VALUES FOR ENVIRONMENTS WITH EXTERNALITIES – THE AVERAGE APPROACH

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Values for Environments with Externalities - The Average Approach*

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Abstract

We propose the "average approach," where the worth of a coalition is a weighted average of its worth for different partitions of the players' set, as a unifying method to extend values for characteristic function form games. Our method allows us to extend the equal division value, the equal surplus value, the consensus value, the λ-egalitarian Shapley value, and the least-square family. For each of the first three extensions, we provide an axiomatic characterization of a particular value for partition function form games, for each of the last two extensions, we find a family of values that satisfy the properties.

JEL classification: D62, C71.

Keywords: Externalities, Sharing the surplus, Average Approach.

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1 Introduction

A central question in game theory is how players can share the gains from cooperation. Shapley (1953) addresses this issue for cooperative games in characteristic function form, where the description of a game specifies the resources every group of players has available for distribution among its members. He proposes the use of a sharing rule, or a value (known as the "Shapley value"), that satisfies the axioms of symmetry, carrier (which amounts to the efficiency plus dummy player axioms), and additivity. The Shapley value has been studied, interpreted, and characterized in many different ways.[1] Its greatest appeal is the fact it arises from apparently distinct and unrelated approaches (axiomatic, marginalistic, potential, dividends). It has also been extremely influential in later proposals for surplus sharing. Many researchers have followed the path he laid out and put forward some modifications of the Shapley axioms to define new values for sharing the surplus generated through cooperation.

A shortcoming of describing a cooperative environment through a characteristic function form game is that it disregards the possible existence of externalities among groups. Externalities in economic or political environments are the norm rather than the exception. For instance, research joint ventures, mergers and acquisitions, international negotiations on environmental issues, and trade agreements all exhibit important cross effects, namely, the gain that a group of agents obtains depends on the groups formed by the other players. A formal description of such settings with externalities is given by Thrall and Lucas (1963) who introduce games in partition function form.

To allow value theory to address environments with externalities, several papers adapted and at times augmented the Shapley axioms of efficiency, symmetry, linearity, and dummy player to partition function form games. This led to several new sharing methods for environments with externalities (see, e.g., Myerson, 1977; Bolger, 1989; Albizuri, Arin, and Rubio, 2005; Macho-Stadler, Pérez-Castrillo, and Wettstein, 2007; Pham Do and Norde, 2007; and McQuillin, 2009).[2]

In this paper, we suggest a unifying method, the "average approach," of extending

[1] See, for instance, Roth (1988).

[2] Different methods to extend the Shapley value are proposed by de Clippel and Serrano (2008), who rely on the marginal approach, and by Dutta, Elhers, and Kar (2010), who use the potential approach.

any value for characteristic function form games that satisfies the above axioms of efficiency, symmetry, and linearity to partition function form games. The average approach associates to each group of players a worth that is some weighted average of what they can obtain for all possible partitions of the other players. This yields a game with no externalities, the value of which determines the value for the original game. The axiomatic basis for this method is given by a natural extension of the symmetry axiom for partition form games, the "strong symmetry axiom," that is introduced by Macho-Stadler, Pérez-Castrillo, and Wettstein (2007) (MPW, 2007, hereafter). The strong symmetry axiom captures the idea that all players with identical influence in a game should receive the same outcome.

We use this approach to propose extensions of several well-known values as well as families of values defined for games without externalities. In addition, we suggest generalizations of the axioms proposed for characteristic function form games (such as the nullifying player, the neutral dummy player, or the coalitional monotonicity axioms), to adapt them to situations with externalities. Our method allows us to extend the *equal division value* (Van den Brink, 2007), the *equal surplus value* (Driessen and Funaki, 1991), the *consensus value* (Ju, Borm, and Ruys, 2007), the λ-*egalitarian Shapley value* (Joosten, 1996), and the *least-square family* (Ruiz, Valenciano, and Zarzuelo, 1998). For each of the first three extensions, we also provide an axiomatic characterization of a particular value for partition function form games. It is worth noting that the extension of the consensus value through the average approach coincides with the one proposed by Ju (2007).[3] For each of the last two extensions, a family of values that satisfy the properties is found.

The paper proceeds as follows. In section 2, we present the environment. Section 3 introduces the basic axioms and the average approach. Each of the sections 4 to 8 presents the extension of one value or family of values. Finally, section 9 briefly concludes.

[3] Besides the previous papers, Hernández-Lamoneda, Sánchez-Pérez, and Sánchez-Sánchez (2009) extend the solidarity value (Nowak and Radzik, 1994) to games with externalities.

2 The environment

The economic environment we study can be described as follows. We denote by $N = \{1, ..., n\}$ the set of players. A coalition S is a group of s players, that is, a non-empty subset of N, $S \subseteq N$. An embedded coalition specifies the coalition as well as the structure of coalitions formed by the other players, that is, an embedded coalition is a pair (S, P), where S is a coalition and $P \ni S$ is a partition of N. Let $\mathcal{P}$ be the set of all partitions of N and ECL the set of embedded coalitions defined as:

$$ECL = \{(S, P) \mid S \in P,\ P \in \mathcal{P}\}.$$

We denote by v a game in *partition function form* (PFF), where $v : ECL \to \mathbb{R}$ is a function that associates a real number with each embedded coalition (S, P). Let $\mathcal{G}^N$ be the set of games in PFF with players in N. We interpret $v(S, P)$ as the worth of coalition S when the players are organized according to the partition P. We assume that players in a coalition can make transfers among them, that is, we consider transferable utility (TU) games. The worth $v(S, P)$ may depend on the partition P. This implies that the organization of the players outside S may create a positive or negative externality on the payoff that players in a coalition S can jointly obtain. We use the convention that the empty set $\varnothing$ is in P for every $P \in \mathcal{P}$, and assume that the function satisfies $v(\varnothing, P) = 0$. We denote by $\mathcal{P}_S = \{P \in \mathcal{P} \mid S \in P\}$ the set of partitions including S. Finally, we denote by $[S]$ the partition of S consisting of all the singleton players in S, that is, $[S] = \left\{\{i\}_{i \in S}\right\}$.

Examples of games with externalities are the games $w_{S,P}$ defined by

$$w_{S,P}(S, P) = w_{S,P}(N, (N, \varnothing)) = 1, \text{ and } w_{S,P}(S', P') = 0 \text{ otherwise.}$$

In the game $w_{S,P} \in \mathcal{G}^N$ there are only two scenarios where a coalition has a positive worth, the first is for the coalition S when the players are organized according to the partition P, and the second is for the grand coalition. The games $w_{S,P}$ constitute a basis for the set $\mathcal{G}^N$.

Some games in $\mathcal{G}^N$ do not have externalities. A game is without externalities if the worth of any coalition S is independent of the way the other players are organized. A game without externalities satisfies $v(S, P) = v(S, P')$ for any $P, P' \in \mathcal{P}_S$ and any coalition

$S \subseteq N$. We denote such a game by $\hat{v}$. Since in this case the worth of a coalition S can be written without reference to the organization of the remaining players, we can write $\hat{v}(S) \equiv \hat{v}(S,P)$ for all $P \in \mathcal{P}_S$ and all $S \subseteq N$. We denote by G^N the set of games without externalities with players in N, which corresponds to the set of TU games in *characteristic function form* (CFF).

A solution concept for PFF games, or *a value*, is a mapping φ which associates with every game $v \in \mathcal{G}^N$ a vector in $\mathbb{R}^n$, specifying the payoff of each player, that satisfies $\sum_{i \in N} \varphi_i(v) = v(N,(N,\varnothing))$. Thus, in this paper, a value always shares the worth of the grand coalition, that is, it satisfies the *efficiency* axiom.

Similarly, *a value for CFF games* is a mapping ψ which associates with every game $\hat{v} \in G^N$ a vector in $\mathbb{R}^n$ such that $\sum_{i \in N} \psi_i(\hat{v}) = \hat{v}(N)$.

3 Basic axioms and the average approach

Shapley (1953) proposes linearity and symmetry as reasonable requirements to impose on values for CFF games. To introduce these axioms, we first define some operations for the set of CFF games.

The *addition* of two games $\hat{v}$ and $\hat{v}'$ in G^N is defined as the game $\hat{v}+\hat{v}'$ where $(\hat{v}+\hat{v}')(S) \equiv \hat{v}(S)+\hat{v}'(S)$ for all $S \subseteq N$. Similarly, given the game $\hat{v}$ and the scalar $\lambda \in \mathbb{R}$, the game $\lambda\hat{v}$ is defined by $(\lambda\hat{v})(S) \equiv \lambda\hat{v}(S)$ for all $S \subseteq N$.

Let σ be a permutation of N. Then the σ *permutation* of the game $\hat{v} \in G^N$ denoted by $\sigma\hat{v}$ is defined by $(\sigma\hat{v})(S) \equiv \hat{v}(\sigma S)$ for all $S \subseteq N$.

C1 *Linearity*: A value for CFF games ψ satisfies the linearity axiom if:

C1.1 For any two games $\hat{v}$ and $\hat{v}'$, $\psi(\hat{v}+\hat{v}') = \psi(\hat{v}) + \psi(\hat{v}')$.

C1.2 For any game $\hat{v}$ and any scalar $\lambda \in \mathbb{R}$, $\psi(\lambda\hat{v}) = \lambda\psi(\hat{v})$.

