TWO-STAGE ELIMINATION CONTESTS WITH OPTIMAL HEAD STARTS

Noam Cohen, Guy Maor and Aner Sela

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Monaster Center for
Economic Research
Ben-Gurion University of the Negev
P.O. Box 653
Beer Sheva, Israel

Fax: 972-8-6472941
Tel: 972-8-6472286

Two-Stage Elimination Contests with Optimal Head Starts

Cohen Noam, Maor Guy and Sela Aner*

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Abstract

We study two-stage elimination Tullock contests. In the first stage all the players compete against each other of which some advance to the second stage while the others are eliminated. The finalists compete against each other in the second stage and one of them wins the prize. To maximize the expected total effort the designer can give a head start to the winner of the first stage when he competes against the other finalists in the second stage. We show that the optimal head start, independent of the number of finalists, always increases the players' expected total effort. We also show how the number of players and finalists affect the value of the optimal head start.

Keywords: Multi-stage contests, Tullock contests, head starts.

JEL classification: C70, D44, L12, O32

*Cohen Maor and Sela: Department of Economics, Ben-Gurion University of the Negev, Beer–Sheva 84105, Israel.

1 Introduction

In elimination contests in each stage some of the contestants are eliminated while the others advance to the next stage until the final stage in which some of the players (usually one of them) win prizes. In elimination tournaments in sport, players or teams usually play pair-wise matches and the winner advances to the next stage while the loser is eliminated from the competition. Examples include the ATP tennis tournaments and the professional playoffs in US-Basketball or World Cup playoffs. In this paper we study two-stage elimination Tullock contests (Tullock 1980)[1] where in the first stage all the n players compete against each other and $k, k \leq n$, players (finalists) proceed to the second (final) stage. The contestants exert their efforts once in the first stage and the finalists are determined sequentially. The first winner is determined by the probability success function considering the efforts of all the players. The second winner is determined by the probability success function considering the efforts of all the contestants excluding the effort of the first winner. This sequential process continues until all the finalists are determined. The winners in the second stage win the prizes.[2] To illustrate, such a two-stage elimination contest can be a preliminary contest in which all the contestants are interviewed for a job and a final stage in which a few finalists are interviewed and tested.[3]

Fu and Lu (2012) studied a multi-stage sequential elimination Tullock contest, and showed that the optimal contest eliminates one contestant at each stage until the final stage. Then, the winner of the final takes the entire prize sum. However, their result to eliminate only one player in the first stage does not hold when the number of stages is limited, especially when there are only two stages. In such a case, the contest designer can decide about the number of players who advance from the first stage to the second one and the incentives he can give them. Such an incentive could be a head start, an exogenously determined mechanism that increases the winning probabilities of some contestants. By using a head start, the contest

[1] A number of studies provided axiomatic justification for the Tullock contest (see, for example, Skaperdas (1996) and Clark and Riis (1998)). Baye and Hoppe (2003) have identified conditions under which a variety of rent-seeking contests, innovation tournaments, and patent-race games are strategically equivalent to the Tullock contest.

[2] Clark and Riis (1996) considered the same method to select the winners in one-stage Tullock contests.

[3] Amegashie et al. (2007) studied such a two-stage elimination all-pay auction with four heterogeneous players and budget constraints. He experimentally showed that players use all of their available budgets in the first stage.

designer can affect the effort levels of the contestants.[4] This was shown by Konrad (2002) in a one-stage two-player all-pay auction under complete information, Kirkegaard (2012) in an asymmetric all-pay auction under incomplete information and Segev and Sela (2014) in a sequential all-pay auction under incomplete information. Franke et al. (2011) studied one-stage Tullock contests with heterogeneous contestants where the designer who wishes to maximize the contestants' total effort may allocate asymmetric success functions (which are equivalent to different head starts) for the contestants. They showed that the optimal contest rule is biased in favor of weaker contestants, i.e., stronger contestants are relatively discouraged while weaker contestants are encouraged both to participate and to exert higher efforts.[5] In other words, head starts should be given to the weak players. While Franke et al. found the optimal head starts in a one-stage Tullock contest which are given according to the players' types, we find the optimal head start in a two-stage Tullock contest which is given to the winner of the first stage according to his performance in that stage.

For simplicity, we begin with the analysis of the most common case in which there are only two finalists in the second (final) stage. We demonstrate that the entire prize sum should be awarded to the winner of the final stage only. Later, we generalize the model for any number of finalists, and we assume that the entire prize sum is awarded to the winner in the final stage. Then, we show that awarding a head start to the winner in the first stage, independent of the number of finalists, always increases the players' total effort. This result implies that the elimination two-stage contest with a head start dominates the one-stage contest with respect to the contestants' total effort.

We also explicitly calculate the optimal head start and show that its value decreases in the number of players. The optimal head start is not monotonic in the number of finalists. We illustrate that it decreases in the number of finalists, obtains its minimal value and then increases again. In particular, the optimal head start has the same value if the number of finalists is either two or equal to the number of players.

Last, we calculate the optimal head start together with the optimal number of finalists. We first show that if the number of players is three then the optimal number of finalists is three as well. We then show that

[4] See Tsoulouhas et al. (2007) for asymmetric rules in labor tournaments, and Epstein et al. (2011) for asymmetric rules in public procurement and lobbying contests.

[5] Nti (2004) determined the optimal contest success function in the one-stage Tullock contest with two players, while Dasgupta and Nti (1998) did this for n homogeneous players.

the optimal number of finalists increases in the value of the head start. By our analysis, we can calculate the combination of the optimal head start and the optimal number of finalists for any number of players. For instance, given the optimal head start, if the number of players is three the optimal number of finalists is three as well (all the players advance to the final stage), but if the number of players is 100, the optimal number of finalists is 27. Our calculations show that, given the optimal head start, when the number of players is relatively large, a change in the number of players yields a very small change in the optimal number of finalists.

