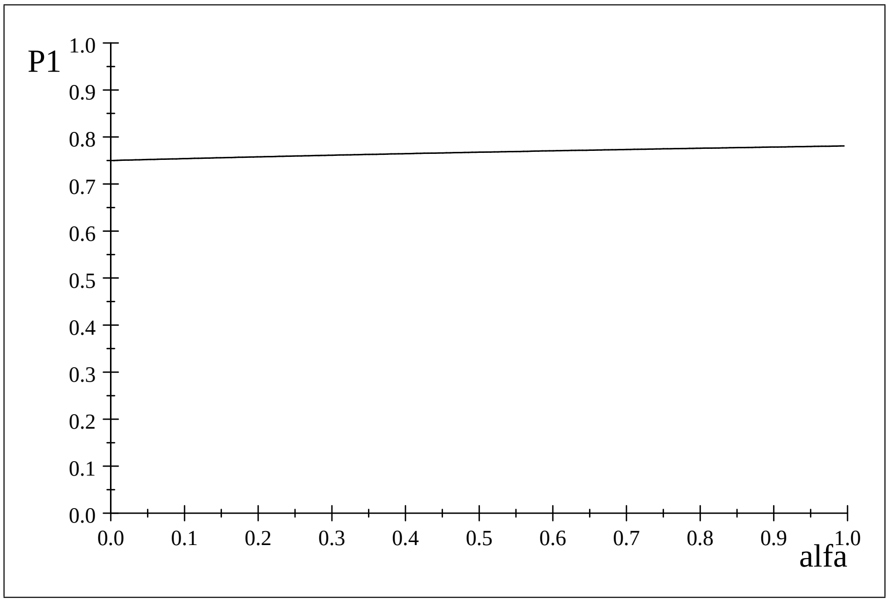

Figure 4: Contestant 1's probability to win as a functions of α.

According to Figure 4, we can see that although contestant 2 has a positive expected payoff in the second stage, contestant 1, the contestant with the higher reward in the first stage, has a higher probability to win in that stage.

4 Results

Based on the analysis in the previous sections we can compare the two-stage all-pay contest under positive and negative synergies. By (4) and (9) we have

Proposition 3 *In the two-stage all-pay contest:*

1. If the contestants are symmetric and there is positive synergy, both contestants have an expected payoff of zero, and if there is negative synergy, both contestants have a positive expected payoff. Then, if the synergy is stronger (larger α) each contestant's expected payoff is higher.

2. If the contestants are asymmetric and there is positive synergy, the contestant with the higher value in the first stage has a positive expected payoff while the other contestant has an expected payoff of zero. Then, if the synergy is stronger, the stronger contestant's (the contestant with the higher value in the first stage) expected payoff is higher. And, if the synergy is negative, both contestants have positive expected payoffs *while the stronger contestant has a higher expected payoff. Then, if the synergy is stronger each contestant's expected payoff is higher.*

By Proposition 3 if the synergy is positive the symmetric contestants have an expected payoff of zero while if the synergy is negative they have a positive expected payoff. Thus, both contestants prefer negative over positive synergy. When contestants are asymmetric, the weaker contestant (the contestant with the lower value in the first stage), like the symmetric contestants, also prefers negative over positive synergy, since under positive synergy he has an expected payoff of zero while under negative synergy he has a positive expected payoff. On the other hand, the stronger contestant (the contestant with the higher value in the first stage) has a positive expected payoff under both positive and negative synergies. In order to compare his expected payoffs for both cases, let $v_1^1 = 2$ and $v_1^2 = 1$. Then, in Figure 5 below we can present the stronger contestant's expected payoffs as a function of the marginal rate α under positive and negative synergies.