C2 *Symmetry*: A value for CFF games ψ satisfies the symmetry axiom if for any permutation σ of N, $\psi(\sigma\hat{v}) = \sigma\psi(\hat{v})$.

In this paper, we consider the family of values ψ that, in addition to efficiency, satisfy symmetry and linearity.

The operations, and the linearity and symmetry axioms can be easily extended to PFF games. The addition of two games v and v' in $\mathcal{G}^N$ is defined as the game $v + v'$ where $(v + v')(S, P) \equiv v(S, P) + v'(S, P)$ for all $(S, P) \in ECL$. Also, given the game v and the scalar $\lambda \in \mathbb{R}$, the game λv is defined by $(\lambda v)(S, P) \equiv \lambda v(S, P)$ for all $(S, P) \in ECL$. Similarly, the σ permutation of the game $v \in \mathcal{G}^N$ denoted by σv is defined by $(\sigma v)(S, P) \equiv v(\sigma S, \sigma P)$ for all $(S, P) \in ECL$.

Then we can define two basic axioms for a value φ:

P1 *Linearity*: A value φ satisfies the linearity axiom if:

P1.1 For any two games v and v', $\varphi(v + v') = \varphi(v) + \varphi(v')$.

P1.2 For any game v and any scalar $\lambda \in \mathbb{R}$, $\varphi(\lambda v) = \lambda \varphi(v)$.

P2 *Symmetry*: A value φ satisfies the symmetry axiom if for any permutation σ of N, $\varphi(\sigma v) = \sigma \varphi(v)$.

As explained in MPW (2007), the symmetry axiom imposes much more structure on a value for CFF games than it does on a value for PFF games. The strong symmetry axiom strengthens the symmetry axiom by requiring that the payoff of a player should not change after permutations in the set of players in $N \backslash S$, for any embedded coalition structure (S, P). It imposes in addition to symmetric treatment of individual players, the symmetric treatment of "externalities" generated by players in a given embedded coalition structure. As a consequence, exchanging the names of the players inducing externalities does not affect the payoff of any player.

Formally, given an embedded coalition (S, P), we denote by $\sigma_{S,P}P$ a new partition such that $S \in \sigma_{S,P}P$, and the other coalitions result from a permutation of the set $N \backslash S$ applied to $P \backslash S$. That is, in the partition $\sigma_{S,P}P$, the players in $N \backslash S$ are reorganized in sets whose size distribution is the same as in $P \backslash S$. Given the permutation $\sigma_{S,P}$, the permutation of the game v denoted by $\sigma_{S,P}v$ is defined by $(\sigma_{S,P}v)(S, P) = v(S, \sigma_{S,P}P)$, $(\sigma_{S,P}v)(S, \sigma_{S,P}P) = v(S, P)$, and $(\sigma_{S,P}v)(R, Q) = v(R, Q)$ for all $(R, Q) \in ECL \backslash \{(S, P), (S, \sigma_{S,P}P)\}$.

P2' *Strong Symmetry*: A value φ satisfies the strong symmetry axiom if:

P2'.1 For any permutation σ of N, $\varphi(\sigma v) = \sigma\varphi(v)$,

P2'.2 For any $(S, P) \in ECL$ and for any permutation $\sigma_{S,P}$, $\varphi(\sigma_{S,P} v) = \varphi(v)$.

For the basic games $w_{S,P}$, the strong symmetry axiom is equivalent to the properties that (a) $\varphi_i(w_{S,P}) = \varphi_j(w_{S,P})$ for all $i, j \in S$ and for all $i, j \in N \backslash S$ and (b) $\varphi_i(w_{S,P})$ only depends on the size of S and the size distribution of the coalitions in P.

Besides the axiomatic approach, MPW (2007) provide an alternative method to go from values for CFF games to values for PFF games. The *average approach* first transforms a PFF game to a CFF game by assigning to any coalition of players an average of the different worths of this group for all the possible organizations of the other players. Then, it uses a value for CFF games (the Shapley value in MPW, 2007) to determine the payoffs of the players.

Formally, the average approach constructs a value φ for PFF games using a value for CFF games ψ as follows. First, for any game $v \in \mathcal{G}^N$, it constructs an average game $\tilde{v}$ by assigning to each $S \subseteq N$ the average worth $\tilde{v}(S) \equiv \sum_{P \in \mathcal{P}_S} \alpha(S, P) v(S, P)$, with $\sum_{P \in \mathcal{P}_S} \alpha(S, P) = 1$. We refer to $\alpha(S, P)$ as the "weight" of the partition P in the computation of the value of coalition $S \in P$. We restrict attention to symmetric and non-negative weights, that is, $\alpha(S, P) \geq 0$ and it depends solely on the size of the coalition S and the size distribution of the coalitions in P.[4] Second, we define $\varphi_i(v) = \psi_i(\tilde{v})$.

The following theorem establishes the relationship between the average approach and the strong symmetry axiom, when we consider symmetric and linear values for CFF.

Theorem 1 *The value φ can be constructed through the average approach using a value for CFF games ψ that satisfies linearity and symmetry if and only if φ satisfies linearity and strong symmetry.*

The proof of Theorem 1 is constructive in part and relies heavily on the linearity axiom that allows to extend properties over a "basis" for $\mathcal{G}^N$, to all of $\mathcal{G}^N$. We also make

[4] The weights $\alpha(S, P)$ can only depend on the sizes of the coalitions if we consider symmetric and linear values. This property allows us, in the Appendix, to sometimes use the expression $\alpha(s, p)$ instead of $\alpha(S, P)$ for the weights, where p is the vector describing the size distribution of the coalitions in the partition P.

use of the fact that the value for CFF games ψ generating a value for PFF games φ is uniquely determined by φ, since both values must coincide for CFF games, when viewed as PFF games (without externalities). Besides its intuitive appeal, as we will see shortly, the average approach provides a structured method of extending CFF values satisfying linearity and symmetry to linear and (strongly) symmetric PFF values.

As discussed in the Introduction, the most prominent value for CFF games that satisfies linearity and symmetry is the Shapley value. MPW (2007) use the average approach to extend the Shapley value to PFF games. Given that we will use this extension in some of the next sections, and also for completeness, we include it here. It requires defining a dummy player and the dummy player axiom.

Player $i \in N$ is a *dummy* player in $\hat{v} \in G^N$ if player i contributes nothing to any coalition, that is, $\hat{v}(S \cup \{i\}) = \hat{v}(S)$ for all $S \subset N$ with $i \in N \backslash S$.

C3 *Dummy player*: A value for CFF games ψ satisfies the dummy player axiom if $\psi_i(\hat{v}) = 0$ whenever i is a dummy player in $\hat{v}$.

Shapley (1953) shows that a value for CFF games ψ satisfies symmetry, linearity, and dummy player if and only if it is the *Shapley value,* ψ^{Sh}:

$$\psi_i^{Sh}(\hat{v}) = \sum_{S \subseteq N} \beta_i(S)\hat{v}(S) \qquad \text{for all } i \in N,$$

where

$$\beta_i(S) = \begin{cases} \frac{(s-1)!(n-s)!}{n!} \text{ for all } S \subseteq N, \text{ if } i \in S \\ -\frac{s!(n-s-1)!}{n!} \text{ for all } S \subseteq N, \text{ if } i \in N \backslash S. \end{cases}$$

Moving now to PFF games, we say that player i is a *dummy* player in $v \in \mathcal{G}^N$ if he alone receives zero for any partition of the other players and furthermore, has no effect on the worth of any coalition S (that is, the worth of S in partition P is constant for all possible assignments of player i to some coalition in P). More formally, player i is a dummy player in $v \in \mathcal{G}^N$ if for every $(S, P) \in ECL$ with $i \in S$ and each $T \in P$, $T \neq S$, $v(S, P) = v(S \backslash \{i\}, \{(S \backslash \{i\}, R \cup \{i\} \cup P \backslash (S, R)\})$.[5]

[5]For $R = \varnothing$, we slightly abuse notation by assuming that the partition $(P \backslash (\varnothing, S)) \cup (\varnothing \cup \{i\}, S \backslash \{i\})$ also includes the empty set.

P3 *Dummy player*: A value φ satisfies the dummy player axiom if for any dummy player i in the game v, $\varphi_i(v) = 0$.

Adopting this dummy player axiom in combination with linearity and symmetry allows identifying a family of values that extend the Shapley value to PFF games.

Theorem 2 *(MPW, 2007) The value φ can be constructed through the average approach with weights that satisfy*

$$\alpha(S \cup i, P) = \sum_{R \in P \backslash (S \cup \{i\})} \alpha(S, (P \backslash (R, (S \cup \{i\}))) \cup (R \cup \{i\}, S)) \qquad (1)$$

for all $i \in N \backslash S$ and for all $((S \cup \{i\}), P) \in ECL$, using a value for CFF games ψ that satisfies linearity, symmetry, and dummy player if and only if φ satisfies linearity, strong symmetry, and dummy player.

4 Extension of the equal division value

The equal division value distributes the worth of the grand coalition equally among all players. This value, which only takes into account the worth of the grand coalition and ignores the worth of any intermediary organization, is characterized by van den Brink (2007) using the notion of "nullifying players." A nullifying player is one whose presence in a coalition implies that the coalition generates zero worth. Formally, player $i \in N$ is a *nullifying* player in $\hat{v} \in G^N$ if $\hat{v}(S) = 0$ for all $S \subseteq N$ with $i \in S$. The following axiom proposed by Van den Brink (2007) captures the idea that such players should get zero in the game:

C4 *Nullifying player*: A value for CFF games ψ satisfies the nullifying player axiom if $\psi_i(\hat{v}) = 0$ whenever i is a nullifying player in $\hat{v}$.