1.1 Related literature

In this paper we assume that the designer can determine the players' contest success functions. However, in the literature on elimination tournaments it is usually assumed that the structure of the tournament is given and the contest designer can determine the players' allocation in the tournament. Rosen (1986), for example, studied an elimination tournament with homogeneous players where the probability of winning a match is a stochastic function of the players' efforts. Gradstein and Konrad (1999) studied a rent-seeking contest à la Tullock (with homogenous players), and Groh et al. (2012) studied an elimination tournament with four asymmetric players where players are matched in the all-pay auction in each of the stages. Groh et al. found optimal seedings for different criteria and, in particular, they showed that total expected effort in the elimination tournament where the two strongest players meet in the final with a probability of one equals the total effort in the all-pay auction where all players compete simultaneously. This result is contrasted with the main finding of Gradstein and Konrad who found that simultaneous contests are strictly superior if the contest's rules are discriminatory enough.

Similarly to previous works we show that it is not profitable for the contest designer to split the prize in the final stage. For example, Moldovanu and Sela (2001) showed that in one-stage all-pay contests under incomplete information when cost functions are linear or concave in effort, it is optimal to allocate the entire prize sum to a single first prize, but when cost functions are convex, several positive prizes may be optimal. Later (2006) these authors studied two-stage all-pay contests with multiple prizes under incomplete information and showed that for a contest designer who maximizes the expected total effort, if the cost functions are linear in effort, it is optimal to allocate the entire prize sum to a single first prize. In Tullock contests, Schweinzer and Segev (2009) demonstrated that the optimal prize structure of symmetric n-player Tullock tournaments assigns the entire prize sum to the winner, provided that a symmetric pure strategy equilibrium exists.

In elimination contests there are several methods different from ours for choosing the finalists who advance to the final stage. Amegashie (1999) and Moldovanu and Sela (2006) split the contestants among several sub-contests in the first stage and then the winners compete against each other in the final stage while the other contestants are eliminated. Amegashie showed that the comparison between one-stage and two-stage Tullock contests is ambiguous and depends on the parameters of the Tullock success functions at each stage. In addition, he found that the players' expected effort in the two-stage elimination Tullock contest and the standard one stage Tullock contest is the same. Moldovanu and Sela also compared between the multi-stage and the one-stage winner-take-all contests from the point of view of a designer who maximizes the expected total effort. For the case of a linear cost of effort, they showed that if the designer maximizes the expected total effort, the optimal architecture is a single grand static contest.

We theoretically show that the two-stage elimination Tullock contest with the optimal head start yields a higher total effort than the one-stage Tullock contest. Similar to our findings, Sheremeta's (2010 a and b) experimental studies showed that the two-stage Tullock contest generates higher aggregate effort than the corresponding one-stage contest.

The rest of the paper is organized as follows. In Section 2 we analyze the subgame perfect equilibrium of the two-stage elimination contest with two finalists and in Section 3 we analyze the general model with any number of finalists. In Section 4 we analyze the two-stage elimination contest with optimal head starts.

2 The model with two finalists

We first study an elimination two-stage Tullock contest with $n \geq 3$ players. In the first stage, all the n players simultaneously exert their efforts and only two of them (so called finalists) advance to the second (final) stage, while all the other players are eliminated. If player $i, i = 1, 2, ..., n$ exerts an effort x_i in the first stage then his probability to be the first to advance to the final stage is

$$\frac{x_i}{\sum_{j=1}^{n} x_j}$$

and his probability to be the second to advance to the final stage is

$$\sum_{\substack{k=1\\k\neq i}}^{n} \frac{x_k}{\sum_{j=1}^{n} x_j} \frac{x_i}{\sum_{\substack{j=1\\j\neq k}}^{n} x_j}$$

Thus, player i's probability to advance to the final stage is

$$\frac{x_i}{\sum_{j=1}^{n} x_j} + \sum_{\substack{k=1\\k\neq i}}^{n} \frac{x_k}{\sum_{j=1}^{n} x_j} \frac{x_i}{\sum_{\substack{j=1\\j\neq k}}^{n} x_j} \tag{1}$$

We assume that the first prize for the winner in the final stage is $\delta > 0.5$ while the second prize for the loser is $1-\delta$. A player has an incentive to be the first to advance to the final stage if he receives a head start that improves his probability to win the first prize in the final stage. Formally, if player i is the first to advance to the final stage and player j is the second, then if they exert efforts of y_i, y_j respectively, player i's probability to win the first prize is $\frac{\alpha y_i}{\alpha y_i + y_j}$ and $1 - \frac{\alpha y_i}{\alpha y_i + y_j}$ to win the second prize, where the head start $\alpha \geq 1$ is a commonly known constant determined by the contest designer. In the following we analyze the subgame perfect equilibrium of the elimination two-stage contest with two finalists. We begin with the analysis of the second stage and go backwards to the first one.

2.1 The second stage

The maximization problem in the second stage of the finalist who won in the first stage is

$$\max_{y_1} \delta \frac{\alpha y_1}{\alpha y_1 + y_2} + (1-\delta) \frac{y_2}{\alpha y_1 + y_2} - y_1$$

where y_2 is the other finalist's effort whose maximization problem is

$$\max_{y_2} \delta \frac{y_2}{\alpha y_1 + y_2} + (1-\delta) \frac{\alpha y_1}{\alpha y_1 + y_2} - y_2$$

The F.O.C. satisfy

$$\frac{\alpha y_2 (2\delta - 1)}{(\alpha y_1 + y_2)^2} = \frac{\alpha y_1 (2\delta - 1)}{(\alpha y_1 + y_2)^2} = 1$$

Thus, the equilibrium efforts in the second stage are

$$y_1 = y_2 = \frac{\alpha(2\delta - 1)}{(1+\alpha)^2} \tag{2}$$

Since $\delta > 0.5$ the finalists' efforts in the second stage satisfy $y_1 = y_2 > 0$. The expected payoff of the winner in the first stage is

$$u_1 = \frac{\alpha^2\delta - 2\alpha\delta + 2\alpha - \delta + 1}{(1+\alpha)^2} \tag{3}$$

and the expected payoff of the other finalist is

$$u_2 = \frac{-\alpha^2\delta - 2\alpha\delta + 2\alpha + \delta + \alpha^2}{(1+\alpha)^2} \tag{4}$$

It can be easily shown that if δ is sufficiently high (i.e., close enough to 1) then $u_1, u_2 > 0$. We can now proceed to the analysis of the equilibrium in the first stage.