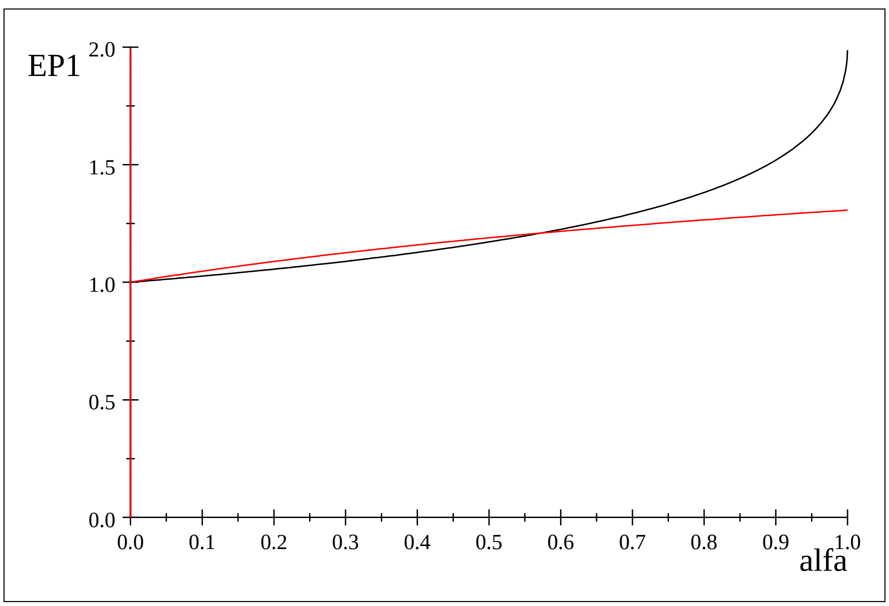

Figure 5: Contestant 1's expected payoff as a function of α under positive synergy (black curve) and negative synergy (red curve).

Figure 5 shows that the stronger's expected payoff could be higher but also lower under positive synergy than under negative synergy. Furthermore, for relatively high values of α the contestants preferences are not identical as the following result shows.

Proposition 4 *In the two-stage all-pay contest,* *1. If the contestants are symmetric, they have higher expected payoffs under negative synergy than under positive synergy.*

2. If the contestants are asymmetric, the weaker contestant (the contestant with the lower value in the first stage) has a higher expected payoff under negative synergy than under positive synergy. On the other hand, if the marginal (increasing/decreasing) rate α is sufficiently high (approaches 1), the stronger contestant has a lower expected payoff under negative synergy than under positive synergy.

Proof. By (10), when the synergy is negative we have that $\frac{dEP_{1-n}}{d\alpha} \geq 0$, and

$$\begin{aligned}\frac{dEP_{1-n}}{d\alpha} &= \frac{1}{\alpha^2}\frac{v_1^2}{\alpha+1}\left(\ln(\alpha+1)(1+\alpha)-\alpha\right) \\ &\leq \frac{1}{\alpha^2}\frac{v_1^2}{\alpha+1}\left(\alpha(1+\alpha)-\alpha\right) = \frac{v_1^2}{\alpha+1} \leq v_1^2\end{aligned}$$

On the other hand, by (5), when the synergy is positive we have that $\frac{dp_{1-n}}{d\alpha} \geq 0$ and

$$\lim_{\alpha\to 1}\frac{dEP_{1-p}}{d\alpha} = \lim_{\alpha\to 1}\frac{1}{\alpha^2}\left(\frac{v_1^1-\alpha v_1^2}{(1-\alpha)^{\frac{v_1^2}{v_1^1}}} - v_1^1\right) = \infty$$

Thus, when α is sufficiently close to 1, we obtain that $EP_{1-p} - EP_{1-n} > 0$. ■

By Proposition 4 and Figure 5, we have values of marginal increasing rate α according to which regardless of whether the contestants are symmetric or asymmetric, they have the same or a higher expected payoff under negative synergy than under positive synergy. This result can be explained as follows: The contestants' expected payoffs in the second stage are decided according to the difference between their efforts in the first stage and not according to the level of these efforts. Thus, when the synergy between the stages is negative, the contestants' expected payoffs in the second stage are not necessarily lower than when the synergy is positive. However, when the synergy is positive, the contestants expected total effort in the second stage is higher than the expected total effort when the synergy is negative.

5 Concluding remarks

We studied two-stage all-pay contests with effort-dependent rewards that either increase (positive synergy) or decrease (negative synergy) in the contestants' efforts. We analyzed the subgame perfect equilibrium and showed that in our model all the contestants, whether symmetric or asymmetric, may have positive expected payoffs. The results also showed a paradoxical behavior: although contestants, whether symmetric or asymmetric, have larger rewards when the effort-dependent rewards are increasing, they all may prefer decreasing effort-dependent rewards over increasing ones.

The above results have been obtained under the assumption that the reward function in the second stage is additively separable, $v_2^i(x_1^i) = v_2 - \alpha x_1^i$ where x_1^i is player i's effort in the first stage. However, these results are robust and hold for other forms of the reward function in this stage as, for example, when the reward function in the second stage is multiplicatively separable, $v_2^i(x_1^i) = v_2 \alpha x_1^i, i = 1, 2$. The reason is that the difference between the contestants' rewards in the second stage is the key parameter for the analysis of the equilibrium in the first stage and that difference in the multiplicative-separable case is equal to that in the additively-separable case multiplied by constant.