He shows that replacing the dummy player axiom in the characterization of the Shapley value with the nullifying player axiom yields the equal division solution: a value ψ for CFF games satisfies symmetry, linearity, and nullifying player if and only if it is the *equal division value* ψ^{ED}, where

$$\psi_i^{ED}(\hat{v}) = \frac{\hat{v}(N)}{n} \qquad \text{for all } i \in N.$$

To extend ψ^{ED} to PFF games, we first adapt the definition of a nullifying player and the nullifying player axiom. We say that player $i \in N$ is a nullifying player in $v \in \mathcal{G}^N$ if any coalition containing player i earns zero independently of the organization of the players outside the coalition. Formally, player $i \in N$ is a *nullifying* player in $v \in \mathcal{G}^N$ if $v(S, P) = 0$ for all $S \subseteq N$ with $i \in S$. The $C4$ axiom can now be stated for PFF games as follows:

P4 *Nullifying player*: A value φ satisfies the nullifying player axiom if $\varphi_i(v) = 0$ whenever i is a nullifying player in v.

Theorem 3 shows that extending the equal division value to PFF games through the average approach is equivalent to requiring that the value satisfies the nullifying player axiom in addition to linearity and strong symmetry.

Theorem 3 *The value φ can be constructed through the average approach using a value for CFF games ψ that satisfies linearity, symmetry, and nullifying player if and only if φ satisfies linearity, strong symmetry, and nullifying player.*

Given that any extension for PFF games of ψ^{ED} using the average approach only takes into account the value of the grand coalition, it is immediate that such an extension leads to the same value φ^{ED} for any system of weights $(\alpha(S, P))_{(S,P)\in ECL}$. This allows us to provide the following extension of the equal division value for PFF games.

Corollary 1 *A value φ satisfies linearity, strong symmetry, and nullifying player if and only if $\varphi = \varphi^{ED}$, where*

$$\varphi_i^{ED} = \frac{v(N, N)}{n} \quad \textit{for all } i \in N.$$

The literature has offered alternative characterizations of the equal division value. In particular, Chameni-Nembua and Andjika (2008) prove that the value ψ for CFF games is linear, symmetric, and non-negative if and only if $\psi = \psi^{ED}$,[6] where non-negativity is defined as follows:

[6] For additional characterizations of the equal division value, see Chun and Park (2012), Casajus and Huettner (2014a), Béal *et al.* (2014), and Béal, Rémila, and Solal (2015).

C5 *Non-negativity*: A value for CFF games ψ satisfies the non-negativity axiom if for all $\hat{v} \in G^N$ such that $\hat{v}(S) \geq 0$ for all $S \subseteq N$, $\psi_i(\hat{v}) \geq 0$ for all $i \in N$ is satisfied.

The natural extension of the non-negativity axiom to PFF games is the following:

P5 *Non-negativity*: A value φ satisfies the non-negativity axiom if for all $v \in \mathcal{G}^N$ such that $v(S, P) \geq 0$ for all $(S, P) \in ECL$, $\varphi_i(v) \geq 0$ for all $i \in N$ is satisfied.

Theorem 4 presents a similar result to Theorem 3 but uses the non-negativity axiom instead of the nullifying player axiom.

Theorem 4 *The value φ can be constructed through the average approach using a value for CFF games ψ that satisfies linearity, symmetry, and non-negativity if and only if φ satisfies linearity, strong symmetry, and non-negativity.*

5 Extension of the equal surplus value

The *equal surplus value* also distributes the "surplus" obtained by the grand coalition equally among all the players. The surplus is defined as the worth of the grand coalition minus the sum of the stand-alone worths of all the players. This value is proposed by Driessen and Funaki (1991), who call it the *Center-of-gravity of the Imputation-Set value* (the *CIS*-value). Since then this value has been given various names such as the *egalitarian value* in Chun and Park (2012) and the equal surplus value in Moulin (2003), which is the terminology that we adopt in this paper.

Several characterizations exist for the equal surplus value. We first use the one provided by Casajus and Huettner (2014a), which is close to the characterizations of the Shapley and the equal division values. Instead of the dummy player or the nullifying player axioms, it introduces a "dummifying player" axiom. The presence of a dummifying player in a coalition implies that there are no gains from cooperation and the worth of the coalition coincides with the sum of the stand-alone worths of its players (a no-cooperation scenario). That is, player $i \in N$ is a *dummifying* player in $\hat{v}$ if $\hat{v}(S) = \sum_{j \in S} \hat{v}(\{j\})$ for all $S \subseteq N$ such that $i \in S$. The following axiom proposed by Casajus and Huettner (2014a) reflects the idea that such players should get their stand-alone worth in the game:

C6 *Dummifying player*: A value for CFF games ψ satisfies the dummifying player axiom if $\psi_i(\hat{v}) = \hat{v}(\{i\})$ whenever i is a dummifying player in $\hat{v}$.

Casajus and Huettner (2014a) show that a value for CFF ψ satisfies symmetry, linearity, and dummifying player if and only if it is the equal surplus value ψ^{ES} formally defined as

$$\psi_i^{ES}(\hat{v}) = \hat{v}(\{i\}) + \frac{1}{n}\left(\hat{v}(N) - \sum_{j\in N} \hat{v}(\{j\})\right) \qquad \text{for all } i \in N.$$

The above expression makes it clear that the equal surplus value uses limited information since it ignores the worth of all intermediate coalitions.

Our definition of a dummifying player in PFF games uses the same logic as its definition in CFF games. In the set of PFF games, $v(\{j\}, [N])$ is the worth that player j obtains if no cooperation exists. Thus, we say that player $i \in N$ is a *dummifying* player in $v \in \mathcal{G}^N$ if $v(S,P) = \sum_{j\in S} v(\{j\}, [N])$ for all $(S,P) \in ECL$ such that $i \in S$. This leads to the following axiom:

P6 *Dummifying player*: A value φ satisfies the dummifying player axiom if $\varphi_i(v) = v(\{i\}, [N])$ whenever i is a dummifying player in v.

As is the case for the dummy player axiom (see Theorem 2), using the average approach to extend the dummifying player axiom imposes restrictions on the weights. Theorem 5 formally states the extension and the restrictions on the weights.

Theorem 5 *The value φ can be constructed through the average approach with weights that satisfy*

$$\alpha(\{i\}, [N]) = 1 \text{ and } \alpha(\{i\}, P) = 0 \text{ for all } P \neq [N] \qquad (2)$$

using a value for CFF games ψ that satisfies linearity, symmetry, and dummifying player if and only if φ satisfies linearity, strong symmetry, and dummifying player.

Following Theorem 5, Corollary 2, whose proof is immediate, presents and characterizes our proposed extension of the equal surplus value to PFF games.

Corollary 2 *A value φ satisfies the linearity, strong symmetry, and dummifying player axioms if and only if $\varphi = \varphi^{ES}$, where*

$$\varphi_i^{ES}(v) = v\left(\{i\}, [N]\right) + \frac{1}{n}\left(v\left(N, N\right) - \sum_{j \in N} v\left(\{j\}, [N]\right)\right).$$

The second characterization we focus on is the one proposed by Ju, Borm, and Ruys (2007) as it can be naturally extended to PFF games.[7] They introduce an axiom involving dummy players which we refer to as the ES dummy player axiom.

C7 *ES dummy player*: A value for CFF games ψ satisfies the ES dummy player axiom if $\psi_i(\hat{v}) = \frac{1}{n}\left(\hat{v}\left(N\right) - \sum_{j \in N} \hat{v}\left(\{j\}\right)\right)$ whenever i is a dummy player in $\hat{v}$.

The alternative characterization we study can be stated as follows. A value ψ satisfies symmetry, linearity, and ES dummy player if and only if it is the equal surplus value ψ^{ES}.

We now extend the ES dummy player axiom to PFF games following the same logic used to extend the dummifying player axiom:

P7 *ES dummy player*: A value φ satisfies the ES dummy player axiom if $\varphi_i(v) = \frac{1}{n}\left(v\left(N, N\right) - \sum_{j \in N} v\left(\{j\}, [N]\right)\right)$ whenever i is a dummy player in v.

The following theorem 6 uses the ES dummy player axiom to characterize the extension of the equal surplus value to PFF games. It requires the same condition on the weights as Theorem 5. It is stated for $n > 3$, as for $n = 3$ it holds without the condition (2) on the weights.

Theorem 6 *Suppose $n > 3$. The value φ can be constructed through the average approach with weights that satisfy (2) using a value for CFF games ψ that satisfies linearity, symmetry, and ES dummy player if and only if φ satisfies linearity, strong symmetry, and ES dummy player.*

[7] For other characterizations of the equal surplus value for CFF game see Driessen and Funaki (1997), van den Brink (2007), van den Brink and Funaki (2009), Chun and Park (2012), Casajus and Heuttner (2014b), and Béal, Rémila, and Solal (2015).