2.2 The first stage

By (1), (3) and (4), the maximization problem of player $i, i = 1, 2, ..., n$ in the first stage is

$$\begin{aligned} &\max_{x_i} \frac{\alpha^2\delta - 2\alpha\delta + 2\alpha - \delta + 1}{(1+\alpha)^2} \frac{x_i}{\sum\limits_{j=1}^{n} x_j} \\ &+ \frac{-\alpha^2\delta - 2\alpha\delta + 2\alpha + \delta + \alpha^2}{(1+\alpha)^2} \sum_{\substack{k=1 \\ k\neq i}}^{n} \frac{x_k}{\sum\limits_{j=1}^{n} x_j} \frac{x_i}{\sum\limits_{\substack{j=1 \\ j\neq k}}^{n} x_j} - x_i \end{aligned}$$

The F.O.C. is

$$
\begin{aligned}
& \frac{\alpha^2\delta - 2\alpha\delta + 2\alpha - \delta + 1}{(1+\alpha)^2} \frac{\sum\limits_{\substack{j=1\\ j\neq i}}^{n} x_j}{(\sum\limits_{j=1}^{n} x_j)^2} \\
& - \frac{-\alpha^2\delta - 2\alpha\delta + 2\alpha + \delta + \alpha^2}{(1+\alpha)^2} \sum_{\substack{k=1\\ k\neq i}}^{n} \frac{x_k}{(\sum\limits_{j=1}^{n} x_j)^2} \frac{x_i}{\sum\limits_{\substack{j=1\\ j\neq k}}^{n} x_j} + \\
& \frac{-\alpha^2\delta - 2\alpha\delta + 2\alpha + \delta + \alpha^2}{(1+\alpha)^2} \sum_{\substack{k=1\\ j\neq i}}^{n} \frac{x_k}{\sum\limits_{j=1}^{n} x_j} \frac{\sum\limits_{\substack{j=1\\ j\neq k,i}}^{n} x_j}{(\sum\limits_{\substack{j=1\\ j\neq k}}^{n} x_j)^2} \\
= \quad & 1
\end{aligned}
$$

By symmetry, we obtain

$$
\frac{\alpha^2\delta - 2\alpha\delta + 2\alpha - \delta + 1}{(1+\alpha)^2} \frac{n-1}{n^2 x} + \frac{-\alpha^2\delta - 2\alpha\delta + 2\alpha + \delta + \alpha^2}{(1+\alpha)^2} \frac{n^2 - 3n + 1}{n^2(n-1)x} = 1
$$

Thus, we have

Proposition 1 *In the subgame perfect equilibrium, the strategy of every player in the first stage of the elimination two-stage contest with two finalists is*

$$
x = \frac{\alpha^2(n^2 + \delta n - 3n + 1) + \alpha(4n^2 - 4\delta n^2 + 10\delta n - 10n - 4\delta + 4) + (n^2 - 2n - \delta n + 1)}{(1+\alpha)^2 n^2 (n-1)}
$$

while the finalists' strategies in the second stage are given by (2).

The players' total effort in both stages is then given by

$$
\begin{aligned}
TE \quad = \quad & nx + 2y = \\
& \frac{\alpha^2(n^2 + \delta n - 3n + 1) + \alpha(4n^2 - 4\delta n^2 + 10\delta n - 10n - 4\delta + 4) + (n^2 - 2n - \delta n + 1)}{(1+\alpha)^2 n (n-1)} \\
& + \frac{2\alpha(2\delta - 1)}{(1+\alpha)^2}
\end{aligned} \tag{5}
$$

Note that

$$
\frac{dTE}{d\delta} = \frac{n(\alpha^2 - 1) + \alpha(6n - 4)}{(1+\alpha)^2 n(n-1)}
$$

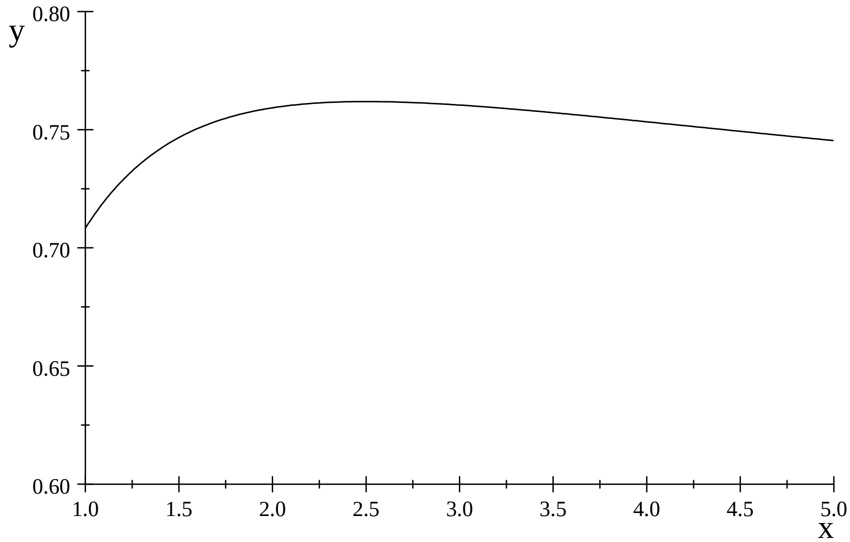

Figure 1: The expected total effort as a function of α for n=3.

Thus, if $\alpha > 1$ we obtain that $\frac{dTE}{d\delta} > 0$, namely, it is optimal to award the entire prize sum (which is equal to 1) to the winner in the second stage. From this point on we will assume that only one prize is awarded, i.e., $\delta = 1$. Then, we can see that when n approaches ∞ we obtain

$$\lim TE_{n\to\infty} = \frac{(1+\alpha)^2}{(1+\alpha)^2} = 1$$

The optimal head start α^* that yields the highest expected total effort is given by

$$\frac{d}{d\alpha} TE(\alpha) = \frac{2}{n\left(\alpha+1\right)^3\left(n-1\right)}\left(2n+\alpha-n\alpha-1\right) = 0$$

This implies that

$$\alpha^* = \frac{2n-1}{n-1} \tag{6}$$

Since $\frac{d\alpha^*}{dn} = -\frac{1}{(n-1)^2} < 0$, the maximal value of the optimal head start α^* is obtained for $n = 3$ ($\alpha^* = 2.5$) and is decreasing in the number of players n. When n approaches infinity it converges to 2. Figure 1 presents the expected total effort as a function of the head start α when the number of players is $n = 3$. [Figure 1 to be here].