A natural extension of our model would be to consider a multi-stage contest in which a contestant's effort will affect not only his reward in the later stages but his opponents' rewards as well. This and other possible extensions indicate the research potential of studying multi-stage contests with effort-dependent rewards.

References

[1] Amegashie, J., Cadsby, C., Song, Y.: Competitive burnout: theory and experimental evidence. *Games and Economic Behavior* 59, 213-239 (2007)

[2] Amman, E., Leininger, W.: Asymmetric all-pay auctions with incomplete information: the two-player case. *Games and Economic Behavior* 14, 1-18 (1996)

[3] Baye, M., Kovenock, D., de Vries, C.: Rigging the lobbying process: An application of the all-pay auction. *American Economic Review* 83, 289-294 (1993)

[4] Baye, M., Kovenock, D., de Vries, C.: The all-pay auction with complete information. *Economic Theory* 8, 291-305 (1996)

[5] Baye, M., Kovenock, D., de Vries, C.: Contests with rank-order spillovers. *Economic Theory* 51(2), 315-350 (2012)

[6] Che, Y-K., Gale, I.: Caps on political lobbying. *American Economic Review* 88, 643-651 (1998)

[7] Cohen, C., Kaplan, T., Sela, A.: Optimal rewards in contests. *Rand Journal of Economics* 39(2), 434-451 (2008)

[8] Gavious, A., Moldovanu, B., Sela, A.: Bid costs and endogenous bid caps. *Rand Journal of Economics* 33(4), 709-722 (2003)

[9] Fu, Q., Lu, J.: The optimal multi-stage contest. *Economic Theory* 51(2), 351-382 (2012)

[10] Gradstein, M., Konrad, K.: Orchestrating rent seeking contests. *Economic Journal* 109, 536-545 (1999)

[11] Groh, C., Moldovanu, B., Sela, A., Sunde, U.: Optimal seedings in elimination tournaments. *Economic Theory* 49, 59-80 (2012)

[12] Harris, C., Vickers, J.: Racing with uncertainty. *Review of Economic Studies* 54(1), 1-21 (1987)

[13] Hart, S.: Discrete Colonel Blotto and general lotto games. *International Journal of Game Theory* 36, 441-460 (2008)

[14] Hilman, A., Riley, J.: Politically contestable rents and transfers. *Economics and Politics* 1, 17-39 (1989)

[15] Kaplan, T., Luski, I., Sela, A., Wettstein, D.: All-pay auctions with variable rewards. *Journal of Industrial Economics* L(4), 417-430 (2002)

[16] Kaplan, T., Luski, I., Wettstein, D.: Innovative activity and sunk costs. *International Journal of Industrial Organization* 21(8), 1111-1133 (2003)

[17] Kaplan, T., Wettstein, D.: The optimal design of rewards in contests. *Review Economic Design* 19(4), 327-339 (2015)

[18] Klumpp, T., Polborn, M.: Primaries and the New Hampshire effect. *Journal of Public Economics* 90, 1073-1114 (2006)

[19] Konrad, K.: Investment in the absence of property rights. *European Economic Review* 46, 1521-1537 (2002)

[20] Konrad, K., Kovenock, D.: Multi-Battle Contests. *Games and Economic Behavior* 66, 256-274 (2009)

[21] Kovenock, D., Roberson, B.: Is the 50-state strategy optimal? *Journal of Theoretical Politics* 21(2), 213-236 (2009)

[22] Krishna, V., Morgan, J.: An analysis of the war of attrition and the all-pay auction. *Journal of Economic Theory* 72(2), 343-362 (1997)

[23] Moldovanu, B., Sela, A.: The optimal allocation of prizes in contests. *American Economic Review* 91, 542-558 (2001)

[24] Moldovanu, B., Sela, A.: Contest architecture. *Journal of Economic Theory* 126(1), 70-97 (2006)

[25] Kvasov, D.: Contests with limited resources. *Journal of Economic Theory* 136, 738-748 (2007)

[26] Roberson, B.: The Colonel Blotto game. *Economic Theory* 29, 1-24 (2006)

[27] Ryvkin, D.: Fatigue in dynamic tournaments. *Journal of Economics and Management Strategy* 20(4), 1011-1041 (2011)

[28] Sela, A.: Sequential two-prize contests. *Economic Theory* 51, 383-395 (2012)

[29] Sela, A., Erez, E.: Dynamic contests with resource constraints. *Social Choice and Welfare* 41(4), 863-882 (2013)

[30] Siegel, R.: All-pay contests. *Econometrica* 77 (1), 71-92 (2009)

[31] Snyder, J.M.: Election goals and the allocation of campaign resources. *Econometrica* 57, 637-660 (1989)