We note that the restriction (2) is due to the structure imposed on the weights by either the dummifying player or the ES dummy player axiom. The restriction does not apply to 3-player games when we use the ES-dummy player axiom because in a 3-player PFF game if one of the players, say player 1, is a dummy player, then this game has no externalities.

6 Extension of the consensus value

Ju, Borm, and Ruys (2007) introduce a value inspired by a sequential two-sided negotiation process through which players proceed to distribute the surplus generated by cooperation. They recursively use the "standard" solution of two player games whereby each of the players receives the equal surplus value to obtain the *consensus value* as a suitably defined average payoff to each of the players in the negotiation process. This value also has an axiomatic characterization using the *neutral dummy player axiom.*

C8 *Neutral dummy player:* A value for CFF games ψ satisfies the neutral dummy player axiom if $\psi_i(\hat{v}) = \frac{1}{2n}\left(\hat{v}(N) - \sum_{j\in N}\hat{v}(\{j\})\right)$ whenever i is a dummy player in $\hat{v}$.

Ju, Borm, and Ruys (2007) prove that a value for CFF games ψ satisfies symmetry, linearity, and neutral dummy player if and only if it is the *consensus* value, denoted by ψ^C. They also prove that the consensus value is the average of the Shapley and equal surplus values, that is,

$$\psi_i^C(\hat{v}) = \frac{1}{2}\psi_i^{Sh}(\hat{v}) + \frac{1}{2}\psi_i^{ES}(\hat{v}) \qquad \text{for all } i \in N.$$

Our extension of the neutral dummy player axiom to PFF games takes a similar approach to the one we used in the extension of the ES dummy player axiom.

P8 *Neutral dummy player:* A value φ satisfies the neutral dummy player axiom if $\varphi_i(v) = \frac{1}{2n}\left(v(N,N) - \sum_{j\in N} v(\{j\},[N])\right)$ whenever i is a dummy player in v.

We now state the result that allows extending the consensus value in Theorem 7 for $n > 3$. Theorem 7 also holds for $n = 3$ but without the condition (3) on the weights.

Theorem 7 *Suppose $n > 3$. The value φ can be constructed through the average approach with weights that satisfy*

$$\alpha(S, (S, [N \backslash S])) = 1 \text{ and } \alpha(S, P) = 0 \text{ for all } P \neq (S, [N \backslash S]) \tag{3}$$

using a value for CFF games ψ that satisfies linearity, symmetry, and neutral dummy player if and only if φ satisfies linearity, strong symmetry, and neutral dummy player.

Given that the consensus value is a combination of the Shapley and the equal division values, the weights need to satisfy both conditions (1) and (2), which amounts to condition (3). Once more, Corollary 3 easily follows from Theorem 7.

Corollary 3 *A value φ satisfies linearity, strong symmetry, and neutral dummy player if and only if $\varphi = \varphi^C$, where*

$$\varphi_i^C(v) = \frac{1}{2}\varphi_i^{Sh1}(v) + \frac{1}{2}\varphi_i^{ES}(v),$$

with $\varphi_i^{Sh1}(v) = \psi_i^{Sh}(\tilde{v})$, where $\tilde{v}$ uses the weights $\alpha(S, (S, [N \backslash S])) = 1$ and $\alpha(S, P) = 0$ for all $P \neq (S, [N \backslash S])$.

It is interesting to note that the value φ^C that is derived from our approach was previously introduced and characterized by Ju (2007) using a different system of axioms. His axioms were based on the "externality-free" extension of the Shapley value to PFF games, φ^{Sh1}, proposed and characterized by Pham Do and Norde (2007).[8]

It is also worth noting that Ju, Borm, and Ruys (2007) extend the consensus value to general convex combinations of the Shapley and the equal surplus values. They define the λ-consensus value as $\psi_i^{\lambda C}(\hat{v}) = (1-\lambda)\psi_i^{Sh}(\hat{v}) + \lambda\psi_i^{ES}(\hat{v})$, for any $\lambda \in [0,1]$. They characterize $\psi^{\lambda C}$ by substituting the neutral player axiom with the λ-dummy axiom which is stated as follows: A value ψ satisfies the λ-dummy player axiom if $\psi_i(\hat{v}) = \frac{\lambda}{n}\left(\hat{v}(N) - \sum_{j \in N} \hat{v}(\{j\})\right)$ whenever i is a dummy player in $\hat{v}$. The definition of the λ-dummy axiom, Theorem 7, and Corollary 3 can be easily generalized for PFF games using the average approach for any $\lambda \in (0,1)$.[9]

[8] An interesting marginality-based axiomatization of φ^{Sh1} can be found in de Clippel and Serrano (2008).

[9] For $\lambda = 0$, $\psi^{\lambda C} = \psi^{Sh}$ and the results hold with condition (1) instead of (3). For $\lambda = 1$, $\psi^{\lambda C} = \psi^{ES}$ and the results hold with condition (2) instead of (3).

7 Extension of the λ-egalitarian Shapley values

Combining properties of the Shapley and the equal division values, Joosten (1996) introduces the λ-*egalitarian Shapley values*. This family of values is obtained by replacing the dummy player axiom used in the Shapley value characterization with the λ-*egalitarian dummy player axiom* (for $\lambda \in [0,1]$).

C9 λ-*egalitarian dummy player*: A value for CFF games ψ satisfies the λ-egalitarian dummy player axiom if $\psi_i(\hat{v}) = \frac{\lambda}{n}\hat{v}(N)$ whenever i is a dummy player in $\hat{v}$.

Joosten (1996) shows that the λ-egalitarian Shapley value for $\lambda \in [0,1]$ can be expressed as $\psi^{\lambda\alpha}(\hat{v}) = (1-\lambda)\psi_i^{Sh}(\hat{v}) + \lambda\psi_i^{ED}(\hat{v})$.

We now extend the λ-egalitarian dummy player axiom to PFF games as follows:

P9 λ-*egalitarian dummy player:* A value φ satisfies the λ-egalitarian dummy player axiom if $\varphi_i(v) = \frac{\lambda}{n}v\,(N,N)$ whenever i is a dummy player in v.

Theorem 8 generalizes the λ-egalitarian Shapley values for $\lambda < 1$ to PFF games. The theorem also holds for $\lambda = 1$, in which case the value corresponds to the equal division value, but without the condition (1) on the weights, as shown in Theorem 3.

Theorem 8 *Let $\lambda \in [0,1)$. The value φ can be constructed through the average approach with weights that satisfy (1) for all $i \in N \backslash S$ and for all $((S \cup \{i\}), P) \in ECL$, using a value for CFF games ψ that satisfies linearity, symmetry, and λ-egalitarian dummy player if and only if φ satisfies linearity, strong symmetry, and λ-egalitarian dummy player.*

In contrast to the extensions of the consensus value or of the general convex combinations of the Shapley and the equal surplus values, Theorem 8 does not characterize a unique value for PFF games for a given λ. Any $\varphi = (1-\lambda)\varphi^{Sh} + \lambda\varphi^{ED}$ extends $\psi^{\lambda\alpha}$, where φ^{Sh} is one of the many values obtained in Theorem 2 satisfying the linearity, strong symmetry, and dummy player axioms. Particular examples for φ^{Sh} are φ^{Sh1}, or the extension of the Shapley value suggested by MPW (2007), where the worth of every embedded coalition matters.

8 Extension of the least square family

The last extension deals with the least-square family introduced by Ruiz, Valenciano, and Zarzuelo (1998). Each value in this family selects the unique efficient payoff vector which minimizes the weighted variance of the excesses of the coalitions.[10] Formally, let the excess vector of the coalition $S \subseteq N$ at an efficient payoff vector $x \in \mathbb{R}^n$ in the game $\hat{v} \in G^N$ be $e(S, x) = \hat{v}(S) - \sum_{i \in S} x_i$. Also, let the average excess at x be $\bar{e}(\hat{v}) = \frac{1}{2^n - 1} \sum_{S \subseteq N} e(S, x)$, where one can use $\bar{e}(\hat{v})$ instead of $\bar{e}(\hat{v}, x)$ because the sum is the same for any efficient payoff vector x. Then, a value for CFF games ψ belongs to the *least square* (LS) *family* if there exist weights $\gamma \in \mathbb{R}^n_+$ such that for all $\hat{v} \in G^N$, $\psi(\hat{v})$ is the solution to

$$Min \sum_{S \subseteq N} \gamma(s) \left(e(S, x) - \bar{e}(\hat{v})\right)^2$$
$$\text{s.t.} \sum_{i \in N} x_i = \hat{v}(N).$$

To introduce the axioms that characterize the LS family, we say that a game $\hat{v} \in G^N$ is *additive* if $\hat{v}(S) = \sum_{i \in S} \hat{v}(\{i\})$ for all $S \subseteq N$ and provide the following two axioms:

C10 *Inessential game*: A value for CFF games ψ satisfies the inessential game axiom if $\psi_i(\hat{v}) = \hat{v}(\{i\})$ whenever $\hat{v}$ is additive.

C11 *Coalitional monotonicity*: A value for CFF games ψ satisfies the coalitional monotonicity axiom if $\psi_i(\hat{v}) \geq \psi_i(\hat{w})$ for all $i \in S$ whenever $\hat{v}$ and $\hat{w}$ are such that $\hat{v}(S) > \hat{w}(S)$ and $\hat{v}(R) = \hat{w}(R)$ for $R \neq S$.