We can see that when $n = 3$, the highest total effort is obtained for $\alpha^* = 2.5$ where the players' total effort increases for every $1 \leq \alpha \leq 2.5$ and decreases for all $\alpha > 2.5$. If we compare the players' performances in our model and the one-stage Tullock model we obtain that

Proposition 2 *For any number of players $n \geq 3$, the expected total effort in the two-stage elimination contest either with or without the optimal head start is larger than in the one-stage contest.*

Proof. The total effort in the standard (one-stage) Tullock contest with n symmetric players and a prize equal to 1 is

$$TE_s = \frac{n-1}{n}$$

By (5), the difference between the total effort in the two-stage elimination Tullock contest with the optimal head start $\alpha^* = 2.5$ and the one-stage contest is

$$\Delta TE(\alpha = 2.5) = \frac{1}{3n^2 - 2n}(n-1) > 0$$

Furthermore, by (5), the difference between the total effort in the two-stage elimination Tullock contest without any head start and the one-stage contest is

$$\Delta TE(\alpha = 1) = \frac{1}{4n^2 - 4n}(n-2) > 0$$

Figure 2 presents the total effort in the two-stage elimination contest and in the standard (one-stage) one. [Figure 2 to be here]. ■

In the next section we generalize the model to any number of finalists k, $2 \leq k \leq n$.

3 The model with $k \geq 2$ finalists

We now study a two-stage Tullock contest when in the first stage all the n players simultaneously exert their efforts and k, $2 \leq k \leq n$ of them advance to the second stage. Following the result in the previous section according to which the entire prize sum should be allocated to the winner in the second stage, we assume that the k finalists compete against each other to win a prize equal to 1, where the winner in the first stage has a head start advantage that increases his winning probability compared to his rivals. By symmetry, we denote the effort of every contestant j, $j \neq i$ by $x_j = x$. Then, if player i exerts an effort x_i his probability to win in the first stage is

$$\frac{x_i}{(n-1)x + x_i}$$

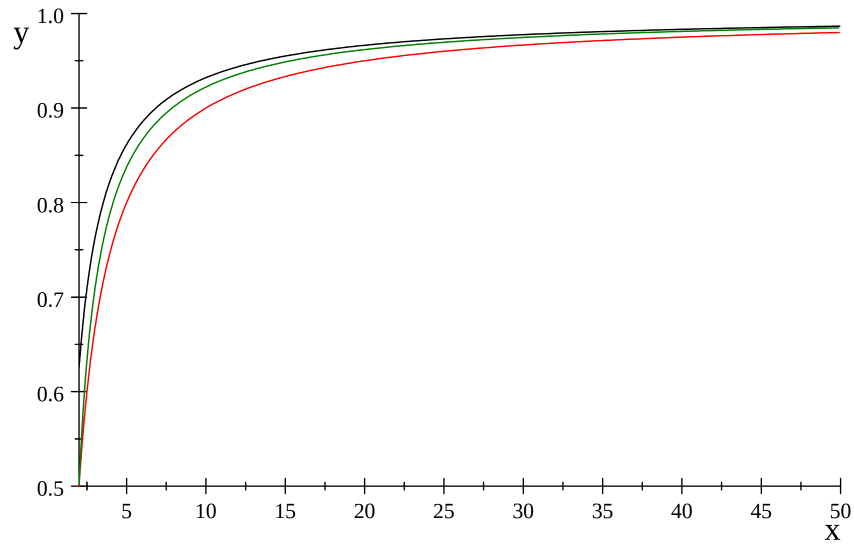

Figure 2: The total effort as a function of the number of players in the one-stage contest (red curve) and in the two-stage contest with the optimal head start (black curve) and without a head start (the green curve).

The probability of player i to be the second to advance to the final is

$$\frac{x(n-1)}{(n-1)x+x_i} \cdot \frac{x_i}{(n-2)x+x_i}$$

Similarly, the probability of player i to be the s-th, $s = 3, 4, ..., k$ to advance to the final is

$$\begin{aligned} & \frac{x(n-1)}{(n-1)x+x_i} \cdot \frac{x(n-2)}{(n-2)x+x_i} \cdot .... \cdot \frac{x(n-(s-1))}{(n-s+1)x+x_i} \frac{x_i}{(n-s)x+x_i} \\ = \quad & x^{s-1} \frac{(n-1)!}{(n-s)!} \prod_{j=2}^{s} \frac{1}{x_i+(n-j+1)x} \cdot \frac{x_i}{x_i+(n-s)x} \end{aligned}$$

In sum, the probability of player i to advance to the final stage is given by

$$\frac{x_i}{(n-1)x+x_i} + \sum_{s=2}^{k} \left[x^{s-1} \frac{(n-1)!}{(n-s)!} \prod_{j=2}^{s} \frac{1}{x_i+(n-j+1)x} \frac{x_i}{x_i+(n-s)x} \right] \tag{7}$$

Each player has an incentive to be the first to advance to the final stage since then he receives a head start that improves his probability to win in the final. Formally, let $y_j = y, 1 \leq j \leq k$ and $j \neq i$, then if player i is the first to advance to the final stage among k finalists and he exerts an effort y_i his probability to win is

$$\frac{\alpha y_i}{\alpha y_i + (k-1)y}$$

where $\alpha \geq 1$ is a commonly known head start determined by the contest designer and $y = y_j, j \neq i$ is the symmetric effort of all the finalists except player i. We begin with the analysis of the second stage and go backwards to the first one.

3.1 The second stage

The maximization problem in the second stage of the finalist (denoted as player 1) who won in the first stage is

$$\max_{y_1} \frac{\alpha y_1}{\alpha y_1 + (k-1)y} - y_1$$

where y is the symmetric effort of all the other finalists. The maximization problem of a finalist $j, j \neq 1$, who did not win in the first stage is

$$\max_{y_j} \frac{y_j}{\alpha y_1 + y_j + (k-2)y} - y_j$$

where y_1 is the effort of the winner in the first stage, and y is the symmetric effort of all the other finalists. Since $y_j = y$, the F.O.C. yield

$$\frac{\alpha(k-1)y}{(\alpha y_1 + (k-1)y)^2} = \frac{\alpha y_1 + (k-2)y}{(\alpha y_1 + (k-1)y)^2} = 1$$

Thus, the equilibrium efforts in the second stage are

$$\begin{aligned} y_1 &= \frac{\alpha(k-1)^2 - (k-1)(k-2)}{(\alpha(k-1)+1)^2} \\ y &= y_j = \frac{\alpha(k-1)}{(\alpha(k-1)+1)^2}, \; j = 2, ..., k \end{aligned} \tag{8}$$

Then, the total effort in the second stage is

$$\begin{aligned} TE_2 &= (k-1)y + y_1 \\ &= \frac{(k-1)(2\alpha(k-1) - (k-2))}{(\alpha(k-1)+1)^2} \end{aligned}$$

The expected payoff of the winner in the first stage is

$$u_1 = \frac{(\alpha(k-1) - (k-2))^2}{(\alpha(k-1)+1)^2} \tag{9}$$

and the expected payoffs of the other finalists are

$$u_j = \frac{1}{(\alpha(k-1)+1)^2} \text{ , } j = 2, ..., k \tag{10}$$

We can now proceed to the analysis of the equilibrium in the first stage.

3.2 The first stage

By (7), (9) and (10), the maximization problem of player $i, i = 1, ..., n$ in the first stage is

$$\begin{aligned} & Max_{x_i} \frac{(\alpha(k-1)-(k-2))^2}{(\alpha(k-1)+1)^2} \frac{x_i}{x_i+(n-1)x} \\ & + \frac{1}{(\alpha(k-1)+1)^2} \sum_{s=2}^{k} \left[x^{s-1} \frac{(n-1)!}{(n-s)!} \prod_{j=2}^{s} \frac{1}{x_i+(n-j+1)x} \frac{x_i}{x_i+(n-s)x} \right] - x_i \end{aligned}$$

where x is the symmetric effort of all the other players. By the symmetry, the F.O.C. yields

$$\begin{aligned} & \frac{(\alpha(k-1)-k+2))^2}{(\alpha(k-1)+1)^2} \frac{n-1}{n^2 x} + \frac{1}{(\alpha(k-1)+1)^2} \sum_{s=2}^{k} \left[x^{s-1} \frac{(n-1)!}{(n-s)!} \frac{(n-s+1)!}{n! x^{s-1}} \frac{n-s}{(n-s+1)^2 x} \right] \\ & - \frac{1}{(\alpha(k-1)+1)^2} \sum_{s=2}^{k} \left[x^{s-1} \frac{(n-1)!}{(n-s)!} \sum_{i=2}^{s} \frac{1}{(n-i+2)x} \frac{(n-s+1)!}{n! x^{s-1}} \frac{1}{(n-s+1)} \right] \\ = & \frac{(\alpha(k-1)-k+2))^2}{(\alpha(k-1)+1)^2} \frac{n-1}{n^2 x} + \frac{1}{(\alpha(k-1)+1)^2} \sum_{s=2}^{k} \left(\frac{n-s}{n(n-s+1)x} - \sum_{i=2}^{s} \frac{1}{n(n-i+2)x} \right) = 1 \end{aligned}$$

Thus, we obtain that

Proposition 3 *In the subgame perfect equilibrium the strategy of every player in the first stage of the elimination two-stage contest with $k \geq 2$ finalists is*

$$x = \frac{(\alpha(k-1)-k+2))^2}{(\alpha(k-1)+1)^2} \frac{n-1}{n^2} + \frac{1}{(\alpha(k-1)+1)^2} \sum_{s=2}^{k} \left(\frac{n-s}{n(n-s+1)} - \sum_{i=2}^{s} \frac{1}{n(n-i+2)} \right)$$

while the finalists' strategies in the second stage are given by (8).

Then, the total effort in the first stage is

$$TE_1 = nx = \frac{(\alpha(k-1)-k+2))^2}{(\alpha(k-1)+1)^2} \frac{n-1}{n} + \frac{1}{(\alpha(k-1)+1)^2} \sum_{s=2}^{k} \left(\frac{n-s}{(n-s+1)} - \sum_{i=2}^{s} \frac{1}{(n-i+2)} \right)$$

and the total effort in both stages is

$$\begin{aligned} TE \;=\; & TE_1 + TE_2 = \\ & \frac{(\alpha(k-1)-k+2))^2}{(\alpha(k-1)+1)^2} \frac{n-1}{n} + \frac{1}{(\alpha(k-1)+1)^2} \sum_{s=2}^{k} \left(\frac{n-s}{(n-s+1)} - \sum_{i=2}^{s} \frac{1}{(n-i+2)} \right) \\ & + \frac{(k-1)(2\alpha(k-1)-k+2)}{(\alpha(k-1)+1)^2} \end{aligned} \tag{11}$$

4 Results

We have assumed that the winner in the first stage receives a head start $\alpha > 1$ in the second stage. The following result shows that awarding a head start in the first stage is an efficient tool for maximizing the players' total effort.