Ruiz, Valenciano, and Zarzuelo (1998) prove that ψ satisfies symmetry, linearity, inessential game, and coalitional monotonicity if and only if it belongs to the LS family.

We now extend the additivity property to games with externalities as follows. A game $v \in \mathcal{G}^N$ is *additive* if $v(S, P) = \sum_{i \in S} v(\{i\}, [N])$ for all $(S, P) \in ECL$. Note that additive games have no externalities. The above two axioms are now generalized for PFF games by

[10]The least square family includes the least square prenucleolus, defined in Ruiz, Valenciano, and Zarzuelo (1996), where all the coalitions have the same weights. It also includes the Shapley value.

P10 *Inessential game*: A value φ satisfies the inessential game axiom if $\varphi_i(v) = v(\{i\}, [N])$ whenever v is additive.

P11 *Coalitional monotonicity*: A value φ satisfies the coalitional monotonicity axiom if $\varphi_i(v) \geq \varphi_i(w)$ for all $i \in S$ whenever v and w are such that $v(S,P) > w(S,P)$ for some $P \in \mathcal{P}_S$ and $v(R,Q) = w(R,Q)$ for $(R,Q) \neq (S,P)$.

An equivalent statement for the $C11$ axiom is that a value φ satisfies the coalitional monotonicity axiom if $\varphi_i(v) \geq \varphi_i(w)$ for all $i \in S$ whenever v and w are such that $v(S,P) \geq w(S,P)$ for all $P \in \mathcal{P}_S$, and $v(R,Q) = w(R,Q)$ for $R \neq S$, $Q \in \mathcal{P}_R$.

Theorem 9 suggests how to extend the LS family of values to PFF games.

Theorem 9 *The value φ can be constructed through the average approach using a value for CFF games ψ that satisfies linearity, symmetry, inessential game, and coalitional monotonicity if and only if φ satisfies linearity, strong symmetry, inessential game, and coalitional monotonicity.*

Given that the LS family in CFF games includes the Shapley value, it is interesting to note that the extensions of the Shapley value characterized in Theorem 2 are also included in the extension of the LS family proposed in Theorem 9. This result is an immediate corollary of the following Proposition 1.

Proposition 1 *If the value φ satisfies linearity, strong symmetry, and dummy player, then it also satisfies inessential game and coalitional monotonicity.*

9 Conclusion

In this paper we have used the "average approach" to extend several surplus sharing methods proposed in the literature for characteristic function form games to partition function form games. Our method can be applied to any value for CFF games that satisfies efficiency, linearity, and symmetry. The axiomatic basis for this approach stems from a natural extension of the symmetry axiom for CFF games to a strong symmetry concept for PFF games.

The extensions of the values make it possible to move from normative and distributive issues pertaining to CFF games to their counterparts in PFF games. They also provide axiomatic characterizations of several values for PFF games. The method may of course be applied to values other than those we discussed in this paper and thus we see it as an expedient link between the analysis of standard transferable utility games and transferable utility games with externalities.

The approach suggested here may also provide a non-cooperative foundation for values similar to the one provided in Macho-Stadler, Pérez-Castrillo, and Wettstein (2006) for the Shapley value extension.

10 Appendix

Proof. of Theorem 1. Suppose that φ can be constructed through the average approach (AA) using a value ψ for CFF games that satisfies linearity and symmetry. Denote by $(\alpha(S,P))_{(S,P)\in ECL}$ the weights used in the AA. Also, we can write $\psi_i(v) = \sum_{S\subseteq N}\theta_i(S)v(S)$ for any game v, where $\theta_i(S)$ is the value ψ assigns to player i in the CFF game where the worth of coalition S is 1 and the worth of any other coalition is 0.

By construction, φ satisfies linearity. We also note that the CFF game $\tilde{w}_{S,P}$ corresponding to $w_{S,P}$ satisfies $\tilde{w}_{S,P}(S) = \alpha(S,P)$, $\tilde{w}_{S,P}(N) = 1$, and $\tilde{w}_{S,P}(S') = 0$ for any S' different from S and N. Therefore, for all $(S,P) \in ECL$, $\varphi_i(w_{S,P}) = \psi_i(\tilde{w}_{S,P}) = \theta_i(S)\alpha(S,P) + \theta_i(N)$ for all $i \in N$. Given that $\theta_i(S)$ are the same for all players $i \in S$, and are also the same for all players $i \in N\backslash S$ by the symmetry of ψ, $\varphi_i(w_{S,P}) = \varphi_j(w_{S,P})$ for all $i,j \in S$ and for all $i,j \in N\backslash S$. Second, $\theta_i(S)$ and $\alpha(S,P)$ only depend on the sizes of S and the rest of coalitions in P; hence, $\varphi_i(w_{S,P})$ also satisfies this property. The two previous properties are equivalent to the strong symmetry axiom for the basic games $w_{S,P}$. Linearity implies that the value φ satisfies the strong symmetry axiom for all games if and only if it satisfies the axiom for the games $w_{S,P}$ for $(S,P) \in ECL$. Hence, φ satisfies the strong symmetry axiom in $\mathcal{G}^N$.

Now assume φ satisfies linearity and strong symmetry. To show it can be constructed via the AA we look first at the values φ assume over games in G^N, and define a value for CFF games ψ by $\psi(\hat{v}) = \varphi(\hat{v})$ for all $\hat{v} \in G^N$. The value for CFF games ψ generates the $\theta_i(S)$ which are the payoff ψ assigns to player i in the CFF game where coalition S receives 1 and all other coalitions receive 0. These are the same for all $i \in S$ and all $i \in N \backslash S$. To recover the candidate weights $\alpha(S, P)$ remember that $\varphi_i(w_{S,P})$ should be equal to $\alpha(S, P)\theta_i(S) + \frac{1}{n}$. Hence, whenever $\theta_i(S) \neq 0$, we obtain $\alpha(S, P) = \frac{\varphi_i(w_{S,P}) - \frac{1}{n}}{\theta_i(S)}$. When $\theta_i(S) = 0$, we take any $(\alpha(S, P))_{P \in \mathcal{P}_S}$ that depend only on the size distribution of coalitions in P and such that $\sum_{P \in \mathcal{P}_S} \alpha(S, P) = 1$.

Since $\theta_i(S)$ are the same for all players $i \in S$, as well as for players $i \in N \backslash S$, and they depend only on the size of S, strong symmetry implies first that the weights $\alpha(S, P)$ are well-defined (that is, independent of i when $\theta_i(S) \neq 0$) and secondly that they depend only on the sizes of S and the rest of coalitions in P. Also, since φ and hence ψ are linear, we obtain $\sum_{P \in \mathcal{P}_S} \alpha(S, P) = 1$ for all $S \subseteq N$. To conclude, we claim the value φ can be constructed through the AA using the value ψ and the weights $\alpha(S, P)$. This claim is valid for any basic game $w_{S,P}$ by construction and by linearity extends to any game in $\mathcal{G}^N$. ■

Proof. of Theorem 3. Given Theorem 1, we only need to prove the nullifying player axiom in both senses of the theorem.

a) Suppose that φ can be constructed through the AA using ψ^{ED}. Let player $i \in N$ be a nullifying player in $v \in \mathcal{G}^N$. By construction, $\varphi_i(v) = \psi_i(\tilde{v})$.

If i is a nullifying player in $v \in \mathcal{G}^N$, then $v(S, P) = 0$ for all $S \subseteq N$ with $i \in S$. Therefore, $\tilde{v}(S) = \sum_{P \ni S, P \in \mathcal{P}} \alpha(S, P)v(S, P) = 0$ for all $S \subseteq N$ with $i \in S$. Consequently, player i is also a nullifying player in the average game $\tilde{v}$ and $\psi_i^{ED}(\tilde{v}) = 0$. Hence, $\varphi_i(v) = 0$. That is, φ satisfies the nullifying player axiom.

b) Suppose now that φ satisfies linearity, strong symmetry, and nullifying player. φ can be obtained through the AA using a value for CFF games ψ and a system of weights $(\alpha(S, P))_{(S,P) \in ECL}$. We note that a nullifying player i in $\hat{v} \in G^N$ is also a nullifying player if we consider $\hat{v}$ as a game of $\mathcal{G}^N$. Given that φ coincides with ψ in games without externalities, $\psi_i(\hat{v}) = \varphi_i(\hat{v}) = 0$, that is, ψ satisfies the nullifying player axiom. ■

Proof. of Theorem 4. a) Suppose that $\varphi^{ED}(v) = \psi^{ED}(\tilde{v})$. If $v \in \mathcal{G}^N$ is non-negative, $v(S, P) \geq 0$ for all $(S, P) \subseteq ECL$. Therefore, $\tilde{v}(S) \geq 0$ for all $S \subseteq N$ and any $(\alpha(S, P))_{(S,P) \in ECL}$. Given that ψ^{ED} satisfies the non-negative axiom, $\psi_i^{ED}(\tilde{v}) \geq 0$ for all $i \in N$; hence, $\varphi_i(v) \geq 0$ for all $i \in N$. That is, φ satisfies the non-negative axiom.

b) Proceeding as in part b) of the proof of Theorem 3, given that φ coincides with ψ in games without externalities, ψ satisfies the non-negative axiom. ■

Proof. of Theorem 5. a) Suppose that φ is constructed through the AA with weights that satisfy (2) using ψ^{ES}. We prove that φ satisfies the dummifying player axiom.