Proposition 4 *In the two-stage elimination contest awarding a head start $\alpha > 1$ to the winner in the first stage, independent of the number of finalists, always increases the players' total effort.*

Proof. The derivative of (11) yields

$$\begin{aligned}\frac{dTE}{d\alpha} &= \frac{-2(k-1)}{(k\alpha-\alpha+1)^3}\Big(\frac{(k-1)(n-1)(k-2-\alpha(1-k))}{n} \\ &\quad +\sum_{s=2}^{k}\left(\frac{n-s}{(n-s+1)}-\sum_{i=2}^{s}\frac{1}{(n-i+2)}\right)+(\alpha-1)(k-1)^2\Big)\end{aligned} \tag{12}$$

When α approaches one we obtain that

$$\lim_{\alpha\to 1}\frac{dTE}{d\alpha}=2\frac{k-1}{k^3}\left(\frac{(k-1)(n-1)}{n}-\sum_{s=2}^{k}\left(\frac{n-s}{(n-s+1)}-\sum_{i=2}^{s}\frac{1}{(n-i+2)}\right)\right)$$

Since

$$\sum_{s=2}^{k}\frac{n-s}{(n-s+1)}\leq(k-1)\frac{n-2}{n-1}$$

We obtain,

$$\lim_{\alpha\to 1}\frac{dTE}{d\alpha}\geq 2\frac{(k-1)^2}{k^3}\Big(\frac{n-1}{n}-\frac{n-2}{n-1}\Big)\geq 0$$

Therefore, the head start $\alpha > 1$, independent of the number of the finalists k, increases the players' total effort. ■

Now, given that a head start is an efficient tool for increasing the players' total effort, it is worthwhile to find out what the optimal head start is and examine how it changes as a function of the number of finalists.

Proposition 5 *In the two-stage elimination contest with n players and k finalists, the optimal head start for the winner in the first stage is*

$$\alpha^*=\frac{1}{(k-1)^2}\sum_{i=2}^{k+1}\Big(n\frac{k-i+2}{n-i+2}-1\Big)+1 \tag{13}$$

Proof. The F.O.C. of the maximization of the total effort given by (11) is

$$\frac{-2(k-1)}{(k\alpha^*-\alpha^*+1)^3}\Big(\frac{(k-1)(n-1)(k-2-\alpha^*(1-k))}{n} + \left(\sum_{s=2}^{k}\Big(\frac{n-s}{(n-s+1)}-\sum_{i=2}^{s}\frac{1}{(n-i+2)}\Big)\right)+(\alpha^*-1)(k-1)^2\Big)=0$$

By some calculations we have

$$\begin{aligned}
\alpha^* &= \frac{-n}{(k-1)^2}\left(\sum_{s=2}^{k}\Big(\frac{n-s}{(n-s+1)}-\sum_{i=2}^{s}\frac{1}{(n-i+2)}\Big)\right)+\frac{n+k-2}{(k-1)} \\
&= \frac{-n}{(k-1)^2}\left((k-1)+\sum_{s=2}^{k}\Big(\frac{-1}{(n-s+1)}-\sum_{i=2}^{s}\frac{1}{(n-i+2)}\Big)\right)+\frac{n+k-2}{(k-1)} \\
&= \frac{n}{(k-1)^2}\left(\sum_{s=3}^{k+1}\Big(\frac{1}{(n-s+2)}+\sum_{i=2}^{s}\frac{1}{(n-i+2)}\Big)\right)+\frac{k-2}{k-1} \\
&= \frac{1}{(k-1)^2}\left(-k+n\sum_{s=2}^{k+1}\sum_{i=2}^{s}\frac{1}{(n-i+2)}\right)+1 \\
&= \frac{1}{(k-1)^2}\Big(-k+n\sum_{i=2}^{k+1}\frac{k-i+2}{(n-i+2)}\Big)+1 \\
&= \frac{1}{(k-1)^2}\sum_{i=2}^{k+1}\Big(n\frac{k-i+2}{n-i+2}-1\Big)+1
\end{aligned}$$

■

The following result shows the effect of the number of players on the optimal head start.

Proposition 6 *In the two-stage elimination contest the value of the optimal heat start α^* decreases in the number of players n.*

Proof. By (13), we obtain that

$$\begin{aligned}
&\alpha^*(n+1)-\alpha^*(n) \\
&= \frac{1}{(k-1)^2}\sum_{i=2}^{k+1}(n+1)\frac{k-i+2}{n-i+3}-n\frac{k-i+2}{n-i+2} \\
&= \frac{1}{(k-1)^2}\sum_{i=2}^{k+1}\frac{k-i+2}{(n-i+3)(n-i+2)}(-i+2)<0
\end{aligned}$$

■

Now we fix the number of the players and change only the number of the finalists. Accordingly, Figure 3 presents the optimal head start α^* as a function of the number of finalists k when the number of players is

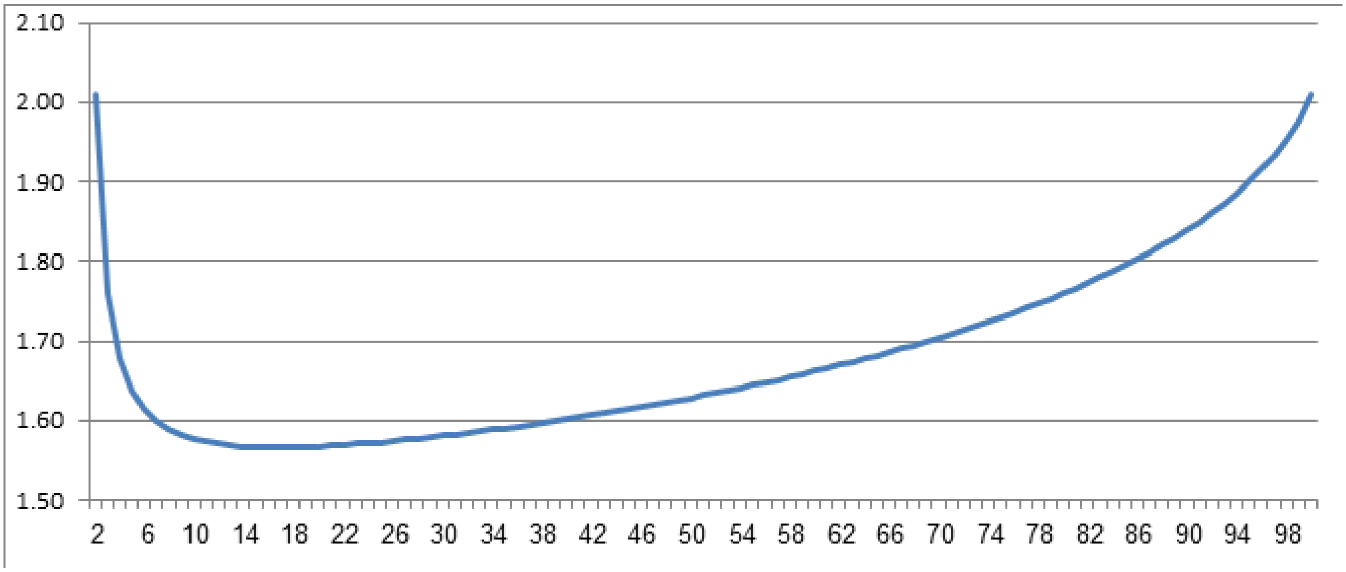

Figure 3: The optimal α^* as a function of the number of finalists k

$n = 100$. [Figure 3 to be here]. We can see by the figure that the optimal head start α^* first decreases and later increases in the number of finalists k. The minimal value of the optimal head start α^* is obtained when the number of finalists is $k = 17$, and the maximal value of the optimal head start α^* is obtained when the number of finalists is either $k = 2$ or $k = n = 100$. The last result can be generalized as follows:

Proposition 7 *In the two-stage elimination contest the optimal head start has the same value if the number of finalists is either two or is equal to the number of players.*

Proof. By (6) we have

$$\alpha^*(k=2) = \frac{2n-1}{n-1}$$

and by (13) we also have

$$\begin{aligned}\alpha^*(k &= n) = \frac{1}{(n-1)^2}\sum_{i=2}^{n+1}(n\frac{n-i+2}{n-i+2}-1)+1 \\ &= \frac{1}{(n-1)^2}\sum_{i=2}^{n+1}(n-1)+1 = \frac{n}{n-1}+1 = \frac{2n-1}{n-1}\end{aligned}$$

■

Figure 3 showed that the optimal head start $\alpha^*(k)$ may either increase or decrease in the number of finalists. However, if the number of players is sufficiently high we obtain that

Proposition 8 *In the two-stage elimination contest if the number of players approaches infinity then the optimal head start* α^* *decreases in the number of finalists.*

Proof. By (13) we have

$$\lim_{n\to\infty} \alpha^* = \lim_{n\to\infty} \frac{1}{(k-1)^2} \sum_{i=2}^{k+1} (n\frac{k-i+2}{n-i+2} - 1) + 1$$

By L'hopital's rule we obtain

$$\begin{aligned} \lim_{n\to\infty} \alpha^* &= \frac{1}{(k-1)^2}(k^2 - 1 - \sum_{i=2}^{k} i) + 1 \\ &= \frac{3k-2}{2k-2} \end{aligned}$$

Thus, when the number of players n converges to infinity, the optimal head start α^* decreases in the number of finalists k. ■

So far we have analyzed the optimal head start α^* for any number of finalists k, but we can also calculate the optimal number of finalists k^* given the optimal head start. Suppose first that the number of players is $n = 3$. Then by Proposition 7 and (6), the optimal head start is the same for either two or three finalists and is equal to $\alpha^* = 2.5$. Thus, by (5) the total effort with two finalists is $TE(k = 2) = 0.7483$, while by (11) the total effort with three finalists is $TE(k = 3) = 0.7546$. Thus we have

Proposition 9 *In the two-stage elimination contest with three players it is optimal to advance all the players to the final stage when the winner in the first stage has the optimal head start* $\alpha^* = 2.5$.

We can now numerically analyze the optimal number of finalists for any number of players $n > 3$. Consider first a two-stage elimination contest without a head start, i.e., $\alpha = 1$. Then, the optimal number of finalists for different values of the number of players n are

n:	3	4	5	6	7	8	9	10	15	20	30	40	50	60	100	150	300	500
k^*:	2	2	3	3	3	4	4	4	5	6	8	10	11	12	16	20	29	37

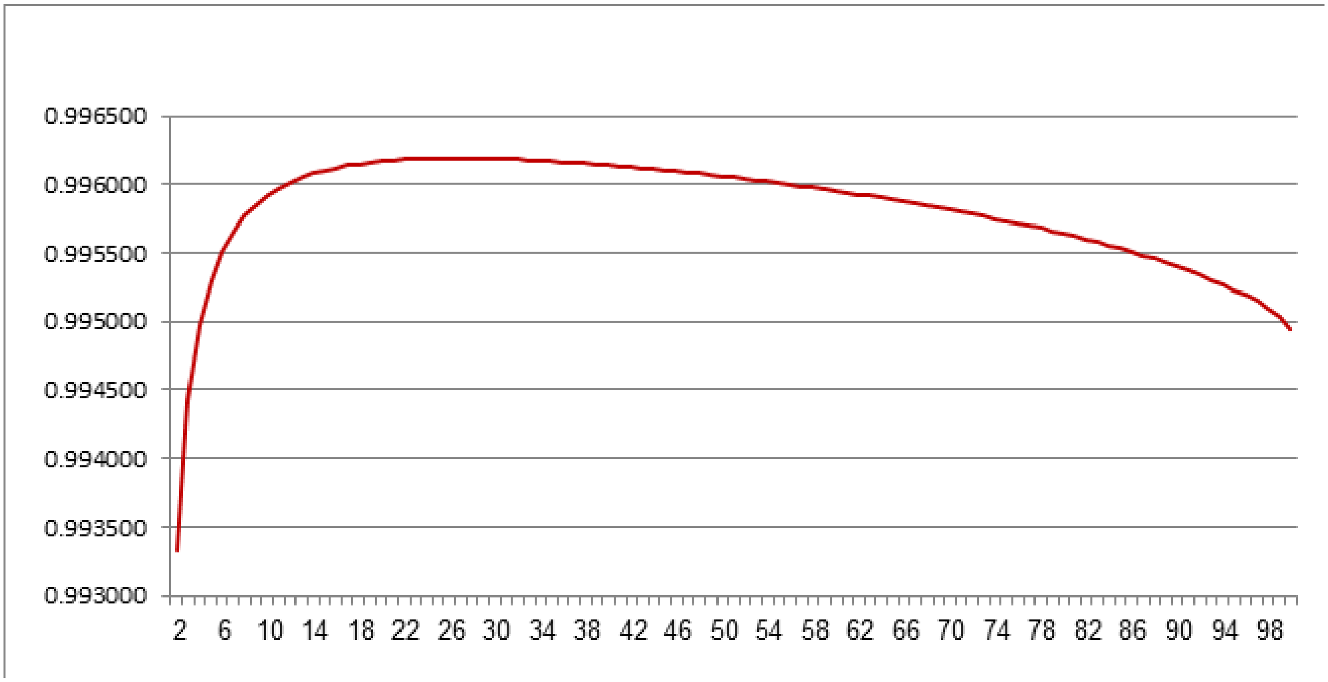

Figure 4: The optimal total effort as a function of the number of finalists k.