For any player $i \in N$, by the AA,

$$\begin{aligned}\varphi_i(v) &= \psi_i^{ES}(\tilde{v}) = \tilde{v}(\{i\}) + \frac{1}{n}\left(\tilde{v}(N) - \sum_{j\in N}\tilde{v}(\{j\})\right) \\ &= \sum_{P\in\mathcal{P}_{\{i\}}}\alpha(\{i\},P)\,v(\{i\},P) + \frac{1}{n}\left(v(N,N) - \sum_{\substack{j\in N\\ P\in\mathcal{P}_{\{j\}}}}\alpha(\{j\},P)\,v(\{j\},P)\right) \quad (4)\end{aligned}$$

Using (2), we can write $\varphi_i(v)$ as

$$\varphi_i(v) = v(\{i\},[N]) + \frac{1}{n}\left(v(N,N) - \sum_{j\in N} v(\{j\},[N])\right). \quad (5)$$

Suppose now that $i \in N$ is a dummifying player in $v \in \mathcal{G}^N$, which implies that $v(N,N) = \sum_{j\in N} v(\{j\},[N])$. Then, $\varphi_i(v) = v(\{i\},[N])$ and φ satisfies the dummifying player axiom.

b) Suppose that φ satisfies linearity, strong symmetry, and dummifying player. φ can be constructed through the AA using a value ψ that satisfies linearity and symmetry. We prove that ψ also satisfies the dummifying player axiom. ψ coincides with φ on G^N. Take a game $\hat{v} \in G^N$. A dummifying player j in $\hat{v}$ is also a dummifying player in $\hat{v} \in \mathcal{G}^N$. Hence, $\psi_j(\hat{v}) = \varphi_j(\hat{v}) = v(\{j\},[N]) = \hat{v}(\{j\})$. Therefore, ψ satisfies the dummifying player axiom.

To show the property that must be satisfied by the weights, we construct the following PFF game w^{ki}, where $k \neq i$: $w^{ki}(k,[N]) = w^{ki}(S,P) = 1$ for any $S \supseteq \{k,i\}$ and $w^{ki}(S,P) = 0$ otherwise. Player i is a dummifying player in this game and hence

$$\varphi_i(w^{ki}) = w^{ki}(\{i\},[N]) = 0.$$

In the corresponding average game $\tilde{w}^{ki}$ we have $\tilde{w}^{ki}(\{k\}) = \alpha(1,(1,...,1))$; $\tilde{w}^{ki}(\{j\}) = 0$ for $j \neq k$ and $\tilde{w}^{ki}(N) = 1$. Since $\psi = \psi^{ES}$ we obtain $\psi_i(\tilde{w}^{ki}) = 0 + \frac{1}{n}(1 - \alpha(1,(1,...,1)))$.

Since $\varphi_i(w^{ki}) = \psi_i(\tilde{w}^{ki})$, we get $\alpha(1,(1,...,1)) = 1$. Given that the weights associated with all embedded coalitions of the form $(\{1\}, P)$, for $P \in \mathcal{P}_{\{1\}}$, are nonnegative and must sum to 1, (2) holds. ■

Proof. of Theorem 6. a) If φ is constructed through the AA with weights that satisfy (2) using ψ^{ES}, then (5) holds. Moreover, if $i \in N$ is a dummy player in $v \in \mathcal{G}^N$, then $v(\{i\},[N]) = 0$ and

$$\varphi_i(v) = \frac{1}{n}\left(v(N,N) - \sum_{j\in N} v(\{j\},[N])\right). \tag{6}$$

Therefore, φ satisfies the ES dummy player axiom.

We note that in a 3-players PFF game if one of the players, say player 1, is a dummy player, then this game has no externalities because $v(\{1\},[N]) = v(\{1\},(\{1\},\{2,3\})) = 0$, $v(\{2\},[N]) = v(\{2\},(\{1,3\},\{2\}))$ and similarly for player 3. Hence, equation (6) directly follows from (4) for any weights.

b) Suppose that φ satisfies linearity, strong symmetry, and ES dummy player. φ can be constructed through the AA using a value ψ that satisfies linearity and symmetry. We now prove that ψ satisfies the ES dummy player axiom. ψ coincides with φ on G^N. Take a game $\hat{v} \in G^N$. A dummy player j in $\hat{v}$ is a dummy player in $\hat{v}$ viewed as a PFF game as well and hence $\psi_j(\hat{v}) = \varphi_j(\hat{v}) = \frac{1}{n}\left(\hat{v}(N,N) - \sum_{j\in N}\hat{v}(\{j\},[N])\right) = \frac{1}{n}\left(\hat{v}(N) - \sum_{j\in N}\hat{v}(\{j\})\right)$, given that $\hat{v}(S,P) \equiv \hat{v}(S)$ for all $(S,P) \in ECL$. Therefore, ψ satisfies the ES dummy player axiom, which implies $\psi = \psi^{ES}$.

To determine the weights we construct the following PFF game v^{ki} defined by $v^{ki}(\{k\},[N]) = v^{ki}(\{k,i\},([N\backslash\{k,i\}],\{k,i\})) = v(\{k\},([N\backslash\{k,i,j\}],\{k\},\{i,j\})) = 1$ for $j \neq k,i$ and $v^{ki}(S,P) = 0$ otherwise. Player i is a dummy player in v^{ki} and hence

$$\varphi_i(v^{ki}) = \frac{1}{n}\left(v^{ki}(N,N) - \sum_{j\in N} v^{ki}(\{j\},[N])\right) = -\frac{1}{n}.$$

In the corresponding average game $\tilde{v}^{ki}$ we have $\tilde{v}^{ki}(\{k\}) = \alpha(1,(1,...,1)) + (n-2)\alpha(1,(2,1,...,1))$; $\tilde{v}^{ki}(\{j\}) = 0$ for $j \neq k$ and $\tilde{v}^{ki}(N) = 0$. Since $\psi = \psi^{ES}$ we obtain $\psi_i(\tilde{v}^{ki}) = -\frac{1}{n}\left(\alpha(1,(1,...,1)) + (n-2)\alpha(1,(2,1,...,1))\right)$. Using that $\varphi_i(v^{ki}) = \psi_i(\tilde{v}^{ki})$, we get $\alpha(1,(1,...,1)) + (n-2)\alpha(1,(2,1,...,1)) = 1$. Moreover, the weights associated with all embedded coalitions of the form $(\{1\}, P)$ where $P \in \mathcal{P}_{\{1\}}$ are nonnegative and must sum to 1 we obtain $\alpha(1,(1,...,1))+\frac{1}{2}(n-1)(n-2)\alpha(1,(2,1,...,1))\leq 1$. Hence, either $\alpha(1,(2,1,...,1))=0$ or $\frac{1}{2}(n-1)(n-2)\leq(n-2)$. However, the last inequality cannot happens for $n>3$. Therefore, $\alpha(1,(2,1,...,1))=0$ and $\alpha(1,(1,...,1))=1$. This implies that (2) holds. ■

Proof. of Theorem 7. a) Suppose $n>3$ and that φ can be constructed through the AA with weights that satisfy (3) using ψ^C. By Theorem 1, φ satisfies linearity and strong symmetry. We now prove that φ satisfies the neutral dummy player axiom.

For any player $i\in N$, by the AA,

$$\begin{aligned}\varphi_i(v) &= \psi_i^C(\tilde{v})=\frac{1}{2}\psi_i^{Sh}(\tilde{v})+\frac{1}{2n}\left(\tilde{v}(N)-\sum_{j\in N}\tilde{v}(\{j\})\right)\\ &= \frac{1}{2}\psi_i^{Sh}(\tilde{v})+\frac{1}{2n}\left(v(N,N)-\sum_{\substack{j\in N\\ P\in\mathcal{P}_{\{j\}}}}\alpha(\{j\},P)\,v(\{j\},P)\right). \qquad (7)\end{aligned}$$

Suppose now that player $i\in N$ is a dummy player in $v\in\mathcal{G}^N$. Given the weights in (3), i is a dummy player in $\tilde{v}$ as well and hence $\psi_i^{Sh}(\tilde{v})=0$. Moreover, we can write (7) as

$$\varphi_i(v)=\frac{1}{2n}\left(v(N,N)-\sum_{j\in N}\alpha(\{j\},[N])\,v(\{j\},[N])\right). \qquad (8)$$

Therefore, φ satisfies the neutral dummy player axiom.

b) Suppose that φ satisfies linearity, strong symmetry, and neutral dummy player. We know that φ can be constructed through the AA using a value ψ that satisfies linearity and symmetry. We now prove that ψ satisfies neutral dummy player. Take a game $\hat{v}$. A dummy player i in $\hat{v}$ is a dummy player in $\hat{v}$ viewed as a PFF game as well and hence

$$\psi_i(\hat{v})=\varphi_i(\hat{v})=\frac{1}{2n}\left(\hat{v}(N,N)-\sum_{j\in N}\hat{v}(\{j\},[N])\right)=\frac{1}{2n}\left(\hat{v}(N)-\sum_{j\in N}\hat{v}(\{j\})\right),$$

given that $\hat{v}(S,P)\equiv\hat{v}(S)$ for all $(S,P)\in ECL$. Therefore, ψ satisfies the neutral dummy player axiom, which implies $\psi=\psi^C$.