We can see that the optimal number of finalists increases in the number of players n. The optimal numbers of finalists k^* for different values of the head start α when the number of players is $n = 100$ are

α :	1	1.1	1.3	1.5	1.7	2	3	4	5	10	15	50	100	200	300
k^* :	16	19	23	26	28	29	32	34	35	36	37	37	37	37	37

We can see that the optimal number of finalists $k^*(\alpha)$ increases in the value of the head start α, but from a relatively large finite value of α, the optimal number of finalists almost does not change. By our analysis, we can find the optimal two-stage elimination contest for any number of players, as shown in Figure 4 where we find the optimal head start and then calculate the players' total effort for any number of finalists. [Figure 4 to be here]. We can see that if the number of players is $n = 100$, the highest total effort is obtained when the number of finalists is 27. Then the optimal head start is $\alpha^* = 1.57$.

5 Concluding remarks

In contrast to the one-stage contest in which the designer can affect the players' performance (effort) mostly by determining the prize structure, in multi-stage contests he has other tools to affect these performances. In our two-stage elimination contest, for instance, the designer determines the number of finalists and the head start given to the winner in the first stage. We showed that, independent of the number of finalists, it is optimal for the designer to maximize the players' total effort by giving a head start to the winner in the first stage such that he will have some advantage in the competition in the final stage. Moreover, we showed that the contest designer can increase the players' total effort by the optimal combination of the number of finalists and the head start given to the winner in the first stage. We found that the optimal head start is monotonically decreasing in the number of players but is not a monotonic function of the number of the finalists, while the number of finalists is monotonically increasing in the number of players. We demonstrated that by controlling some structural parameters of the two-stage elimination contest the designer can significantly increase the contestants' performance compared to the one-stage contest.

References

[1] Amegashie, J.: The design of rent-seeking competitions: committees, preliminary and final contests. *Public Choice* 99, 63-76 (1999)

[2] Amegashie, J., Cadsby, C., Song, Y.: Competitive burnout: theory and experimental evidence. *Games and Economic Behavior* 59, 213-239 (2007)

[3] Baye, M., Hoppe, H.: The strategic equivalence of rent-seeking, innovation, and patent-race games. *Games and Economic Behavior* 44(2), 217-226 (2003)

[4] Clark, D., Riis, C.: A multi-winner nested rent-seeking contest. *Public Choice* 77, 437-443 (1996)

[5] Clark, D., Riis, C.: Contest success functions: an extension. *Economic Theory* 11, 201-204 (1998)

[6] Corns A., Schotter, A.: Can affirmative action be cost effective? an experimental examination of price-preference auctions. *American Economic Review* 89, 291-305 (1999)

[7] Dasgupta, A., Nti, K.O.: Designing an optimal contest. *European Journal of Political Economy* 14, 587–603 (1998)[

[8] Epstein, G., Mealem, Y., Nitzan, S.: Political culture and discrimination in contests. *Journal of Public Economics* 1-2, 88-93 (2011)

[9] Franke, J., Kanzow, C., Leininger, W., Schwartz, A.: Effort maximization in asymmetric contest games with heterogeneous contestants. *Economic Theory* 52(2), 586-630 (2013)

[10] Fu, Q., Lu, J.: The optimal multi-stage contest. *Economic Theory* 51(2), 351-382 (2012)

[11] Gradstein, M., Konrad, K.: Orchestrating rent seeking contests. *The Economic Journal* 109, 536-545 (1999)

[12] Groh, C., Moldovanu, B., Sela, A., Sunde, U.: Optimal seedings in elimination tournaments. *Economic Theory* 49, 59-80 (2012)

[13] Kirkegaard, R.: Favoritism in contests: head starts and handicaps. *Games and Economic Behavior* 76 (1), 226-248 (2012).

[14] Konrad, K.: Investment in the absence of property rights: the role of incumbency advantages. *European Economic Review* 46(8), 563-575 (2002)

[15] Kovenock, D., Roberson, B.: Is the 50-state strategy optimal? *Journal of Theoretical Politics* 21(2), 213-236 (2009)

[16] Moldovanu, B., Sela, A.: The optimal allocation of prizes in contests. *American Economic Review* 91, 542-558 (2001)

[17] Moldovanu, B., Sela, A.: Contest architecture. *Journal of Economic Theory* 126(1), 70-97 (2006)

[18] Nti, K.O.: Maximum efforts in contests with asymmetric valuations. *European Journal of Political Economy* 20, 1059–1066 (2004)

[19] Rosen, S.: Prizes and incentives in elimination tournaments. *American Economic Review* 76, 701-715 (1986)

[20] Schweinzer, P., Segev, E.: The optimal prize structure of symmetric Tullock contests. *Public Choice* 153(1), 69-82 (2012).

[21] Segev, E., Sela, A.: Sequential all-pay auctions with head starts. *Social Choice and Welfare* 43(4), 893-923 (2014)

[22] Sheremeta, R.: Expenditures and information disclosure in two-stage political contests. *Journal of Conflict Resolution* 54, 771–798 (2010a)

[23] Sheremeta, R.: Experimental comparison of multi-stage and one-stage contests. *Games and Economic Behavior* 68, 731–747 (2010b)

[24] Skaperdas, S.: Contest success functions. *Economic Theory* 7, 283-290 (1996)

[25] Tsoulouhas, T., Knoeber, C.R., Agrawal, A.: Contests to become CEO: Incentives, selection and handicaps. *Economic Theory* 30, 195-221 (2007)

[26] Tullock, G.: Efficient rent-seeking, in J.M. Buchanan, R.D. Tollison and G. Tullock (Eds.), Toward a theory of rent-seeking society. College Station: Texas A.&M. University Press (1980).