We now prove that equation (3) holds if $n>3$. First, we define the game v^k as follows: $v^k(\{k\},[N])=v^k(\{k,i\},([N\backslash\{k,i\}],\{k,i\}))=v^k(\{k\},([N\backslash\{i,j\}],\{i,j\}))=0$ for any $j\in N\backslash\{k,i\}$; $v^k(S,P)=1$ for any $S\ni k$ and any $P\in\mathcal{P}_S$ different from $[N]$ and

$([N\backslash\{i,j\}],\{i,j\})$, $([N\backslash\{k,i\}],\{k,i\})$; $v^k(S,P)=0$ if $k\notin S$. Player i is a dummy player in v^k. Therefore,

$$\varphi_i(v^k)=\frac{1}{2n}\left(v^k(N,N)-\sum_{j\in N}v^k(\{j\},[N])\right)=\frac{1}{2n}.$$

We now proceed to construct the corresponding average game $\tilde{v}^k$. Note that $\tilde{v}^k(\{k\})$ is the sum of weights of all partitions of the $n-1$ players other than k for which v^k assumes the value of 1. Since all weights sum up to 1 we obtain $\tilde{v}^k(\{k\})=1-\alpha(1,(1,...,1))-(n-2)\alpha(1,(2,1,...,1))$. Similarly $\tilde{v}^k(\{k,i\})$ is the sum of weights of all partitions of the $n-2$ players other than k and i for which v assumes the value of 1. Then, $\tilde{v}^k(\{k,i\})=1-\alpha(2,(2,1,...,1))$. For all other embedded coalitions (S,P), $\tilde{v}^k$ assumes the values 1 when $k\in S$ and 0 when $k\notin S$. Player i's marginal contribution to any coalition other than $\{k\}$ is zero whereas i's marginal contribution to $\{k\}$ is $\alpha(1,(1,...,1))+(n-2)\alpha(1,(2,1,...,1))-\alpha(2,(2,1,...,1))$.

We know that $\psi_i^C(\tilde{v}^k)=\frac{1}{2}\psi_i^{Sh}(\tilde{v}^k)+\frac{1}{2n}(1-0)$. Since φ can be constructed by the AA using ψ^C, $\frac{1}{2n}=\varphi_i(v^k)=\psi_i^C(\tilde{v}^k)=\frac{1}{2}\psi_i^{Sh}(\tilde{v}^k)+\frac{1}{2n}$. Therefore, it must be the case that $\psi_i^{Sh}(\tilde{v}^k)=0$. This implies

$$\alpha(1,(1,...,1))+(n-2)\,\alpha(1,(2,1,...,1))-\alpha(2,(2,1,...,1))=0. \tag{9}$$

Second, take $S\subset N$ and $i\in N\backslash S$. We define the game w^S as follows: $w^S(S,([N\backslash S],S))=w^S(S\cup\{i\},([N\backslash(S\cup\{i\})],S\cup\{i\}))=w^S(S,([N\backslash(S\cup\{i,j\})],S,\{i,j\}))=1$ for any $j\in N\backslash(S\cup\{i\})$; $w^S(R,P)=0$ for any other (R,P). Player i is a dummy player in game w^S.

Suppose $s>1$. Then,

$$\varphi_i(w^S)=\frac{1}{2n}\left(w^S(N,N)-\sum_{j\in N}w^S(\{j\},[N])\right)=0.$$

We again proceed to construct the corresponding average game. $\tilde{w}^S(S)=\alpha(s,(s,1,...,1))+(n-s-1)\alpha(s,(s,2,1,...,1))$ and $\tilde{w}^S(S\cup\{i\})=\alpha(s+1,(s+1,1,...,1))$. Player i's marginal contribution to any coalition other than S is zero whereas i's marginal contribution to S is $\alpha(s+1,(s+1,1,...,1))-(\alpha(s,(s,1,...,1))+(n-s-1)\alpha(s,(s,2,1,...,1)))$. Given that $\psi_i^C(\tilde{w}^S) = \frac{1}{2}\psi_i^{Sh}(\tilde{w}^S) + \frac{1}{2n}(0-0)$ and $\psi_i^C(\tilde{w}^S) = \varphi_i(\tilde{w}^S)$, it must be the case that $\psi_i^{Sh}(\tilde{w}^S) = 0$, which implies

$$\alpha\left(s+1,(s+1,1,...,1)\right) = \alpha\left(s,(s,1,...,1)\right) + (n-s-1)\,\alpha\left(s,(s,2,1,...,1)\right). \qquad (10)$$

Now suppose $s=1$ and $S=\{1\}$. Then,

$$\varphi_i(w^{\{1\}}) = \frac{1}{2n}\left(w^{\{1\}}\left(N,N\right) - \sum_{j\in N} w^{\{1\}}\left(\{j\},[N]\right)\right) = -\frac{1}{2n}.$$

Going again through the corresponding average game we obtain. $\tilde{w}^{\{1\}}(\{1\}) = \alpha(1,(1,...,1)) + (n-2)\alpha(1,(2,1,...,1))$ and $\tilde{w}^{\{1\}}(\{1,i\}) = \alpha(2,(2,1,...,1))$. Player i's marginal contribution to any coalition other than $\{1\}$ is zero whereas i's marginal contribution to $\{1\}$ is $\alpha(2,(2,1,...,1)) - (\alpha(1,(1,...,1)) + (n-2)\alpha(1,(2,1,...,1)))$. Given that $\psi_i^C\left(\tilde{w}^{\{1\}}\right) = \frac{1}{2}\varphi_i^{Sh}(\tilde{w}^{\{1\}}) + \frac{1}{2n}(-\alpha(1,(1,...,1)) - (n-2)\alpha(1,(2,1,...,1)) = \varphi_i(w^{\{1\}})$, it must be the case that

$$\frac{1}{2}\left(\frac{(n-2)!}{n!}(\alpha(2,(2,1,...,1)) - \alpha(1,(1,...,1)) - (n-2)\alpha(1,(2,1,...,1)\right)$$
$$+\frac{1}{2n}\left(-\alpha(1,(1,...,1)) - (n-2)\alpha(1,(2,1,...,1)\right) = -\frac{1}{2n}.$$

Easy calculations lead to

$$\alpha\left(2,(2,1,...,1)\right) + (n-1) = n\alpha\left(1,(1,...,1)\right) + n\left(n-2\right)\alpha\left(1,(2,1,...,1)\right). \qquad (11)$$

From (9) and (11) we obtain

$$\alpha\left(2,(2,1,...,1)\right) + (n-1) = n\alpha(2,(2,1,...,1)) - n\left(n-2\right)\alpha(1,(2,1,...,1)) + n\left(n-2\right)\alpha\left(1,(2,1,...,1)\right),$$

that is, $\alpha(2,(2,1,...,1)) = 1$. This implies that $\alpha(2,p) = 0$ for any partition p different from $(2,1,...,1)$. Also, using (10) recursively, we have $\alpha\left(3,(3,1,...,1)\right) = \alpha\left(2,(2,1,...,1)\right) + (n-3)\,\alpha\left(2,(2,2,1,...,1)\right) = 1$. Thus,

$$\alpha\left(r,(r,1,...,1)\right) = 1$$

for any $r > 1$. Also, $\alpha(2,(2,1,...,1)) = 1$ together with (9) imply

$$\alpha(1,(1,...,1)) + (n-2)\,\alpha(1,(2,1,...,1)) = 1.$$

Moreover

$$\begin{aligned} 1 &= \sum_p \alpha(1,p) \geq \alpha(1,(1,...,1)) + \frac{1}{2}(n-1)(n-2)\,\alpha(1,(2,1,...,1)) \\ &= 1-(n-2)\,\alpha(1,(2,1,...,1)) + \frac{1}{2}(n-1)(n-2)\,\alpha(1,(2,1,...,1)) \\ &= 1+\frac{(n-2)}{2}(n-3)\,\alpha(1,(2,1,...,1)) \end{aligned}$$

Therefore, if $n > 3$ it is necessarily the case that $\alpha(1,(2,1,...,1)) = 0$ and thus $\alpha(1,(1,...,1)) = 0$ as well and equation (3) holds. (If $n = 3$ then $\alpha(1,(1,1,1)) + \alpha(1,(2,1)) = 1$ always.)

■

Proof. of Theorem 8. a) Suppose that φ can be constructed through the AA with weights that satisfy (1) for all $i \in N\backslash S$ and for all $((S \cup \{i\}), P) \in ECL$ using $\psi^{\lambda E}$. By Theorem 1, φ satisfies linearity and strong symmetry. We now prove that φ satisfies the λ-egalitarian dummy player axiom.

For any player $i \in N$, by the AA,

$$\varphi_i(v) = \psi_i^{\lambda ED}(\tilde{v}) = (1-\lambda)\psi_i^{Sh}(\tilde{v}) + \frac{\lambda}{n}\tilde{v}(N). \tag{12}$$

Suppose now that player $i \in N$ is a dummy player in $v \in \mathcal{G}^N$. Then, given (1), i is a dummy player in $\tilde{v}$ as well (see proof of Theorem 1 in MPW, 2007) and hence $\psi_i^{Sh}(\tilde{v}) = 0$. Since $\tilde{v}(N) = v(N,N)$, (12) implies φ satisfies the λ-egalitarian dummy player axiom.

b) Suppose that φ satisfies linearity, strong symmetry, and λ-egalitarian dummy player. We know that φ can be constructed through the AA using a linear and symmetric value ψ and we prove that ψ satisfies λ-egalitarian dummy player. Take a game $\hat{v} \in G^N$. A dummy player j in $\hat{v}$ is a dummy player in $\hat{v}$ viewed as a PFF game as well and hence

$$\psi_j(\hat{v}) = \varphi_j(\hat{v}) = \frac{\lambda}{n}\hat{v}(N,N).$$

Therefore, ψ satisfies the λ-egalitarian dummy player axiom, which implies $\psi = \psi^{\lambda ED}$.

We now prove that (1) holds for all $i \in N\backslash S$ and for all $((S \cup \{i\}), P) \in ECL$. We define the game $w^{(S\cup\{i\}),P}$ for $s < n-1$, where $i \notin S$ by $w^{(S\cup\{i\}),P}(S',P') = 1$ for $(S',P') = ((S\cup\{i\}),P)$ and for all $(S',P') = (S,(P\backslash((S\cup\{i\}),R))\cup(R\cup\{i\},S))$ where $R \in P\backslash(S\cup\{i\})$, otherwise $w^{(S\cup\{i\}),P}(S',P') = 0$. Player i is a dummy player in the game

$w^{(S\cup\{i\}),P}$ and hence,

$$\varphi_i(w^{(S\cup\{i\}),P}) = \frac{\lambda}{n} w^{(S\cup\{i\}),P}(N,N) = 0.$$

We construct the corresponding average game. $\tilde{w}^{(S\cup\{i\}),P}(S\cup\{i\}) = \alpha((S\cup\{i\}),P)$, $\tilde{w}^{(S\cup\{i\}),P}(S) = \sum_{R\in P\setminus(S\cup\{i\})} \alpha(S,(P\setminus(R,(S\cup\{i\})))\cup((R\cup\{i\}),S))$ and for all coalitions $T \neq (S\cup\{i\}), S$, $\tilde{w}^{(S\cup\{i\}),P}(T) = 0$. For the value of player i in the average game to be zero we must have its Shapley value equal to zero (recall that $\tilde{w}^{S,P}(N) = 0$).

Player i's marginal contribution to any coalition other than S is zero whereas i's marginal contribution to S is $\alpha((S\cup\{i\}),P) - \sum_{R\in P\setminus S} \alpha(S,(P\setminus(R,S))\cup(R\cup\{i\},S))$. Hence, it must be the case that $\alpha((S\cup\{i\}),P) = \sum_{R\in P\setminus S} \alpha(S,(P\setminus(R,S))\cup((R\cup\{i\}),S))$.

Finally, we note that (1) trivially holds if $s = n-1$. ■

Proof. of Theorem 9. a) Suppose that φ is constructed through the AA using ψ from the LS family. We prove that φ satisfies the inessential game and coalitional monotonicity axioms.

First, if $v \in \mathcal{G}^N$ is additive, then

$$\begin{aligned}
\tilde{v}(S) &= \sum_{P\in\mathcal{P}_S} \alpha(S,P)v(S,P) = \sum_{P\in\mathcal{P}_S} \alpha(S,P) \sum_{i\in S} v(\{i\},(P\setminus S,[S])) \\
&= \sum_{i\in S} v(\{i\},(P\setminus S,[S])) \sum_{P\in\mathcal{P}_S} \alpha(S,P) = \sum_{i\in S} v(\{i\},(P\setminus S,[S])) \\
&= \sum_{P\in\mathcal{P}_{\{i\}}} \alpha(\{i\},P) \sum_{i\in S} v(\{i\},(P\setminus S,[S])) = \sum_{i\in S} \sum_{P\in\mathcal{P}_{\{i\}}} \alpha(\{i\},P)v(\{i\},P) = \sum_{i\in S} \tilde{v}(\{i\}).
\end{aligned}$$

Hence, $\tilde{v}$ is an additive game in G^N and $\varphi_i(v) = \psi_i(\tilde{v}) = \tilde{v}(\{i\}) = \sum_{P\in\mathcal{P}_{\{i\}}} \alpha(\{i\},P)v(\{i\},P) = \sum_{P\in\mathcal{P}_{\{i\}}} \alpha(\{i\},P)v(\{i\},[N]) = v(\{i\},[N])$ for all $i \in N$. This implies that φ satisfies the inessential game axiom.

Second, consider v and w such that $v(S,P) > w(S,P)$ for some $P \in \mathcal{P}_S$ and $v(R,Q) = w(R,Q)$ for $(R,Q) \neq (S,P)$. $\tilde{v}(R) = \tilde{w}(R)$ for all $R \neq S$ and either $\tilde{v}(S) = \tilde{w}(S)$ (if $\alpha(S,P) = 0$) or $\tilde{v}(S) > \tilde{w}(S)$. If $\tilde{v}(S) = \tilde{w}(S)$, then $\varphi(v) = \psi(\tilde{v}) = \psi(\tilde{w}) = \varphi(w)$. If $\tilde{v}(S) > \tilde{w}(S)$, then given that ψ satisfies coalitional monotonicity, $\varphi(v) = \psi(\tilde{v}) \geq \psi(\tilde{w}) = \varphi(w)$. In both cases, $\varphi(v) \geq \varphi(w)$ and φ also satisfies the coalitional monotonicity axiom.

b) Suppose that φ satisfies linearity, strong symmetry, inessential game, and coalitional monotonicity. φ can be constructed through the AA using a linear and symmetric value ψ. We now prove that ψ satisfies inessential game and coalitional monotonicity.

Suppose that ψ does not satisfy the coalitional monotonicity axiom. Then, there exist games $\hat{v}, \hat{w} \in G^N$, coalition $S \in N$ and player $i \in S$ such $\hat{v}(S) > \hat{w}(S)$, $\hat{v}(R) = \hat{w}(R)$ for all $R \neq S$, and $\psi_i(\hat{v}) < \psi_i(\hat{w})$. Consider now $\hat{v}$ and $\hat{w}$ as games without externalities in $\mathcal{G}^N$. They satisfy $\hat{v}(S, P) \geq \hat{w}(S, P)$ for all $P \in \mathcal{P}_S$, and $\hat{v}(R, Q) = \hat{w}(R, Q)$ for $R \neq S$, $Q \in \mathcal{P}_R$. Given that φ satisfies coalitional monotonicity, $\varphi_i(\hat{v}) \geq \varphi_i(\hat{w})$. However, this contradicts $\varphi_i(\hat{v}) = \psi_i(\hat{v}) < \psi_i(\hat{w}) = \varphi_i(\hat{w})$.

Suppose that ψ does not satisfy the inessential game axiom. Then, there exist a game $\hat{v} \in G^N$ and a player $i \in N$ such $\hat{v}(S) = \sum_{j \in S} \hat{v}(\{j\})$ and $\psi_i(\hat{v}) \neq \hat{v}(\{i\})$. Consider now $\hat{v}$ as a game without externalities in $\mathcal{G}^N$. $\hat{v}$ is also an additive game in $\mathcal{G}^N$ because $\hat{v}(S, P) = \hat{v}(S) = \sum_{j \in S} \hat{v}(\{j\}) = \sum_{j \in S} \hat{v}(\{j\}, [N])$ for all $(S, P) \in ECL$. Given that φ satisfies the inessential game axiom, $\varphi_i(\hat{v}) = \hat{v}(\{i\}, [N])$. However, this contradicts $\psi_i(\hat{v}) \neq \hat{v}(\{i\})$ since $\psi_i(\hat{v}) = \varphi_i(\hat{v}) = \hat{v}(\{i\}, [N]) = \hat{v}(\{i\})$. ■

Proof. of Proposition 1. Suppose that φ satisfies linearity, strong symmetry, and dummy player. Let v be an additive game, that is, $v(S, P) = \sum_{i \in S} v(\{i\}, [N])$ for all $(S, P) \in ECL$. We can write $v = \sum_{j \in N} v^j$, where v^j is defined by

$$\begin{aligned} v^j(S, P) &= v(\{j\}, [N]) \text{ if } j \in S \\ &= 0 \text{ if } j \notin S. \end{aligned}$$

Any player $k \neq j$ is a dummy player in v^j. Then, $\varphi_k(v^j) = 0$ for any $k \neq j$ and $\varphi_j(v^j) = v(N, N) = v(\{j\}, [N])$. Therefore,

$$\varphi_i(v) = \sum_{j \in N} \varphi_i\left(v^j\right) = v(\{i\}, [N])$$

for any $i \in N$. Thus, the value φ satisfies the inessential game axiom.

To prove that φ also satisfies coalitional monotonicity, let v and w be such that $v(S, P) > w(S, P)$ and $v(R, Q) = w(R, Q)$ for $(R, Q) \neq (S, P)$. Theorem 2 ensures that φ can be constructed through the AA using some vector of weights α. Let $\tilde{v}$ and $\tilde{w}$ be the average games that use the weights α. It is immediate that $\tilde{v}(R) = \tilde{w}(R)$ for all $R \neq S$ and $\tilde{v}(S) \geq \tilde{w}(S)$. Since ψ^{Sh} satisfies coalitional monotonicity in CFF games,

$$\varphi_i(v) = \psi_i^{Sh}(\tilde{v}) \geq \psi_i^{Sh}(\tilde{w}) = \varphi_i(w)$$

for any $i \in S$. Therefore, φ satisfies the axiom of coalitional monotonicity. ■

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